Modeling Ambiguity Aversion as Aversion to Utility Dispersion Caused by Ambiguous Events

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Abstract: In choice under ambiguity a decision maker may know the objective probabilities of some but not all events. The phenomenon of ambiguity aversion is observed when the decision maker prefers to bet on events with known (rather than unknown) objective probabilities. This paper proposes a novel way of modeling ambiguity aversion. In the proposed model the ambiguousness of an act is determined by the dispersion of the utility of its outcomes across states with unknown objective probabilities. The proposed model can rationalize Ellsberg (1963) example and Machina (2009) reflection example. Behavioral characterization of the model is provided.

Keywords: Subjective Uncertainty, Ambiguity, Ambiguity Aversion

JEL Classification Codes: D81

Introduction

We consider monetary bets (acts) on two independent random variables: 1) a toss of a fair coin with a 50%-50% chance of heads or tails to come up; 2) a draw of a ball from an urn that contains only black and white balls in unknown proportion. For example, an act f_1 on figure 1 yields $4000 if heads come up and nothing if tails come up. An act g_1 on figure 1 yields $4000 if a black ball is drawn and nothing if a white ball is drawn.

Act f_1

	BLACK	WHITE
HEADS	$4000	$4000
TAILS	$0	$0

Act g_1

	BLACK	WHITE
HEADS	$4000	$0
TAILS	$4000	$0

Figure 1 Ellsberg (1961) two-color example

Ellsberg (1961) argued that people prefer f_1 over g_1 and this preference holds for any permutation of rows and/or columns. Ellsberg (1961) showed that such preference falsifies Savage (1954) subjective expected utility theory. Ellsberg (1961) example motivated the development of numerous models of ambiguity aversion.

Act f_2

	BLACK	WHITE
HEADS	$4000	$4000
TAILS	$8000	$0

Act g_2

	BLACK	WHITE
HEADS	$4000	$8000
TAILS	$4000	$0

Figure 2 A variant of Machina (2009) reflection example

Consider now a variant of Machina (2009) reflection example presented on figure 2.[1] Machina (2009) conjectured that people prefer f_2 over g_2 and this preference holds for any permutation of rows and/or columns.[2] Machina (2009) demonstrated that such preference falsifies Choquet expected utility (Gilboa, 1987; Schmeidler, 1989). It also falsifies Tversky and Kahneman (1992) cumulative prospect theory which coincides with Choquet expected utility in this example (when all outcomes are nonnegative).

[1] This example corresponds to the example in table 5 in Machina (2009) with an additional restriction that Machina's events E_1 and E_3 are equally likely (which implies that events E_2 and E_4 are equally likely as well).

[2] L'Haridon and Placido (2010) present experimental evidence on Machina (2009) reflection example confirming his conjecture.

Baillon et al. (2011) showed that the preference typically revealed in Machina (2009) reflection example also falsifies Cerreia-Vioglio et al. (2009) model or uncertainty averse preferences which includes as special cases:

a) Gilboa and Schmeidler (1989) maxmin expected utility or multiple priors;
b) Hansen and Sargent (2001) and Strzalecki (2011) multiplier preferences or relative entropy;
c) Klibanoff et al. (2005) smooth model with concave φ;
d) Maccheroni et al. (2006) variational preferences;
e) Ahn (2008) model with concave φ; and
f) Chateauneuf and Faro (2009) confidence function model.

Additionally, Baillon et al. (2011) showed that Ghirardato et al. (2004) α-maxmin theory can accommodate either Ellsberg (1961) example or Machina (2009) reflection example but not both at the same time. This conclusion also holds for Olszewski (2007) model that is de facto equivalent to α-maxmin in the present context.

Siniscalchi (2009) vector expected utility (VEU) can accommodate either the example on figure 1 or on figure 2 but not both at the same time. This novel result is demonstrated in the appendix. Intuitively, VEU is built on the premise that the ambiguousness of complementary uncertain events "cancels out". This cancelation is done on the expected utility basis. For example, f_1 is not ambiguous according to VEU because two complementary uncertain events (drawing a black ball and drawing a white ball) yield the same expected utility. On the other hand, g_1 is ambiguous because drawing a black ball yields \$4000 whereas drawing a white ball yields nothing. Similarly, VEU predicts that g_2 is not ambiguous for a risk-neutral individual (with a linear Bernoulli utility function) because drawing a black ball yields exactly the same expected value (\$4000) as drawing a white ball. However, f_2 is always ambiguous according to VEU because drawing a black ball yields a strictly higher utility than drawing a white ball. Thus, "cancelation" in VEU works in the opposite directions in examples 1 and 2.

The model of Nau (2006)[3] can rationalize either the example on figure 1 or on figure 2 but not both at the same time. This novel result is formally shown in the appendix.

[3] Neilson (2010) second-order expected utility theory coincides with model II in Nau (2006, p. 143).

Intuitively, Nau (2006) generates ambiguity aversion through an additive function of expected utilities in uncertain events. If this function is concave, then two uncertain events each yielding a 50% chance of \$4000 (as in f_1) are preferred over one uncertain event yielding \$4000 for sure (as in g_1) due to Jensen's inequality. For the same reason, a risk-neutral individual prefers two uncertain events each with an expected value of \$4000 (as in g_2) over two uncertain events with expected values of \$6000 and \$2000 (as in f_2). Thus, concavity in Nau (2006) model works in the opposite directions in examples 1 and 2.

Jaffray (1989) linear utility theory for belief functions can rationalize either the example on figure 1 or on figure 2 but not both at the same time. This novel result is shown in the appendix. This result also holds for Gul and Pesendorfer (2010) expected uncertain utility theory which coincides with Jaffray (1989) model in the present context of a finite state space (*cf*. theorem 2 in Jaffray (1989) and formula (6) in Gul and Pesendorfer (2010)).

The only model in the existing literature that can account for both examples presented on figures 1 and 2 is Ergin and Gul (2009) second-order probabilistically sophisticated (SPS) preferences. Yet, two representations that Ergin and Gul (2009) considered within this general class of preferences can rationalize only one example at most. The first representation (SPS expected utility) is equivalent to Klibanoff et al. (2005) smooth model. This model can rationalize either Ellsberg (1961) example (with concave φ) or Machina (2009) reflection example (with convex φ) but not both at the same time (see Baillon et al. (2011)). The analogous result for the second representation (SPS Choquet expected utility) is demonstrated in the appendix.

This paper presents a new model of ambiguity aversion that can rationalize Machina's reflection example. As in Knight (1921), the premise of our model is the distinction between events with known and unknown objective probabilities. Unlike many contemporary models of ambiguity aversion, we do not assume that a decision maker behaves as if making probability judgments on the relative likelihood of ambiguous events. Though such judgments can be incorporated into the model (as demonstrated in the concluding section), we model ambiguity aversion as aversion to utility dispersion caused by ambiguous events.

The paper is organized as follows. Section 1 introduces notation and a new model. Sections 2 and 3 illustrate how the proposed model can rationalize the Ellsberg paradox and the Machina's reflection example correspondingly. Section 4 provides a behavioral characterization of the model. Section 5 concludes.

1. Notation and Model

There is a non-empty set *S*. The elements of *S* are called states of the world. There is a sigma-algebra *Σ* of the subsets of *S* that are called events. The objective probability of event $E \in \Sigma$, if known, is denoted by *P*(*E*). Obviously, $P(\varnothing)=0$ and $P(S)=1$. Set *S* is partitioned into $n \in \mathbb{N}$ disjoint events E_i [4] such that:

1) $P(E_i)$ is known for all $i \in \{1,...,n\}$;

2) $P(E)$ is known for all $E \subset E_1$; and

3) the objective probability is not known for any event $E \subset E_i$, $i \in \{2,...,n\}$.

In the framework of pure subjective uncertainty the objective probabilities of all events are unknown, i.e. $E_1=\varnothing$ and $E_i=S$ for some $i \in \{2,...,n\}$. In choice under risk with a given state space the objective probabilities of all events are known, i.e. $E_1=S$ and $E_i=\varnothing$ for all $i \in \{2,...,n\}$. In choice under ambiguity, analyzed in this paper, the objective probabilities of some but not all events are known, i.e. there is $i \in \{2,...,n\}$ such that $E_i \neq \varnothing$ and $E_i \neq S$.

There is a connected and separable set *X*. The elements of *X* are called outcomes. An act $f: S \to X$ is a *Σ*-measurable function from *S* to *X*. The set of all acts is denoted by *F*.

A decision maker has a preference relation $\succeq$ on *F*. As usual, the symmetric part of $\succeq$ is denoted by $\sim$ and the asymmetric part of $\succeq$ is denoted by $\succ$. Preferences are represented by utility function $U: F \to \mathbb{R}$ when $f \succeq g$ if and only if $U(f) \geq U(g)$ for all $f, g \in F$. In this paper we consider utility function (1).

(1) $$U(f)=\sum_{s\in S} u_s \circ f(s)+0.5\sum_{i=2}^{n}\sum_{s\in E_i}\sum_{t\in E_i}\varphi_{st}\left(\left|u\circ f(s)-u\circ f(t)\right|\right)$$

In formula (1), a standard state-contingent utility function $u_s: X \to \mathbb{R}$ captures the desirability of outcome $x \in X$ contingent on the state $s \in S$. (Bernoulli) utility function $u: X \to \mathbb{R}$ represents state-uniform preferences under risk. Functions $u_s(.)$ and $u(.)$ satisfy the restriction $\sum_{s\in E} u_s(x)=P(E)\cdot u(x)$ for all $x \in X$ and all $E \in \Sigma$ such that $P(E)$ is

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[4] *i.e.*, $\bigcup_{i=1}^{n} E_i = S$ and $E_i \cap E_j = \varnothing$ for all $i, j \in \{1,...,n\}$, $i \neq j$.

known.[5] Functions $u_s : X \to \mathbb{R}$ and $u : X \to \mathbb{R}$ are continuous and determined up to an increasing linear transformation.

Function $\varphi_{st} : \mathbb{R}_+ \to \mathbb{R}$ captures the attitude of a decision maker to utility dispersion across ambiguous states $s,t \in E_i$. Function $\varphi_{st}(.)$ satisfies two natural restrictions. First, $\varphi_{st}(0)=0$, i.e., ambiguous states that yield the same utility are *de facto* not ambiguous. Second, $\varphi_{st}(v)=\varphi_{ts}(v)$ for all $v \in \mathbb{R}_+$, i.e., ambiguity attitudes do not depend on the labeling of ambiguous states. Function $\varphi_{st}(.)$ is continuous and determined up to an increasing linear transformation. Savage (1954) subjective expected utility theory is a special case of representation (1) when function $\varphi_{st}(.)$ is always equal to zero and a decision maker has state-uniform preferences.

2. Ellsberg (1963) three-color example

In the well-known Ellsberg (1963) three-color example $S=\{R,B,Y\}$, $P(\{R\})=1/3$, $P(\{B,Y\})=2/3$ and objective probabilities of the states *B* and *Y* are unknown. Thus, we have a partition $E_1=\{R\}$ and $E_2=\{B,Y\}$. Four acts are presented in Table 1.

Acts	**States of the world**		
	R	*B*	*Y*
f	\$100	\$0	\$0
g	\$0	\$100	\$0
h	\$100	\$0	\$100
l	\$0	\$100	\$100

Table 1 Four acts in Ellsberg (1963) three-color example

People often reveal a choice pattern $f \succ g$ and $l \succ h$. This revealed choice pattern is known as ambiguity aversion. The opposite choice pattern $g \succ f$ and $h \succ l$ is known as ambiguity seeking. According to representation (1) a decision maker reveals preference $f \succ g$ if and only if condition (2) holds.

(2) $$u_R(\$100)-u_R(\$0)+u_B(\$0)-u_B(\$100) > \varphi_{BY}\left(u(\$100)-u(\$0)\right)$$

Similarly, a decision maker reveals preference $l \succ h$ if and only if condition (3) holds.

[5] This restriction insures that a decision maker has state-uniform preferences under risk.

(3) $u_R(\$100)-u_R(\$0)+u_B(\$0)-u_B(\$100)<-\varphi_{BY}\left(u(\$100)-u(\$0)\right)$

Inequalities (2) and (3) cannot hold simultaneously if $\varphi_{BY}\left(u(\$100)-u(\$0)\right)=0$. Thus, Savage (1954) subjective expected utility theory cannot explain ambiguity aversion. Inequalities (2) and (3) can hold simultaneously only if $\varphi_{BY}\left(u(\$100)-u(\$0)\right)<0$. Thus, ambiguity aversion is captured by a negative function $\varphi_{st}(.)$ in representation (1). Similarly, ambiguity seeking corresponds to a positive function $\varphi_{st}(.)$.

3. Machina (2009) reflection example

In Machina (2009) reflection example there are four states of the world that we label as *R*, *B* , *Y* and *G*. Objective probabilities of these four states are unknown. Yet, it is known that $P(\{R,B\})=0.5$ and $P(\{Y,G\})=0.5$. Thus, we have a partition $E_1=\varnothing$, $E_2=\{R,B\}$ and $E_3=\{Y,G\}$. Four acts are presented in Table 2.

Acts	**States of the world**			
	R	*B*	*Y*	*G*
f	\$4000	\$4000	\$8000	\$0
g	\$4000	\$8000	\$4000	\$0
h	\$0	\$4000	\$8000	\$4000
l	\$0	\$8000	\$4000	\$4000

Table 2 Four acts in Machina (2009) reflection example

L'Haridon and Placido (2010) found that people often reveal a choice pattern $f \succ g$ and $l \succ h$ in Machina (2009) reflection example. Let $a=u(\$4000)-u(\$0)$ and $b=u(\$8000)-u(\$4000)$. According to representation (1) a decision maker then reveals preference $f \succ g$ if and only if condition (4) holds.

(4) $u_B(\$4k)-u_B(\$8k)+u_Y(\$8k)-u_Y(\$4k)>\varphi_{RB}(b)+\varphi_{YG}(a)-\varphi_{YG}(a+b)$

Similarly, a decision maker reveals preference $l \succ h$ if and only if condition (5) holds.

(5) $u_B(\$4k)-u_B(\$8k)+u_Y(\$8k)-u_Y(\$4k)<\varphi_{RB}(a+b)-\varphi_{RB}(a)-\varphi_{YG}(b)$

Inequalities (4) and (5) can hold simultaneously only if condition (6) is satisfied.

(6) $\varphi_{RB}(a+b)+\varphi_{YG}(a+b)>\varphi_{RB}(a)+\varphi_{RB}(b)+\varphi_{YG}(a)+\varphi_{YG}(b)$

Thus, if function $\varphi_{st}(.)$ in representation (1) is convex then inequality (6) is satisfied and a decision maker can reveal a choice pattern $f \succ g$ and $l \succ h$. On the other hand, if function $\varphi_{st}(.)$ is concave then inequality (6) is violated and a decision maker can reveal the opposite choice pattern $g \succ f$ and $h \succ l$. Only when function $\varphi_{st}(.)$ is linear the right-hand side of inequality (4) is exactly equal to the right-hand side of inequality (5) and a decision maker cannot reveal a switching choice pattern in Machina (2009) reflection example.

4. Behavioral characterization of the model

We now present the list of axioms to be imposed on the preference relation $\succeq$ so that it admits representation (1). For compact notation, let fEg denote an act that yields outcome $f(s)$ in states $s \in E$ and outcome $g(s)$ in states $s \in S \setminus E$ for some event $E \in \Sigma$.

Axiom 1 (*Completeness*) For all $f, g \in F$ either $f \succeq g$ or $g \succeq f$ (or both).

Axiom 2 (*Separability for Risk*) If $f_1 E g_2 \succeq f_2 E g_1$ and $f_2 E g_3 \succeq f_3 E g_2$ then $f_1 E g_3 \succeq f_3 E g_1$ for all $f_1, f_2, f_3 : E \to X$, all $g_1, g_2, g_3 : S \setminus E \to X$ and all $E \in \Sigma$ such that $P(E)$ is known.[6]

Axiom 3 (*Continuity*) For all $f \in F$ the sets $\{g \in F : g \succeq f\}$ and $\{g \in F : f \succeq g\}$ are closed.

Proposition 1 (*Debreu, 1960*) Preference relation $\succeq$ satisfies axioms 1-3 if and only if

(7) $U(f)=\sum_{s\in E_1} u_s \circ f(s)+\sum_{i=2}^{n} u_{E_i}\left(f(s_{i1}),...,f(s_{im(i)})\right),$

where utility functions $u_s : X \to \mathbb{R}$ and $u_{E_i} : X^{m(i)} \to \mathbb{R}$ are continuous and determined up to an increasing linear transformation; and the elements of E_i are numbered as $s_{i1},...,s_{im(i)}$.

Proof: If $E_i = S$ for some $i \in \{2,...,n\}$ then equation (7) holds trivially by setting $U(f)=u_{E_i}(f)$. If $E_i \neq S$ for all $i \in \{2,...,n\}$ then there is at least one event $E \in \Sigma$ such that $P(E)$ is known and $E \neq \varnothing, S$. Proposition 1 then follows from Theorems 1 and 3 in Debreu (1960). *Q.E.D.*

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[6] When $E = S$ axiom 2 becomes a standard transitivity axiom: if $f_1 \succeq f_2$ and $f_2 \succeq f_3$ then $f_1 \succeq f_3$ for all $f_1, f_2, f_3 \in F$.

Proposition 1 establishes standard separability for risk. The next step is to obtain separability for ambiguity. Intuitively, for ambiguous events we want to weaken axiom 2 so that it holds only when each pair of acts has the same utility dispersion across ambiguous states. Thus, we first need a behavioral characterization of utility dispersion. We will use the tradeoff technique (*cf*. chapter 4 in Wakker, 2010).

In choice under ambiguity, as already mentioned above, there is $i \in \{2,...,n\}$ such that $E_i \neq \varnothing$ and $E_i \neq S$. Let xE_iz denote an act that yields outcome $x \in X$ in states $s \in E_i$ and outcome $z \in X$ in states $s \in S \setminus E_i$. If $x_1E_iz_1 \sim y_1E_iz_2$ and $x_2E_iz_1 \sim y_2E_iz_2$ then proposition 1 implies that $u_{E_i}(x_1) - u_{E_i}(x_2) = u_{E_i}(y_1) - u_{E_i}(y_2)$.[7] This provides us with a simple behavioral characterization of utility difference. In general, this behavioral characterization may depend on specific event E_i if a decision maker has state-dependent preferences under risk. Thus, it is necessary to assume state-uniform preferences for all events with known objective probabilities. This is achieved in the following axiom.

Axiom 4 (*State-Uniform Preferences Under Risk*) If $x_1Ez_1 \sim y_1Ez_2$, $x_2Ez_1 \sim y_2Ez_2$ and $x_1E'z_3 \sim y_1E'z_4$ then $x_2E'z_3 \sim y_2E'z_4$ for all $x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4 \in X$ and for all $E, E' \in \Sigma$ such that $P(E)$ and $P(E')$ are known.

With axiom 4 at hand, if $x_1E_iz_1 \sim y_1E_iz_2$ and $x_2E_iz_1 \sim y_2E_iz_2$ then we can simply write that $u(x_1) - u(x_2) = u(y_1) - u(y_2)$, i.e., we can drop the subscript of utility function $u_{E_i}(x)$. With this behavioral characterization of utility dispersion we are now ready to impose separability for ambiguous events. For compact notation, let $[xtf]E_ig$ denote an act that yields outcome $x \in X$ is state $t \in E_i$, outcome $f(s)$ in states $s \in E_i \setminus \{t\}$ and outcome $g(s)$ in states $s \in S \setminus E_i$ for some $i \in \{2,...,n\}$.

Axiom 5 (*Separability for Ambiguity*) If $[x_1tf_2]E_ig \succeq [x_2tf_1]E_ig$ and $[x_2tf_3]E_ig \succeq [x_3tf_2]E_ig$ then $[x_1tf_3]E_ig \succeq [x_3tf_1]E_ig$ for all $x_1, x_2, x_3 \in X$, all $f_1, f_2, f_3 : E_i \setminus \{t\} \to X$, all $g : S \setminus E_i \to X$, all $t \in E_i$ and all $i \in \{2,...,n\}$ such that

(8) $x_1Ez_1 \sim x_2Ez_2$ and $f_2(s)Ez_1 \sim f_1(s)Ez_2$ for all $s \in E_i \setminus \{t\}$;

[7] We slightly abused notation by abbreviating $u_{E_i}(x,...,x)$ into $u_{E_i}(x)$ for all $x \in X$.

(9) $x_2 E z_3 \sim x_3 E z_4$ and $f_3(s) E z_3 \sim f_2(s) E z_4$ for all $s \in E_i \setminus \{t\}$;

for some $z_1, z_2, z_3, z_4 \in X$ and some $E \in \Sigma$ such that $P(E)$ is known and $E \neq \varnothing, S$.

Proposition 2 Preference relation $\succeq$ satisfies axioms 1-5 if and only if it admits representation (1).

Proof is presented in the appendix.

5. Conclusion

This paper proposes a novel way of modeling the phenomenon of ambiguity aversion. The preferences of a decision maker are represented by a standard separable utility function that includes extra terms for utility dispersion across ambiguous states. This modeling approach captures attitudes to ambiguity through subjective functions on the set of outcomes (not the state space). Such modeling approach avoids the necessity of considering mixture acts and extending the preferences of a decision maker over the set of mixture acts. In the baseline model (1) a decision maker does not have subjective beliefs on the likelihood of events with unknown objective probabilities.

In practical applications, beyond simple examples considered in sections 2 and 3, however, the model with state-contingent utility functions may be too general and a more parsimonious specification is desirable. A natural special case to consider is the case of state-uniform preferences.[8] In this case, state-contingent utility functions $u_s : X \to \mathbb{R}$ differ only by a positive multiplicative constant that is conventionally interpreted as (objective or subjective) probability $P(s)$ of state $s \in S$. Using a specific functional form $\varphi_{st}(v) = P(s) \cdot P(t) \cdot \varphi(v)$ where $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is an arbitrary function,[9] we obtain a parsimonious model (10).

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[8] I.e., we impose axiom 4 on all events (not only those with known objective probabilities).

[9] Ellsberg (1963) three-color example and Machina (2009) reflection example suggest that people have a negative and convex function $\varphi(.)$. If function $\varphi(.)$ is differentiable, monotonicity requires $-1 \leq \varphi'(v) \leq 1$ for all $v \in \mathbb{R}_+$.

(10) $U(f)=\sum_{s\in S}P(s)\cdot u\circ f(s)+0.5\sum_{i=2}^{n}\sum_{s\in E_i}\sum_{t\in E_i}P(s)\cdot P(t)\cdot\varphi\left(\left|u\circ f(s)-u\circ f(t)\right|\right)$

Model (10) evaluates an act through a linear tradeoff between its subjective expected utility and its utility dispersion across ambiguous states. Model (10) resembles the theory of disappointment without prior expectation proposed by Delquié and Cilo (2006) for choice under risk. When outcomes are monetary, utility function $u(.)$ is linear and function $\varphi(.)$ is quadratic, model (10) becomes the analog of Markowitz (1952) mean-variance approach for decision under ambiguity. The second term on the right hand side of equation (10) (or, more generally, of equation (1)) can be interpreted as a subjective measure of ambiguousness of an act. Model (10) also resembles Jaffray (1989) if the focal set of a belief function in Jaffray (1989) contains only singletons and pairs of ambiguous states.

The model presented in this paper separates only between states with known and unknown objective probabilities. A natural extension is to consider various subjective degrees of ambiguity and source preference.

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Appendix

Vector expected utility and the example presented on figure 2

We normalize von-Neumann-Morgenstern utility function so that $u(\$0)=0$, $u(\$4000)=v$, $v\in(0,1)$, and $u(\$8000)=1$. Let $p\in[0,1]$ denote a baseline probability that a black ball is drawn. Let ζ denote the adjustment factor for the event when a black ball is drawn. The adjustment factor for the complementary event (when a white ball is drawn) is then $-\zeta p/(1-p)$. The utility of f_2 and g_2 can be written as (A1) and (A2) correspondingly.

(A1) $VEU(f_2)=0.5(v+p)+\mathrm{A}(0.5\zeta p)$

(A2) $VEU(g_2)=vp+0.5(1-p)+\mathrm{A}(\zeta p[v-0.5])$

A decision maker then prefers f_2 over g_2 if inequality (A3) is satisfied.

(A3) $(p-0.5)(1-v) > \mathrm{A}(\zeta p[v-0.5]) - \mathrm{A}(0.5\zeta p)$

Similarly, we can show that the preference for f_2 is preserved after a permutation of rows and columns in figure 2 if inequality (A4) is satisfied.

(A4) $\mathrm{A}(-0.5\zeta p) - \mathrm{A}(\zeta p[0.5-v]) > (p-0.5)(1-v)$

In vector expected utility function A(.) is symmetric, *i.e.* $\mathrm{A}(-\varphi)=\mathrm{A}(\varphi)$ for all φ. Using this fact it is straightforward to show that inequalities (A3) and (A4) can hold simultaneously only if inequality (A5) is satisfied.

(A5) $\mathrm{A}(0.5\zeta p) > \mathrm{A}(\zeta p|v-0.5|)$

Since $v \in (0,1)$, the argument of function A(.) on the left-hand-side of (A5) is always greater than that on the right-hand-side of (A5). Hence, inequality (A5) effectively states that function A(.) is increasing on the positive orthant of Euclidean space. Since A(0)=0 by definition, we conclude that function A(.) is positive. Yet, to explain the Ellsberg (1961) paradox function A(.) must be nonpositive (*cf*. Siniscalchi, 2009, section 4.3).

Nau (2006) model and the example presented on figure 2

Nau (2006) general model I allows for state-dependent utility. Let subscript $i \in \{HB,HW,TB,TW\}$ denote each of four possible states (*e.g.*, HB denotes a state when Heads come up and a Black ball is drawn). In each state i we normalize standard state-dependent utility function so that the utility of \$0 is zero, the utility of \$4000 is $v_i \in (0,1)$ and the utility of \$8000 is one. Let functions $u_B : \mathbb{R} \rightarrow \mathbb{R}$ and $u_W : \mathbb{R} \rightarrow \mathbb{R}$ denote second-order utilities (unique up to a positive affine transformation). We normalize second-order utilities so that $u_B(v_{TB}) = u_W(v_{HW})$ and $u_B(1) = u_W(1)$.

A decision maker prefers f_2 over g_2 if inequality (A6) is satisfied.

(A6) $u_B(1+v_{HB}) + u_W(v_{HW}) > u_B(v_{HB}+v_{TB}) + u_W(1)$

Using our normalization of second-order utilities, inequality (A6) can be rewritten as (A7).

(A7) $u_B(1+v_{HB}) + u_B(v_{TB}) > u_B(v_{HB}+v_{TB}) + u_B(1)$

Inequality (A7) is Karamata's majorization inequality (Karamata, 1932) which generalizes Jensen's inequality and characterizes a convex function $u_B(.)$. Yet, to explain the Ellsberg (1961) paradox function $u_B(.)$ must be concave (*cf*. theorem 1 in Nau, 2006).

Jaffray (1989) model and examples presented on figure 1-2

Let p_i denote the Möbius inverse of a belief function for a state $i \in \{HB,HW,TB,TW\}$ (*e.g.*, HB denotes a state when Heads come up and a Black ball is drawn). Let p_{-i} denote the Möbius inverse of a belief function for an event that is complementary to state $i \in \{HB,HW,TB,TW\}$. Finally, let p_j denote the Möbius inverse of a belief function for an event $j \in \{H,T,B,W\}$ (*e.g.*, H denotes an event when Heads come up). Let $u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ denote utility function.

A decision maker then prefers f_1 over g_1 if inequality (A8) is satisfied.

$$\text{(A8)}\quad \begin{aligned}&(p_{HW}-p_{TB})\left[u(\$4000,\$4000)-u(\$0,\$0)\right]+\\&+(p_H-p_B)\left[u(\$4000,\$4000)-u(\$0,\$4000)\right]+\\&+(p_W-p_T)\left[u(\$0,\$4000)-u(\$0,\$0)\right]>0\end{aligned}$$

The preference for f_1 is preserved after a permutation of rows and columns in figure 1 if inequality (A9) is satisfied.

$$\text{(A9)}\quad \begin{aligned}&(p_{TB}-p_{HW})\left[u(\$4000,\$4000)-u(\$0,\$0)\right]+\\&+(p_T-p_W)\left[u(\$4000,\$4000)-u(\$0,\$4000)\right]+\\&+(p_B-p_H)\left[u(\$0,\$4000)-u(\$0,\$0)\right]>0\end{aligned}$$

Adding inequalities (A8) and (A9) together we get condition (A10).

$$\text{(A10)}\quad (p_T+p_H-p_B-p_W)\left[u(\$4000,\$4000)+u(\$0,\$0)-2u(\$0,\$4000)\right]>0$$

Thus, Jaffray (1989) model can account for Ellsberg (1961) two-color example only when either 1) $p_T+p_H>p_B+p_W$ and utility function $u(.,.)$ is strictly supermodular; or 2) $p_T+p_H<p_B+p_W$ and utility function $u(.,.)$ is strictly submodular.

A decision maker prefers f_2 over g_2 if inequality (A11) is satisfied.

$$\text{(A11)}\quad \begin{aligned}&(p_{TB}-p_{HW})\left[u(\$8000,\$8000)-u(\$4000,\$4000)\right]+\\&+(p_B-p_H)\left[u(\$4000,\$8000)-u(\$4000,\$4000)\right]+\\&+(p_T-p_W+p_{-HW}-p_{-TB})\left[u(\$0,\$8000)-u(\$0,\$4000)\right]>0\end{aligned}$$

The preference for f_2 is preserved after a permutation of rows and columns in figure 2 if inequality (A12) is satisfied.

$$\text{(A12)}\quad \begin{aligned}&(p_{HW}-p_{TB})\left[u(\$8000,\$8000)-u(\$4000,\$4000)\right]+\\&+(p_W-p_T)\left[u(\$4000,\$8000)-u(\$4000,\$4000)\right]+\\&+(p_H-p_B+p_{-TB}-p_{-HW})\left[u(\$0,\$8000)-u(\$0,\$4000)\right]>0\end{aligned}$$

Adding inequalities (A11) and (A12) together we get condition (A13).

$$\text{(A13)}\quad (p_T+p_H-p_B-p_W)\left[u(\$4000,\$4000)-u(\$4000,\$8000)+u(\$0,\$8000)-u(\$0,\$4000)\right]>0$$

Thus, Jaffray (1989) model can account for the example presented on figure 2 only when either 1) $p_T+p_H>p_B+p_W$ and utility function $u(.,.)$ is strictly submodular; or 2) $p_T+p_H<p_B+p_W$ and utility function $u(.,.)$ is strictly supermodular. This means that Jaffray (1989) model can rationalize either the example on figure 1 or on figure 2 but not both at the same time.

Ergin and Gul (2009) second-order probabilistically sophisticated Choquet expected utility and examples presented on figure 1-2

We normalize von-Neumann-Morgenstern utility function so that *u*(\$0)=0, *u*(\$4000)=*v*, *v*∈(0,1), and *u*(\$8000)=1. Let *c*(*p*) denote capacity of probability *p*∈[0,1] that a black ball is drawn. A decision maker then prefers f_1 over g_1 if inequality (A14) is satisfied.

(A14) $$\int_0^1 p\,dc(p) < 0.5$$

A decision maker prefers f_2 over g_2 if inequality (A15) is satisfied.

(A15) $$\int_0^1 [0.5p + 0.5v]\,dc(p) > \int_0^1 \left[0.5(1-p) + pv\right]dc(p)$$

Simplifying inequality (A15) yields inequality (A14) with a reversed sign. Thus, Ergin and Gul (2009) second-order probabilistically sophisticated Choquet expected utility can rationalize either the example on figure 1 or on figure 2 but not both at the same time.

Proof of proposition 2

The necessity of axioms 1-5 is relatively straightforward to show. We shall prove only their sufficiency. Proposition 1 implies that conditions (8)-(9) in axiom 5 can be rewritten as (11)-(12) for some state $s \in E_i \setminus \{t\}$.

(11) $$u(x_1) - u \circ f_2(s) = u(x_2) - u \circ f_1(s)$$

(12) $$u(x_2) - u \circ f_3(s) = u(x_3) - u \circ f_2(s)$$

Note that conditions (11)-(12) together imply condition (13)

(13) $$u(x_1) - u \circ f_3(s) = u(x_3) - u \circ f_1(s)$$

Axiom 5 then guarantees that the conditions of Thomsen-Blaschke theorem (Blaschke, 1928) are satisfied (see Figure 1). Thus there is a topological transformation carrying the family of indifference curves depicted on Figure 3 into a family of parallel straight lines $u_t(x) + u_s \circ f(s) + \delta_{ts}(u(x) - u \circ f(s)) = const$, where functions $u_t : X \to \mathbb{R}$, $u_s : X \to \mathbb{R}$ and $\delta_{ts} : \mathbb{R} \to \mathbb{R}$ are continuous and determined up to an increasing linear transformation..

By a symmetry argument for every other state $s' \in E_i \setminus \{t\}$ we obtain (14).

(14) $$u_{E_i}\left(f(s_{i1}), \ldots, f(s_{im(i)})\right) = \sum_{s \in E_i} u_s \circ f(s) + \sum_{s \in E_i} \sum_{t \in E_i} \delta_{st}\left(u \circ f(s) - u \circ f(t)\right)$$

In formula (14) utility dispersion across each pair of ambiguous states $s,t \in E_i$ is considered twice (state *s* is first compared to state *t* and then vice versa). For parsimony, we can introduce another function $\varphi_{st}(|v|) \equiv \varphi_{ts}(|v|) = \delta_{st}(v) + \delta_{ts}(-v)$ for all $v \in \mathbb{R}$. Plugging this new function into equation (14) and equation (14) in its turn into representation (7) yields (1). *Q.E.D*.

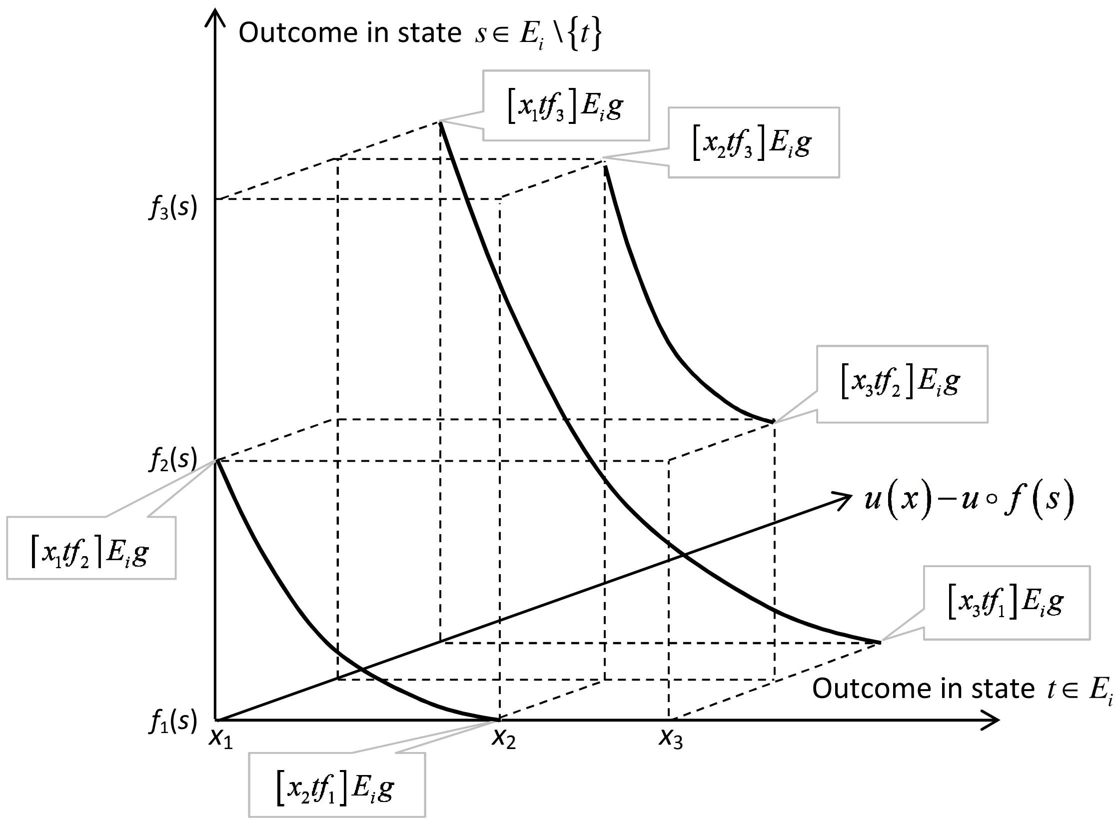

Figure 3 Graphical illustration of axiom 5