Equilibria in Continuous Time Preemption Games with Spectrally-Negative Markovian Payoffs*

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Abstract

This paper studies timing games in continuous time where payoffs are stochastic, strongly Markovian, and spectrally negative. The main interest is in characterizing equilibria where players preempt each other along almost every sample path as opposed to equilibria where one of the players acts as if she were the (exogenously determined) leader in a Stackelberg game. It is found that the existence of such preemptive and Stackelberg equilibria depends crucially on whether there is a coordination mechanism that allows for rent equalization or not. Such a coordination mechanism is introduced and embedded in the timing game.

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1 Introduction

In many competitive timing situations the first mover has an advantage: the firm that first adopts a new technology, the first developer of a new real-estate opportunity, etc. However, if the leader role is not exogenously determined, then the competition to become the leader may erode that first mover advantage. Indeed, the standard prediction from the literature on timing games with a first mover advantage is that preemption equalizes the expected payoffs of the first and second mover. This point has been made ever since such early contributions as, for example, Posner (1975).

Preemption is often analyzed in a continuous time framework. This can lead to a coordination problem, because in continuous time it is impossible to distinguish between simultaneous action and immediate reaction.[1] Fudenberg and Tirole (1985) solve this problem by using a technique from optimal control which allows them to show that in symmetric two-player deterministic timing games with a first mover advantage there always exists an equilibrium in which rents are equalized. Unfortunately, their equilibrium concept is rather complicated as it involves players to choose a distribution function describing the probability with which they act before any point in time, as well as an "intensity over an interval of atoms" when players wish to act at the same time. It is this device that allows the derivation of a preemption equilibrium where (i) players do not act simultaneously when this is not optimal (an outcome they refer to as a "coordination failure") and (ii) symmetric players each act first with probability 1/2 at the preemption point.

In most real-life preemptive situations the future is not known with certainty and, therefore, a deterministic timing game may not be the best modeling tool. Using techniques from optimal stopping theory, several papers have studied timing games in which players' payoffs are subject to random shocks.[2] This introduces an "option value of waiting" into the payoffs which, in general, delays stopping. The addition of uncertainty complicates the game theoretic analysis even further. As in the deterministic case this is mainly due to the difficulty of solving the coordination problem that arises when for each player it is a best response to stop, but only if just one player actually succeeds. In the literature this is often solved by making fairly ad-hoc assumptions based on Fudenberg and Tirole (1985). It is not at all clear, however, that this is appropriate. In addition, this approach could not deal with

[1] In fact, this problem also arises in discrete time. There, however, it is fairly straightforward to model time for coordination, see also Section 4.

[2] See, for example, Boyer et al. (2004), Pawlina and Kort (2006, 2010), Bouis et al. (2009), Roques and Savva (2009), Mason and Weeds (2010), and Thijssen (2010). Chevalier-Roignant and Trigeorgis (2011) give an overview of the recent literature.

asymmetries. Thijssen et al. (2002) extend the Fudenberg and Tirole (1985) concepts in an appropriate way to a stochastic setting for symmetric games. They show that, qualitatively, the subgame perfect equilibria are not changed by the introduction of uncertainty. It is not clear, however, how restrictive their assumptions are. In addition, the level of technicality required to derive the results may hamper their applicability and their extension to asymmetric games.

In this paper a different approach to the problem of preemption in a real options model is taken. The contribution is three-fold. First, the analysis presented here separates, as much as possible, the timing and coordination issues involved in preemption models. This makes it easier to prove equilibrium existence and also makes it clearer how the different assumptions needed to guarantee existence of equilibrium interact. The simplicity of the arguments is based on an exploitation of the strong Markovian nature of the underlying stochastic process, which allows one to take the range of the process as the state space, rather than time itself. As a result, this paper presents a more complete picture of all equilibria that can exist in preemption games. In particular, rather than using subgame perfect equilibrium, I resort to Markovian equilibria. The equilibria that are derived in this paper rely on the condition that players can act exactly at the point in time in which they wish to do so. In our set-up this means that the underlying stochastic process can not exhibit upward jumps. In the case of upward jumps, namely, it is possible that players "overshoot". Still, this assumption allows for many more processes than are typically used in the existing literature.

Second, the paper explicitly models the Fudenberg and Tirole (1985) "interval of atoms" as a mixed strategy in a (possibly countably often) repeated "grab-the-dollar" game.[3] This is done in a set-up introduced by Dutta and Rustichini (1995), which allows for instantaneous reactions in a continuous time setting. This makes the modeling more intuitive and shows how coordination can be achieved as the outcome of a non-cooperative procedure.

Finally, this paper starts from fairly general assumptions on primitives, which enables one to see how economic primitives interact with equilibrium existence. This also allows for a comprehensive analysis of asymmetric games and a large class of underlying stochastic processes. The existing literature almost exclusively relies on geometric Brownian motion. This set-up in this paper makes it possible to analyse other processes such as mean-reverting diffusions and Lévy processes with downward jumps. In fact, any strong Markovian process without upward jumps is allowed.

The equilibria that are found exhibit the same qualitative properties, regardless

[3]Fudenberg and Tirole (1985) themselves appeal to the "grab-the-dollar" game to underpin their equilibrium concept, even though it is not formally embedded in the game.

of the underlying stochastic process. It turns out that the most important determinant in what kind of equilibrium arises, is whether there is time for a coordination game to equalize (*ex ante* expected) rents between the first and the second mover. If that is impossible, then only *Stackelberg equilibria* are possible. These are equilibria where one of the players moves at the same point in time at which she would have moved if she had (exogenously) been assigned to be the (Stackelberg) leader. If rent equalization is possible, then *preemptive equilibria* arise. These are equilibria where one of the players moves at a sub-optimal time in order to prevent being preempted. Interestingly, this dichotomy implies that competitive pressure between the players is stronger only if the stochastic evolution of the environment is slow enough to allow players to coordinate. In fast-moving environments preemptive pressure may not exist to the same extent. This happens because if there is no time for coordination no player will be willing to take the risk to move, because if the other player does so as well at the same time it is guaranteed that a so-called "coordination failure" takes place. This is a situation where both players end up moving while it is optimal for only one player to do so. When coordination time is available players have an instrument to balance the first mover advantage against the loss that a coordination failure would entail.

The paper is organized as follows. The basic ingredients of the model are described in Section 2. Section 3 introduces the main strategy and equilibrium concepts, as well as the equilibrium results regarding existence of different types of equilibria. The arguments in Section 3 are kept as simple as possible by assuming that the coordination problem is solved exogenously. A non-cooperative defense of why the assumption of rent-equalization is reasonable is given in Section 4. Implications of the theory for predictions on preemptive behavior under different stochastic processes are presented in Section 5. Numerical examples show that, both for spectrally negative exponential jump-diffusions and exponential mean-reverting diffusions the preemption region is increasing in volatility. This indicates that preemptive behavior can be expected to be observed more often in situations with higher levels of uncertainty. Some concluding remarks are made in Section 6.

2 The Basic Set-Up

Consider a situation where two players $i \in \{1, 2\}$ have to decide on a stopping time over an infinite time horizon. Their payoffs are influenced by a state-variable which takes values in $E = (a, b) \subseteq \mathbb{R}$. Let $\bar{E}$ denote the closure of E (in the standard topology on $\mathbb{R}$). For each $y \in E$, the state variable follows a strong Markovian càdlàg (right-continuous with left-limits) semimartingale $(Y_t)_{t \geq 0}$ on a probability space $(\Omega, \mathscr{F}, P_y)$, endowed with a filtration $(\mathscr{F}_t)_{t\geq 0}$, with $Y_0 = y$, P_y-a.s. The process $(Y_t)_{t\geq 0}$ is assumed to be adapted to $(\mathscr{F}_t)_{t\geq 0}$.

It is assumed that both players discount payoffs at the constant and common rate $r > 0$. All strategies in this paper are based on *hitting times*. These are stopping times of the form $\tau(Y^*) := \inf\{t \geq 0 | Y_t \geq Y^*\}$, for some $Y^* \in \bar{E}$, where $\tau_y(a) = 0$ and $\tau_y(b) = \infty$, P_y-a.s, for all $y \in E$. For $y \in E$ and $Y^* \in \bar{E}$, let $\nu_y(Y^*)$ denote the Laplace transform of $\tau(Y^*)$ (under P_y) evaluated at r, i.e.

$$\nu_y(Y^*) := \mathbb{E}^{P_y}\left[e^{-r\tau(Y^*)}\right].$$

Note that $\nu_y(Y^*) = 1$ for all $y \geq Y^*$, that $\nu_y(b) = 0$, for all $y \in E$, and that, because of the strong Markov property, it holds that $\nu_y(Y^2) = \nu_y(Y^1)\nu_{Y^1}(Y^2)$, for all $y < Y^1 < Y^2$.

The following assumption is made on the stochastic environment. A stochastic process $(Y_t)_{t\geq 0}$ is called *spectrally-negative* if it has no upward jumps (i.e. $Y_t - Y_{t-} \leq 0$ for all $t > 0$, P_y-a.s.).

Assumption 1. The process $(Y_t)_{t\geq 0}$ is spectrally-negative and the function $\nu_y(\cdot)$ is continuous for all $y \in E$.

This assumption allows, in particular, for all processes with continuous sample paths, but also for jump-diffusions as long as the magnitude of jumps is negative a.s. The requirement that $(Y_t)_{t\geq 0}$ has no upward jumps is equivalent to saying that the supremum process $\left(\bar{Y}_t\right)_{t\geq 0}$, defined by $\bar{Y}_t = \sup_{0\leq s\leq t} Y_s$, has continuous sample paths. This assumption simplifies finding solutions to the optimal stopping problems below considerably. It also often makes deriving $\nu_y(\cdot)$ fairly straightforward (see Section 5 for some examples).

The payoffs accruing to the players depend on their "stopping status" $k \in \{0, 1\}$, which indicates whether a player has stopped ($k = 1$) or not ($k = 0$). Let $D^i_{k\ell}(y)$, $y \in \bar{E}$, denote the expected present value of stopping (under P_y) to Player i if her stopping status is k, the stopping status of Player j, $j \neq i$, is ℓ, and the state variable has value y. In addition, it is assumed that stopping entails incurring a once off sunk cost $I^i > 0$.

Assumption 2. For all $i \in \{1, 2\}$, it holds that

1. $D^i_{k\ell}$, $k, \ell = 0, 1$, is continuous on E;
2. $D^i_{10}(a) - D^i_{00}(a) < I^i$ and $D^i_{10}(b) - D^i_{00}(b) > I^i$;
3. $D^i_{10}(\cdot) - D^i_{00}(\cdot)$ is strictly increasing;

4. $D_{11}^i(\cdot) - D_{01}^i(\cdot)$ is non-decreasing;

5. $D_{10}^i(\cdot) - D_{00}^i(\cdot) > D_{11}^i(\cdot) - D_{01}^i(\cdot) \geq D_{11}^i(\cdot) - D_{00}^i(\cdot) \geq 0$.

Assumptions 2.2 and 2.3 ensure that there is a unique threshold for the state variable where the net present value of being the first player to stop becomes positive. Assumption 2.4 allows for the possibility that it is never optimal to become the second mover. Assumption 2.5 implies that there is a first-mover advantage and that $D_{10}^i(\cdot) > D_{11}^i(\cdot) \geq D_{00}^i(\cdot) \geq D_{01}^i(\cdot)$.

The final assumption that is made ensures that waiting forever renders the option valueless. This assumption essentially rules out speculative bubbles.

Assumption 3. The functions $\nu_y(\cdot)$, and $D_{k\ell}^i(\cdot)$ are such that $\lim_{Y^*\uparrow b} \nu_y(Y^*)[D_{1k}^i(Y^*) - D_{0k}^i(Y^*) - I^i] = 0$, for all $k \in \{0, 1\}$.

In essence the timing game described in this paper is a game between two players to determine who is the first and who is the second mover. These roles will from here on be referred to as the *leader* and *follower* roles, respectively. The players, therefore, care about the payoffs $D_{k\ell}^i(\cdot)$ only insofar as they are the building blocks of the expected payoffs of each of these two roles. In order to derive these, first note that if Player i stops at a time where the state variable equals $y \in E$ and the other player has already stopped, then her payoff will be

$$M^i(y) = D_{11}^i(y) - I^i. \tag{1}$$

Before analyzing the game where players vie for the leader role, let's first study the standard Stackelberg model applied to a situation where players have to choose stopping times. Assume that Player i is the leader in this game. Player j, hence, is the follower. The strategies in this game are going to be the thresholds at which the players move. We want to allow for the possibilities that Player i stops immediately, no matter what the value of the state variable, and that Player j never stops. So, the strategy space is taken to be $\bar{E}$. The follower reacts to the timing decision of the leader. So, the follower's strategy is independent of that of the leader. The follower's optimal stopping problem, therefore, equals

$$\begin{aligned} F^j(y) =& D_{01}^j(y) + \sup_\tau \mathbb{E}_y \left[e^{-r\tau} \left(D_{11}^j(Y_\tau) - D_{01}^j(Y_\tau) - I^j \right) \right] \\ \overset{(*)}{=}& D_{01}^j(y) + \max_{Y^j} \nu_y(Y^j) \left[D_{11}^j(Y^j) - D_{01}^j(Y^j) - I^j \right], \end{aligned} \tag{2}$$

where $(*)$ follows from the strong Markov and spectral negativity properties of $(Y_t)_{t\geq 0}$. After all, these properties imply that one can write

$$\begin{aligned} &\mathbb{E}_y \left[e^{-r\tau(Y^*)} (D_{1k}^i(Y_{\tau(Y^*)}) - D_{0k}^i(Y_{\tau(Y^*)}) - I^i) \right] \\ &\qquad = \mathbb{E}_y \left[e^{-r\tau(Y^*)} \right] (D_{1k}^i(Y^*) - D_{0k}^i(Y^*) - I^i), \end{aligned}$$

for all $Y^* \in \bar{E}$.

In any meaningful notion of equilibrium, the leader will take into account that the follower will move at the stopping time $\tau(Y_F^j)$. Therefore, if the leader moves at $y \in E$, then her expected payoff is

$$L^i(y) = D_{10}^i(y) + \nu_y(Y_F^j)\left[D_{11}^i(Y_F^j) - D_{10}^i(Y_F^j)\right] - I^i, \tag{3}$$

which is a continuous function. So, the leader's optimal stopping problem becomes

$$\begin{aligned}\hat{L}^i(y) :=& D_{00}^i(y) + \sup_\tau \mathbb{E}_y\left[e^{-r\tau}\left(L^i(Y_\tau) - D_{00}^i(Y_\tau)\right)\right]\\ =& D_{00}^i(y) + \max_{Y^i} \nu_y(Y^i)\left[L^i(Y^i) - D_{00}^i(Y^i)\right].\end{aligned} \tag{4}$$

The following lemma describes the solutions to the maximization problems faced by the players.

Lemma 1. *Suppose that Assumptions 1–3 hold. The functions $\hat{L}^i(\cdot)$ and $F^i(\cdot)$ have unique maximizers $Y_L^i \in E$ and $Y_F^j \in \bar{E}$, respectively, which satisfy $Y_L^i \leq Y_F^j$.*

Proof. Existence of Y_L^i and Y_F^j follows trivially from the continuity of $D_{kl}^i(\cdot)$ and $\nu_y(\cdot)$, and the fact that $\bar{E}$ is a compact set. Due to Assumptions 2.2, 2.3, and 3, $L_y^i(\cdot)$ attains it maximum on E. The fact that $Y_L^i \leq Y_F^j$ follows from Assumptions 2.3–5.

Uniqueness of Y_F^j is established as follows. Define $g^j(\cdot) := D_{11}^j(\cdot) - D_{01}^j(\cdot) - I^j$. Suppose that Y_1 and Y_2 are two distinct maximizers of $F^j(\cdot)$, such that (wlog) $Y_1 < Y_2$. First, assume that $Y_2 < b$. Then it holds that $g^j(Y_1) = \nu_{Y_1}(Y_2)g^j(Y_2)$, and, thus, that $g^j(Y_2) > g^j(Y_1)$. Continuity of $g^j(\cdot)$ implies that there exists $Y_3 \in (Y_1, Y_2)$, such that $g^j(Y_3) > g^j(Y_1)$. Therefore, it holds that

$$\begin{aligned}&\nu_{Y_1}(Y_3)\nu_{Y_3}(Y_2) = \nu_{Y_1}(Y_2) = \frac{g^j(Y_1)}{g^j(Y_2)} < \frac{g^j(Y_3)}{g^j(Y_2)}\\ \overset{*}{\iff}\ &\nu_{Y_3}(Y_2)g^j(Y_2) < \nu_{Y_1}(Y_3)g^j(Y_3)\\ \iff\ &\nu_{Y_1}(Y_2)g^j(Y_2) = g^j(Y_1) < \nu_{Y_1}(Y_3)g^j(Y_3),\end{aligned}$$

where $(*)$ follows from $g^j(Y_2) > 0$, since $Y_2 < b$. But this is a contradiction to Y_1 being a maximizer.

Finally, if $Y_1 = b$ is a maximizer, uniqueness follows from the fact that there is no Y^* for which $g^j(Y^*) > 0$. After all, if there were, then

$$\nu_y(Y^*)g^j(Y^*) > \nu_y(b)g^j(b) = 0.$$

A similar reasoning shows uniqueness of Y_L^i. ■

3 Equilibrium Analysis

In games with a first mover advantage players may try to preempt each other. Such preemptive situations arise whenever the value of becoming the leader exceeds the value of being the follower, while it is not optimal for either player to stop. Since the purpose of the paper is to investigate this competition and since any reasonable concept of equilibrium must have the follower stopping at Y_F^i, it will be implicitly assumed in the remainder that the follower's strategy is to stop at that threshold.

At each point in time a player has to balance current payoffs to the payoffs of becoming leader or follower at some later date. Therefore, we denote the present values of becoming the leader or follower at the first hitting time of some trigger Y^* under P_y by

$$L_y^i(Y^*) := D_{00}^i + \nu_y(Y^*)\left[L^i(Y^*) - D_{00}^i(Y^*)\right] \quad \text{and}$$
$$F_y^i(Y^*) := D_{00}^i + \nu_y(Y^*)\left[F^i(Y^*) - D_{00}^i(Y^*)\right],$$

respectively, for all $y \leq Y^* \in E$. Note that $L_y^i(Y_L^i) = \hat{L}^i(y)$. It can easily be seen that $L_y^i(Y^*) \geq F_y^i(Y^*)$ iff $L^i(Y^*) \geq F^i(Y^*)$, for all $y \in E$. Since $L^i(\cdot)$ and $F^i(\cdot)$ are continuous, there exists $Y_P^i < Y_L^i$ such that $L^i(Y_P^i) = F^i(Y_P^i)$. In fact, due to the monotonicity assumptions in Assumption 2, Y_P^i is unique. This point is called Player i's *preemption point* and it is the lowest value for y at which Player i would want to preempt Player j. Hence, the region in which Player i would wish to preempt Player j is $[Y_P^i, Y_F^i)$. Let the *preemption region* be defined as $S_P := [Y_P^1 \vee Y_P^2, Y_F^1 \vee Y_F^2)$. We will focus on games with $S_P \neq \emptyset$. Obviously, Player i will only want to preempt Player j if there is a threat that Player j might preempt, i.e. when $y \in S_P$.

Combining this with the results from the previous section, the payoff structure of the game can be summarized as follows.

Lemma 2. *Under Assumptions 1–3 it holds that for every player $i \in \{1,2\}$ there exist unique thresholds*

1. *$Y_F^i \in \bar{E}$ such that $M^i(y) = L^i(y) = F^i(y)$, for all $y \geq Y_F^i$;*
2. *$Y_L^i < Y_F^i$ such that $\max_{Y^*} L_y^i(Y^*) = L^i(y)$, for all $y \geq Y_L^i$;*
3. *$Y_P^i < Y_L^i$ such that $L^i(y) \geq F^i(y)$, for all $y \geq Y_P^i$.*

A plot of typical value functions in an asymmetric case is given in Figure 1.

3.1 Strategies and Payoffs

The main difference between deterministic timing games – where time is the state variable – and games where the state variable is a stochastic process is that in

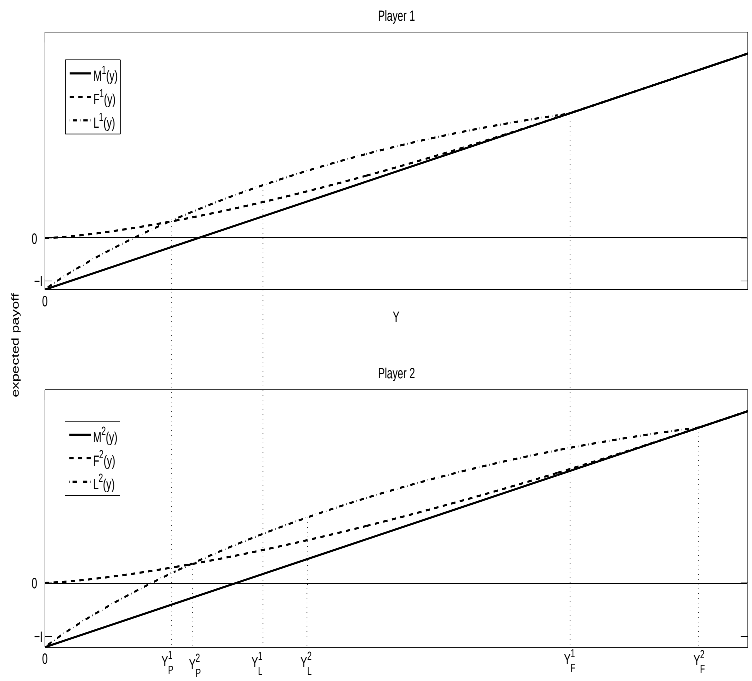


Figure 1: Value functions for leader, follower, and simultaneous stopping. The underlying stochastic process is a geometric Brownian motion.

the latter case an agent's strategy, however defined, can not be forward looking. In its most general form a strategy for Player i will be a process $\left(X^i_t\right)_{t\geq 0}$ taking values in $[0,1]$, where X^i_t is the probability with which Player i has stopped up to and including time t. Obviously, $\left(X^i_t\right)_{t\geq 0}$ has to be a non-decreasing process. In addition, we want to rule out that players can act on information that has not yet been released. That is, "insider trading" should not be allowed. These two considerations lead us to conclude that $\left(X^i_t\right)_{t\geq 0}$ must (i) be adapted to $(\mathscr{F}_t)_{t\geq 0}$ and (ii) have càglàd (left-continuous with right-limits) sample paths. Due to the process $(Y_t)_{t\geq 0}$ being a càdlàg semimartingale and $\left(X^i_t\right)_{t\geq 0}$ being càglàd, stochastic integrals of the form $\int X^i dY$ are well-defined (see Protter (2004)).

For our purposes it suffices to restrict attention to strategies that are driven by stopping times. Let τ be a stopping time (relative to the filtration $(\mathscr{F}_t)_{t\geq 0}$). The *stopping strategy induced by the stopping time* τ is given by

$$X^i(\tau) := \begin{cases} 0 & \text{if } t < \tau, \\ 1 & \text{if } t \geq \tau. \end{cases}$$

As already remarked, given the strong Markovian nature of $(Y_t)_{t\geq 0}$, the infinite time horizon and the no-positive-jump assumption, all optimal stopping problems in Section 2 take the form of trigger policies. Therefore, it stands to reason to focus on *threshold strategies*. These consist of a single threshold $Y^i \in \bar{E}$, with the convention that Player i stops at the induced first-hitting time $\tau(Y^i)$. So, the strategy space for each player is $\bar{E}$.

Let $(Y^1, Y^2) \in \bar{E} \times \bar{E}$ and define $\tau := \tau(Y^1) \wedge \tau(Y^2)$. Then the expected payoff to Player i of the pair of thresholds (Y^i, Y^j) is given by

$$\begin{aligned} V_y^i(Y^i, Y^j) =& D_{00}^i(y) + \mathbb{E}^{P_y}\Big[e^{-r\tau}\Big(1_{[\tau(Y^i)<\tau(Y^j)]}L^i(Y_\tau) + 1_{[\tau(Y^i)>\tau(Y^j)]}F^i(Y_\tau) \\ &+ 1_{[\tau(Y^i)=\tau(Y^j)]}W^i(Y_\tau) - D_{00}^i(Y_\tau)\Big)\Big], \end{aligned} \tag{5}$$

for all $y \in E$. Here $W^i(\cdot)$ is a tie-breaking rule giving the expected payoff if both players stop at the same time.

The tie-breaking is introduced to allow for coordination between players if both wish to stop simultaneously. To allow for some generality, this function is assumed to be given by

$$W^i(y) = p_i(y)L^i(y) + p_j(y)F^i(y) + p_3(y)M^i(y), \quad \text{all } y \in E,$$

for some $(p_1(y), p_2(y), p_3(y)) \geq 0$, with $p_1(y) + p_2(y) + p_3(y) = 1$. In this set-up the probability with which Player i is the only one who stops is $p_i(y)$ and the probability that both players stop simultaneously is $p_3(y)$. This formulation encompasses most contributions in the literature. For example, Murto (2004) does not allow coordination and, thus, assumes $p_3(y) = 1$, all $y \in E$; Weeds (2002) assumes that $p_1(y) = p_2(y) = 1/2$, $y \in S_P$, and $p_3(y) = 1$ otherwise; and Thijssen et al. (2002) argue, based on an argument by Fudenberg and Tirole (1985) for deterministic games, that $p_3(y) = 0$, all $y \in S_P$, but that $p_1(y)$ and $p_2(y)$ are such that $W^i(y) = F^i(y)$. If $W^i(y) = F^i(y)$, all $y \in S_P$, then the game is said to have the *rent-equalization* property. The importance of this property for the existence of equilibria is established in the next section. Note that, if $Y_P^1 = Y_P^2 \equiv Y_P$, rent-equalization requires that $p_3(Y_P) = 0$, but any $p_1(Y_P) \in [0, 1]$ and $p_2(Y_P) = 1 - p_1(Y_P)$ would work. In particular, the following assumption is made.

Assumption 4. If $Y_P^1 = Y_P^2 \equiv Y_P$, then $p_1(Y_P) = p_2(Y_P) = 1/2$.

Finally if $Y_P^i < Y_P^j$, then, again, rent-equalization requires that $p_3(Y_P^j) = 0$. This, in turn, means that for Player i rent-equalization is consistent with $p_i(Y_P^j) = 0$ and $p_j(Y_P^j) = 1$. This is unsatisfactory, because at Y_P^j Player i strictly prefers to be the leader, whereas Player j is indifferent. Therefore, the following assumption is made.

Assumption 5. If $Y_P^i < Y_P^j$, then $p_i(Y_P^j) = 1$.

This assumption ensures that in asymmetric games the player with the lower preemption threshold can wait until the preemption threshold of the other player is reached without running the risk of being preempted. The assumption is made mainly for technical reasons. If one is unwilling to make this assumption, some of the results below have to be reformulated in terms of ε-equilibrium, which would increase notational burden without adding much in the way of economic insight. It will be argued in Section 4 that Assumptions 4 and 5 are consistent with non-cooperative coordination.

To summarize, a *stopping game* is a tuple $\Gamma = \left\langle \bar{E}, (V_y^i)_{y \in E} \right\rangle_{i \in \{1,2\}}$. A stopping game Γ is *preemptive* if $S_P \neq \emptyset$. A *Markov equilibrium* in the stopping game Γ is a pair of thresholds $(\bar{Y}^1, \bar{Y}^2) \in \bar{E} \times \bar{E}$, such that $V_y^i(\bar{Y}^i, \bar{Y}^j) \geq V_y^i(Y^i, \bar{Y}^j)$, for all $Y^i \in \bar{E}$, all $y \in E$, and all $i \in \{1, 2\}$.

In the remainder of the paper the focus will be on three different types of Markov equilibria. A Markov equilibrium $(\bar{Y}^1, \bar{Y}^2)$ is called *preemptive* if $\bar{Y}^i < Y_L^i$, for at least one $i \in \{1, 2\}$. A Markov equilibrium $(\bar{Y}^1, \bar{Y}^2)$ is called a *Stackelberg* equilibrium if $\bar{Y}^i = Y_L^i$, for at least one $i \in \{1, 2\}$. Finally, a Markov equilibrium $(\bar{Y}^1, \bar{Y}^2)$ is called *collusive* if $\bar{Y}^i > Y_F^i$ and $\bar{Y}^1 = \bar{Y}^2$. In such equilibria no player stops in the preemption region, not even at their optimal (Stackelberg) threshold. Such equilibria are, therefore, indicative of the presence of (tacit) collusion.

3.2 Preemptive and Stackelberg Equilibria

Throughout this subsection it will be assumed (wlog) that $Y_L^1 \leq Y_L^2$. The existence of equilibria depends crucially on the ordering of Y_L^1 and Y_P^2, as well as the tie-breaking rule $W^i(\cdot)$ in the preemption region. In the remainder, let $Y_F := Y_F^1 \vee Y_F^2$ be the larger of the two follower thresholds.

The following lemma shows that in preemptive and Stackelberg equilibria both players choose a trigger no larger than their Y_F. In other words, such equilibria can never be collusive.

Lemma 3. *Let Γ be a preemptive stopping game satisfying Assumptions 1–3. If Player j plays a strategy $Y^j \leq Y_F$, then it is weakly dominant for Player i to choose a strategy $Y^i \leq Y_F$. Furthermore, all equilibria are either preemptive or Stackelberg.*

Proof. Suppose that Player j plays $Y^j \leq Y_F$ and that Player i chooses a strategy $Y^i \in \bar{E}$, with $Y^i > Y_F^i$. Let $y \in [Y_F^i, Y^i)$. Consider the following two cases.

1. $Y^j \leq y$.

In this case Player j becomes the leader and, therefore, Player i's expected payoff is

$F^i(y)$. Deviating to $Y^i = Y_F^i$ would lead to the expected payoff $M^i(y) = F^i(y)$.
2. $Y^j > y$.
In this case Player i can become the leader and get an expected payoff $L^i(y) > F^i(y)$ if $y < Y_F^j$. Otherwise, Player i gets the payoff $M^i(y) = F^i(y)$.

Finally, suppose that there exists an equilibrium with $\bar{Y}^i \leq Y_F^i$, all $i = 1, 2$, which is not preemptive or Stackelberg. This implies that $\bar{Y}^i > Y_L^i$, all $i = 1, 2$. Suppose (wlog) that $Y_L^1 \leq Y_L^2$. Then Player 1 has an incentive to deviate to $\hat{Y}^1 = Y_L^1$, which would leads to a Stackelberg equilibrium. ■

In particular this lemma shows that there are no collusive equilibria when $\bar{Y}^i \leq Y_F^i$, all $i = 1, 2$. Such equilibria will be studied in Section 3.3.

Proposition 1. *Let Γ be a preemptive stopping game satisfying Assumptions 1–3. Assume that the rent-equalization property does not hold.*

1. *If $Y_P^2 \geq Y_L^1$, then (Y_L^1, Y_F^2) is the unique Stackelberg equilibrium. Furthermore, there are no preemptive equilibria.*
2. *If $Y_P^2 < Y_L^1$, then no Stackelberg or preemptive equilibria exist.*

Proof.

1. Consider the following cases.
   i. $y \geq Y_F^2$
   In this region $F^i(y) = M^i(y) = L^i(y)$, for both players. Therefore, it is optimal for both to stop.
   ii. $Y_L^1 \leq y < Y_F^2$
   Given that Player 1 stops and $W^2(y) < F^2(y)$, Player 2 has no incentive to deviate. Conversely, given that Player 2 does not stop immediately it is optimal for Player 1 to stop.
   iii. $y < Y_L^1$
   Given that Player 2 does not stop before Y_F^2 is hit it is optimal for Player 1 to wait until Y_L^1 is reached. Conversely, since $Y_L^2 \geq Y_L^1$ and $Y_P^2 \geq Y_L^1$ it holds for all $y \leq \hat{Y} < Y_L^1$ that

$$\nu_{\hat{Y}}(Y_L^1)[F^2(Y_L^1) - D_{00}^2(Y_L^1)] \geq F^2(\hat{Y}) - D_{00}^2(\hat{Y}) > L^2(\hat{Y}) - D_{00}^2(\hat{Y}),$$

   which, in turn, implies that $L_y^2(\hat{Y}) < F_y^2(Y_L^1)$. So, Player 2 prefers to become the follower at Y_L^1 rather than to preempt and become leader at some $y \leq \hat{Y} < Y_L^1$. Finally, since $W_y^2(Y_L^1) < F^2(Y_L^1)$, it holds that $F_y^2(Y_L^1) > W_y^2(Y_L^1)$. Hence, Player 2 has no incentive to deviate to any $y \leq \hat{Y} \leq Y_L^1$. For any $\hat{Y} > Y_L^1$, it holds that $V_y^2(\hat{Y}, Y_L^1) = V_y^2(Y_F^2, Y_L^1) = F_y^2(Y_L^1)$.

To show that there are no preemptive equilibria, suppose, on the contrary that $\bar{Y}^1 < Y_L^1$. Then Player 2's best response is $\bar{Y}^2 = Y_F^2$. But then Player 1 is better off by deviating to $\hat{Y}^1 = Y_L^1$, because $L^1(\bar{Y}^1) < L^1_{\bar{Y}^1}(Y_L^1)$.

2. Let $y \in (Y_P^2, Y_L^1)$. Suppose, by contradiction, that $(\bar{Y}^1, \bar{Y}^2)$ is a preemption or Stackelberg equilibrium. If $y \geq \bar{Y}^i$, $i = 1, 2$, then both players stop simultaneously and $V_y^2(\bar{Y}^2, \bar{Y}^1) = W^2(y) < F^2(y)$, which implies that Player 2 wants to deviate. If $\bar{Y}^1 \leq y < \bar{Y}^2$, then there exists $\hat{Y}^1 \in (y, \bar{Y}^2 \wedge Y_L^1)$ such that $L_y^1(\hat{Y}^1) > L_y^1(\bar{Y}^1) = L^1(y)$. This holds because $L_y^1(\cdot)$ is increasing on $(y, \bar{Y}^2 \wedge Y_L^1)$. So, Player 1 wishes to deviate. A similar reasoning applies to Player 2 if $\bar{Y}^2 \leq y < \bar{Y}^1$.

■

If a preemption game satisfies the rent-equalization property, the picture looks very different.

Proposition 2. *Let Γ be a preemptive stopping game satisfying Assumptions 1–3 and the rent-equalization property, such that Assumption 5 holds.*

1. *Suppose that $Y_P^2 \leq Y_L^1$, and that $Y_P^1 \neq Y_P^2$. The following holds:*

   (a) if $Y_P^1 < Y_P^2$, then the unique preemptive equilibrium is (Y_P^2, Y_P^2);

   (b) if $Y_P^2 < Y_P^1$, then the unique preemptive equilibrium is (Y_P^1, Y_P^1).

   Furthermore, no Stackelberg equilibria exist.

2. *If $Y_P^2 > Y_L^1$, then all Stackelberg equilibria are of the form (Y_L^1, Y^2), for any $Y^2 \geq Y_P^2$. Furthermore, there are no preemptive equilibria.*

3. *If $Y_P^1 = Y_P^2 \equiv Y_P$, then (Y_P, Y_P) is the unique preemptive equilibrium. Furthermore, no Stackelberg equilibria exist.*

Proof. First note that it is obvious that the statements are true for $y \geq Y_F^1 \vee Y_F^2$.

1. We first show that no Stackelberg equilibria exist. Since $Y_L^1 \leq Y_L^2$ it suffices to argue that (Y_L^1, Y_F^2) is not an (Stackelberg) equilibrium. Given the pair of strategies (Y_L^1, Y_F^2), Player 2 has an incentive to deviate to $\hat{Y}^2 = Y_L^1 - \varepsilon$, for some small $\varepsilon > 0$, for which it holds that $L^2(\hat{Y}^2) > F^2_{\hat{Y}^2}(Y_L^1)$. Such an ε exists because of continuity.

(a) The preemptive equilibrium is established as follows. Suppose that $Y_F^i \leq Y_F^j$. Note that $Y_P^2 < Y_F^i$, since $S_P \neq \emptyset$. Take any of the proposed equilibrium strategy pairs $(\bar{Y}^1, \bar{Y}^2)$. Consider the following cases.

i. $y \in [Y_F^i, Y_F^j)$.
In this case $V_y^i(\bar{Y}^i, \bar{Y}^j) = L^i(y) = M^i(y) = F^i(y)$, and $V_y^j(\bar{Y}^j, \bar{Y}^i) = W^j(y) = F^j(y)$. Player i has no incentive to deviate, since any deviation to $\hat{Y}^i > y$ would lead to a payoff $F^i(y)$. The same holds for Player j.
ii. $y \in [Y_P^2, Y_F^i)$.
Note that $V_y^k(\bar{Y}^k, \bar{Y}^\ell) = W^k(y) = F^k(y)$, for $k = 1, 2$. So, for neither player would a deviation lead to a higher payoff.
iii. $y < Y_P^2$.
As in case (ii), Player 2 has no incentive to deviate to any $\hat{Y}^2 > Y_P^2$. Let $\hat{Y}^2 < Y_P^2$. Because of the no-positive-jump assumption it holds that

$$L_y^2(\hat{Y}^2) \leq L_y^2(Y_P^2) = F_y^2(Y_P^2),$$

where the first inequality holds because $\hat{Y}^2 < \bar{Y}^1 < Y_L^2$ and $L_y^2(\cdot)$ is non-decreasing on (a, Y_L^2), and the second equality holds by definition. Because $\bar{Y}^1 < Y_L^1$ it holds that $L_y^1(\hat{Y}^1) < L_y^1(\bar{Y}^1)$ for any $\hat{Y}^1 < \bar{Y}^1$. Since $Y_L^1 \geq Y_P^2 > Y_P^1$, it also holds that $F_y^1(Y_P^2) < L_y^1(Y_P^2)$. However, Assumption 5 ensures that $V_y^1(Y_P^2, Y_P^2) = L_y^i(Y_P^2)$.

(b) The proof is analogous to that for the previous statement.

2. It is dominant for Player 1 to stop whenever $y \geq Y_L^1$. Given that $Y_P^2 > Y_L^1$, it is weakly dominant for Player 2 not to preempt Player 1, since for any $y \leq \hat{Y}^2 \leq Y_L^1$, it holds that

$$L_y^2(\hat{Y}^2) \leq L_y^2(Y_L^1) \leq F_y^2(Y_L^1).$$

So, any Markov equilibrium $(\bar{Y}^1, \bar{Y}^2)$ must have $\bar{Y}^1 = Y_L^1$ and $\bar{Y}^2 > Y_L^2$. Therefore, there are no preemptive equilibria. In addition, for any $y \in [Y_L^1, Y_P^2)$ it is optimal for Player 2 to become follower rather than leader since $W^2(y) < F^2(y)$. So, $\bar{Y}^2 \geq Y_P^2$. For all $y \geq Y_P^2$, however, Player 2 is indifferent between stopping immediately and not stopping immediately because $W^2(y) = F^2(y)$ due to rent equalization. So, any $\bar{Y}^2 \geq Y_P^2$ leads to a Stackelberg equilibrium.

3. Because of rent equalization, at Y_P each player is indifferent between stopping immediately and waiting. If one player chooses Y_P, then the other player has no incentive to deviate from Y_P because for all $\hat{Y}^i < Y_P$, it holds that

$$L_y^i(\hat{Y}^i) < F_y^i(\hat{Y}^i) \leq F_y^i(Y_P).$$

So, (Y_P, Y_P) is a preemption equilibrium.

Suppose, however, that there is another preemption equilibrium $(\bar{Y}^1, \bar{Y}^2)$, with, say, $Y_P < \bar{Y}^1 \leq \bar{Y}^2$. Let $y \leq Y_P$. Because of continuity of $L_y^2(\cdot)$ and $F_y^2(\cdot)$, Player 2 can find $\delta > 0$, such that

$$\nu_{\bar{Y}^1-\delta}(\bar{Y}^1)[F^2(\bar{Y}^1) - D_{00}^2(\bar{Y}^1)] < L^2(\bar{Y}^1 - \delta) - D_{00}^2(\bar{Y}^1 - \delta).$$

This, in turn, implies that $L_y^2(\bar{Y}^1 - \delta) > F_y^2(\bar{Y}^1)$, and, thus, that Player 2 should deviate.

■

3.3 Collusive Equilibria

Apart from preemptive and Stackelberg equilibria, there may exist equilibria in which both players stop simultaneously. As is made clear in Section 3.2, such equilibria can never be preemptive. In fact, they only exist if the value of becoming the leader at any point in the preemption region is exceeded by the expected payoff of simultaneous stopping at some later date. Define, for $Y^* > Y_F^i$,

$$M_y^i(Y^*) = D_{00}^i(y) + \nu_y(Y^*)\left[M^i(Y^*) - D_{00}^i(Y^*)\right]. \tag{6}$$

By a similar reasoning as in Lemma 1 it can easily be shown that (6) has a unique maximizer Y_M^i.

Using this notation the following equilibrium can exist in some cases.

Proposition 3. *Let Γ be a preemptive stopping game satisfying Assumptions 1–3. If $Y^* > Y_F^1 \vee Y_F^2$ is such that*

$$L^i(y) \leq M_y^i(Y^*),$$

for all $y \in S_P^i$ and $i \in \{1,2\}$, then (Y^*, Y^*) is a (collusive) Markov equilibrium.

Proof. Consider the following cases.
1. $y > Y^*$.
Given that layer j stops immediately, the best response of Player i is to stop immediately as well, since $Y^* > Y_F^i$.
2. $y \leq Y^*$.
Suppose that Player i deviates to $\hat{Y}^i > Y^*$. Then Player j will stop at time $\tau(Y^*)$. Since $Y^* > Y_F^i$, Player i will then stop immediately as well. So, $V_y^i(\hat{Y}^i, Y^*) = V_y^i(Y^*, Y^*)$. Conversely, if Player i deviates to $Y_P^i \leq \hat{Y}^i < Y^*$, then either

$$V_y^i(\hat{Y}^i, Y^*) = L^i(y) \leq M_y^i(Y^*) = V_y^i(Y^*, Y^*),$$

if $y \geq \hat{Y}^i$, or

$$\begin{aligned}
V_y^i(\hat{Y}^i, Y^*) &= L_y^i(\hat{Y}^i) \\
&= D_{00}^i(y) + \nu_y(\hat{Y}^i)\left[L^i(\hat{Y}^i) - D_{00}^i(\hat{Y}^i)\right] \\
&\leq D_{00}^i(y) + \nu_y(\hat{Y}^i)\left[M^i(\hat{Y}^i, Y^*) - D_{00}^i(\hat{Y}^i)\right] \\
&= M^i(y, Y^*) = V_y^i(Y^*, Y^*),
\end{aligned}$$

if $y < \hat{Y}^i$. ■

4 Non-Cooperative Coordination and Rent-Equalization

If one views the use of continuous time simply as a modeling tool that opens up the toolkit of stochastic calculus, then it is no great step to allow players to coordinate "in between two instantaneous points in time". In discrete time this can be modeled fairly straightforwardly. If the stochastic process moves along discrete points in time in $\mathbb{Z}_+$, then one can allow for coordination in between instances of time. So, for every $t \in \mathbb{Z}_+$ coordination can take place at times $[t, t+1) \cap \mathbb{Q}$. This idea can be extended to continuous time by using the techniques introduced in Dutta and Rustichini (1995). They view time as the two-dimensional set $\mathscr{T} = \mathbb{R}_+ \times \mathbb{Z}_+$, endowed with the lexicographic ordering, denoted by $\geq_L$, and the standard topology induced by $\geq_L$. That is, a typical time element is a pair $s = (t, z) \in \mathscr{T}$, which consists of a continuous and a discrete part. In the remainder, t refers to the continuous part and z to the discrete component. One can think of the continuous part of time as "process time" in which the stochastic environment evolves and the discrete part as "coordination time" in which players coordinate their actions. The great advantage of using this set-up is that it allows each part of the model to be analyzed in its most suitable way: stochastic evolution in continuous time and strategic interaction discrete time.

Obviously, the stochastic structure that has been used so far needs to be adapted to this new definition. Since we essentially want to keep the stochastic process $(Y_t)_{t\geq 0}$ defined on the continuous part of time only, this is a fairly straightforward exercise. A filtration on $(\Omega, \mathscr{F})$ is now a sequence of σ-fields, $\left(\mathscr{F}_{(t,z)}\right)_{(t,z)\geq_L (0,0)}$, such that

$$\mathscr{F}_{(t,z)} \subseteq \mathscr{F}_{(t',z')} \subseteq \mathscr{F},$$

whenever $(t, z) \leq_L (t', z')$. For all $y \in \mathbb{R}$, let P_y be a probability measure on $(\Omega, \mathscr{F})$ and define the process $\left(Y_{(t,z)}\right)_{(t,z)\geq_L (0,0)}$ such that $Y_{(t,z)} = Y_t$, for all $t \in \mathbb{R}_+$ and $z \in \mathbb{Z}_+$. So, the extended process only moves in "process time" and is constant in "coordination time". This way, stochastic integrals can also be extended trivially to operate on $\mathscr{T}$.

In this framework, the threshold strategies introduced in Section 3.1 are not so much the thresholds at which players stop, but the thresholds at which they are willing to engage in a coordination game. As argued by Fudenberg and Tirole (1985), this coordination game is most conveniently modeled as a "grab–the–dollar" game. This is an infinitely repeated game the stage game of which is as depicted in Figure 2. That is, play continues until at least one player "grabs the dollar". We

	Grab	Don't grab
Grab	$M^1(y), M^2(y)$	$L^1(y), F^2(y)$
Don't grab	$F^1(y), L^2(y)$	play again

Figure 2: The coordination game.

allow for mixed strategies in the stage game. Given that the "grab–the–dollar" game is (potentially) infinitely repeated we can restrict attention to stationary strategies and denote the probability with which Player i grabs the dollar in the stage game by α^i. The payoff to Player i in this repeated game depends on the probability that she is the first to grab the dollar. For a given pair (α^1, α^2), the probability that Player i grabs the dollar first is denoted by $p_i(y)$ and is equal to

$$\begin{aligned} p_i(y) =&\alpha^i(1-\alpha^j) + \alpha^i(1-\alpha^i)(1-\alpha^j)^2 + \cdots \\ =&\alpha^i \sum_{z=1}^{\infty} (1-\alpha^i)^{z-1}(1-\alpha^j)^z \\ =&\frac{\alpha^i(1-\alpha^j)}{\alpha^i + \alpha^j - \alpha^i\alpha^j}. \end{aligned} \tag{7}$$

Similar computations show that the probabilities that Player j grabs the dollar first, denoted by $p_j(y)$, and that both players grab the dollar simultaneously, denoted by $p_3(y)$, are equal to

$$p_j(y) = \frac{\alpha^j(1-\alpha^i)}{\alpha^i + \alpha^j - \alpha^i\alpha^j}, \quad \text{and} \quad p_3(y) = \frac{\alpha^i\alpha^j}{\alpha^i + \alpha^j - \alpha^i\alpha^j}, \tag{8}$$

respectively.

The expected payoff to Player i in the repeated game then equals

$$W_y^i(\alpha^i, \alpha^j) = p_i(y)L^i(y) + p_j(y)F^i(y) + p_3(y)M^i(y). \tag{9}$$

It is obvious that it is a weakly dominant strategy to set $\alpha^i = 1$ whenever $y \geq Y_F^i$ and $\alpha^i = 0$, whenever $y < Y_P^i$. Furthermore, for each $y \in S_P \setminus \partial S_P$, there is a unique mixed strategy equilibrium where

$$\bar{\alpha}^i = \frac{L^j(y) - F^j(y)}{L^j(y) - M^j(y)}. \tag{10}$$

The expected payoffs in this equilibrium are easily confirmed to be $W_y^i(\bar{\alpha}^i, \bar{\alpha}^j) = F^i(y)$. So, in $S_P \setminus \partial S_P$, this way of modelling the coordination process automatically leads to rent-equalization.

The situation is different for $y \in \partial S_P$. First suppose that $Y_P^i < Y_P^j$. The mixed strategy derived above would give that $\alpha^i(Y_P^j) = 0$ and $\alpha^j(Y_P^j) > 0$, which results in $p_i(Y_P^j) = 0$ and $p_j(Y_P^j) = 1$, which is a very unsatisfactory outcome. However, the pair $(\alpha^i(Y_P^j), \alpha^j(Y_P^j)) = (1, 0)$ also constitutes a Nash equilibrium. This Nash equilibrium would give $p_i(Y_P^j) = 1$ and $p_j(Y_P^j) = 0$. This gives a justification for Assumption 5.

A fully degenerate case occurs when $Y_P^1 = Y_P^2 \equiv Y_P$. Then $\alpha^1(Y_P) = \alpha^2(Y_P) = 0$ and $p_1(Y_P) = p_2(Y_P) = p_3(Y_P) = 0/0$. However, if $D_{k\ell}^i(\cdot)$ and $\nu_y(\cdot)$ are C^1, then an application of L'Hotital's rule shows that

$$p_1(Y_P) = p_2(Y_P) = 1/2, \quad \text{and} \quad p_3(Y_P) = 0.$$

This can be thought of as a limiting case where both players use the same infinitesimally small probability $\varepsilon > 0$ of grabbing the dollar. Since $p_3(Y_P)$ is of order ε^2 and $p_1(Y_P)$ and $p_2(Y_P)$ are of order ε, the probability of both players stopping simultaneously vanishes at a faster rate. This gives some justification for Assumption 4.

5 Examples with Spectrally Negative Lévy Processes

In this section some examples are given that illustrate the applicability of the results derived so far. Attention is mainly focussed on preemptive and Stackelberg equilibria in games that satisfy the rent-equalization property. A *Lévy process* is an adapted process with independent and stationary increments.[4] Each Lévy process has a càdlàg version, which is the one we will work with. For Borel sets U with $0 \notin \bar{U}$, the *Poisson random measure* of $(Y_t)_{t\geq 0}$ is given by $N(t, U) := \sum_{0<s\leq t} 1_U(\Delta Y_s)$. So, $N(t, U)$ is the number of jumps with a jump size in U. The corresponding compensated Poisson random measure is denoted by $\tilde{N}(t, U)$, i.e.

$$\tilde{N}(t, U) = N(t, U) - m(U)t, \quad \text{where} \quad m(U) = \mathbb{E}_y[N(1, U)],$$

is the *Lévy measure* of $(Y_t)_{t\geq 0}$. In differential form a time homogeneous Lévy process can be written as

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dB_t + \int_{\mathbb{R}} \gamma(Y_t, z)\tilde{N}(dz, dt), \tag{11}$$

[4] See, for example, Øksendal and Sulem (2007).

where $(B_t)_{t\geq 0}$ is a standard Brownian motion. Assume that $(Y_t)_{t\geq 0}$ takes values in $E = (a, b)$ and that

$$a - y < \gamma(y, z) \leq 0, \quad \text{for all } (y, z) \in E \times \mathbb{R}. \tag{12}$$

This assumption ensures that $(Y_t)_{t\geq 0}$ has no upwards jumps and is, hence, spectrally negative. Such processes are useful to model situations where "success comes on foot and leaves on horseback".

The generator of the process $(Y_t)_{t\geq 0}$, killed at rate $r > 0$, as defined for $f \in C^2$, is given by

$$\begin{aligned}\mathscr{L}_Y(g) =& \frac{1}{2}\sigma^2(y)g''(y) + \mu(y)g'(y) \\ &+ \lambda \int_{\mathbb{R}} \left[g(y + \gamma(y, z)) - g(y) - g'(y)\gamma(y, z)\right] m(dz) - rg(y),\end{aligned} \tag{13}$$

where λ is the intensity of the Poisson process that governs the occurrence of the jumps of $(Y_t)_{t\geq 0}$. Suppose that (13) has an increasing C^2 solution $g(\cdot)$, such that $g(a) = 0$. For $Y^* \geq y$, Dynkin's formula (Øksendal and Sulem (2007, Theorem 1.22)) then gives

$$\begin{aligned}&\mathbb{E}_y\left[e^{-r\tau(Y^*)}g(Y_{\tau(Y^*)})\right] = g(y) + \mathbb{E}_y\left[\int_0^{\tau(Y^*)} \mathscr{L}_Y g(Y_t)dt\right] \\ \iff &\mathbb{E}_y\left[e^{-r\tau(Y^*)}\right] = \frac{g(y)}{g(Y^*)}.\end{aligned}$$

All diffusions, i.e. Lévy processes without jumps, are spectrally negative. For several well-known classes of spectrally negative Lévy processes the stochastic discount factor $\nu_y(\cdot)$ can be computed explicitly. First consider arithmetic Brownian motion (ABM). This is a Lévy process of which the evolution is described by the stochastic differential equation (11) with $\mu(y) = \mu \in \mathbb{R}$, $\sigma(y) = \sigma > 0$, and $\lambda = 0$. For this process it holds that

$$\nu_y(Y^*) = e^{\beta_1(y - Y^*)},$$

where $\beta_1 > 0$ is the positive root of the quadratic equation

$$\frac{1}{2}\sigma^2\beta^2 + \mu\beta - r = 0.$$

A second example is the geometric Brownian motion (GBM), which takes $\mu(y) = \mu y$, $\sigma(y) = \sigma y$, $\lambda = 0$ in (11). In addition it is assumed that $r > \mu$. It then holds that

$$\nu_y(Y^*) = \left(\frac{y}{Y^*}\right)^{\beta_1},$$

where $\beta_1 > 1$ is the positive root of the quadratic equation

$$\frac{1}{2}\sigma^2\beta(\beta-1)+\mu\beta-r=0.$$

If one adds Beta distributed negative jumps to a GBM, i.e. $\lambda > 0$ and

$$m'(z)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}z^{a-1}(1-z)^{b-1}, \quad a,b>0,$$

where $\Gamma(\cdot)$ is the Gamma function, then it follows that (see Alvarez and Rakkolainen (2010))

$$\nu_y(Y^*)=\left(\frac{y}{Y^*}\right)^{\beta_1},$$

where $\beta_1 > 0$ is the positive root of the equation

$$\frac{1}{2}\sigma^2\beta(\beta-1)+\left(\mu+\frac{\lambda a}{a+b}\right)\beta-(r+\lambda)+\lambda\frac{\Gamma(a+b)\Gamma(b+\beta)}{\Gamma(b)\Gamma(a+b+\beta)}=0.$$

The results in this paper also apply to processes that exhibit mean-reversion. Consider, for example, the diffusion

$$\frac{dY}{Y}=\eta(\bar{Y}-Y)dt+\sigma dB,$$

on $\mathbb{R}_+$, where $\bar{Y}$ is the long-run value of Y and η determines the speed of mean-reversion. The generator of this process is

$$\mathscr{L}_Y g=\frac{1}{2}\sigma^2Y^2g''+\eta(\bar{Y}-Y)g'-rg.$$

The solution to $\mathscr{L}_Y g=0$ is (see, for example, Dixit and Pindyck (1994))

$$g(y)=Ay^{\theta_1}H\left(\frac{2\eta}{\sigma^2}y;\theta_1,b(\theta_1)\right)+By^{\theta_2}H\left(\frac{2\eta}{\sigma^2}y;\theta_2,b(\theta_2)\right),$$

where

$$H(x;\theta,b)=\sum_{n=0}^{\infty}\frac{\Gamma(\theta+n)/\Gamma(\theta)}{\Gamma(b+n)/\Gamma(b)}\frac{x^n}{n!},$$

is the generalized hypergeometric function, $\theta_1 > 0$ and $\theta_2 < 0$ are the roots of the quadratic equation

$$\frac{1}{2}\sigma^2\theta(\theta-1)+\eta\bar{Y}\theta-r=0,$$

and $b(\theta)=2(\theta+(\eta/\sigma^2)\bar{Y})$. Since we are looking for solutions with $g(0)=0$, it must hold that $B=0$. Since g is increasing if $A>0$, we get that

$$\nu_y(Y^*)=\left(\frac{y}{Y^*}\right)^{\theta_1}\frac{H\left(\frac{2\eta}{\sigma^2}y;\theta_1,b(\theta_1)\right)}{H\left(\frac{2\eta}{\sigma^2}Y^*;\theta_1,b(\theta_1)\right)}.$$

These results can be used to find the optimal thresholds Y_F^i and Y_L^i. For $k = 0, 1$, namely, these thresholds are obtained as the solution to the optimization problem

$$\max_{Y^*} \nu_y(Y^*)[D_{1k}^i(Y^*) - D_{0k}^i(Y^*) - I^i],$$

which leads to the first order condition (assuming differentiability of $D_{k\ell}^i(\cdot)$):

$$\frac{\partial \nu_y(Y^*)}{\partial Y^*}[D_{1k}^i(Y^*) - D_{0k}^i(Y^*) - I^i] + \nu_y(Y^*)\frac{\partial}{\partial Y^*}[D_{1k}^i(Y^*) - D_{0k}^i(Y^*)] = 0. \quad (14)$$

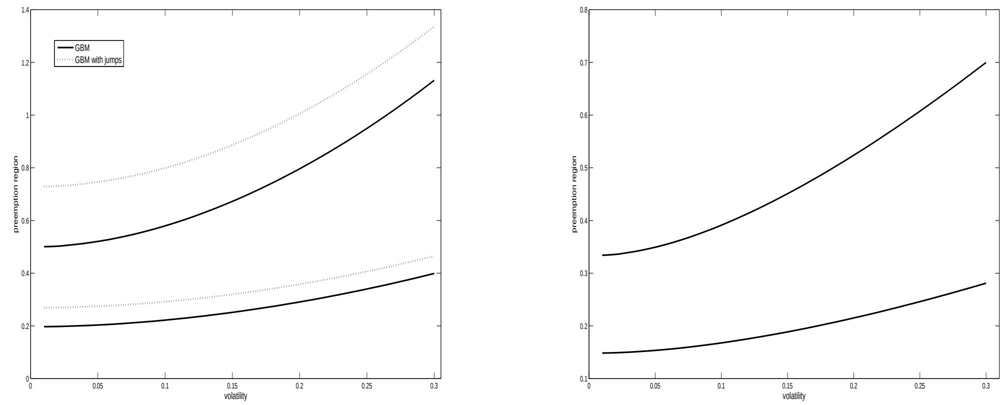
For ABM, GBM with beta distributed negative jumps, and mean-reversion we have

$$\frac{\partial \nu_y(Y^*)}{\partial Y^*} = -\beta_1 \nu_y(Y^*), \quad \frac{\partial \nu_y(Y^*)}{\partial Y^*} = -\frac{\beta_1}{Y^*}\nu_y(Y^*), \quad \text{and}$$
$$\frac{\partial \nu_y(Y^*)}{\partial Y^*} = \frac{2\eta}{\sigma^2}\frac{\theta}{b}H\left(\frac{2\eta}{\sigma^2}Y^*; \theta_1 + 1, b(\theta_1) + 1\right),$$

respectively.

For several different stochastic processes the preemption region is plotted in Figure 3 as a function of volatility. The net present values are taken to be linear in the underlying shock: $D_{k\ell}^i(y) = D_{k\ell}y$, where $D_{k\ell}$ are constants such that $D_{10} > D_{11} \geq D_{00} \geq D_{01}$ and $D_{10} - D_{00} > D_{11} - D_{01}$. As can be seen, the preemption region tends to get wider in the case of higher volatility. This happens because a higher volatility does not influence the present values whereas it increases option values. These option values have a bigger impact on the follower threshold than on the preemption threshold. After all, preemptive pressure erodes the option value for the leader. This implies that in games with higher levels of uncertainty, it is more likely that a preemptive situation occurs.

Another useful comparison to make is to look at the regions in which Stackelberg or preemptive equilibria occur. From Proposition 2 it is clear that these equilibria can not occur simultaneously. The ordering of the thresholds Y_P^1, Y_P^2, and Y_L^1 determines what type of equilibrium exists. These thresholds, in turn, are determined by the underlying present value functions $D_{k\ell}^i(\cdot)$. Figure 4 uses the same setting and numerical parameter values as the one used to generate Figure 3. The volatility is now kept fixed at $\sigma = .15$, and, instead, D_{10}^1 and D_{01}^2 are varied. The equilibrium regions are plotted in Figure 4. This figure assumes that $y < Y_P^1 \wedge Y_P^2$. As can be seen the qualitative picture is the same for the three different stochastic processes. Note that $Y_L^1 < Y_L^2$ for all values of D_{10}^1 and D_{01}^2. So, in a Stackelberg equilibrium Player 1 would stop first at the first hitting time of Y_L^1 and Player 2 would follow as soon as Y_F^2 is hit. For small values of D_{10}^1 and D_{01}^2, the first mover advantage is biggest for Player 2, which implies that $Y_P^2 < Y_P^1$ and, thus, that Player 2 preempts Player 1 at the first hitting time of Y_P^1. For larger values of D_{10}^1 and D_{01}^2 the case



(a) (b)

Figure 3: Preemption region as function of volatility σ for different stochastic processes. General parameter values are $I = 1$, $r = .1$, $D_{10} = 10$, $D_{11} = 3$, $D_{00} = 2$, and $D_{01} = 1$. (a) GBM (solid lines) and GBM with negative Beta jumps (dashed lines) with $\mu = .03$, $\lambda = .1$, $a = 1.5$, $b = 2$; (b) exponential mean-reversion with $\bar{Y} = 2$ and $\eta = .015$.

is reversed ($Y_P^1 < Y_P^2 < Y_L^1$) and Player 1 preempts Player 2 at the first hitting time of Y_P^2. In both cases the equilibria are preemptive, even though there is a clear prediction which player is the first to stop. Finally, for large values of D_{10}^1 the first mover advantage of Player 1 is so large that it dominates the preemptive pressure that Player 2 provides ($Y_L^1 < Y_P^2$) and, thus, Player 1 acts as a Stackelberg leader in these instances and stops at the first hitting time of Y_L^1. This shows that in a competitive situation one should not necessarily expect to see preemption. A final region of interest is where D_{10}^1 and D_{01}^2 are such that $Y_P^1 = Y_P^2 \equiv Y_P$. In such situations the coordination game described in Section 4 is played. Since this happens in the degenerate case where $\alpha^1(Y_P) = \alpha^2(Y_P)$ both players are the first to stop with probability 1/2. Also note that, in the space of parameters, the set where this happens has measure zero.

Note that the nature of the stochastic process only determines the relative size of these equilibrium regions. In particular, the case of mean reversion makes it most likely that a preemptive equilibrium prevails, whereas the case of a geometric Brownian motion makes it most likely that a Stackelberg equilibrium prevails. This is intuitively clear since, for the same values of $D_{k\ell}^i$ the geometric Brownian motion gives a much higher present value than the mean reverting process due to its exponential growth rate.

If $y \in S_P = (Y_P^1 \vee Y_P^2, Y_F^1 \vee Y_F^2)$, then the coordination game described in Section 4

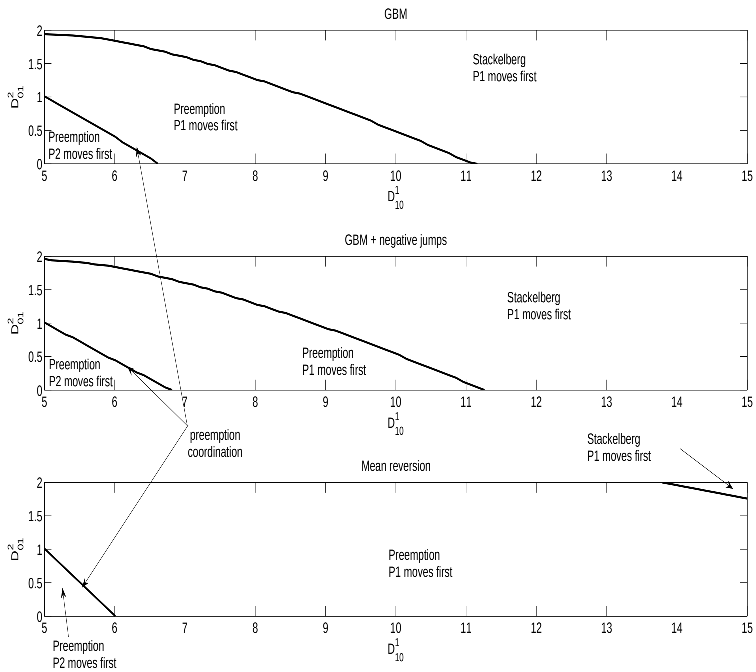


Figure 4: Equilibrium regions for different stochastic processes.

gets played immediately. The probabilities with which each player stops are such that $p_3(y) > 0$, which implies that a "coordination failure" can occur that is consistent with equilibrium. For each player, namely, the rents extracted upon becoming the leader outweigh the risk of simultaneous stopping. Finally, if $y > Y_F^1 \wedge Y_F^2$, then both players stop immediately.

6 Concluding Remarks

This paper presents equilibrium results for a large class of timing games in which there is a first-mover advantage. The approach exploits the Markovian nature of many often-used stochastic processes. This allows for a more straightforward analysis than in most existing contributions. In addition, the issue of rent-equalization has been studied separately and embedded in the timing game, which makes it easier to see how it arises and whether its presence is reasonable. In some applications where the environment changes very rapidly – such as financial markets for high frequency traded products – it might not be. In that case, no preemptive equilibria exist and only Stackelberg equilibria or collusive equilibria can be obtained. In fact, no equilibria may exist at all. If rent equalization is possible, then there always exists a Stackelberg or preemptive equilibrium.

Another advantage of the set-up in this paper is that the ideas can easily be adapted to games in which there is a second mover advantage (wars of attrition). Also, games with both a first mover advantage on the upside and a second mover advantage on the downside can be analyzed using this framework. In particular, this opens up the possibility of a systematic analysis of the investment and disinvestment behavior of competing firms.

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