

# ITERATIVELY STABLE CHEAP TALK\*

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ABSTRACT: We propose a new selection criterion from the set of Bayesian-Nash equilibria in cheap talk signalling games à la Crawford and Sobel (1982). A candidate strategy profile is stable if the infinite iteration of the composed best-response mapping from any neighboring perturbation of the candidate profile converges back to this same profile. Under some conditions, any game in which the maximal number of actions taken in equilibrium is some maximal integer  $\kappa_{\max}$ , has a unique stable equilibrium. If the sender's bias is either "upward bias at the top" or "downward at the bottom," the selected equilibrium induces  $\kappa_{\max}$  actions and it is maximal for some partial order over interval partitions, with respect to which the composed best-response is increasing. If the sender's bias is "inward," the selected equilibrium may induce either  $\kappa_{\max}$  or  $\kappa_{\max} - 1$  actions. In particular, if the game is symmetric with respect to the central type, the selected equilibrium induces  $\kappa_{\max} - 1$  actions. We compare our concept to Chen, Kartik and Sobel's (2008) NITS criterion.

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# 1 Introduction

We consider Crawford and Sobel’s (1982) one-dimensional model of cheap talk when the sender and the receiver do not share the same interests, i.e. the sender is biased. Crawford and Sobel (1982) show that, when the bias is upward (the sender prefers an action higher than the receiver, regardless his type), there is a finite upper bound,  $\kappa_{\max}$ , to the number of distinct actions that the receiver takes in equilibrium. Moreover, for each  $\kappa = 1, \dots, \kappa_{\max}$ , there is at least one equilibrium in which the receiver takes  $\kappa$  actions. Crawford and Sobel (1982) also provide a sufficient condition, Condition (M), under which there is a unique equilibrium outcome in which the receiver takes  $\kappa$  distinct actions, and the ex ante expected payoffs for both sender and receiver are strictly increasing in  $\kappa$ . For this reason, the equilibrium that induces the highest number of actions is sometimes called the “most informative equilibrium” and is the one that is typically considered in applications.

Gordon (2010) generalizes the analysis to games where the direction of the sender’s bias, upward or downward, is not necessarily the same for all sender types and shows that, in this case, the maximal number of actions taken in equilibrium may not be bounded. In particular, it is unbounded when the bias is outward, which means that the sender likes higher actions than the receiver does when the sender’s type is the highest and lower actions than the receiver does when the sender’s type is the lowest.

In general, these games have many equilibria. In this paper, we propose a new selection criterion to select among them.

Some of the early criteria for selecting equilibria in signalling games (Banks and Sobel, 1987; Cho and Kreps, 1987) fail to reduce the set of equilibria in cheap talk games. The reason is that these refinements operate by restricting beliefs off-the-equilibrium path. But in cheap talk games, any equilibrium outcome can be supported by a signalling strategy that uses all messages, so that there are no off-the-equilibrium paths. Other refinements are too strong (Faell, 1993), eliminating all equilibria.

Recently, Chen, Kartik and Sobel (2008) introduce a new criterion, *no incentive to separate* (NITS). An equilibrium satisfies NITS if the sender of the “bad” type weakly prefers the equilibrium outcome to credibly revealing his type. The “bad” type is the type that no other type wishes to imitate. When the sender has a strict

upward bias, the “bad” type is the lowest type (here, 0). These authors show that an equilibrium satisfying NITS always exists, when the bias is upward (or downward). They also prove that if condition M holds, the most informative equilibrium is uniquely selected.<sup>1</sup>

The justification for NITS is that, when the bias is upward, and for certain perturbations of the game that have an economic interpretation (Kartik, 2009; Chen 2011), the set of equilibria that satisfy certain monotonicity conditions converge in the limit to the set of equilibria that satisfy NITS.

However, for games with other types of biases, the limiting behavior of the perturbed games is not known and also not particularly easy to conjecture. For example, when the bias is inward, which means that the sender likes higher actions than the receiver does when the sender’s type is the lowest and lower actions than the receiver does when the sender’s type is the highest, there are two candidate “bad” types. The highest and the lowest (here types 0 and 1). Which of these types should have no incentive to separate in the selected equilibrium?<sup>2</sup> In contrast with the upward case considered by Chen, Kartik and Sobel (2008), when the bias is general, which type is the “bad” one is in a sense endogenous.

In this paper, we contribute to the literature on cheap talk communication in three ways. First, we study equilibrium selection in games where the bias is not necessarily upward. Second, we introduce iterative stability, a selection criterion based on an idea that is quite different from the criteria that have been studied, not only in the cheap talk, but more broadly in the signalling literature. Third, even for games where the bias is upward, we refine the NITS criterion in games where it does not make a unique prediction (for games that violate Condition M).

Iterative stability works as follows. Gordon (2010) studies the equilibria as the fixed-points of the composed best-response mapping. In this paper, we use this same mapping as a tool to select among equilibria, by interpreting it as an adaptative

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<sup>1</sup>More generally, Gordon (2011) shows that, when the bias is upward, and without assuming Condition M, the set of equilibria that satisfy NITS contains the set of ex ante Pareto undominated equilibria. This result is established using the same iterative dynamics as the one studied in this paper.

<sup>2</sup>There are examples of games with an inward bias where one of the two candidate “bad” types, either type 0 or type 1, has an incentive to separate in any equilibrium.

learning behavior. Heuristically, given an equilibrium, the stability test considers a situation where the players (i) initially play strategies that are similar to the candidate equilibrium strategies, but are not necessarily exactly these strategies, (ii) play at each stage a best-response to the other player's previous period strategy. If, in the long run, such a process eventually returns to a particular equilibrium, provided that the strategies that are played initially are sufficiently close to this equilibrium, we say that this equilibrium is stable. In more mathematical terms, an equilibrium is iteratively stable if the infinite iteration of the composed best-response from any initial perturbation of the equilibrium strategies converges to the same equilibrium.

We show that when the bias is either upward at the top (the sender prefers a higher action than the receiver when his type is the highest) or downward at the bottom, and the number of intervals in equilibrium is bounded, there is a unique iteratively stable equilibrium. The unique surviving candidate for iterative stability is the greatest (respectively, the least) element of the set of all equilibria. Moreover, for generic games, this equilibrium is stable, so that there is indeed a unique iteratively stable equilibrium.<sup>3</sup> In particular, for games with an upward bias in which NITS does not make a unique prediction, iterative stability uniquely selects from NITS.

When the bias is upward at the top and the number of intervals in equilibrium is unbounded, no finite equilibrium is iteratively stable. In particular, when the bias is outward (at extreme types, the sender's preferred action is more extreme than the receiver's), no equilibrium with finitely many intervals is stable.<sup>4</sup>

Finally, when the bias is inward and the number of intervals in equilibrium is bounded, there is at least one iteratively stable equilibrium, which has either the maximal cardinality or this cardinality minus one. If there is at most one equilibrium of each cardinality, there is at most one iteratively stable equilibrium and it either the maximal cardinality or this cardinality minus one. In the subclass of games that are symmetric around the central type  $t = \frac{1}{2}$ , the equilibrium with maximal cardinality

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<sup>3</sup>Roughly speaking, the genericity condition requires Condition  $M$  to hold locally around the equilibrium. In particular, games that satisfy Condition  $M$  satisfy the genericity condition. Thus Condition  $M$  is a sufficient condition for existence of an iteratively stable equilibrium in games, whose equilibria have a bounded maximal number of actions.

<sup>4</sup>We do not define the stability of an infinite equilibrium. The question of selection in games with infinite equilibria remains open for future research.

minus one is always the unique iteratively stable equilibrium of the game.

## 2 The model

There are two players, the sender  $S$  and the receiver  $R$ . Only the sender has payoff-relevant private information, his type. The sender observes his type, and sends a message to the receiver. The receiver then reads this message, and takes an action. Talking is “cheap”, in the sense that messages do not directly affect payoffs.

Let  $[0, 1]$  be the sender’s set of types, with typical element  $t$ . The sender’s type is drawn from a continuous probability distribution, with density  $f(\cdot)$ , supported on  $[0, 1]$ . Let  $\mathbb{R}$  be the set of receiver’s possible actions, with typical element  $a$ . The sender and the receiver each have twice continuously differentiable von Neuman-Morgenstern utility functions respectively denoted by  $U^S(a, t)$  and  $U^R(a, t)$ . For each  $a \in \mathbb{R}$ , each  $t \in [0, 1]$ , and all  $i \in \{R, S\}$ , we assume that  $U_{aa}^i(a, t) < 0$ , and  $U_{at}^i(a, t) > 0$ . For each  $t \in [0, 1]$ , and all  $i \in \{R, S\}$ , we assume that  $\max_{a \in \mathbb{R}} U^i(a, t)$  has a unique solution in  $\mathbb{R}$ , which we denote by  $a^i(t)$ . Everything is common knowledge, except the type, which is known only by the sender.

In general, a pure strategy for the sender can be described by an arbitrary partition  $(I_m)_{m \in M}$  of  $[0, 1]$  indexed by a (possibly infinite) message space  $M$ . The interpretation is that the types that belong to the the element of the partition labelled  $m$  all send the same message  $m$ . A pure strategy of the receiver is a list of actions  $(a_m)_{m \in M}$ . The interpretation is that the receiver takes the action associated with  $m$  when he receives message  $m$ . The outcome generated by the strategies  $(I_m)$  and  $(a_m)$  is the function

$$\gamma(t) = \sum_{m \in M} a_m 1_{I_m}(t)$$

where  $1_{I_m}(t)$  denotes the indicator function of the set  $I_m$ . For each  $t \in [0, 1]$ , the action  $\gamma(t)$  is chosen by the receiver when he receives the message sent by type  $t$ , under these strategies. An equilibrium is a pair  $((I_m)_{m \in M}, (a_m)_{m \in M})$  such that for all  $m, n \in M$ , all  $t \in I_m$ , we have

$$U^S(a_m, t) \geq U^S(a_n, t)$$

(i.e. the sender plays a best reply) and for each  $m \in M$ , the action  $a_m$  maximizes the expected utility of the sender conditional on the information that  $t$  is in  $I_m$ .

Abusing notation, for each interval  $I \subseteq [0, 1]$ , let  $a^R(I)$  be the action that maximizes the receiver's expected utility conditional on  $I$ . From Crawford and Sobel (1982), we know that all equilibria of the game have the following properties. First, all  $I_m$  are intervals. Second, for all  $m \in M$ , we have  $a_m = a^R(I)$ . Third, if  $I_m$  and  $I_n$  are consecutive intervals in the partition, where  $t$  is the boundary type between them, then the boundary type sender is indifferent between the two actions induced by types in the intervals  $I_m$  and  $I_n$ .

$$U^S(a^R(I_m), t) = U^S(a^R(I_n), t)$$

### 3 Equilibria and Iterative Stability

Further abusing notations, we let, for all  $t, t' \in [0, 1]$ ,  $a^R(t, t') := a^R([t, t'])$ . It is well known that any equilibrium has the property that any action taken by the receiver is induced by a (possibly singleton) interval of sender types. Thus any equilibrium outcome induces an interval partition of the type space. Some games have equilibria where infinitely many actions are induced. In this paper, however, we will only study the stability of finite equilibria, and will therefore restrict attention to finite interval partitions.<sup>5</sup>

In general, an interval partition with exactly  $\kappa$  intervals can also be described by a vector  $z$  :

$$0 = z_0 \leq z_1 \leq \dots \leq z_\kappa = 1.$$

Let  $Z_\kappa$  be the set of all interval partitions with  $\kappa$  intervals, and let

$$Z \equiv \bigcup_{\kappa \geq 1} Z_\kappa$$

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<sup>5</sup>The outcome of any strategy profile in which the sender plays a best response to a pure strategy of the receiver induces an interval partition of the type space. It is thus natural to restrict attention to interval partition outcome profiles. The finiteness assumption has no bite to the extent that our results apply to finite equilibria of games that may have infinite equilibria. It does, however, prevent us from studying the iterative stability of infinite equilibria.

be the set of all interval partitions. For all  $z \in Z$ , let  $\kappa(z)$  be such that  $z \in Z_{\kappa(z)}$ .

We now define the composed best-response mapping, where first the receiver best responds, then the sender,  $\theta : Z \rightarrow Z$  as follows. Let  $z \in Z$ . Let  $k_1$  be the highest index (if any), such that  $U^S(a^R(z_{k-1}, z_k), 0) < U^S(a^R(z_k, z_{k+1}), 0)$ . If there is no such index, let  $k_1 \equiv 0$ . Similarly, let  $k_2$  be the lowest index (if any), such that  $U^S(a^R(z_{k-1}, z_k), 1) > U^S(a^R(z_k, z_{k+1}), 1)$ . If there is no such index, let  $k_2 \equiv \kappa$ . Necessarily,  $k_2 - k_1 > 0$ . If  $k_2 - k_1 > 1$ , then for each  $k = 1, \dots, k_2 - k_1 - 1$ , let  $z'_k$  be the unique type such that

$$U^S(a^R(z_{k_1+k-1}, z_{k_1+k}), z'_k) = U^S(a^R(z_{k_1+k}, z_{k_1+k+1}), z'_k).$$

and let  $\theta(z) \equiv z' = (0, z_1, \dots, z_{k_2-k_1-1}, 1)$ . If instead  $k_2 - k_1 = 1$ , let  $z' \equiv (0, 1)$ . Clearly the mapping  $\theta(\cdot)$  is well-defined on  $Z$ .

The following taxonomy of biases is useful.

DEFINITION 1: *The bias is UPWARD if  $a^R(t) < a^S(t)$  for all  $t \in [0, 1]$ , DOWNWARD if  $a^S(t) < a^R(t)$  for all  $t \in [0, 1]$ , UPWARD AT THE TOP if  $a^R(1) \leq a^S(1)$ , DOWNWARD AT THE BOTTOM if  $a^S(0) \leq a^R(0)$ , OUTWARD if  $a^S(0) \leq a^R(0)$  and  $a^R(1) \leq a^S(1)$  and INWARD if  $a^R(0) < a^S(0)$  and  $a^S(1) < a^R(1)$ .*

A consequence of results in Gordon (2010) is that fixed-points of the mapping  $\theta(\cdot)$  are the finite equilibria of the game.<sup>6</sup>

LEMMA 1 (Gordon, 2010): *The partition  $z$  is an equilibrium partition if and only if  $z = \theta(z)$ . When this is the case, we have  $z_1 < \dots < z_{\kappa-1}$ . In addition, if the bias is upward at the top, we have  $z_{\kappa-1} < z_{\kappa}$ , and if the bias is downward at the bottom, we have  $z_0 < z_1$ .*

A positive integer  $k$  is an equilibrium cardinality for a game if this game has an equilibrium where exactly  $k$  actions are induced. Crawford and Sobel (1982) and Gordon (2010) establish several results on the set of equilibrium cardinalities. In particular, they relate the cardinality of the set of equilibrium cardinalities to the form

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<sup>6</sup>One difference is that Gordon (2010) considers a different mapping for each fixed number  $\kappa$  of intervals, while here we have one mapping that handles all interval partitions. However this paper's Lemma 1 and Theorem 1 easily follow from the results he establishes.

of the bias. Crawford and Sobel (1982) established that when the bias of the sender is upward, then the set of equilibrium cardinalities is of the form  $\{1, \dots, \kappa_{\max}\}$ , with  $\kappa_{\max} < +\infty$ . Gordon (2010) shows that for any bias, the set of equilibrium cardinalities is either of the form  $\{1, \dots, \kappa_{\max}\}$  with  $\kappa_{\max} < +\infty$ , or it is  $\mathbb{N}$ . In particular, if the bias is outward, the set of equilibrium cardinalities is  $\mathbb{N}$ . We summarize these results in Theorem 1.

**THEOREM 1** (Crawford and Sobel, 1982; Gordon, 2010): *The set of equilibrium cardinalities is either of the form  $\{1, \dots, \kappa_{\max}\}$  for some integer  $\kappa_{\max}$  or it is  $\mathbb{N}$ . If the bias is either upward or downward, this set is  $\{1, \dots, \kappa_{\max}\}$ . If the bias is outward, this set is  $\mathbb{N}$ .*

Thus in general, there are many equilibria. The objective of this paper is to select among these equilibria. Let  $\theta^1(\cdot) \equiv \theta(\cdot)$  and for all  $n \geq 1$ , let  $\theta^n(\cdot)$  be the mapping obtained by composing the mapping  $n$  times. For any interval partition  $z \in Z$ , the sequence  $(z^n)$  of vectors in  $Z$  such that  $z^0 \equiv z$  and for all  $n \geq 1$ ,  $z^n \equiv \theta^n(z)$  is the ITERATION FROM  $z$ .

We now define the convergence in an Euclidean sense of a sequence in  $Z$ . For all  $z, z' \in Z_\kappa$ , let

$$d(z, z') \equiv \sum_{k=0}^{\kappa} |z_k - z'_k|.$$

We say that a sequence  $z^n$  converges to  $z^*$  in an Euclidean sense if there is  $\kappa$  such that  $z^* \in Z_\kappa$  and  $n_0$  such that for all  $n \geq n_0$ , we have  $z^n \in Z_\kappa$  and

$$\lim_{n \rightarrow +\infty} d(z_n, z^*) = 0.$$

Unless specified otherwise, the convergence of a sequence  $(z^n)$  in  $Z$  will always be in an Euclidean sense. Next, for any partition  $z$ , first let

$$\begin{aligned} \langle 0, z \rangle &\equiv (0, z_0, \dots, z_\kappa) \in Z_{\kappa+1}. \\ \langle z, 1 \rangle &\equiv (z_0, \dots, z_\kappa, 1) \in Z_{\kappa+1}. \\ \langle 0, z, 1 \rangle &\equiv (0, z_0, \dots, z_\kappa, 1) \in Z_{\kappa+2}. \end{aligned}$$

be the partitions obtained by adding to  $z$  a singleton interval either at 0, at 1 or at both



ends. Second, let  $\lceil z \rceil$  be the unique partition obtain by deleting all cutpoints equal to 0 or 1, except for  $z_0$  and  $z_\kappa$ , that it the unique partition such that  $\lceil z \rceil_1 < \dots < \lceil z \rceil_{\kappa-1}$  and for some integers  $h, k, l$  such that  $h + k + l = \kappa$ , we have  $\lceil z \rceil \in Z_k$ , and

$$z = \left( \underbrace{0, \dots, 0}_{h \text{ times}}, \lceil z \rceil_0, \dots, \lceil z \rceil_\kappa, \underbrace{1, \dots, 1}_{l \text{ times}} \right).$$

Last, let

$$B(z) \equiv \{\lceil z \rceil, \langle 0, \lceil z \rceil \rangle, \langle \lceil z \rceil, 1 \rangle, \langle 0, \lceil z \rceil, 1 \rangle\}$$

be the set of partitions obtained from  $z$  be removing all cutpoints equal to 0 or 1, except for  $z_0$  and  $z_\kappa$  and adding at most one cutpoint at 0 (possibly none) and at most one cutpoint at 1 (possibly none). We are interested in the following iterative stability criteria.

Roughly speaking, we say that a partition  $z^*$  is globally iteratively stable if the iteration from any interval partition  $z^0$  converges to some partition in  $B(z^*)$  in an Euclidean sense. Except in a small set of cases, a globally stable partition does not exist, which motivates a weaker stability condition. We say that  $z^*$  is locally iteratively stable if the iteration from any interval partition  $z^0$  that is “similar enough” to  $z^*$  converges to some partition in  $B(z^*)$  in an Euclidean sense,

**DEFINITION 2: ITERATIVE STABILITY.** *An interval partition  $z^*$  is GLOBALLY STABLE if for any  $z^0$  in  $Z$ , the iteration from  $z^0$  converges to some partition in  $B(z^*)$ , (in an Euclidean sense). Let  $\mathcal{T}$  be a topology on  $Z$ . An interval partition  $z^*$  is  $\mathcal{T}$ -STABLE if there is a  $\mathcal{T}$ -neighborhood  $W(z^*)$  of  $z^*$  in  $Z$ , and such that for any  $z^0$  in  $W(z^*)$ , the iteration from  $z^0$  converges to some partition in  $B(z^*)$ , (in an Euclidean sense).*

Two remarks are in order. First, note that the definition of iterative stability allows different iterations that start from different initial partitions  $z^0$  to converge to different partitions in  $B(z^*)$ . Second, although for an arbitrary partition, we may have  $z \notin B(z)$ , by Lemma 1, any equilibrium  $z^* \in Z_\kappa$  satisfies  $0 \leq z^*_1 < \dots < z^*_{\kappa-1} \leq 1$ , and therefore satisfies  $z^* \in B(z^*)$ .

The topology  $\mathcal{T}$  formalizes the idea of similarity among partitions, and determines

only which perturbations or mistakes each of the equilibria candidates for being stable will be tested against. Assumptions on  $\mathcal{T}$  will be introduced in Section 5. Note that the topology  $\mathcal{T}$  plays no role in the convergence of the sequence of iterations towards the limit, since this convergence is required to be in an Euclidean sense, (to some element in  $B(z^*)$ ).

## 4 The structure of the set of equilibria

In this section we present various properties of the set of equilibria, that play an important role in the analysis of the stability of the equilibria. First, as Gordon (2010) established, the mapping  $\theta(\cdot)$  is nondecreasing on the set  $Z_\kappa \cap \theta^{-1}(Z_\kappa)$ . For any  $z, z' \in Z_\kappa$ , let  $z \leq z'$  if  $z_h \leq z'_h$  for all  $h$ , let  $z < z'$  if  $z \leq z'$  and  $z \neq z'$  and let  $z \ll z'$  if  $z \leq z'$  and  $z_h < z'_h$  for all  $h = 1, \dots, \kappa - 1$ . We say that a mapping  $\varphi : Z_\kappa \rightarrow Z_\kappa$  is nondecreasing if for all  $z, z' \in Z$ ,  $z < z' \implies \varphi(z) \leq \varphi(z')$  and that it is increasing if for all  $z, z' \in Z$ ,  $z < z' \implies \varphi(z) < \varphi(z')$ . We say that a sequence  $(z^n)$  in  $Z_\kappa$  is nondecreasing if  $z^n \leq z^{n+1}$  for all  $n$ , that it is increasing if  $z^n < z^{n+1}$  for all  $n$  and that it is strongly increasing if  $z^n \ll z^{n+1}$  for all  $n$ . Similarly, we define nonincreasing, decreasing and strongly decreasing.

LEMMA 2 (Gordon, 2010): *The mapping  $\theta(\cdot)$  is increasing on  $Z_\kappa \cap \theta^{-1}(Z_\kappa)$ .*

Second, since the mapping  $\theta(\cdot)$  can only remove cutpoints, not add new ones, any iteration from some initial vector  $z$  has the property that the number of cutpoints of  $z^n$  is nonincreasing in  $n$ . Since this number is a positive integer for all  $n$ , it follows that when  $n$  is large enough, the number of cutpoints of the vector  $z^n$  is constant.

LEMMA 3: *Let  $z^0 \in Z$ . Let*

$$\kappa^\infty \equiv \min \{k \in \mathbb{N} : \theta^n(z^0) \in Z_k \text{ for some } n \in \mathbb{N}\}$$

and

$$n_0 \equiv \min \{n \in \mathbb{N} : \theta^n(z^0) \in Z_{\kappa^\infty}\}.$$

*For all  $n \geq n_0$ , we have  $\theta^n(z^0) \in Z_{\kappa^\infty}$ .*

The remaining properties hold in the case where the bias is upward at the top (symmetric properties hold when the bias is downward at the bottom). We will now introduce a weak partial order on  $Z$  that will be helpful for games with such biases. For vectors of the same cardinality, the partial order coincides with the usual vector “coordinate by coordinate” partial order. Vectors of different cardinalities are comparable if the greater cardinality vector is comparable with the vector of equal cardinality obtained by completing the smaller cardinality vector by adding the required number of cutpoints at 0.

For all  $\kappa, \kappa' \geq 0$ , such that  $\kappa \leq \kappa'$ , and all  $z \in Z_\kappa$  and  $z' \in Z_{\kappa'}$ , let

$$z \leq_1 z'$$

if and only if

$$\left( \underbrace{0, \dots, 0}_{\kappa' - \kappa \text{ times}}, z_0, \dots, z_\kappa \right) \leq z'.$$

Also, for all  $z, z' \in Z$ , let  $z <_1 z'$  if  $z \leq_1 z'$  and  $z \neq z'$ , and let  $z \ll_1 z'$  if

$$\left( \underbrace{0, \dots, 0}_{\kappa' - \kappa \text{ times}}, z_0, \dots, z_\kappa \right) \ll z'.$$

The definition of nondecreasing, nonincreasing, decreasing, increasing, strongly decreasing and strongly increasing are extended in the natural way. For games such that the bias is upward at the top, Lemma 2 generalizes as follows.

LEMMA 4: *Suppose that the bias is upward at the top. The mapping  $\theta(\cdot)$  is non-decreasing on  $Z$ .*

**Proof.** This follows directly from Lemma 3 and the fact that when the bias is upward at the top, for all  $z \in Z_\kappa$ , the set of indices such that

$$U^S(a^R(z_{k_1+k-1}, z_{k_1+k}), \theta(z)_k) = U^S(a^R(z_{k_1+k}, z_{k_1+k+1}), \theta(z)_k)$$

is either empty or of the form  $\{k_1 + 1, \dots, \kappa - 1\}$ . Next, let  $z, z'$  be such that  $z \leq_1 z'$ .

Then

$$\theta(z) \leq_1 \theta(0, \dots, 0, z_0, \dots, z_\kappa) \leq_1 \theta(z').$$

■

The following result is an immediate implication of Lemma 4.

**THEOREM 2:** *Suppose that the bias is upward at the top and that there is a finite maximal equilibrium cardinality  $\kappa_{\max} \geq 1$ . There is an equilibrium  $z^G$  such that for all equilibrium  $z^*$ , we have  $z^* \leq_1 z^G$ . The cardinality of  $z^G$  is  $\kappa_{\max}$ .*

**Proof.** The set  $L \equiv Z_1 \cup \dots \cup Z_{\kappa_{\max}}$  is a lattice<sup>7</sup> that satisfies  $\theta(L) \subseteq L$ . By assumption, this set contains all the equilibria of the game, which are the fixed-points of the mapping  $\theta(\cdot)$  in  $L$ . The set of fixed-points of  $\theta(\cdot)$  in  $L$  is not empty, since the vector  $(0, 1)$ , which represents the babbling equilibrium, is an element of  $Z_1$ . By Tarski's fixed-point Theorem, the set of equilibria has a greatest element  $z^G$ , whose cardinality is necessarily  $\kappa_{\max}$ . ■

The assumption of the Theorem are satisfied in particular if the bias is upward. Which yields the following Corollary.

**COROLLARY 1:** *Suppose that the bias is upward. There is an equilibrium  $z^G$  such that for all equilibrium  $z^*$ , we have  $z^* \leq_1 z^G$ . The cardinality of  $z^G$  is  $\kappa_{\max}$ .*

## 5 Perturbations

The set of perturbations that a partition  $z$  will be tested against when testing for its stability is formalized by a perturbation topology  $\mathcal{T}$  on  $Z$ . A partition  $z$  is stable if it is robust to all perturbations in at least one  $\mathcal{T}$ -open set containing  $z$ .

For any two topologies  $\mathcal{T}$  and  $\mathcal{T}'$  on a set  $A$ , the topology  $\mathcal{T}$  is said to be **COARSER** than  $\mathcal{T}'$ , or equivalently  $\mathcal{T}'$  is said to be **FINER** than  $\mathcal{T}$ , if each  $\mathcal{T}$ -open set is also  $\mathcal{T}'$ -open. The coarser the topology on  $Z$  is, the more perturbations each candidate stable

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<sup>7</sup>A subset  $L \subseteq Z$  is a **complete lattice** if, for each nonempty subset  $H \subseteq L$ , the set  $\{z \in L : z \leq z' \text{ for all } z' \in H\}$  is nonempty and has a greatest element in  $L$ , the infimum of  $H$  in  $L$ ; and the set  $\{z \in L : z \geq z' \text{ for all } z' \in H\}$  is nonempty and has a least element in  $L$ , the supremum of  $H$  in  $L$ . In particular, a nonempty complete lattice  $L$  has a least element and a greatest element.

set is tested against, which makes the stability condition harder to satisfy. The coarser the topology  $\mathcal{T}$  is, the hardest it is for an equilibrium to be stable, and the easiest it is for an equilibrium to be unstable.

We introduce two coarseness conditions (C1) and (C2) and two refinement conditions (F1) and (F2) of the form “ $\mathcal{T}$  is coarser / finer than... .” Some of the conditions are not needed for some of the results, but conditions (C1), (C2) and (F2) are sufficient for all the results in the paper. Our first condition says that the topology is coarse enough for the addition of a singleton interval at 0 or 1 to always be an admissible perturbation. To formalize this idea, for any  $z, z' \in Z$ , we say that  $z$  and  $z'$  are  $\mathcal{T}$ -INDISTINGUISHABLE under some topology  $\mathcal{T}$  on  $Z$  if any  $\mathcal{T}$ -open set that contains one of these vectors also contains the other. Requiring that a topology makes two given partitions indistinguishable is requiring it to be coarse enough not to distinguish between them. Our assumption requires two partitions that differ only by a singleton interval at either 0 or 1 to be indistinguishable.

(C1) For all  $z \in Z$ , the partitions  $z, \langle 0, z \rangle$  and  $\langle z, 1 \rangle$  are indistinguishable.

Our second coarseness condition compares the topology induced by  $\mathcal{T}$  on  $Z_\kappa$  with the Euclidean topology on this set. Let  $\mathcal{E}_\kappa$  be the Euclidean metric topology on  $Z_\kappa$ . Condition (C2) says that the topology of  $Z$  should induce on  $Z_\kappa$  a topology coarser than  $\mathcal{E}_\kappa$ .

(C2) For all  $\kappa \geq 1$ , The topology  $\mathcal{T}_\kappa$  is coarser than  $\mathcal{E}_\kappa$ .

Under (C2), two vectors which are “close” for the Euclidean distance will also be considered “neighbors” by a perturbation topology  $\mathcal{T}$ . The converse is not true. For example, suppose that  $\mathcal{T}$  satisfies both (C1) and (C2). Then, by (C1), the sequence  $(0, 1 - \frac{1}{n}, 1)$  converges to  $(0, 0, 1)$  for  $\mathcal{T}$  but not in an Euclidean sense.

Our third condition requires  $\mathcal{T}$  to be fine enough to contain a certain family of open sets.

(F1) For all  $z^\circ \in Z$  and all  $z^*$  such that  $z^* \in Z_{\kappa(z^\circ)}$  and  $z^* \ll [z^\circ]$ , the set  $\{z \in Z : z^* \leq_1 z\}$  contains some  $\mathcal{T}$ -open set that contains  $z^\circ$ .

For some results, we will need the following stronger refinement condition.

(F2) For all  $z^\circ \in Z_{\kappa^\circ}$  and all  $\varepsilon > 0$ , the set of partitions  $z$  in  $Z$  such that  $\kappa(z) \geq \kappa^\circ$  and

$$\min_{h, \in \{0, \dots, \kappa(z) - \kappa^\circ\}} \left\{ d \left( (z_0, \dots, z_{\kappa(z)}), \left( \underbrace{0, \dots, 0}_h, z_0^\circ, \dots, z_{\kappa^\circ}^\circ, \underbrace{1, \dots, 1}_{\kappa(z) - \kappa^\circ - h} \right) \right) \right\} < \varepsilon$$

contains some  $\mathcal{T}$ -open set that contains  $z^\circ$ .

We now provide examples of pseudometric topologies<sup>8</sup> that satisfy the assumptions. Our first example satisfies (C1), (C2), (F1) and (F2).

EXAMPLE 1: For all  $\kappa, \kappa' \geq 1$  and all  $z \in Z_\kappa$  and  $z' \in Z_{\kappa'}$ , let

$$\mu(z, z') \equiv \min_{(h, k) \in \{0, \dots, \kappa'\} \times \{0, \dots, \kappa\}} \left\{ d \left( \left( \underbrace{0, \dots, 0}_h, z_0, \dots, z_\kappa, \underbrace{1, \dots, 1}_{\kappa' - h} \right), \left( \underbrace{0, \dots, 0}_k, z'_0, \dots, z'_{\kappa'}, \underbrace{1, \dots, 1}_{\kappa - k} \right) \right) \right\}.$$

Clearly  $\mu(\cdot, \cdot)$  satisfies the axioms for a pseudometric on  $Z$ , but it is not a metric, since it satisfies (C1). A rough intuitive description of the  $\mu(\cdot, \cdot)$  pseudometric is the following. Two partitions are close under the  $\mu(\cdot, \cdot)$  pseudometric if there is a small  $\varepsilon > 0$  and a bijection between the cutpoints of both partitions, which are in the interval  $(\varepsilon, 1 - \varepsilon)$ , such that cutpoints that are paired together are  $\varepsilon$ -close. The remaining unmatched cutpoints are confined to lie either on  $[0^-, \varepsilon]$  or on  $[1 - \varepsilon, 1^+]$ . The intervals whose ends are paired can be interpreted as “similar” sets of types that transmit “similar” information and induce “similar” actions. The unpaired intervals represent types that behave differently in the two partitions, but their measure is small and they are confined near the ends of the type space. The topology induced by  $\mu$  satisfies all four assumptions.

Our second example satisfies (C1), (C2) and (F1) but not (F2).

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<sup>8</sup>The function  $\mu : Z^2 \rightarrow \mathbb{R}_+$  is a pseudometric if for all  $z, z', z'' \in Z$ ,  $\mu(z, z) = 0$ ,  $\mu(z, z') = \mu(z', z)$  and  $\mu(z, z'') \leq \mu(z, z') + \mu(z, z'')$ . A set  $Y \subseteq Z$  is open for the topology associated with  $\mu$  if, for all  $z \in Y$ , there is some  $\varepsilon > 0$  such that the open ball centered at  $z$  with radius  $\varepsilon$  is contained in  $Y$ .

EXAMPLE 2: (*Hausdorff pseudometric*). For all  $z, z' \in Z$ , let

$$\rho(z, z') := \max \left\{ \max_{i \in \{0, \dots, \kappa\}} \min_{h \in \{0, \dots, \kappa\}} |z_i - z'_h|, \max_{j \in \{0, \dots, \kappa\}} \min_{h \in \{0, \dots, \kappa\}} |z'_j - z_h| \right\}.$$

In other words,  $\rho(z, z')$  is the smallest radius  $r \geq 0$ , such that, for each  $z_i$ , with  $i \in \{0, \dots, \kappa\}$ , there is at least one coordinate of  $z'$  within  $r$  of  $z_i$  for the pseudometric, and for each  $z'_j$ , with  $j \in \{0, \dots, \kappa\}$ , there is at least one coordinate of  $z$  within  $r$  of  $z'_j$ .

A rough intuitive description of the Hausdorff pseudometric is the following. Two partitions are close under the Hausdorff pseudometric if there is a bijection between the “large” intervals in the two partitions, such that intervals that are paired together are similar (in terms of their lower and upper bounds), and the remaining unmatched intervals in the two partitions are “small”. The “large” intervals, which are matched, can be interpreted as “similar” sets of types that transmit “similar” information and induce “similar” actions. The unpaired intervals represent types that behave differently in the two partitions, but their measure is small.

## 6 Stable equilibria

We now analyze the existence and uniqueness of a stable equilibrium. We first present a genericity condition on games and other assumptions. We then consider two classes of games. In the first class, we study games that have either an upward bias at the top (or symmetrically, a downward bias at the bottom). This class contains in particular, the case considered by Crawford and Sobel (1982), and games with an outward bias. In the second class, we study games that have an inward bias.

### 6.1 Genericity and other assumptions

Some of our results hold for all games, except for some knife-edge cases. The purpose of the genericity condition we present here is to rule out these cases. We first introduce some notation. For all  $t \in [0, 1]$ , and all  $\kappa \geq 2$ , a  $\kappa$ -forward solution at  $t$  is

a sequence  $0 = z_0 \leq z_1(t) \leq \dots \leq z_\kappa(t) \leq 1$  such that for all  $h = 1, \dots, \kappa - 1$ , we have

$$U^S(a^R(z_h(t), z_{h-1}(t)), z_h(t)) = U^S(a^R(z_{h+1}(t), z_h(t)), z_h(t)),$$

with the initial conditions  $z_0(t) = 0$  and  $z_1(t) = t$ . As Crawford and Sobel (1982) point out, a partition  $z^*$  with  $\kappa \geq 2$  intervals is a finite equilibrium if it is a  $\kappa$ -forward solution (at  $t = z_1^*$ ) that satisfies  $z_\kappa = 1$ . Consider the function, defined for all  $\kappa \geq 1$  and all  $t$  in some subset of  $[0, 1]$ , by

$$\Delta_\kappa(t) = U^S(a^R(z_{\kappa-1}(t), 1), z_{\kappa-1}(t)) - U^S(a^R(z_{\kappa-1}(t), z_{\kappa-2}(t)), z_{\kappa-1}(t)).$$

Then  $z_1^* \in [0, 1]$  is the first cutpoint of some equilibrium with  $\kappa$  intervals if  $\Delta_\kappa(\cdot)$  is well-defined at  $z_1^*$  and  $\Delta_\kappa(z_1^*) = 0$ . Note that if  $z^*$  is interior, then the function  $\Delta_\kappa(\cdot)$  is well-defined in some neighborhood of  $z_1^*$ . We say that  $z^*$  is REGULAR if  $z_1^*$  is not a local minimum of the function  $\Delta_\kappa(\cdot)$ . Moreover, we say that  $z^*$  is ISOLATED if there is some neighborhood of  $z_1^*$  in the domain of  $\Delta_\kappa(\cdot)$  where  $z_1^*$  is the unique solution of the equation  $\Delta_\kappa(z_1^*) = 0$ .

Note that if  $z^*$  is an equilibrium in  $Z_\kappa$  that is either irregular or non isolated, and the function  $\Delta_\kappa(\cdot)$  is differentiable on its domain,  $z_1^*$  is a solution in  $[0, 1]$  of the following system

$$\begin{cases} \Delta_\kappa(z_1^*) = 0 \\ \frac{d\Delta_\kappa}{dt}(z_1^*) = 0 \end{cases}$$

which motivates our second genericity condition.

(G1) *All the equilibria of the game are regular and isolated.*

We have the following useful result on isolated and regular equilibria of a game.

LEMMA 5: *Let  $z^* \in Z_\kappa$  be an interior, isolated and regular equilibrium of some game, such that  $\lceil z^* \rceil \neq (0, 1)$ . Suppose that  $(z_n)$  is a decreasing (increasing) iteration from  $z^0 \in Z_\kappa$  that converges to  $z^*$ . Then there exists an increasing (decreasing) iteration  $(y_n)$  from some  $y^0 \in Z_{\kappa(\lceil z^* \rceil)}$  that converges to  $\lceil z^* \rceil$ . Moreover  $(z_n)$  is strongly decreasing (increasing) and  $(y_n)$  is strongly increasing (decreasing) for  $n$  large enough.*



**Proof.** Since the iteration from  $z^0$  is decreasing and converges to  $z^*$ , with  $z^0, z^* \in Z_\kappa$ , it must be that  $z^0$  is strongly decreasing for  $n$  large enough. Therefore  $z^* \ll z^0$ . Next, if  $z_1^* = 0$ , redefine  $z^*$  to be  $\lceil z^* \rceil$ , so that  $z_0^* < z_1^*$  and  $\kappa$  to be such that  $z^* \in Z_\kappa$ . Since  $z^*$  satisfies  $z_1^* > 0$  and  $z_{\kappa-1}^* < 1$  and is isolated, let  $W(z_1^*)$  be an open interval around  $z_1^*$  such that  $\Delta_\kappa(\cdot)$  is well-defined on  $W(z_1^*)$  and satisfies  $\Delta_\kappa(t) \neq 0$  for all  $t \in W(z_1^*) \setminus \{z_1^*\}$ . Recall that  $(0, z_1^*, z_2(z_1^*), \dots, z_{\kappa-1}(z_1^*), 1) = z^*$ . Since  $z^*$  is regular, there is some  $y_1^0$  close enough to  $z_1^*$  so that  $y^0 \equiv (0, y_1^0, z_2(y_1^0), \dots, z_{\kappa-1}(y_1^0), 1) \ll z^0$ ,  $y_1^0 \in W(z_1^*)$  and  $\Delta_\kappa(z_1) < 0$ . By continuity of  $\Delta_\kappa(\cdot)$ , it must be that  $\Delta_\kappa(t) < 0$  for all  $t \in W(z_1^*) \setminus \{z_1^*\}$ . The inequality  $\Delta_\kappa(y_1^0) < 0$  implies that the iteration from  $y^0$  is increasing. Since this iteration is also bounded above by the iteration from  $z^0$ , it converges to some equilibrium  $y^*$  such that  $y^0 < y^* \leq z^*$ . But this implies that  $y_1^* \in [y_1^0, z_1^*] \subseteq W(z_1^*)$ . Since  $y^*$  is an equilibrium, we have  $\Delta_\kappa(y_1^*) = 0$ . By the definition of  $W(z_1^*)$ , necessarily  $y_1^* = z_1^*$ , which implies that  $y^* = z^*$ . Using again the same argument as in the first sentence of the proof, we obtain that  $y^0 \ll z^*$ . ■

For some results, we will assume that there is at most one equilibrium of any cardinality.

(U) For each  $\kappa \geq 1$ , there is at most one equilibrium with  $\kappa$  intervals.

Our Conditions (G1) and (U) are related to Condition (M), introduced by Crawford and Sobel (1982), which plays an important role in the literature. We do not need this condition for any result in the paper, but we mention it here to shed some light on the content of our Condition (G1). Condition (M), says that for all  $\kappa \geq 2$ , the  $\kappa$ -forward solution is increasing on its domain.

(M) For each  $\kappa \geq 2$ , and all  $z_1 < z_1'$  such that  $z_\kappa(\cdot)$  is well-defined at  $z_1$  and  $z_1'$ , we have  $z_\kappa(z_1) < z_\kappa(z_1')$ .

As Crawford and Sobel (1982) observed, Condition (M) implies Condition (U). They also show that Condition (M) implies that the ex ante expected payoffs of the sender and the receiver are strictly increasing in the size of the equilibrium. Our genericity condition (G1) is both a weaker and local version of (M), as it says that at any equilibrium  $z^*$  with  $\kappa$  intervals, the forward solution  $z_\kappa(\cdot)$  may increase or

decrease, but does not reach a local minimum at  $z_1^*$ . Note that (G1) is implied by (M), therefore any result that holds under (G1) will also hold if (G1) is replaced by (M).

Finally, for some results we will require the game to be such that the maximal number of intervals in equilibrium is finite.

(B) *There is  $\kappa_{\max} < +\infty$  such that the maximal number of intervals in any equilibrium of the game is  $\kappa_{\max}$ .*

From Crawford and Sobel (Lemma 1, 1982), we know that this condition is satisfied by all games in which the bias is upward, but from Gordon (2010), we know that it is satisfied by no game in which the bias is outward.

## 6.2 Upward bias at the top

Throughout this section, we assume that the bias is upward at the top. From Theorem 2, any game in this class that in addition satisfies (B) is such that the set of equilibria has a greatest element  $z^G$ , which has cardinality  $\kappa_{\max}$ . For all  $\kappa \geq 1$ , let  $1^\kappa$  be the greatest element of the set  $Z_\kappa$ , i.e.  $1^\kappa \equiv (0, 1, \dots, 1)$ .

LEMMA 6: *Let  $\mathcal{T}$  satisfy (C1). Suppose that the bias is upward at the top. If  $z$  and  $z'$  are equilibrium partitions such that  $z <_1 z'$  and  $z' \notin B(z)$ , the partition  $z$  is not  $\mathcal{T}$ -stable.*

**Proof.** Let  $\kappa, \kappa' \geq 1$  be such that  $z \in Z_\kappa$  and  $z \in Z_{\kappa'}$ . Necessarily,  $\kappa \leq \kappa'$ . Let  $z^0 \equiv (z_0, \dots, z_\kappa, 1, \dots, 1) \in Z_{\kappa+\kappa'}$ . From (C1), the vector  $z^0$  is  $\mathcal{T}$ -indistinguishable from the vector  $z$ . Thus  $z^0$  is an element of any neighborhood of  $z$  in  $Z$ . We will prove that the iteration from  $z^0$  does not converge to  $z$ . The set

$$L \equiv \{y \in Z_1 \cup \dots \cup Z_{\kappa+\kappa'} : y \leq z^0\}$$

is a complete lattice and satisfies  $\theta_\kappa(L) \subseteq L$ . Thus the set of fixed-points of  $\theta(\cdot)$  in  $L$  has a greatest element  $z^M$ . Since  $z' \in L$ , we have  $z' \leq z^M$ . We will now show that

the iteration from  $z^0$  converges to  $z^M$ .<sup>9</sup>

Because  $a^R(1) \leq a^S(1)$  and  $z$  is an equilibrium, we have  $\theta(z^0) \leq z^0$ . Since the mapping  $\theta^n(\cdot)$  is nondecreasing for all  $n$ , then  $\theta^n(\theta(z^0)) \leq \theta^n(z^0)$ , i.e.  $\theta^{n+1}(z^0) \leq \theta^n(z^0)$  for all  $n \geq 0$ . Therefore, the iteration from  $z^0$  is nonincreasing. Moreover  $z^M \leq z^0$ . Since the mapping  $\theta^n(\cdot)$  is nondecreasing for all  $n$ , then  $\theta^n(z^G) \leq \theta^n(z^0)$ , i.e.  $z^G \leq \theta^n(z^0)$  for all  $n \geq 0$ . Therefore the iteration from  $z^0$  is also bounded below by  $z^M$ . There is  $n_0 \geq 1$  and  $\kappa''$  such that for all  $n \geq n_0$ ,  $\theta^n(z^0) \in Z_{\kappa''}$ . Thus the iteration from  $z^0$  converges in an Euclidean sense to some  $z^* \in Z_{\kappa''}$ , such that  $z^M \leq z^* \leq z^0$ . By continuity of the mapping  $\theta(\cdot)$  on  $Z_{\kappa''}$  for the Euclidean topology, the limit  $z^*$  is a fixed-point of  $\theta(\cdot)$  in  $Z_{\kappa''}$ , thus in  $L$ . By definition of  $z^M$  and since we already observed that  $z^M \leq z^*$ , it must be that  $z^* = z^M$ . This implies that the iteration from  $z^0$  converges to  $z^M$ . Since  $z' <_1 z^0$ , by monotonicity of  $\theta(\cdot)$ , we have  $z' \leq_1 z^M$ . Since the bias is outward, we have  $z_{\kappa(z^M)-1}^M < 1$ , thus  $z^M \notin \{\langle [z], 1 \rangle, \langle 0, [z], 1 \rangle\}$ . Since  $z <_1 z' \leq_1 z^M$  and  $z' \notin B(z)$ , it follows that  $z^M \notin \{[z], \langle 0, [z] \rangle\}$ . Therefore  $z^M \notin B(z)$ , which proves that the iteration from  $z^0$  does not converge to any element of  $B(z)$  and therefore that  $z$  is not  $\mathcal{T}$ -stable, the desired conclusion. ■

The following result is a direct implication of Lemma 6. For games such that  $\kappa_{\max} < +\infty$ , we can apply Lemma 6 to any candidate equilibrium  $z$  and to  $z' = z^G$ , the greatest equilibrium. For games for which the maximal number of actions taken in equilibrium is unbounded, we can apply Lemma 6 to any candidate finite equilibrium  $z$  and to the greatest element  $z'$  of the set of equilibria that have exactly three more cutpoints than  $z$ , which implies that  $z' \notin B(z)$ . This yields the following result.<sup>10</sup>

**THEOREM 3:** *Let  $\mathcal{T}$  satisfy (C1). Suppose that the game satisfies (B) and that the bias is upward at the top. Then no equilibrium that is not in  $B(z^G)$  is  $\mathcal{T}$ -stable. In particular, if the bias is upward, no equilibrium that is not in  $B(z^G)$  is  $\mathcal{T}$ -stable. If the set of equilibrium cardinalities is unbounded, no finite equilibrium is  $\mathcal{T}$ -stable. In particular, if the bias is outward, no finite equilibrium partition is  $\mathcal{T}$ -stable.*

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<sup>9</sup>One can also show that if  $z^1 \in L$  is such that  $z^G \leq z^1 \leq z^0$ , then the iteration from  $z^1$  also converges to  $z^M$ .

<sup>10</sup>One can prove the following stronger version of the first sentence in Theorem 3. In any  $\mathcal{T}$ -open set of  $Z$ , there exists some  $z^0$  such that the iteration from  $z^0$  converges to  $z^G$ .

By Theorem 3, at most one equilibrium may be  $\mathcal{T}$ -stable. If equilibrium cardinalities are bounded, the unique candidate is the greatest equilibrium  $z^G$ . We now show that, generically, this equilibrium is indeed iteratively  $\mathcal{T}$ -stable, provided that  $\mathcal{T}$  satisfies (F1).

**THEOREM 4:** *Let  $\mathcal{T}$  satisfy (F1). Suppose that the game satisfies (G1), (B), and that the sender's bias is upward at the top. Then the greatest equilibrium  $z^G$  is  $\mathcal{T}$ -stable. If in addition,  $z^G$  is either the partition  $(0, 1)$  or  $(0, 0, 1)$ ,  $z^G$  is even globally stable.*

**Proof.** We distinguish two cases, depending on whether  $\lceil z^G \rceil = (0, 1)$  or not.

**Case 1:**  $\lceil z^G \rceil = (0, 1)$ .

Since the sender's bias is upward at the top, necessarily  $z^G \neq \langle 0, \lceil z^G \rceil, 1 \rangle$  and  $z^G \neq \langle \lceil z^G \rceil, 1 \rangle$ . Then the iteration from  $1^\kappa$  for any  $\kappa \geq 2$  is a decreasing sequence that converges to  $z_G$ . Therefore, the iteration from any  $z^0 \in Z_\kappa$ , for any  $\kappa \geq 2$  is a decreasing sequence that converges to  $z_G$ . Moreover, the iteration from  $(0, 1)$  converges to  $(0, 1) = \lceil z^G \rceil$ . Therefore  $z^G$  is globally stable.

**Case 2:**  $\lceil z^G \rceil \neq (0, 1)$ .

Since the iteration from  $1^\kappa$  for some  $\kappa$  large enough is a decreasing sequence that converges to  $z^G$  and  $\lceil z^G \rceil \neq (0, 1)$ , Lemma 5 ensures that there is a vector  $y^0 \ll z^G$  such that the iteration from  $y^0$  is increasing and converges to  $\lceil z^G \rceil$ . Since this iteration is strongly increasing for  $n$  large enough, we have  $y^0 \ll z^G$ . From (F1), the set  $W(z^G) := \{z \in Z_\kappa : y^0 \leq z\}$  is a  $\mathcal{T}$ -neighborhood of  $z^G$ . Consider now an arbitrary  $z^0 \in W(z^G)$  and the iteration from  $z^0$ . Then, we have  $y^0 \leq_1 z^0 \leq_1 1_\kappa$ , for some  $\kappa$ . The iteration from  $y^0$  converges to  $\lceil z^G \rceil$  and the iteration from  $1_\kappa$  converges to  $z^G$ . Since the bias is upward, we have  $z_{\kappa-1}^G < 1$  so that either  $z^G = \lceil z^G \rceil$  or  $z^G = \langle 0, \lceil z^G \rceil \rangle$ . Let

$$\kappa^\infty \equiv \min \{k \in \mathbb{N} : \theta^n(z^0) \in Z_k \text{ for some } n \in \mathbb{N}\}.$$

Either  $\kappa^\infty = \kappa(\lceil z^G \rceil)$  or  $\kappa^\infty = \kappa(\langle 0, \lceil z^G \rceil \rangle)$ . In the first case, the iteration converges to  $\lceil z^G \rceil$ , in the second to  $\langle 0, \lceil z^G \rceil \rangle$ . Therefore  $z^G$  is  $\mathcal{T}$ -stable, the desired conclusion. ■

Since Condition (M) implies (G1) and that there is at most one equilibrium of

each cardinality, we obtain the following Corollary.

**COROLLARY 2:** *Let  $\mathcal{T}$  satisfy (F1). Suppose that the bias is upward and that Condition (M) holds. Then the unique equilibrium with  $\kappa_{\max}$  intervals is the unique  $\mathcal{T}$ -stable equilibrium.*

## 6.3 Inward bias

In this section we consider the class of games with an inward bias whose equilibria have a bounded number of actions. Then, we focus on games that are symmetric around the central type  $\frac{1}{2}$ .

### 6.3.1 General results

We will say that an equilibrium  $z^* \in Z_\kappa$  is INTERIOR if the inequalities

$$\begin{aligned} U^S(a^R(0, z_1^*)) &\neq U^S(a^R(0, 0)) \\ U^S(a^R(z_{\kappa-1}^*, 1)) &\neq U^S(a^R(1, 1)) \end{aligned}$$

hold. Note that if an equilibrium  $z^*$  is interior, then  $\langle 0, z^* \rangle$  and  $\langle z^*, 1 \rangle$  are not equilibria. Games in which some equilibria violate this condition are knife-edge cases. In this section, we restrict attention to game such that all equilibria are interior, our second genericity condition.

(G2) *All equilibria of the game are interior.*

Note that in a game that satisfies (G2), all equilibria  $z^* \in Z_\kappa$  are such that  $z_1^* > 0$  and  $z_{\kappa-1}^* < 1$ .

Chen, Kartik and Sobel (2008) introduce the following concept for the lowest type in the upward bias case. Here, in the inward bias case, it is natural to define it on both ends.

**DEFINITION 3: INCENTIVE TO SEPARATE.** *Let  $t \in \{0, 1\}$ . We say that type 0 has an incentive to separate at equilibrium  $z^*$  if  $U^S(a^R(0, z_1^*), 0) < U^S(a^R(0, 0), 0)$  that it has no incentive to separate (NITS) at equilibrium  $z^*$  if  $U^S(a^R(0, z_1^*), 0) >$*

$U^S(a^R(0,0),0)$ . Similarly we say that type 1 has an incentive to separate at equilibrium  $z^*$  if  $U^S(a^R(z_{\kappa-1}^*,1)) < U^S(a^R(1,1))$  that it has no incentive to separate (NITS) at equilibrium  $z^*$  if  $U^S(a^R(z_{\kappa-1}^*,1)) > U^S(a^R(1,1))$ .

**DEFINITION 4: S-STABILITY.** We say that a partition equilibrium  $z^* \in Z_\kappa$  is  $S$ -stable if there are partitions  $z', z'' \in Z_\kappa$  such that  $z' \ll z^* \ll z''$  and the iterations from both  $z'$  and  $z''$  converge to  $z^*$  (in an Euclidean sense).

Note that by Lemma 5, under (G1), a sufficient condition for  $z^*$  to be  $S$ -stable is that there is a partition  $z^\circ \in Z_\kappa$  such that the iteration from  $z^\circ$  in either increasing or decreasing and converge to  $z^*$  (in an Euclidean sense). The following two Lemmas provides a simple test for  $\mathcal{T}$ -stability of an equilibrium, under assumptions (C1), (C2) and (F2).

**LEMMA 7:** Let  $\mathcal{T}$  satisfy (C1), (C2). If an equilibrium  $z$  is  $\mathcal{T}$ -stable, then it is  $S$ -stable and exactly one of the following properties holds. (i) It satisfies NITS on both ends. (ii) Only type 0 has an incentive to separate and the iteration from  $\langle 0, z \rangle$  converges to  $z$ . (iii) Only type 1 has an incentive to separate and the iteration from  $\langle z, 1 \rangle$  converges to  $z$ .

**Proof.** First  $\mathcal{T}$ -stability implies  $S$ -stability. This is because in the (Euclidean) neighborhood of  $z$  in  $Z_\kappa$ , there is always some partition such that iterations of  $\theta$  from this partition are monotone, therefore convergent. If the convergence is not towards  $z$ , the equilibrium  $z$  is not  $\mathcal{T}$ -stable. Second,  $\mathcal{T}$ -stability also implies that one of the conditions (i), (ii) or (iii) holds. Indeed, if  $z$  violates NITS on both sides, the iterations from from  $\langle 0, z \rangle$  and  $\langle z, 1 \rangle$  converge to some equilibrium with one more interval than  $z$ . If only type 0 has an incentive to separate, then the iteration from  $\langle 0, z \rangle$  is monotone, therefore converges. But if it does not converge to  $z$ , the equilibrium is not  $\mathcal{T}$ -stable. ■

In particular, if an equilibrium is  $\mathcal{T}$ -stable, (and  $\mathcal{T}$  satisfies (C1)) it must satisfy NITS at least on one end. We have the following converse implication.

**LEMMA 8:** Let  $\mathcal{T}$  satisfy (F2). Suppose that an equilibrium  $z$  is  $S$ -stable and satisfies exactly one of the following properties (i) It satisfies NITS on both ends. (ii)

Only type 0 has an incentive to separate and the iteration from  $\langle 0, z \rangle$  converges to  $z$ .  
 (iii) Only type 1 has an incentive to separate and the iteration from  $\langle z, 1 \rangle$  converges to  $z$ . Then  $z$  is  $\mathcal{T}$ -stable

**Proof.** If an equilibrium is  $S$ -stable, and either (i), (ii) and (iii), then for  $\varepsilon > 0$  small enough, the iteration from any partition  $z \in Z$  such that  $\kappa(z) \geq \kappa^\circ$  and

$$\min_{h, \in \{0, \dots, \kappa(z) - \kappa^\circ\}} \left\{ d \left( (z_0, \dots, z_{\kappa(z)}), \left( \underbrace{0, \dots, 0}_h, z_0^\circ, \dots, z_{\kappa}^\circ, \underbrace{1, \dots, 1}_{\kappa(z) - \kappa^\circ - h} \right) \right) \right\} < \varepsilon$$

converge to  $z$ , and under (F2), this set is an  $\mathcal{T}$ -neighborhood of  $z$ . ■

**THEOREM 5:** *Let  $\mathcal{T}$  satisfy (F2). Suppose that the game satisfies (G1), (G2) and (B) and that the bias is inward. Then there is a  $\mathcal{T}$ -stable equilibrium, which either has the maximal number of intervals or this number minus one.*

**Proof.** Let  $m$  be the maximal number of intervals in any equilibrium. We distinguish two cases, depending on whether there is at least one equilibrium with  $m$  intervals that satisfies NITS on exactly one end or not.

**Case 1:** We first consider the case where at least one equilibrium  $y^0$  with  $m$  intervals that satisfies NITS on exactly one end. Let  $\Upsilon$  be the set of equilibria with  $m$  intervals. All equilibria in  $\Upsilon$  satisfy NITS on at least one end, otherwise there would be an equilibrium with  $m + 1$  intervals, which contradicts the definition of  $m$ . Next, construct a sequence of equilibria in  $\Upsilon$  as follows. Take  $y^0$  as the initial term of the sequence, which we define by induction. For all  $n \geq 0$ , we define  $y^{n+1}$  and we distinguish three cases. First, if  $y^n$  satisfies NITS on both ends, let  $y^{n+1} \equiv y^n$ . Second, if  $y^n$  is such that type 0 has an incentive to separate, but not type 1, consider the iteration from  $\langle 0, y^n \rangle$  and let  $y^{n+1}$  be the limit of the iteration. Third, if  $y^n$  is such that type 1 has an incentive to separate, but not type 0, consider the iteration from  $\langle y^n, 1 \rangle$  and let  $y^{n+1}$  be the limit of the iteration. The sequence  $(y^n)_n$  is well defined, because in the second and third cases, the iteration is monotone and converges in  $\Upsilon$ .

Since  $\Upsilon$  is finite, the sequence  $(y^n)_n$  must be cyclical, i.e. there is some integer  $k \geq 1$  such that for all  $n$  large enough, we have  $y^{n+k} = y^n$ . We now show that the

cycle is of length 1. Indeed, suppose that this is not the case. This implies that there is an index  $n \geq 0$  such that at  $y^n$ , type 0 has an incentive to separate, but not type 1; and at  $y^{n+1}$ , type 1 has an incentive to separate, but not type 0. But this would imply that there is an equilibrium with  $m + 1$  intervals, which contradicts the definition of  $m$ . Thus, the cycle has length one, and there is  $y^\infty \in \Upsilon$  such that  $y_n = y^\infty$  for all  $n$  large enough. But then  $y^\infty$  is a  $\mathcal{T}$ -stable equilibrium with  $m$  intervals.

**Case 2:** Suppose now that no equilibrium with  $m$  intervals satisfies NITS on exactly one end. This implies that there exists an equilibrium  $y^0$  with  $m$  intervals that satisfies NITS on both ends. If this equilibrium is  $S$ -stable, it is then  $\mathcal{T}$ -stable. Otherwise, there is an  $S$ -perturbation of it that generates a monotone iteration that converges to an  $S$ -stable equilibrium  $y^1$  with either  $m$  or  $m - 1$  intervals. Either way, if  $y^1$  satisfies NITS on both ends, it is then an  $\mathcal{T}$ -stable equilibrium with either  $m - 1$  or  $m$  intervals. Suppose that at least one end type has an incentive to separate at  $y^1$ , for example type 0. By assumption, this implies that  $y^1$  has  $m - 1$  intervals. The iteration from  $\langle 0, y^1 \rangle$  converges to some  $S$ -stable equilibrium  $y^2$  with  $m$  intervals. By assumption,  $y^2$  satisfies NITS on both ends, thus it is  $\mathcal{T}$ -stable.

In all cases, we found a  $\mathcal{T}$ -stable equilibrium, with either  $m$  or  $m - 1$  intervals. ■

### 6.3.2 Games that have a unique equilibrium with $\kappa$ actions.

We restrict now attention to games that have an inward bias, whose equilibria have a bounded number of actions, i.e. they satisfy Condition (B), and in addition, have at most one equilibrium with  $\kappa$  intervals, for all  $\kappa \geq 1$ , i.e. they satisfy Condition (U).

Our next result says under these assumptions, all equilibria, except perhaps the equilibrium with  $\kappa_{\max}$  intervals, are  $S$ -stable.

**LEMMA 9:** *Suppose that the game satisfies (U) and that the bias is inward. Then any equilibrium whose cardinality is not maximal is  $S$ -stable.*

**Proof.** An implication of results by Gordon (2010) and Condition (U) is that the equilibria are nested, which means that if  $\kappa \geq 1$  and if  $z^* = (z_0^*, \dots, z_{\kappa+1}^*)$  and  $z^\circ = (z_0^\circ, \dots, z_\kappa^\circ)$  are respectively the equilibria with  $\kappa + 1$  and  $\kappa$  intervals then



$$(0, z_1^*, \dots, z_{\kappa-1}^*, 1) \ll z^\circ \ll (0, z_2^*, \dots, z_\kappa^*, 1).$$

Moreover the iterations of  $\theta$  from  $(0, z_1^*, \dots, z_{\kappa-1}^*, 1)$  and  $(0, z_2^*, \dots, z_\kappa^*, 1)$  converge monotonically to  $z^\circ$ . Therefore  $z^\circ$  is  $S$ -stable, the desired conclusion. ■

Our next result says that, under an inward bias and Conditions  $(B)$  and  $(U)$ , if some equilibrium with less than  $\kappa_{\max}$  intervals satisfies NITS on at least one end, the equilibrium with one more interval, is not  $S$ -stable.

**LEMMA 10:** *Suppose that the game satisfies  $(B)$  and  $(U)$  and that the bias is inward. Then if for some  $\kappa \leq \kappa_{\max} - 1$ , the equilibrium with  $\kappa$  intervals satisfies NITS (at least at one end), then the equilibrium with  $\kappa + 1$  intervals is not  $S$ -stable.*

**Proof.** Let  $y^*$  be the equilibrium with  $\kappa$  intervals and, without loss of generality, suppose that  $y^*$  satisfies NITS at 1. Let  $z^*$  be the equilibrium with  $\kappa + 1$  intervals. The  $\kappa$ -forward solution function  $z_\kappa(\cdot)$  is well-defined at  $y_1^*$  and  $z_1^*$  and  $\Delta_{\kappa+1}(y_1^*) < 0$  since  $y^*$  satisfies NITS at 1, and  $\Delta_{\kappa+1}(z_1^*) = 0$ , since  $z^*$  is an equilibrium with  $\kappa + 1$  intervals. We will show that the  $\kappa$ -forward solution function is well-defined on some right neighborhood of  $z_1^*$  and that  $\Delta_{\kappa+1}(\cdot)$  is negative in this neighborhood. Let

$$w_1^* \equiv \sup \{x_1 \in (z_1^*, y_1^*] : \text{the function } z_\kappa(\cdot) \text{ is well-defined on } [z_1^*, x_1]\}.$$

Since  $0 < z_1^* < \dots < z_{\kappa-1}^* < 1$ , the function  $z_1^*$  is well-defined in some neighborhood of  $z_1^*$ . Therefore  $w_1^* > z_1^*$ . By continuity,  $z_\kappa(\cdot)$  is well-defined at  $w_1^*$ . We now distinguish two cases,  $w_1^* < y_1^*$  and  $w_1^* = y_1^*$ . In both, cases, we will show that  $\Delta_{\kappa+1}(w_1^*) < 0$ .

*Case 1:*  $w_1^* < y_1^*$ . In this case, by assumption  $(U)$ ,  $z_\kappa(w_1^*) < 1$ . Therefore  $z_\kappa(w_1^*) = z_{\kappa-1}(w_1^*)$ , which implies that  $\Delta_{\kappa+1}(w_1^*) < 0$ .

*Case 2:*  $w_1^* = y_1^*$ , so that  $z_\kappa(w_1^*) = 1$  and  $\Delta_{\kappa+1}(w_1^*) < 0$  since  $y^*$  satisfies NITS at 1.

Next, there cannot be any  $x_1 \in (z_1^*, w_1^*)$  such that  $\Delta_\kappa(x_1^*) = 0$  since then we would have an equilibrium with  $\kappa + 1$  intervals distinct from  $z^*$ , which would contradict  $(U)$ . Thus by continuity  $\Delta_{\kappa+1}(\cdot)$  is negative in  $(z_1^*, w_1^*)$ .

This implies that the iteration from any partition  $(0, x_1, z_2(x_1), \dots, z_\kappa(x_1), 1)$  with  $x_1 \in (z_1^*, w_1^*)$  is increasing and does not converge to  $z^*$ , since  $z_1^* < x_1^*$ . Such a starting

point for the iteration can be found in any neighborhood of  $z^*$ , by taking  $x_1^*$  close enough to  $z_1^*$ . Thus  $z^*$  is not  $S$ -stable. ■

Lemmas 9 and 10 permit a very precise description of the set of  $\mathcal{T}$ -stable equilibria for games with an inward bias and that satisfy  $(B)$ ,  $(U)$ ,  $(G1)$  and  $(G2)$ .

Indeed, the equilibrium  $z^*$  with  $\kappa_{\max}$  intervals satisfies NITS at least at one end. Otherwise, the set  $Y$  of partitions  $z \in Z_{\kappa_{\max}+1}$  such that  $\langle 0, z^* \rangle \leq z \leq \langle z^*, 1 \rangle$ , which is a complete sublattice, would be such that  $\theta(Y) \subseteq Y$ , which by Tarski's fixed-point Theorem, would imply that there is an equilibrium in  $Z_{\kappa_{\max}+1}$ , which would contradict the definition of  $\kappa_{\max}$ . Therefore there is some smallest integer  $\kappa^*$  such that the equilibrium with  $\kappa$  intervals satisfies NITS at one end (at least). Lemmas 9 and 10 imply that  $\kappa^*$  is either  $\kappa_{\max}$  or  $\kappa_{\max} - 1$ .

By definition of  $\kappa^*$ , no equilibrium with less than  $\kappa^*$  intervals satisfies NITS. Under  $(U)$ , the equilibria are nested, which implies that the length of the first and last interval of the equilibrium with  $\kappa$  intervals is decreasing in  $\kappa$ . Therefore the equilibria with at least  $\kappa^*$  intervals are the ones that satisfy NITS (at least at one end, the same for all these equilibria).

If  $\kappa^* = \kappa_{\max}$ , the equilibria with less than  $\kappa_{\max}$  intervals are  $\mathcal{T}$ -unstable (under  $(C1)$ ) because they do not satisfy NITS on any side. In this case, the equilibrium with  $\kappa_{\max}$  intervals is the only remaining candidate for stability.

If  $\kappa^* = \kappa_{\max} - 1$ , the equilibria with less than  $\kappa_{\max} - 1$  intervals are  $\mathcal{T}$ -unstable (under  $(C1)$ ) because they do not satisfy NITS on any side. Moreover, the equilibrium with  $\kappa_{\max}$  intervals is not  $\mathcal{T}$ -unstable (under  $(C2)$ ) because it is not  $S$ -stable. In this case, the equilibrium with  $\kappa_{\max} - 1$  intervals is the only remaining candidate for stability.

Thus, under  $(B)$ ,  $(U)$ ,  $(C1)$  and  $(C2)$ , there is at most one  $\mathcal{T}$ -stable equilibrium, and any  $\mathcal{T}$ -stable equilibrium has either  $\kappa_{\max} - 1$  or  $\kappa_{\max}$  intervals. Moreover, under  $(B)$ ,  $(F2)$ ,  $(G1)$  and  $(G2)$ , there exists a  $\mathcal{T}$ -stable equilibrium. We summarize these findings in the following result.

**THEOREM 6:** *Let  $\mathcal{T}$  satisfy  $(C1)$ ,  $(C2)$  and  $(F2)$ . Suppose that the game satisfies  $(G1)$ ,  $(G2)$ ,  $(B)$  and  $(U)$  and that the bias is inward. Then there exists a unique  $\mathcal{T}$ -stable equilibrium. It has either  $\kappa_{\max}$  or  $\kappa_{\max} - 1$  intervals. If it has  $\kappa_{\max}$  intervals,*

it satisfies NITS on at least one end, and the equilibrium with  $\kappa_{\max} - 1$  intervals does not satisfy NITS at any end. If it has  $\kappa_{\max} - 1$  intervals, it satisfies NITS at both ends, and the equilibrium with  $\kappa_{\max}$  intervals is not  $S$ -stable.

### 6.3.3 Symmetric games

The statement of Theorem 6 can be made even more precise for games that, in addition to the assumptions in the Theorem 6, are also symmetric around the central type  $t = \frac{1}{2}$ . For such games, we show that the second term of the alternative always holds. In particular, the equilibrium with  $\kappa_{\max} - 1$  intervals is always  $\mathcal{T}$ -stable, under (F2) (Condition (G1) will be implied by the other conditions, as we shall see) and no other equilibrium is  $\mathcal{T}$ -stable under (C1) and (C2). We say that a game is SYMMETRIC AROUND  $t = \frac{1}{2}$  if there is some action  $a^*$  such that for all  $i \in \{R, S\}$ , all  $t \in [0, \frac{1}{2}]$  and all  $a \in \mathbb{R}$ , we have

$$U^i \left( a^* - a, \frac{1}{2} - t \right) = U^i \left( a^* + a, \frac{1}{2} + t \right)$$

and

$$f \left( \frac{1}{2} - t \right) = f \left( \frac{1}{2} + t \right).$$

(S) The game is symmetric around  $t = \frac{1}{2}$ .

Symmetry and uniqueness of the equilibrium with  $\kappa$  intervals imply that all equilibria are symmetric around  $\frac{1}{2}$ . Therefore, any equilibrium either satisfies NITS on both sides or on none of them. Moreover, if some equilibrium  $z$  with  $\kappa$  intervals fails NITS, one can construct at least one equilibrium with  $\kappa + 2$  intervals. This is done by iterating the mapping  $\theta$  from the partition  $\langle \langle 0, z \rangle, 1 \rangle$  obtained from  $z$  by adding a cutpoint at 0 and another at 1. This iteration converges to some partition with  $\kappa + 2$  intervals, which is necessarily an equilibrium. This implies that both equilibria with  $\kappa_{\max} - 1$  and  $\kappa_{\max}$  intervals satisfy NITS on both sides and in particular that the equilibrium with  $\kappa_{\max} - 1$  intervals is the unique  $\mathcal{T}$ -stable equilibrium. Finally Conditions (S) and (U) imply (G1). We summarize these results next.

THEOREM 7: Suppose that the game satisfies (B), (U), (S), (G2) and that the bias is inward. Then  $\kappa_{\max} \geq 2$ . If  $\mathcal{T}$  satisfies (C1) and (C2), no equilibrium different from the equilibrium with  $\kappa_{\max} - 1$  intervals is  $\mathcal{T}$ -stable. If  $\mathcal{T}$  satisfies (F2), this equilibrium is  $\mathcal{T}$ -stable.

To illustrate, we provide a class of examples of symmetric games with an inward bias. As the Theorem predicts, under (C1), (C2) and (F2), the equilibrium with the maximal number of intervals minus one is uniquely  $\mathcal{T}$ -stable.

EXAMPLE 3: Let  $t$  be uniformly distributed on  $[0, 1]$ ,  $U^R(a, t) = -(a - t)^2$ , and  $U^S(a, t) = -(a - [(1 - 2b)t + b])$ , with  $b \in [0, \frac{1}{2}]$ . Thus the sender and the receiver have the same preferred action  $a = \frac{1}{2}$  when  $t = \frac{1}{2}$  and the problem is symmetric around the type  $t = \frac{1}{2}$  and the action  $a^* = \frac{1}{2}$ . When his type is less than  $\frac{1}{2}$ , the sender has an upward bias, when his type is greater than  $\frac{1}{2}$ , he has a downward bias. The absolute value of the difference between the preferred actions of the two players is maximized for types 0 and 1, where it equals  $b$ . For any value of the parameter  $b$ , there exists some positive integer  $\kappa_{\max}(b)$ , such that the game has exactly one equilibrium with  $\kappa$  intervals for  $\kappa = 1, \dots, \kappa_{\max}(b)$ . All equilibria are symmetric around the type  $\frac{1}{2}$ . The positive integer  $\kappa_{\max}(b)$  is weakly decreasing in  $b$ . The partitions  $(0, 1)$  and  $(0, \frac{1}{2}, 1)$  are equilibria for all  $b \in [0, \frac{1}{2}]$ , so that  $\kappa_{\max}(b) \geq 2$ . The limit of  $\kappa_{\max}(b)$  as  $b$  goes to 0 is  $+\infty$ . The values of  $\kappa_{\max}$  for different values of  $b$  are given in the following table.

$b$	$\rightarrow 0^+$	...	$(1 - \frac{\sqrt{2}}{8}, \frac{1}{8})$	$(\frac{1}{8}, \frac{1}{4})$	$(\frac{1}{4}, \frac{1}{2})$
$\kappa_{\max}$	$+\infty$	...	4	3	2

For any generic value of  $b \in (0, \frac{1}{2})$ , that differs from a boundary value at which some new equilibrium appears, the only stable equilibrium is the one that has  $\kappa_{\max} - 1$  actions. Because the equilibria are symmetric around the type  $\frac{1}{2}$ , any equilibrium either satisfies NITS on both sides or in neither of them. The only equilibria that satisfy NITS (on both sides) are the equilibria with either  $\kappa_{\max}$  or  $\kappa_{\max} - 1$  intervals. But the only equilibria that satisfy S-stability are the equilibria with at most  $\kappa_{\max} - 1$  intervals. As a result, under (C1), (C2) and (F2), the only  $\mathcal{T}$ -stable equilibrium is

the one with exactly  $\kappa_{\max} - 1$  intervals. In particular, for  $b \in (\frac{1}{4}, \frac{1}{2})$ , the only  $T$ -stable equilibrium is the one with a single interval, where no information is transmitted, even though the partition  $(0, \frac{1}{2}, 1)$  is also an equilibrium.

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