A General Equilibrium Explanation for Financial Markets Anomalies: Belief Heterogeneity under Limited Enforceability

Pablo F. BEKER[*]

Department of Economics
University of Warwick

Emilio ESPINO[†]

Department of Economics
Universidad Torcuato Di Tella

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Abstract

We consider an exchange economy with many infinitely lived investors who believe that the true data generating process is a Markov chain and update their (heterogeneous) priors according to Bayes' rule. We assume there is limited enforceability a la Alvarez and Jermann [2] version of Kehoe and Levine [18] and we provide a recursive characterization of the set of constrained efficient allocations and a version of the principle of optimality. Then, we analyze the dynamics of asset prices and asset trading of competitive equilibria that decentralizes constrained efficient allocations. We show that asset prices might display short-term momentum and long-term reversal and portfolios fluctuations do not vanish. In particular, these features are displayed by a calibrated version of the model. We also show that in any competitive equilibrium that sustain Pareto optimal allocations the assets return neither display short-term momentum nor long-term reversal in the long run and that (for almost all priors) the volume of trade vanishes. Our work shows that the aforementioned empirical regularities might be evidence against the lack of Pareto efficiency of markets but not against the subjective expected utility hypothesis.

Keywords: heterogeneous beliefs, endogeneous incomplete markets.

[*]University of Warwick, Department of Economics, Warwick, Coventry CV4 7AL, UK. E-mail: Pablo.Beker@warwick.ac.uk.

[†]Universidad Torcuato Di Tella, Department of Economics, Saenz Valiente 1010 (C1428BIJ), Buenos Aires, Argentina. E-mail: eespino@utdt.edu

1 Introduction

Over the last several years, a large volume of empirical work has documented a variety of ways in which asset returns can be predicted based on publicly available information. Many of these results belong to one of two categories. On the one hand, returns appear to exhibit *short-term momentum*, that is positive autocorrelation, in the short to medium run. On the other hand, they also exhibit *long-term reversal*, that is negative autocorrelation, in the long run. Moreover, the empirical literature has also documented that asset trading volume (around the date that information is released) is extremely large across virtually all developed stock markets, and many of the most interesting patterns in prices and returns are tightly linked to movements in volume.

These facts have been qualified as "anomalies" since it has been surprisingly difficult either to "rationalize" or to explain using plausible assumptions on the existing models.[1] The conventional wisdom nowadays is that these patterns of returns cannot be explain using standard models of rational choice. Indeed, Barberis and Thaler [5] write

> "The traditional finance paradigm, which underlies many of the other articles in this handbook, seeks to understand financial markets using models in which agents are "rational". Rationality means two things. First, when they receive new information, agents update their beliefs correctly, in the manner described by Bayes' law. Second, given their beliefs, agents make choices that are normatively acceptable, in the sense that they are consistent with Savage's notion of Subjective Expected Utility (SEU). This traditional framework is appealingly simple, and it would be very satisfying if its predictions were confirmed in the data. Unfortunately, after years of effort, it has become clear that basic facts about the aggregate stock market, the cross-section of average returns and individual trading behavior are not easily understood in this framework."

To the contrary, this paper argues that the the aforementioned empirical facts are not necessarily incompatible with the predictions derived from the assumption that agents are subjective expected utility maximizers who update their priors according to Bayes' rule.

[1] De Bondt and Thaler [13] write: "Economics can be distinguished from other social sciences by the belief that most (all?) behaviour can be explained by assuming that agents have stable, well-defined preferences and make rational choices consistent with those preferences in markets that (eventually) clear. An empirical result qualiifes as an anomaly if it is difficult to "rationalize", or if implausible assumptions are necessary to explain it within the paradigm."

We consider an exchange economy populated by infinitely-lived agents who are subjective utility maximizers. These agents believe that the true process of the states of nature (and that of the individual endowments) is generated by iid draws from a fixed distribution and update their priors in a Bayesian fashion as data unfolds[2].

We first show that in any competitive equilibrium that sustain Pareto optimal allocations the return of the assets neither display short-term momentum nor long-term reversal in the long run and that (for almost all priors) the volume of trade vanishes. This result is to the best of our knowledge the first formalization of the conventional wisdom mentioned above and that lead many researchers to clasify these empirical regularites of asset returns as anomalies. However, this result still allows for the possibility that the concept of Pareto efficiency is what is rejected in the data and not rationality in the sense of Bayesian updating coupled with the subjective expected utility hypothesis.

One of the usual suspects for preventing the market to achieve a Pareto efficient allocation of resources is the existence of borrowing constraints. These constraints might arise for several reasons. We follow Alvarez and Jerman [2] who consider the case where borrowing constraints are endogenously caused by the lack of perfect enforceability. They analyze optimal allocations that are constrained to be not only feasible but also enforceable in the sense of Kehoe and Levine [18]. They show that if agents have homogeneous beliefs, then portfolios that support a constrained efficient allocation need not be constant over time. They are silent, however, about short-term momentum and long-run reversals.

We consider constrained efficient allocations in the Alvarez and Jermann sense when agents have heterogeneous beliefs. We provide a recursive characterization of the set of constrained efficient allocations and a version of the principle of optimality for these economies. Later, we analyze the dynamics of asset prices and asset trading for competitive equilibria that decentralize constrained efficient allocations. We show that if some agent does not have the truth in the support of his prior, then (generically) asset trading does not vanish. This is because limited enforceability prevents the agents to place bets that would make her wealth converge to its lower bound with positive probability. Furthermore, we also provide conditions under which competitive equilibrium asset prices display short-term momentum and long-run reversal. Unlike Daniel et al [12] and Barberis et al [4], we do not need to assume cognitive biases to explain these patterns of asset prices. The driving force of our price dynamics, instead, are the changes in the wealth distribution associated with the interaction among agents with heterogeneous beliefs who are restricted to trade in competitive

[2] All our results hold true if one assumes the agents believe the data generating process is a first-order Markov process. The iid assumption is for simplicity.

markets that provide limited risk sharing.

To the best of our knowledge, this is the first attempt to evaluate the ability of standard general equilibrium models to explain short-term momentum and long-term reversal. However some work has been done related to asset trading volume in general equilibrium. Judd *et al.* [17] considered a stationary Markovian economy where heterogeneous agents have homogeneous degenerate beliefs and showed that each investor's equilibrium portfolios is constant along time and across states after an initial trading stage. Thus, for instance, differences in risk aversion, in financial wealth, etc. cannot explain why investors change their portfolios over time if the competitive equilibrium allocation is Pareto optimal. In a recent paper, Beker and Espino [6] considered the case of heterogeneous beliefs. They define the $\mathcal{B}$-margin of heterogeneity as the likelihood ratio of the agents' beliefs and they argue that if the equilibrium allocation is Pareto optimal, then portfolios converge if and only if the $\mathcal{B}$-margin converges. They show that this margin converges under the standard assumptions made in the literature, namely, (i) all agents believe the data generating process (henceforth, dgp) consist of iid from a fixed distribution but disagree on the probabilities of the state of nature and (ii) at least one agent has the true distribution in the support of her priors.[3] On the other hand, they show that portfolios might not be constant if some agent makes mistakes rarely and no agent learns fast enough. Overall, their results can be read as saying, in a robust way, that if the equilibrium allocation is efficient and some agent's beliefs are closer (in a sense that can be made precise) to the truth than the rest, belief heterogeneity cannot explain non-vanishing asset trading. Thus, Beker and Espino's [6] results suggest that some form of inefficiency of financial markets needs to be incorporated to the analysis.

2 The Model

We consider an infinite horizon pure exchange economy with one good. In this section we establish the basic notation and describe the main assumptions.

2.1 The Environment

Time is discrete and indexed by $t = 0, 1, 2, ...$ The set of possible states of nature is $S \equiv \{1, ..., K\}$. The state of nature at date zero is known and denoted by $s_0 \in S$. The set of partial histories up to date $t \geq 1$, S^t, is the $t-$Cartesian product of S with typical element $s^t = (s_1, ..., s_t)$. S^∞ is the set of infinite sequences of the states of nature and $s = (s_1, s_2, \cdots)$, called a path, is a typical element. For every

[3] Their result extends to the case in which the agents believe the dgp is a Markov process of any finite order.

partial history s^t, $t \geq 1$, a *cylinder* with base on s^t is the set $C(s^t) \equiv \{\widetilde{s} \in S^\infty : \widetilde{s} = (s^t, \widetilde{s}_{t+1}, \cdots)\}$ of all paths whose t initial elements coincide with s^t. Let $\mathcal{F}_t$ be the σ-algebra that consists of all finite unions of the sets $C(s^t)$. The σ-algebras $\mathcal{F}_t$ define a filtration $\mathcal{F}_0 \subset ... \subset \mathcal{F}_t \subset ... \subset \mathcal{F}$ where $\mathcal{F}_0 \equiv \{\emptyset, S^\infty\}$ is the trivial $\sigma-$algebra and $\mathcal{F}$ is the σ-algebra generated by the algebra $\bigcup\limits_{t=1}^{\infty} \mathcal{F}_t$.

Let Δ^{K-1} be the $K-1$ dimensional unit simplex in $\Re^K$. We say that $\pi : S \times S \to [0,1]$ is a transition probability matrix if $\pi(\cdot\,|\xi\,) \in \Delta^{K-1}$ for all $\xi \in S$. If $\{s_t\}$ follows a first-order stationary Markov process with a $K \times K$ transition probability matrix π, then P^π denotes the probability measure on $(S^\infty, \mathcal{F})$ uniquely induced by π. Let Π^K denote the set of $K \times K$ transition probability matrices, $\mathcal{B}\left(\Pi^K\right)$ be its corresponding Borel sets and $\mathcal{P}(\Pi^K)$ be the set of probability measures on $\left(\Pi^K, \mathcal{B}\left(\Pi^K\right)\right)$. For the characterization of the dinamics in Sections **??**, 5-**??**, we need to be explicit about the true data generating process (henceforth, dgp). We will assume that

A.0 The true dgp is given by P^{π^*} for some $\pi^* \in int\left(\Pi^K\right)$; i.e., $\pi^*(\xi'\,|\xi\,) > 0$ for all $(\xi, \xi') \in S \times S$.

2.2 The Economy

There is a single perishable consumption good every period. The economy is populated by I (types of) infinitely-lived agents where $i \in \mathcal{I} = \{1, ..., I\}$ denotes an agent's name. A consumption plan is a sequence of functions $\{c_t\}_{t=0}^{\infty}$ such that $c_0 \in \mathbb{R}_+$ and $c_t : S^\infty \to \mathbb{R}_+$ is $\mathcal{F}_t-$measurable for all $t \geq 1$ and $\sup_{(t,s)} c_t(s) < \infty$. Given s_0, the agent's consumption set, $\mathbb{C}(s_0)$, is the set of all consumption plans.

2.2.1 Beliefs

P_i is the probability measure on $(S^\infty, \mathcal{F})$ that represents agent i's prior. Throughout this paper, we assume that each agent i assigns positive probability to every partial history s^t; i.e., $P_i(C(s^t)) > 0$ for all s^t. We say that agent i believes the dgp consists of draws from a fixed transition probability matrix if for every event $A \in \mathcal{F}$ is

$$P_i\left(A\right) = \int_{\Pi^K} P^\pi\left(A\right) \mu_i\left(d\pi\right), \tag{1}$$

where $\mu_i \in \mathcal{P}(\Pi^K)$ is agent i's *belief* over the unknown parameters.

The following assumption will be used to obtain the recursive characterization of equilibrium portfolios.

A1 Agent i believes the true dgp consists of draws from a fixed transition probability matrix.

a. μ_i has countable support.

b. μ_i has density f_i with respect to Lebesgue that is continuous.

We want to emphasize that the assumption that agents i and j satisfy $A1$ does not imply that they have the same priors. Indeed, although $A1$ implies that they agree the dgp consists of draws from a fixed transition probability matrix, it allows for disagreement about the distribution itself (i.e., $\mu_i \neq \mu_j$).

The following assumptions, when coupled with $A1$, impose more structure on the agent's prior.[4]

A2 Agent i has the true transition probability matrix in the support of her prior. That is, either

a. $\mu_i(\pi^*) > 0$.

b. $f_i(\pi^*) > 0$.

Remark 1: It is ubiquitous in the learning literature related to asset pricing to assume both that (i) every agent knows the dgp and (ii) some agent learns the true conditional probability of the states. In our setting, the latter is guaranteed by assuming $A0$ and strengthening $A1$ with $A2$ for some agent so that the true parameter, π^*, is in the support of some agent's prior.[5]

Now we consider the case in which agents have heterogeneous prior beliefs. Let $\Theta_n(\pi) = \left\{\widetilde{\pi} \in \Pi^K : \|\pi - \widetilde{\pi}\| = \sup_{(\xi,\xi')} \left|\pi(\xi' \,|\xi) - \widetilde{\pi}(\xi' \,|\xi)\right| < \frac{1}{n}\right\}$. The following assumption, which will be used in Proposition 4, says that the Radon-Nikodym derivative of i's prior belief with respect to j's prior belief, $\frac{d\mu_i}{d\mu_j}$, converges to zero as it approaches some $\overline{\pi}$.

A3 There exists $\overline{\pi} \in \Pi^K$ with the property that for every $\varepsilon > 0$ there is n such that $\pi \in \Theta_n(\overline{\pi})$ implies $\frac{d\mu_i}{d\mu_j}(\pi) \leq \varepsilon$.

To see the scope of assumption A3, note that it is satisfied if agents' prior beliefs are not mutually equivalent (so that there exists Borel set where one agent puts positive probability but the other does not).[6]

[4] We adopt the convention of writting $\mu_i(\{\theta\})$ as $\mu_i(\theta)$ for all θ.

[5] The finiteness of S_t is fundamental for this learning result.

[6] Moreover, if the prior beliefs are mutually equivalent, assumption 3 still holds if beliefs have continuous densities and there is a point where the density of the prior belief of only one of the agents is zero.

We say that agent i is *dogmatic* if his belief is a point mass probability measure on some $\pi_i \in \Pi^K$; i.e., $\mu^{\pi_i} : \mathcal{B}\left(\Pi^K\right) \to [0,1]$ is given by

$$\mu^{\pi_i}(B) \equiv \begin{cases} 1 & \text{if } \pi_i \in B \\ 0 & \text{otherwise.} \end{cases}$$

and μ^{π} denotes the vector $(\mu^{\pi_1}, ..., \mu^{\pi_I})$.

2.2.2 Preferences

Agents' preferences have a subjective expected utility representation that is time separable and a common discount factor; i.e., for every $c_i \in \mathbb{C}(s_0)$ her preferences are represented by

$$U_i^{P_i}(c_i) = E^{P_i}\left(\sum_{t=0}^{\infty} \rho_t \; u_i(c_{i,t})\right).$$

We assume that $u_i : \mathbb{R}_+ \to \{-\infty\} \cup \mathbb{R}$ is continuously differentiable, strictly increasing, strictly concave and $\lim_{x \to 0} \frac{\partial u_i(x)}{\partial x} = +\infty$ for all i. Notice that we allow for utility functions unbounded from below. The multi-period stochastic discount factors are defined recursively using the one-period state contingent discount factor $\beta : S \to (0,1)$ such that

$$\rho_{t+1}(s^t) = \rho_t(s^{t-1})\beta(s_t)$$

for all $t \geq 0$ and all $s^t = (s^{t-1}, s_t)$. For instance, the standard case with $\beta(\xi) = \beta$ for all ξ implies that $\rho_t(s^{t-1}) = \beta^t$ for all $t \geq 0$ and all s^{t-1}. We say that the volatility of the stochastic discount rate vanishes if $\max_{\xi, \widetilde{\xi}} \left|\beta(\xi) - \beta(\widetilde{\xi})\right| \longrightarrow 0$.

2.2.3 Feasibility, Enforceability and Constrained Optimality

Agent i's endowment at date t is a time-homogeneous function of the current state of nature, that is $y_i(s_t) > 0$ for all $s_t \in \{1, ..., K\}$ and the aggregate endowment is $y(s_t) \equiv \sum_{i=1}^{I} y_i(s_t) \leq \overline{y} < \infty$. An allocation $\{c_i\}_{i=1}^{I} \in \mathbb{C}(s_0)^I$ is *feasible* if $c_i \in \mathbb{C}(s_0)$ for all i and $\sum_{i=1}^{I} c_{i,t}(s) \leq y(s_t)$ for all $s \in S^\infty$. Let $Y^\infty(s_0)$ denote the set of feasible allocations given s_0.

Define

$$\begin{aligned} U_i(c_i)(s^t) &= u_i(c_i(s^t)) + \beta(s_t) \sum_{s_{t+1}} \pi_{\mu_{i,s^t}}(s_{t+1} \,|s_t) U_i(c_i)(s^t, s_{t+1}) \\ U_i(y_i)(s^t) &= u_i(y_i(s_t)) + \beta(s_t) \sum_{s_{t+1}} \pi_{\mu_{i,s^t}}(s_{t+1} \,|s_t) U_i(y_i)(s^t, s_{t+1}) \end{aligned}$$

where $\pi_{\mu_{i,s^t}}(s_t, s_{t+1}) = \int \pi(s_{t+1} | s_t) \; \mu_{i,s^t}(d\pi)$ and Bayes' rule implies that prior beliefs evolve according to

$$\mu_{i,(s^t,s_{t+1})}(d\pi) = \frac{\pi(s_{t+1} | s_t) \; \mu_{i,s^t}(d\pi)}{\int \pi(s_{t+1} | s_t) \; \mu_{i,s^t}(d\pi)}, \tag{2}$$

where $\mu_{i,s^0} = \mu_{i,0} \in \mathcal{P}(\Pi^K)$ is given at date 0.

For each i, we write $U_i(y_i)(s^t) = U_i(s_t, \mu_{i,s^t})$ to make clear that the utility attained from consuming the individual endowment can be expressed as a function that depends only on s_t and μ_{i,s^t}.

A feasible allocation $\{c_i\}_{i=1}^I$ is *enforceable* if the following participation constraints are satisfied for every agent i

$$U_i(c_i)(s^t) \geq U_i(s_t, \mu_{i,s^t}), \text{ for all } t \text{ and all } s^t.$$

Let $Y_E^\infty(s_0, \mu_0) \subset Y^\infty(s_0)$ be the set of feasible enforceable allocations. A feasible allocation $\{c_i\}_{i=1}^I$ is *Pareto optimal (PO)* if there is no alternative feasible allocation $\{\widehat{c}_i\}_{i=1}^I \in Y^\infty(s_0)$ such that $U_i^{P_i}(\widehat{c}_i) > U_i^{P_i}(c_i^*)$ for all $i \in \mathcal{I}$. A feasible enforceable allocation $\{c_i^*\}_{i=1}^I$ is *Constrained Pareto optimal (CPO)* if there is no alternative feasible enforceable allocation $\{\widehat{c}_i\}_{i=1}^I \in Y_E^\infty(s_0)$ such that $U_i^{P_i}(\widehat{c}_i) > U_i^{P_i}(c_i^*)$ for all $i \in \mathcal{I}$.

Given the state of nature and prior beliefs at date zero, s_0 and $\mu_0 \equiv (\mu_{1,0}, ..., \mu_{I,0})$, define the *utility possibility correspondence* by

$$\mathcal{U}^{FB}(s_0, \mu_0) = \{u \in \mathbb{R}^I : \exists \; \{c_i\}_{i=1}^I \in Y^\infty(s_0), \; u_i \leq U_i^{P_i}(c_i) \quad \forall i\},$$

and the *constrained utility possibility correspondence* by

$$\mathcal{U}^E(s_0, \mu_0) = \{u \in \mathbb{R}^I : \exists \; \{c_i\}_{i=1}^I \in Y_E^\infty(s_0, \mu_0), \; U_i(s_0, \mu_0) \leq u_i \leq U_i^{P_i}(c_i) \quad \forall i\}.$$

Lemma 19 in the Appendix makes evident that the set of CPO allocations can be characterized as the solution to the following planner's problem. Given μ_0, s_0 and welfare weights $\alpha \in \mathbb{R}_+^I$, define

$$v^*(s_0, \mu_0, \alpha) \equiv \sup_{\{c_i\}_{i=1}^I \in Y_E^\infty(s_0,\mu_0)} \sum_{i=1}^I \alpha_i \; E^{P_i}\left(\sum_t \rho_t \; u_i(c_{i,t})\right). \tag{3}$$

It is straightforward to prove that (3) can be rewritten as

$$v^*(s_0, \mu_0, \alpha) = \sup_{u \,\in\, \mathcal{U}^E(s_0,\mu_0)} \sum_{i=1}^I \alpha_i \; u_i, \tag{4}$$

The maximum in (4) is attained since the problem consists in maximizing a continuous function on a set that is compact by Lemma 19. The next result shows shows how to characterize the utility possibility set.

Lemma 1 *$u \in \mathcal{U}(\xi,\mu)$ if and only if $u_i \geq U_i(\xi,\mu_i)$ for all i and*

$$\min_{\widetilde{\alpha}\in\Delta^{I-1}}\left[v^*(\xi,\mu,\widetilde{\alpha})-\sum_{i=1}^{I}\widetilde{\alpha}_i\ u_i\right]\geq 0.$$

Let $\partial\mathcal{U}(\xi,\mu)$ be a utility possibility frontier at (ξ,μ). Let $\alpha_{\partial\mathcal{U}(\xi,\mu)}(U_i(\xi,\mu_i),u_{-i}) \in \Delta^{I-1}$ be the welfare weights corresponding to $(U_i(\xi,\mu_i),u_{-i}) \in \partial\mathcal{U}(\xi,\mu)$; i.e., the welfare weight parameterizing the supporting hyperplanes of $\partial\mathcal{U}(\xi,\mu)$ for the utility levels at the frontier for which agent i attains the reservation value $U_i(\xi,\mu_i)$. Since u_i is strictly concave and continuously differentiable, it follows from (Rockafellar, 1970) that $\alpha_{\partial\mathcal{U}}(.,.)$ is a continuous function and $\alpha_{i,\mathcal{FA}}(u_i,u_{-i})$ is increasing with respect to u_i. Define

$$\underline{\alpha}_i(\xi,\mu)=\max_{u_{-i}}\quad \alpha_{\partial\mathcal{U}(\xi,\mu)}(U_i(\xi,\mu_i),u_{-i}), \tag{5}$$

subject to $(U_i(\xi,\mu_i),u_{-i}) \in \partial\mathcal{U}(\xi,\mu)$. Since $\partial\mathcal{U}(\xi,\mu)$ defines a continuous function, it follows from the Theorem of the Maximum that $\underline{\alpha}_i(.)$ is continuous with respect to (ξ,μ).

Define $\mathcal{U}^{FB}(\mu)=\{(u(1),...,u(K))\in\left(\mathbb{R}^I_+\right)^K : u(\xi)\in\mathcal{U}^{FB}(\xi,\mu)\}$; the set of utility levels that are attainable for any alternative shock.

3 A Recursive Approach to Constrained Pareto Optimality

In this section, we provide a recursive characterization of the set of Pareto optimal allocations and a version of the Principle of Optimality for economies with heterogeneous prior beliefs and limited enforceability.

3.1 The Recursive Planner's Problem

In Appendix B we show that v^* solves the functional equation[7]

$$v^*(\xi,\mu,\alpha)=\max_{(c,w'(\xi'))}\sum_{i=1}^{I}\alpha_i\left\{u_i(c_i)+\beta(\xi)\sum_{\xi'}\pi_{\mu_i}(\xi'\,|\xi)\ w_i'(\xi')\right\}, \tag{6}$$

subject to

$$\sum_{i=1}^{I}c_i=y(\xi)\qquad\text{for all }\xi,\qquad c_i\geq 0, \tag{7}$$

$$u_i(c_i)+\beta(\xi)\sum_{\xi'}\pi_{\mu_i}(\xi'\,|\xi)\ w_i'(\xi')\geq U_i(\xi,\mu_i), \tag{8}$$

[7] In sections 3.1 and **??**, we abuse notation and let c to be a non-negative vector and c_i its i^{th} component.

$$w_i'(\xi') \geq U_i(\xi', \mu_i'(\xi,\mu)(\xi')) \qquad \text{for all } \xi', \tag{9}$$

$$\min_{\widetilde{\alpha} \in \Delta^{I-1}} \left[v^*(\xi', \mu'(\xi,\mu)(\xi'), \alpha') - \sum_{i=1}^{I} \alpha_i' \; w_i'(\xi') \right] \geq 0 \text{ for all } \xi', \tag{10}$$

where $\mu'(\xi,\mu) = (\mu_1'(\xi,\mu_1)...\mu_I'(\xi,\mu_I))$,

$$\mu_i'(\xi,\mu_i)(\xi')\,(B) = \frac{\int_B \pi(\xi' \,|\xi) \; \mu_i\,(d\pi)}{\int \pi(\xi' \,|\xi) \; \mu_i\,(d\pi)} \text{ for any } B \in \mathcal{B}(\Pi^K), \tag{11}$$

and $\alpha'(\xi')$ solves problem (10) for all ξ'.

In the recursive dynamic program defined by (6) - (11), the current state of nature, ξ, captures the impact of changes in aggregate output while (α,μ) summarizes and isolates the $\mathcal{B}$-margin of heterogeneity and its law of motion. The planner takes as given (ξ,α,μ) and allocates current consumption and continuation utility levels among agents. That is, instead of allocating consumption from tomorrow on, the planner assigns to each agent the utility level associated with the corresponding continuation sequence of consumption. Indeed, the optimization problem defined in condition (10) characterizes the set of continuation utility levels attainable at $(\xi', \mu'(\xi,\mu))$ (see Lemma 1 in Appendix B).[8] The weights $\alpha'(\xi')$ that attain the minimum in (10) will then be the new weights used in selecting tomorrow's allocation.

Define $\Delta(\xi,\mu) \equiv \{\alpha \in \Delta : \alpha_i \geq \underline{\alpha}_i(\xi,\mu) \text{ for all } i\}$. The (normalized) law of motion for the welfare weights, $\alpha_i'(\xi,\alpha,\mu)(\xi')$, follows from the first order conditions with respect to the continuation utility levels for each individual. It follows by standard arguments that the corresponding consumption policy function, $c_i(\xi,\alpha)$, is the unique solution to

$$c_i(\xi,\alpha) + \sum_{h \neq i} \left(\frac{\partial u_h}{\partial c_h} \right)^{-1} \left(\frac{\alpha_i}{\alpha_h} \; \frac{\partial u_i(c_i(\xi,\alpha))}{\partial c_i} \right) = y(\xi). \tag{12}$$

for each i, where $\left(\frac{\partial u_h}{\partial c_h}\right)^{-1}$ denotes the inverse function of $\frac{\partial u_h}{\partial c_h}$.

Any (c,α') that satisfies (7) - (10) will be referred as a *feasible enforceable recursive allocation.* Given (s_0,α_0,μ_0), we say the policy functions (c,α') coupled with μ'

[8] To understand condition (10) notice that the utility possibility set, i.e. the set of expected lifetime utility levels that are attainable by mean of feasible allocations, is convex, compact and contains its corresponding frontier. The frontier of a convex set can always be parametrized by supporting hyperplanes. Moreover, under our assumptions, the corresponding parameters can be restricted to lie in the unit simplex and, therefore, they can be interpreted as welfare weights. Thus, a utility level vector w is in the utility possibility set if and only if for every welfare weight α the hyperplane parametrized by α and passing through w, αw, lies below the hyperplane generated by the utility levels attained by the PO allocation corresponding to that welfare weight α, attaining the value $v(\xi,\alpha,\mu)$. This is why we must have $\alpha w \leq v(\xi,\alpha,\mu)$ for all α or, equivalently, $\min_{\widetilde{\alpha}} [v(\xi,\widetilde{\alpha},\mu) - \widetilde{\alpha} w] \geq 0$. See Appendix B for technical details.

generates an allocation $\widehat{c} \in \mathbb{C}(s_0)^I$

$$\begin{aligned} \widehat{c}_{i,t}(s) &= c_i(s_t, \alpha_t(s)), \\ \alpha_{t+1}(s) &= \alpha'_{cpo}(s_t, \mu_{s^t}, \alpha_t(s))(s_{t+1}), \\ \mu_{(s^t, s_{t+1})} &= \mu'(s_t, \mu_{s^t})(s_{t+1}), \end{aligned} \tag{13}$$

for all i, $t \geq 0$ and $s \in S^\infty$, where $\alpha_0(s) = \alpha_0$ and $\mu_{i,s^0} = \mu_{i,0}$.

Proposition 2 *An allocation $(c_i^*)_{i=1}^I$ is CPO given (ξ, μ, α) if and only if it is generated by the set of policy functions solving (6) - (11) evaluated at v^*.*

Thus, the value of any plan that can be attained with a feasible enforceable allocation c can also be attained by a feasible enforceable recursive allocation that delivers to each agent consumption for today and promising to each agent some contingent levels of expected utility from tomorrow on that satisfy the enforceability constraints, attached to the corresponding welfare weight.

3.2 Limited Enforceability and the The Extend of Risk Sharing

This section considers a two-state economy and ask when enforceability constraints make full risk sharing forever impossible, that is when the set of PO allocations does not intersects the set of CPO allocations.

For the case in which agents have homogeneous priors, Alvarez and Jermann [3, Proposition 1] gave necessary and sufficient conditions on the fundamentals for full risk sharing to be possible. The following Proposition, instead, provides a necessary and sufficient condition in terms of the minimum enforeceable welfare weights.

Proposition 3 *Suppose $A1$ holds for both agents and $\mu_1 = \mu_2 = \mu^\pi$. Then, no CPO allocation is PO if and only if $\underline{\alpha}_1(1, \mu^\pi) > 1 - \underline{\alpha}_2(2, \mu^\pi)$.*

For the case in which agents have heterogenous priors, the following Proposition shows that when agents' prior beliefs satisfy $A1$ and $A3$, CPO allocations are *never* PO.

Proposition 4 *Suppose $A1$ holds for both agents and $A3$ holds. Then, no CPO allocation is PO.*

To understand the intuition behind this result, let's assume agent i is relatively more optimistic about state of nature i. Since the efficient allocation requires the welfare weight of agent i to strictly decrease if state of nature i occurs, then for any pre-specified lower bound there is a finite string of realizations of state of nature i

such that the continuation utility of agent i is below such bound. We conclude that the efficient allocation violates the participation constraint of both agents in finite time with positive probability.

3.3 The Dynamic of the Welfare Weights Distribution

In this section we assume that every agent has dogmatic beliefs, that is $\mu^{\pi} = (\mu^{\pi_1}...\mu^{\pi_I})$. Let 2^S be the power set of S and $\mathcal{B}(\Delta(\xi,\mu^{\pi}))$ be the Borel sets of $\Delta(\xi,\mu^{\pi})$. $\Omega = S \times \Delta(\xi,\mu^{\pi})$ is the state space and $\mathcal{G} = 2^S \times \mathcal{B}(\Delta(\xi,\mu^{\pi}))$ is the $\sigma-$algebra on Ω. For $t \geq 0$, Ω^t is the $t-$cartesian product of Ω with typical element $\omega^t = (s_0,\alpha_0,...,s_t,\alpha_t)$ and $\Omega^{\infty} = \Omega \times \Omega \times ...$is the infinite product of the state space with typical element $\omega = (\omega_0,\omega_1,...)$. $\mathcal{G}_{-1} \equiv \{\varnothing,\Omega^{\infty}\}$ is the trivial $\sigma-$algebra, $\mathcal{G}_t$ is the $\sigma-$algebra that consists of all the cylinder sets of length t. The $\sigma-$algebras $\mathcal{G}_t$ define a filtration $\mathcal{G}_{-1} \subset \mathcal{G}_0 \subset ... \subset \mathcal{G}_t \subset ... \subset \mathcal{G}^{\infty}$, where $\mathcal{G}^{\infty} \equiv \mathcal{G} \times \mathcal{G} \times ...$ is the $\sigma-$algebra on Ω^{∞}. The endogenous law of motion for the welfare weights derived above, α'_{cpo}, together with the transition probability matrix π^* defines a time-homogeneous transition function on states of nature and welfare weights:

$$F_{cpo} : \Omega \times \mathcal{G} \rightarrow [0,1]$$

where for any $(\mathcal{S} \times \mathcal{A}) \in \mathcal{G}$,

$$F_{cpo}\left[(\xi,\alpha),\mathcal{S}\times\mathcal{A}\right] = \sum_{\xi'\in\mathcal{S},\ \alpha'_{cpo}(\xi,\alpha)(\xi')\in\mathcal{A}} \pi^*\left(\xi'\,|\xi\right)$$

Finally, note that the transition function F_{cpo} together with a probability measure ψ on $(\Omega,\mathcal{G})$ induces a unique probability measure $P^{F_{cpo}}(\psi,\cdot)$ on $(\Omega^{\infty},\mathcal{G}^{\infty})$. In the special case in which ψ is a point mass on ω_0 for some $\omega_0 \in \Omega$, we denote the latter by $P^{F_{cpo}}(\omega_0,\cdot)$.

Next, define an operator on the space of probability measures over $(\Omega,\mathcal{G})$ as

$$T^*\psi(\mathcal{S},\mathcal{A}) = \int F_{cpo}\left((\xi,\alpha),\mathcal{S}\times\mathcal{A}\right)d\psi, \quad P^{F_{cpo}}(\omega_0,\cdot) - a.s.$$

where $P^{F_{cpo}}(\omega_0,\cdot)$ is the probability measure induced by the transition F_{cpo} and an initial probability measure that is a point mass on ω_0.

Standard arguments can be used to show that if there are only two agents and they have dogmatic heterogeneous beliefs there exists a unique invariant measure over $(\Omega,\mathcal{G})$ and that the distribution of states of nature and welfare weights converges weakly to that invariant measure.

Proposition 5 *Suppose A.0 and A1 holds for every agent. If each agent i and j has dogmatic beliefs μ^{π_i} and $\pi_i \neq \pi_j$, then there exists a unique invariant distribution $\psi_{cpo} : \mathcal{G} \rightarrow [0,1]$.*

3.4 Computation

For many purposes it is important to have an algorithm capable of finding the value function v^*. Let $\widehat{v}$ be the value function solving the recursive problem when the enforceability constraints are ignored while the corresponding operator is denoted $\widehat{T}$ (i.e. the value function and the operator stemming from Beker and Espino (2010)). Evidently, $v^*(\xi, \mu, \alpha) \leq \widehat{v}(\xi, \mu, \alpha)$ for all (ξ, μ, α).

Proposition 6 *Let $v_0 = \widehat{v}$ and $v_n = T^n(\widehat{v})$. Then, $\{v_n\}$ is a monotone decreasing sequence and $\lim_{n\to\infty} v_n = v^*$.*

3.5 Discussion

There are at least two alternative approaches to state recursively the dynamic program defined by (4). To simplify the exposition of these alternative approache, let's assume there are only two agents. The first alternative was developed by Thomas and Worral [29] and Kocherlakota [19].[9] Instead of parametrizing allocations with welfare weights, the planner chooses current feasible consumption and continuation utilities for both agents in order to maximize the utility of agent 1 subject to three restrictions: (i) the utility of agent 2 is above some pre-specified level (the so-called promise keeping constraint); (ii) allocations are period-by-period enforceable and (ii) continuation utility levels lie in the next period utility possibility correspondence. Very importantly, this last two conditions implies that the corresponding value function defines the constraint set. The second alternative, developed in Beker and Espino [6], studies directly the operator defined by (6) - (10).

Since both approaches use the value function to define the constraint set, it is not clear that any of the associated operators satisfies Blackwell's discounting (sufficient) condition for a contraction. If enforceability constraints are ignored, Beker and Espino [6] show that discounting is satisfied if the operator is restricted to F_H. With enforceability constraints, however, this approach cannot be applied. From a conceptual point of view the problem can be explained as follows. For any function v that defines the constraint set, there might be some positive constant, $a > 0$ such that $v + a$ enlarges the feasible set of choices of continuation utilities with respect to v. Although $v + a$ is still an affine linear transformation of v, it gives some room to deal with enforceability and that conflicts with discounting. As a matter of fact,

[9] In a partial equilibrium setting, Thomas and Worral [29] study the efficent distribution of risk between a risk-neutral firm and a risk-averse worker in an enviroment without commitment. This simplified framework lets them to describe the sequential Pareto frontier recursively. Kocherlakota [19] extends their setting to study a general equilibirium version and he claims that their same technique can be applied to his problem.

uniqueness is not satisfied since the function $f(\xi,\mu,\alpha)=\sum_{i=1}^{I}\alpha_i U_i(\xi,\mu_i)$ is also a fixed point of the operator defined by the right hand side of (6) - (11).

Our strategy relates to the seminal idea pioneered by Abreu, Pearce and Stacchetti [1] (discussed in Alvarez and Jermann [3] in a setting without commitment). They construct an operator that iterates directly on the utility possibility correspondence and then the value function (and the corresponding policy functions) are recovered from the frontier of the fixed point of that operator on utility correspondences. Our approach follows their idea but it iterates directly on the utility posibility frontier parametrized by welfare weights. To implement this strategy, it is key that the utility possibility correspondence is convex-valued (see Lemma 19), a property that can be absent in Abreu, Pearce and Stacchetti [1] since they are particularly interested in non-convex problems (e.g., they study bang-bang solutions stemming from non-convexities generated by other incentive constraints).

4 Equilibrium with Endogenous Solvency Constraints

Consider a market economy in which the following trading opportunities are available. Every period t and after having observed s^t, agents meet in spot markets to trade the consumption good and a complete set of Arrow securities in zero net supply. Arrow security ξ' issued at date t on path s pays one unit of consumption if next period's state of nature is ξ' and 0 otherwise. Let $q_t^{\xi'}(s)$ and $a_{i,t}^{\xi'}(s)$ denote the price of security ξ' issued at date t and agent i's holdings of it at date t, respectively. All prices are in units of the date–t consumption good, $a_{i,t}=\left(a_{i,t}^{1},...,a_{i,t}^{K}\right)$ and $a_{i,0}=0$ for all i.

Trading strategies are restricted by solvency constraints which prohibits agents from holding large amounts of contingent debt, hence preventing default. Indeed, type i agents face a state contingent solvency constraint, such that $B_{i,t+1}^{\xi'}(s)$ limits security ξ' holdings chose at date t. We write q, a_i and B_i to denote the corresponding stochastic processes.

Given a q and B_i, agent i's problem is

$$\max_{(c_i,a_i)} \quad E^{P_i}\left(\sum_{t=0}^{\infty}\rho_t\ u_i(c_{i,t})\right), \tag{14}$$

subject to

$$c_{i,t}(s)+\sum_{\xi'}q_t^{\xi'}(s)a_{i,t+1}^{\xi'}(s)=y_i(s_t)+a_{i,t}(s), \tag{15}$$

$$a_{i,t+1}^{\xi'}(s)\geq B_{i,t+1}^{\xi'}(s), \tag{16}$$

$$c_{i,t}(s)\geq 0 \text{ and } a_{i,0}=0, \tag{17}$$

for all s and all t.

Definition. *A competitive equilibrium with solvency constraints* $\{B_i\}$ *is an allocation* $\{c_i\}$*, portfolios* $\{a_i\}$ *and a price system* q *such that*
(CE 1) Given q *and* B_i*,* $\{c_i, a_i\}$ *solves agent* i*'s problem (14-17).*
(CE 2) All markets clear. For all s *and all* t

$$\sum_{i=1}^{I} c_{i,t}(s) = y(s_t), \tag{18}$$

$$\sum_{i=1}^{I} a_{i,t+1}^{\xi'}(s) = 0 \qquad \text{for all } \xi'. \tag{19}$$

It is important to mention that our equilibrium concept does not rely on solvency constraints that are not too tight, as discussed in Alvarez and Jermann [2], [3]. Indeed, under our decentralization, individual asset holdings are always at the solvency constraints by construction. However, as discussed in [3, pp 1131], some of these are "false corner"; i.e., if the solvency constraints were relaxed a bit, the agent would not change the optimal choice of consumption and asset holdings. Thus, for those states, his intertemporal marginal rate of substitution equals the Arrow security price and then his constraint is not effectively binding. By construction, whenever the intertemporal marginal rate of substitution falls short of the Arrow security price (i.e., the implicit Lagrange multiplier for that particular Arrow security is positive), the agent's enforceability constraint is indeed binding.

Consequently, , we can conclude that equilibrium prices satisfy

$$q_t(s)(\xi') = \beta(s_t) \max_i \left\{ \pi_{\mu_{i,s^t}}(\xi' \,|s_t) \frac{u_i'(c_{i,t+1}(s))}{u_i'(c_{i,t}(s))} \right\}.$$

The following proposition follows immediately from Alvarez and Jermann [2, Proposition 3.1].

Proposition 7 *Suppose allocation* $(c_h^*)_{h=1}^I$ *is CPO given* (ξ, μ, α)*. Suppose that agent* i *participation constraint does not bind at* ξ'*. Then,*

$$q_t(s)(\xi') = \beta(\xi)\ \pi_{\mu_{i,s^t}}(\xi' \,|s_t) \frac{u_i'(c_{i,t+1}(s'))}{u_i'(c_{i,t}(s))} \text{ for any } s' \in C\left(s^t, \xi'\right).$$

4.1 Decentralization

Here we study the determinants of the financial wealth distribution that supports CPO allocations in a dynamically complete markets equilibrium with endogenous solvency constraints. First, we characterize individual financial wealth recursively as a time invariant function of the states (ξ, μ, α). Later, we employ a version of the Negishi's approach to pin down the CPO allocation that can be decentralized as a competitive equilibrium without transfers in which solvency constraints are determined endogenously.

We begin by defining $A_i(\xi, \alpha, \mu)$ as the solution to the functional equation

$$A_i(\xi, \alpha, \mu) = c_i(\xi, \alpha) - y_i(\xi) + \sum_{\xi'} Q(\xi, \alpha, \mu)(\xi') \; A_i(\xi', \alpha', \mu'), \tag{20}$$

where
$$Q(\xi, \mu, \alpha)(\xi') = \beta(\xi) \; \max_h \left\{ \pi_{\mu_h}(\xi' \,|\xi) \; \frac{\partial u_h(c_h(\xi', \alpha'(\xi, \mu, \alpha)(\xi'))/\partial c_h}{\partial u_h(c_h(\xi, \alpha))/\partial c_h} \right\}, \tag{21}$$

is the state price. Expression (20) computes recursively the present discounted value of agent i's excess demand at the PO allocation priced by (21). Intuitively, $A_i(\xi, \alpha, \mu)$ is the transfer needed to support as a competitive equilibrium the PO allocation parametrized by α given (ξ, μ) (see Espino and Hintermaier [15] for further discussion.)

Define the state price by and the (implicit) risk-free interest rate, R^{RF}, is defined

$$\left(R^{RF}(\xi, \mu, \alpha)\right)^{-1} = \sum_{\xi'} Q(\xi, \mu, \alpha)(\xi').$$

Definition: *We say that a PO allocation generates positive risk-free interest rates if* $R^{RF}(\xi, \mu, \alpha) > 1$ *for all* (ξ, μ, α).

In Theorem 8, we show that A_i is well-defined. Furthermore, we show that there exist a welfare weight α_0 such that A_i is zero for every i. The allocation parametrized by α_0 is the natural candidate to be decentralized as a competitive equilibrium.

Theorem 8 *Suppose that* A1 *holds for all agents and the PO allocation generates positive risk-free interest rates. There is a unique continuous function* A_i *solving (20). Moreover, for each* (s_0, μ_0) *there exists* $\alpha_0 = \alpha(s_0, \mu_0) \in \mathbb{R}^I_+$ *such that* $A_i(s_0, \mu_0, \alpha_0) = 0$ *for all* i.

Now we proceed to implement the Negishi's approach to decentralize the PO allocation parametrized by α_0 as competitive equilibrium with solvency constraints. For each s, t and ξ', use (21) and (20) to define recursively

$$\widehat{a}^{\xi'}_{i,t+1}(s) = A_i\left(\xi', \mu_{(s^t, \xi')}, \alpha_{t+1}(s)\right) \tag{22}$$

$$\widehat{q}^{\xi'}_t(s) = Q(s_t, \mu_{s^t}, \alpha_t(s))(\xi') \tag{23}$$

$$\widehat{B}^{\xi'}_{i,t+1}(s) = A_i\left(\xi', \mu_{(s^t, \xi')}, \alpha_{t+1}(s)\right), \tag{24}$$

with $\mu_{s^0} = \mu_0$, α_t is generated by (13) and $\alpha_0 = \alpha(s_0, \mu_0)$.

In a decentralized competitive setting with sequential trading, $A_i(s_t, \alpha_t(s), \mu_{s^{t-1}})$ can be interpreted as the financial wealth that agent i needs to have at date t on path s to afford the consumption bundle corresponding to the PO allocation parametrized by $\alpha_t(s)$ given $(s_t, \mu_{s^{t-1}})$.

Remark 2: Observe that if agents have both homogeneous and dogmatic beliefs (i.e., $\mu_{i,0} = \mu_i^\pi$ for all i and for some $\pi \in \Delta^{K-1}$), it follows immediately that for each i, $\alpha_{i,t+1}(s) = \alpha_{i,0}$ and $\mu_{i,s^t} = \mu_i^\pi$ for all s and all $t \geq 0$. Therefore, $c_{i,t}(s) = c_i(s_t, \alpha_0)$ for all s and all $t \geq 0$. Very importantly, $(A_i(1, \alpha_0, \mu^\pi), ..., A_i(K, \alpha_0, \mu^\pi)) \in \mathbb{R}^K$ and satisfies $A_i(s_0, \alpha_0, \mu^\pi) = 0$ for all i.

Theorem 9 *The unique CPO allocation $\widehat{c}$ solving (3) for $\alpha_0 = \alpha(s_0, \mu_0)$ can be decentralized as a competitive equilibrium with solvency constraints $\{\widehat{B}_i\}$, portfolios $\{\widehat{a}_i\}$ and Arrow security prices $\widehat{q}$.*

5 Dynamic Asset Trading

In this section we study the dynamic behavior of welfare weights and its implications for asset trading.

It is well known that there exists some $\pi_i \in \Pi^K$ such that μ_{i,s^t} converges weakly to μ^{π_i} for $P^{\pi_i}-$almost all $s \in S^\infty$ (for $\mu_{i,0}-$ almost all $\pi \in \Pi^K$) and for all i. Therefore, if one is interested in the asymptotic dynamics of the welfare weights, the restriction to dogmatic beliefs is without loss in generality.

5.1 The Fixed Equilibrium Portfolio Property

Definition: Let $A_{i,t}(s) \equiv A_i(s_t, \alpha_t(s), \mu_{s^{t-1}})$ be agent i's financial wealth at date t on path s. We say that the fixed equilibrium portfolio (FEP hereafter) property holds on s if for each i, there exists $\{\widetilde{a}_i(1), ..., \widetilde{a}_i(K)\} \in \Re^K$ such that $A_{i,t}(s) = \widetilde{a}_i(s_t)$ for all $t \geq 0$. The FEP property holds asymptotically on s if for each i, there exists $\{\widetilde{a}_i(1), ..., \widetilde{a}_i(K)\} \in \Re^K$ such that $A_{i,t}(s) \to \widetilde{a}_i(s_t)$.

If the *FEP* property holds on s, any portfolio that decentralizes a PO allocation with a fixed set of non-redundant assets is constant over time.

Judd *et al.* [17] show that the *FEP* property is always satisfied after a once-and-for all initial rebalancing when agents have homogeneous priors (the $\mathcal{B}$-margin of heterogeneity is constant) and degenerate beliefs. Specifically, $\widetilde{a}_i(\xi) = A_i(\xi, \mu^\pi, \alpha_0)$ for all ξ since in their setting $\mu_{i,0} = \mu_i^\pi$ for some $\pi \in \Pi^K$ and for all i. Therefore, the agents' financial wealth is a vector in $\mathbb{R}^K$ in any dynamically complete markets equilibrium. When agents have homogeneous but non-degenerate beliefs, the welfare weights are constant along time, $\alpha_t(s) = \alpha_0$, and the distribution of consumption is given by $c_i(\xi) = c_i(\xi, \alpha_0)$ for each i and so it remains unchanged as time and uncertainty unfold. However, the financial wealth distribution, $A_{i,t}(s) = A_i(s_t, \alpha_0, \mu_{s^{t-1}})$, is still history dependent because the agents' learning process make state prices history dependent. Consequently, the FEP property does not necessarily hold. See Beker and Espino [6] for a thorough characterization of this case.

In the context of a heterogenous belief economy, Beker and Espino [6] argue that the portfolios that decentralize Pareto optimal allocations change over time because the changes in the likelihood ratios affects the dynamics of the wealth. However, they show that under assumptions $A0$, $A1$ and $A2$ the portfolios that support any PO allocation converge (generically) and so genuine asset trading generated by beliefs heterogeneity vanishes in any dynamically complete markets equilibrium.[10]

Alvarez and Jermann [3] consider an economy where agents have homogeneous dogmatic beliefs, i.e. $\mu_{i,0} = \mu^{\pi}$ for $\pi \in \Pi^K$, with limited enforceability and show that the limiting distribution of asset trading depends on the endowment distribution. If the distribution of endowments is such that some PO allocations are also CPO, then the wealth distribution converges to a degenerate distribution and so the FEP property holds asymptotically. On the other hand, if the distribution of endowments is such that no PO allocation is CPO, then the wealth distribution converges to a non-degenerate distribution and so the FEP property does not hold.

One concludes that (i) regardless of whether agents have homogeneous or heterogeneous beliefs (generically) one cannot have non-vanishing asset trading generated in a stationary dynamically complete markets equilibrium and(ii) asset trading is persistent only for some endowment distributions if allocations are CPO.

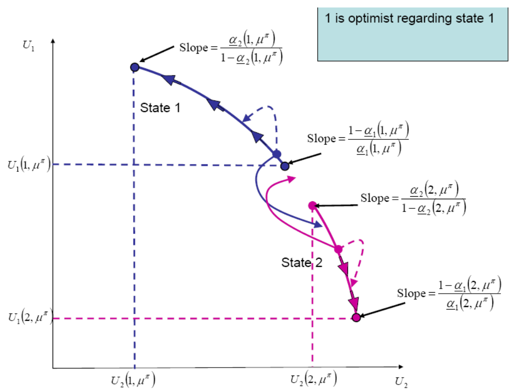
In what follows we combine Beker and Espino [6] and Alvarez and Jermann [3] to study the implications of the interaction between belief heterogeneity and limited enforceability. Our recursive approach permits to conclude that when agents have heterogeneous dogmatic beliefs, changes in portfolios never vanish and thus the FEP property can never holds, regardless of the endowment distribution and the entropy of beliefs. Moreover, not only asset trading does not vanish, but also the size of the change in portfolios is not directly related to the size of the heterogeneity of beliefs. Indeed, we show that for *any* degree of belief heterogeneity causes large fluctuations in the wealth distribution, and so large changes in portfolios, in any CPO equilibrium allocation.[11]

[10] If agents have different dogmatic prior beliefs with the same entropy, portfolios can fluctuate forever and the FEP property does not hold. But the assumption of identical entropy makes these examples non-generic. Generically, only the welfare weight of the agents whose prior is closest to the true distribution (according to entropy) does not converge to zero (see Sandroni [24]) and, therefore, the limiting economy reduces to the economy studied by Beker and Espino [6].

[11] If agents have different non-degenerate prior beliefs for which the support contains the truth,

More precisely, we say that agents have $\varepsilon-$heterogeneous dogmatic beliefs if $\min_{i,h} \left|\pi_i(\xi' \,|\xi\,) - \pi_h(\xi' \,|\xi\,)\right| \geq \varepsilon$ for all (ξ, ξ').

Proposition 10 *Suppose A0 holds. If agents have $\varepsilon-$heterogeneous dogmatic beliefs then, $P^{\pi^*}-$a.s., $\alpha_{1,t}(s)$ fluctuates between $\underline{\alpha}_1(2, \mu^\pi)$ and $1-\underline{\alpha}_2(1, \mu^\pi)$ and so the FEP property does not hold asymptotically.*



6 Short-Term Momentum and Long-Term Reversal

In Section 6.1 we introduce a formal definition of short-term momentum and long-term reversal in terms of the empirical autocorrelations of the assets excess returns. In Section 6.2, we argue that in any PO or CPO equilibrium, the empirical autocorrelations can be approximated using the population autocorrelations. In Section 6.3 we provide a statistical charaterization of the autocorelations in terms of the changes on the true conditional expectation of the assets excess returns induced by the arrival of new information. Finally, in Section 6.4 we write the asset price as a conditional expectation with respect to a probability measure that we call the market belief and we characterize the changes on the true conditional expectation of excess returns upon the arrival of new information in term of how the bias about the probability of a positive return changes as the market updates its belief. We show that a necessary condition is that the market belief has the following peculiar dynamic with positive probability: first it has to be less biased after a positive than a negative abnormal return in period 1 but later on it has to be more biased after a positive than a negative positive return.

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the analysis is more intricate. More later.

6.1 Definitions

Let $d_{t+k}(s)$ and $p_{t+k}(s)$ be the date $t+k$ ex-dividend price and the dividend of an asset, respectively, on path s. For $k \geq 1$, let

$$R_{t+k}(s) = \frac{p_{t+k}(s) + d_{t+k}(s)}{r_{t+k-1}(s)} - p_{t+k-1}(s)$$

be the one-period net excess return (the return hereafter) between dates $t+k-1$ and $t+k$.

We imagine an econometrician who collects data on asset returns. A sample point beginning at date t is a vector $\left(R_{t+1}, ..., R_{t+\overline{k}}\right)$, where $\overline{k} > 1$, consisting of the returns at $\overline{k}$ consecutive dates. D is the (finite) set of dates when sample points begin. Let T be the sample size. For each date t in D and for some $1 \leq k \leq \overline{k}$, let

$$\overline{R}_{T,k}(s) \equiv \frac{1}{T}\sum_{t\in D} R_{t+k}(s) \text{ and } \sigma^2_{T,k} \equiv \frac{1}{T}\sum_{t\in D}\left(R_{t+k}(s) - \overline{R}_k(s)\right)^2$$

be the empirical average return and variance of the asset between periods k and $k-1$. Let

$$cov_{T,k}(s) \equiv \frac{1}{T}\sum_{t\in D}\left(R_{t+1}(s) - \overline{R}_1(s)\right)\left(R_{t+k}(s) - \overline{R}_k(s)\right) \quad \text{and}$$

$$\rho_{T,k}(s) \equiv \frac{cov_{T,k}(s)}{\sigma_{T,1}(s)\ \sigma_{T,k}(s)}$$

be the empirical autocovariance and empirical correlation coefficient of order k.

Now we are ready to give a formal definition of the so-called financial markets anomalies that we are interested to explain.

Definition *An asset displays short-term momentum on a path s if* $\lim_{T\to\infty} \rho_{T,2}(s) > 0$*. An asset displays long-term reversal on a path s if* $\lim_{T\to\infty} \rho_{T,3}(s) < 0$*.*

Short-term momentum occurs if an increase in the asset's return between dates t and $t+1$ is followed, on average, by another increase between days $t+1$ and $t+2$. Long term reversal occurs, instead, if an increase in the asset's return between dates t and $t+1$ is followed, on average, by a decrease between days $t+2$ and $t+3$.

If allocations are PO or CPO, the history can be summarized by the state variables (ξ, α, μ). We consider a class of assets for which the equilibrium return in period $k+1$ depends in a continuous fashion on the state of nature in period $k+1$, the period k's state variables and the cumulated dividend during the last k periods.[12] That is , for each $e \in \{PO, CPO\}$ and for every $k \geq 0$ there exists a continuous function

[12] This class is broad enough to include Arrow securities, Lucas trees, risk free bonds, etc.

$R_{k+1,e} : \Omega \times R \times S \rightarrow R$ such that

$$R_{t+k+1}(s) = R_{k+1,e}\left(s_{t+k}, \alpha_{t+k}, \widehat{d}_{t+k}\right)(s_{t+k+1})$$

where $\widehat{d}_{t+k}$ is the cumulative dividend announcement between dates t and $t+k$. Furthermore, we assume that $\widehat{d}_t(s) = 0$ for every $t \in D$ and for any $k \geq 1$ there exists a function $\delta_k : S^k \rightarrow \Re$ such that

$$\widehat{d}_{t+k}(s) = \delta_k(s_{t+1}, ..., s_{t+k})$$

In the particular case in which $k \in \{0, 1\}$, we abuse notation and simply write

$$R_{t+k+1}(s) = R_{k+1,e}(s_{t+k}, \alpha_{t+k})(s_{t+k+1})$$

6.2 Asymptotic Approximation

We are interested in characterizing the asymptotic behavior of the empirical autocorrelations. Note that the empirical autocorrelations are continuous functions of the rates of return and (PO or CPO) equilibrium rates of return are continuous functions of a Markov process with transition F_e on $(\Omega, \mathcal{G})$. If one argues that the Markov process is ergodic with invariant distribution ψ_e, then standard arguments show that the following asymptotic approximation holds

$$\lim_{T \rightarrow \infty} cov_{T,k}(s) = cov^{P_e}(R_{1,e}, R_{k,e}), \qquad P^{\pi^*} - a.s., \tag{25}$$

where $P_e \equiv P^{F_e}(\psi_e, \cdot)$.

The ergodicity of the Markov process of the states in a PO or CPO equilibrium allocations can be intuitively explained as follows. For the case in which allocations are PO and the dgp is iid, Beker and Espino [6] show that if A1 holds for every agent and A2 holds for some agent, then the vector of welfare weights associated with a PO allocation converges to a fixed vector almost surely. An analogous result can be proved in the case that the dgp is generated by draws from a time-homogeneous transition matrix as in this paper. In particular, the vector of welfare weights converge $P^{\pi *} - a.s.$ to $\alpha_\infty = \left(\frac{\alpha_{1,0}\mu_1(\pi^*)}{\sum_{i \in I} \alpha_{i,0}\mu_i(\pi^*)}, ..., \frac{\alpha_{I,0}\mu_I(\pi^*)}{\sum_{i \in I} \alpha_{i,0}\mu_i(\pi^*)}\right)$ if A2.a holds for all agents or to $\alpha_\infty = \left(\frac{\alpha_{1,0}f_1(\pi^*)}{\sum_{i \in I} \alpha_{i,0}f_i(\pi^*)}, ..., \frac{\alpha_{I,0}f_I(\pi^*)}{\sum_{i \in I} \alpha_{i,0}f_i(\pi^*)}\right)$ if A2.b holds for all agents. This result coupled with the well-known consistency property of Bayesian learning implies that there exists a unique invariant distribution ψ^{PO} over $(\Omega, \mathcal{G})$. For the case in which allocations are CPO and agents have dogmatic priors, Proposition 5 shows that there exists a unique invariant distribution ψ^{cpo} over $(\Omega, \mathcal{G})$. We summarize this results in the following Theorem.

Theorem 11 *Assume A.0 holds and A.1 holds for every agent. Then the asymptotic approximation holds if*
(a) Allocations are PO or
(b) Allocations are CPO and agents have dogmatic beliefs.

6.3 Statistical Characterization

In order to get our characterization of short-term momentum and long-term reversal, it is important to note that by the law of iterated expectations,

$$cov^{P_e}(R_{1,e}, R_{k,e}) = E^{P_e}\left[\left(R_{1,e} - E^{P_e}(R_{1,e})\right) E^{P_e}(R_{k,e}|\, R_{1,e})\right], \tag{26}$$

where $E^{P_e}(R_{k,e}|\, R_{1,e})$ denotes the expectation of the return in period k conditional to the $\sigma-$algebra generated by the return in period 1.

Definition: *Returns are unpredictable if the expected return is constant over time, i.e. for every* $k \in \{1,2\}$, $E^{P_e}(R_{k,e}|\, \mathcal{G}_\tau)$ *is* $\mathcal{G}_{-1}-$*measurable for all* $\tau \leq k$.

Our first result follows immediately from (26) and the definition of unpredictable return.

Proposition 12 *If returns are unpredictable, then the asset does not display financial markets anomalies.*

The following definitions will be used to characterize the cases in the expected return is predictable, that is $E^{P_e}(R_{k,e}|\, \mathcal{G}_\tau)$ varies with τ.

Now we are interested in characterizing how the first announcement affects the expectation of the returns at the second and third announcement. An announcement is a a measurable $\mathcal{G}_\tau$ function $R^*_{\tau,e} : \Omega \times S \to \Re$. For any $k > \tau$ and state $(\xi, \alpha) \in \Omega$, let

$$E^{P_e}\left(R_{k,e}|\, R^*_{\tau,e} = R\right) = \sum_{(\xi,\alpha,\xi'):R^*_{\tau,e}(\xi,\alpha)(\xi')=R} \frac{\pi^*(\xi'|\xi)}{P^e\left(R^*_{\tau,e}(\xi,\alpha)(\xi')=R\middle|\xi,\alpha\right)} E^{P_e}(R_{k,e}|\, \mathcal{G}_\tau)\left(\xi', \alpha'(\xi,\alpha)(\xi')\right)$$

Definition *State* $\left(\widetilde{\xi}, \widetilde{\alpha}\right)$ is *consistent with an announcement* $R^*_{\tau,e} = R$ *if there exists a state* (ξ, α) *such that* $R^*_{\tau,e}(\xi,\alpha)\left(\widetilde{\xi}\right) = R$ *and* $\alpha'(\xi,\alpha)\left(\widetilde{\xi}\right) = \widetilde{\alpha}$. *A state* $\left(\widehat{\xi}, \widehat{\alpha}\right)$ *is similar to* $\left(\widetilde{\xi}, \widetilde{\alpha}\right)$ *if it is consistent with the same realization of the announcement than* $\left(\widetilde{\xi}, \widetilde{\alpha}\right)$.

A pair of states $\left(\widetilde{\xi},\widetilde{\alpha}\right)$ *and* $\left(\widehat{\xi},\widehat{\alpha}\right)$ *is consistent with an announcement* $R^*_{\tau,e}$ *if state* $\left(\widetilde{\xi},\widetilde{\alpha}\right)$ *is consistent with a positive realization of announcement* $R^*_{\tau,e}$ *and state* $\left(\widehat{\xi},\widehat{\alpha}\right)$ *is consistent with a negative realization of announcement* $R^*_{\tau,e}$*.*

Definition: *For any* $\tau < k$*, the return in period* k *underreacts to an announcement* $R^*_{\tau,e}$ *at the consistent pair of states* $\left(\widetilde{\xi},\widetilde{\alpha}\right)$ *and* $\left(\widehat{\xi},\widehat{\alpha}\right)$ *if*

$$E^{P_e}\left(R_{k,e}\middle|\,\widetilde{R}^*_{\tau,e}\right) > E^{P_e}\left(R_{k,e}\middle|\,\widehat{R}^*_{\tau,e}\right).$$

where $\widetilde{R}^*_{\tau,e}$ *and* $\widehat{R}^*_{\tau,e}$ *are the realizations consistent with states* $\left(\widetilde{\xi},\widetilde{\alpha}\right)$ *and* $\left(\widehat{\xi},\widehat{\alpha}\right)$*, respectively.*

The return in period k *overreacts to an announcement* $R^*_{\tau,e}$ *at a consistent pair of states* $\left(\widetilde{\xi},\widetilde{\alpha}\right)$ *and* $\left(\widehat{\xi},\widehat{\alpha}\right)$ *if the reverse inequality holds.*

The return in period k *underreacts (overreacts) to announcement* $R^*_{\tau,e}$ *if,* P_e-*a.s., it underreacts (overreacts) to every consistent pair of states* $\left(\widetilde{\xi},\widetilde{\alpha}\right)$ *and* $\left(\widehat{\xi},\widehat{\alpha}\right)$*.*

When $R^*_{\tau,e} = R_{1,e} - E^{P_e}\left(R_{1,e}\right)$ we call announcement $R^*_{\tau,e}$ abnormal returns and we say that the return in period k underreacts or overreacts to abnormal returns. This Proposition imposes a weak simmetry condition on the distribution of the return in period 1 and provides a sufficient condition, for the financial market anomalies to occur, that follows immediately from (26) and the definitions above.

Proposition 13 *Suppose the return in period 1 is symmetrically distributed. If the return in period 2 underreacts to abnormal returns in period* 1*, then the asset displays short-term momentum. If the return in period 3 overreacts to abnormal returns, then the asset displays long-term reversal.*

The following Corollary provides a necessary condition for having short-term momentum followed by long-term reversal.

Corollary 14 *Suppose the return in period 1 is symmetrically distributed. If the return in period 2 overreacts to abnormal returns in period 1, then the asset does not display short-term momentum. If the return in period 3 underreacts to abnormal returns, then the asset does not display long-term reversal.*

6.4 The Economics Behind Underreaction and Overreaction

Both from an intuitive and an analytical point of view, a good way to understand the economics behind underreaction and overreaction is to consider the case of two states of nature. Since the effects at work in this particular case are likely to operate to some extent also in a more general case, we restrict to $S=2$ in this section.

Definition: *A state of nature $\xi' = g$ is unambiguously good news for announcement $R_{k,e}$ if, $\psi^e - a.s$,*

$$R_{k,e}(\xi,\alpha,d)(g) > R_{k,e}(\xi,\alpha,d)(b) \text{ for all } d \in \{0,1,..,k\}.$$

A state of nature $\xi' = g$ is unambiguously good news with constant dispersion if it is unambiguously good news and $R_{k,e}(\xi,\alpha,d)(g) - R_{k,e}(\xi,\alpha,d)(b)$ is independent of (ξ,α,d).

We reinterpret the equivalent martingale measure

$$m(g\,|\mathcal{G}_k)(\xi,\alpha) = \frac{Q(\xi,\alpha)(g)}{Q(\xi,\alpha)(g)+Q(\xi,\alpha)(b)} > 0$$

as the *market belief* about the states of nature the next period. Proposition 13 characterizes the financial anomalies in term of the behavior of the conditional expectation of the asset return. Now we would like to find conditions on the market belief that are sufficient for the conditional expectation to behave as in the hypothesis of Proposition 13.

Since $E^m(R_{k+1,e}|\,\mathcal{G}_k) = 0$, it follows that the return is always positive in one state of nature and negative in another. Moreover, one can write the true expectation as

$$E^{P_e}(R_{k+1,e}|\,\mathcal{G}_k)(\cdot) = E^{P_e}(R_{k+1,e}|\,\mathcal{G}_k)(\cdot) - E^m(R_{k+1,e}|\,\mathcal{G}_k)(\cdot) \qquad (27)$$

The market belief's bias towards an event is the difference between the probability that the true measure and the market belief place on that event. Condition (27) underscores that the true expected return is a decreasing function of the market belief's bias towards a positive return. Therefore, the return in period $k+1$ underreacts to abnormal returns in period k if the market belief in period k is less bias towards a positive return after a positive abnormal return than after a negative one. The following definitions will be useful to characterize how the bias towards a positive return reacts to announcements.

Definition *The market belief in period k is less biased towards state of nature g at state $(g,\widetilde{\alpha})$ than at state $(b,\widehat{\alpha})$ if*

$$\pi^*(g|\,g) - m(g\,|g,\widetilde{\alpha}) > \pi^*(g|\,b) - m(g\,|b,\widehat{\alpha}).$$

The market belief in period k is more biased towards state of nature g at state $(g,\widetilde{\alpha})$ than at state $(b,\widehat{\alpha})$ if the opposite inequality holds.

Remark 3**:** Note that in a two-state economy, the market belief in period k is less biased towards state of nature g at state $(g, \widetilde{\alpha})$ than at state $(b, \widehat{\alpha})$ if and only if it is less biased towards state of nature b at state $(b, \widehat{\alpha})$ than at state $(g, \widetilde{\alpha})$.

As the following proposition shows, one can tie the way the market updates its belief about a positive return in the next period, after an announcement is made, to the behavior of the true expectation of the asset return conditional on that announcement.

Proposition 15 *Suppose $S = 2$ and state of nature g is unambiguously good news with constant dispersion for the return in period $k+1$. Consider a pair of states $(g, \widetilde{\alpha})$ and $(b, \widehat{\alpha})$ that is consistent with an announcement in period k. If the market belief in period k is less (more) biased towards state of nature g at every state similar to $(g, \widetilde{\alpha})$ than at every state similar to $(b, \widehat{\alpha})$, then the return in period $k+1$ underreacts (overreacts) to an announcement in period k at the consistent pair of states $(g, \widetilde{\alpha})$ and $(b, \widehat{\alpha})$.*

To understand the intuition behind Proposition 15, recall that the return in period 2 underreacts (overreacts) if its conditional expectation after a positive announcement is larger (smaller) than after a negative announcement. If the market belief in period k is less biased towards state of nature g at state $(g, \widetilde{\alpha})$ than at state $(b, \widehat{\alpha})$, the true expectation is larger if state $(g, \widetilde{\alpha})$ rather than state $(b, \widehat{\alpha})$ occurs in period k.

Corollary 14 and Proposition 15, together, help us understand what evolution of the market belief can produce short-term momentum followed by long-term reversal. Indeed, a necessary condition is that the market belief in period 1 is less biased towards state of nature g at some pair of states consistent with abnormal returns in period 1 and the market belief in period 2 is more biased towards state of nature g at some pair of states consistent with a positive return in period 2. This *peculiar dynamics* requires some strong history dependence of the market belief. In the rest of the paper we will identify which models can have equilibria with this peculiar dynamics of the market belief.

7 Identifying the Class of Standard Models

In this section we identify a class of allocations that fail to generate either short-term momentum or long-term reversal. In Section 7.1 we consider PO allocations and we show (Theorem 16) that, regardless of whether beliefs are homogeneous or heterogeneous, neither they display short-term momentum nor long-term reversal. In section 7.2, we consider two-state two-agent economies with persistent shocks, similar discount rates and homogeneous beliefs. We show that even though CPO allocations of those economies might display short-term momentum they never display long-term reversal. Theorem 16 and 17 constitute, to the best of our knowledge, the first formal attempt to identify the class of models that cannot account for the aforementioned empirical regularities, the so-called "standard" models.

7.1 PO Allocations

The following Theorem shows that any models where equilibrium allocations are Pareto optimal cannot account for short-term momentum and long-term reversals.

Theorem 16 *Suppose A0 holds, that every agent has beliefs satisfying either A1.a or A1.b and some agent satisfies A2. Then, there is no dynamically complete markets equilibrium in which a long-lived asset displays short-term momentum or long-term reversal.*

The intuition behind this result is simple. In the long run (and also in finite time if agents have homogeneous beliefs) the distribution of welfare weights is degenerate. Since there is at least one agent who has the true dgp in the support of her priors and there is no aggregate risk, it follows that the market belief converges to π^* and so, by a no-arbitrage argument, the conditional expectation of any excesss net return is always zero and, therefore, returns are unpredictable. The desired result follows from Proposition 12.

7.2 CPO Allocations: Homogeneous Beliefs

In this section, we consider CPO allocations in a two-agent, two-state economy where agents have dogmatic beliefs. By Proposition 3 and Theorem 16, we need to consider only the case in which $\underline{\alpha}_1(1) > 1-\underline{\alpha}_2(2)$. We show that even though CPO allocations might display short-term momentum, they never display long-term reversal if the shocks are persistent.

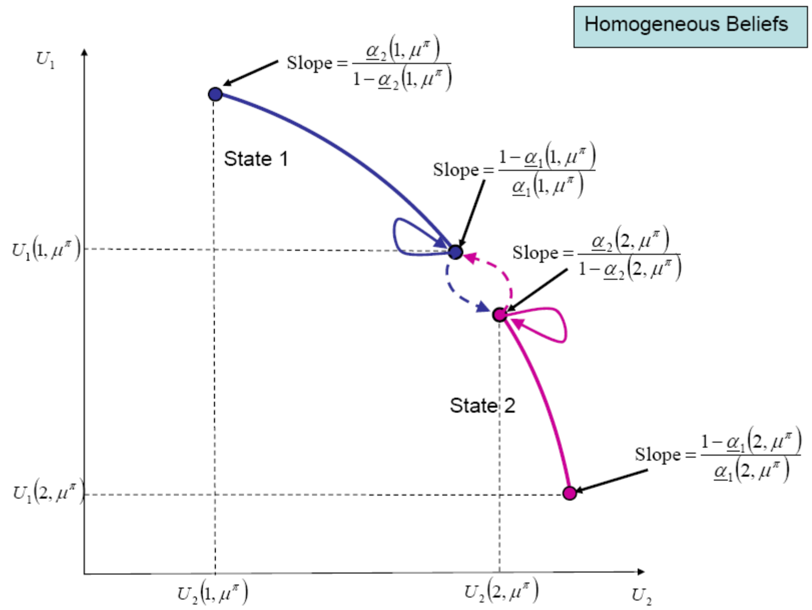
To simplify the analysis, we consider a simple family of four-period-lived assets. The set D consists of the dates the asset is issued and $\overline{k} = 3$. Dividends are paid at date $t+3$ and consists in the accumulated announcements between dates $t+1$ and $t+3$. Asset ξ' in this family, announces a unit dividend in period $k \in \{1, 2, 3\}$ if state of nature ξ' occurs that period. That is, the cumulative dividend of asset ξ' up to period k is:

$$\delta_k\left(\xi_1, ..., \xi_k\right) \equiv \sum_{\tau=1}^{k} 1_{\xi_\tau=\xi'} \tag{28}$$

where $1_{\xi_\tau=\xi'}$ is the indicator function of state of nature ξ' in period τ.[13]

[13] This peculiar security is similar to the one studied by Barberis et al [4] and so we use it to make our results immediately comparable to theirs.

The following figure illustrates the welfare weights law of motion for this case:



That is, agent i is constrained in state of nature i after a reversal and nobody is constrained upon persistence. Our analysis generalizes Alvarez and Jermann [2] in that we do not impose symmetry.

As in Alvarez and Jermann [2], the welfare weights converge $P^{\pi^*}-a.s.$ to a simple random vector that takes value $\underline{\alpha}(1) \equiv (\underline{\alpha}_1(1), 1-\underline{\alpha}_1(1))$ in state of nature 1 and $\underline{\alpha}(2) \equiv (1-\underline{\alpha}_1(2), \underline{\alpha}_2(2))$ in state of nature 2. It is easy to see that the invariant ψ_{cpo} on $(\Omega, \mathcal{G})$ has only two points in its support, namely $(1, \underline{\alpha}(1))$ and $(2, \underline{\alpha}(2))$.

The Arrow security prices $\widehat{q}_t^{\xi'}(s)$ converge to a simple random variable $\beta(s_t)\pi\left(\xi' \,|s_t\right)\theta\left(\xi' \middle| s_t\right)$ where

$$\theta\left(\xi' \middle| s_t\right) = \begin{cases} \frac{u'_{\xi'}\left(c_{\xi'}(\xi', \underline{\alpha}_{\xi'}(\xi'))\right)}{u'_{s_t}\left(c_{s_t}(s_t, 1-\underline{\alpha}_{s_t}(s_t))\right)} > 1 & \text{if } \xi' \neq s_t \\ 1 & \text{otherwise} \end{cases}$$

and the market belief about state of nature ξ' converges to a simple random variable taking values

$$m^*\left(\xi' \,|s_t\right) = \frac{\pi^*\left(\xi' \,|s_t\right)\theta\left(\xi' \middle| s_t\right)}{\pi^*\left(1\,|s_t\right)\theta\left(1| \, s_t\right) + \pi^*\left(2\,|s_t\right)\theta\left(2| \, s_t\right)} \tag{29}$$

One concludes immediately from (29) that

$$\pi^*\left(1\,|1\right) - m^*\left(1\,|1, \underline{\alpha}(1)\right) > 0 > \pi^*\left(1\,|2\right) - m^*\left(1\,|2, \underline{\alpha}(2)\right)$$

and so the market belief in period 1 is less biased towards state of nature 1 at state $(1, \underline{\alpha}(1))$ than at state $(2, \underline{\alpha}(2))$.

The following Theorem is the main result of this section. On the one hand, it shows in a robust way that homogeneous symmetric economies might display short-term momentum. On the other hand, it shows that homogenous symmetric economies with persistent shocks (i.e. $\pi^*\left(\xi' \left|\xi'\right.\right) > \frac{1}{2}$ for all ξ') cannot display long-term reversal as the volatility of the stochastic discount factor vanishes.

Theorem 17 *Suppose $S = 2$, A0 holds and agents have homogeneous dogmatic beliefs satisfying A1 and A2. Assume state of nature ξ' is unambiguously good news for the returns in period 1 and 2. Then,*
(a) Asset ξ' displays short-term momentum.
(b) Asset ξ' does not display long-term reversal as the volatility of the stochastic discount rates vanishes if shocks are persistent.

For the result on short-term momentum we rely on Propositions 13 and 15. To be able to apply Proposition 13 we first note that the distribution of the return in period 1 is symmetric in any symmetric economy and so the pair of states $(1, \underline{\alpha}(1))$ and $(2, \underline{\alpha}(2))$ is consistent with abnormal returns. To show that the return in period 2 underreacts to abnormal returns we use Proposition 15. In order to do so, we first argue that state of nature ξ' is unambiguously good news with constant dispersion because the interest rate is constant in any symmetric economy. The desired result follows because, as we noted above, the market belief in period 1 is less biased towards state of nature 1 at state $(1, \underline{\alpha}(1))$ than at state $(2, \underline{\alpha}(2))$. To understand the intuition behind the result on long-term reversal let's assume state of nature $\xi' = g$. Therefore, the abnormal return is positive if state of nature g occurs in period 1 and negative otherwise. We need to show that the expected return in period 3 is larger if state of nature g rather than b occurs in period 1. The law of iterated expectations implies that each of those expectations equals the expectation in period 1 of the expected return in period 3 conditional on the state in period 2. Since the market belief underreacts to the consistent pair of states $(1, \underline{\alpha}(1))$ and $(2, \underline{\alpha}(2))$ in period 2 and state of nature g is unambiguously good news with constant dispersion in period 3, then Proposition 15 implies that the return in period 3 underreacts to the return in period 2. This means that the expectation of the return in period 3 is larger when state of nature g rather than b occurs in period 2 and, because of the persistency assumption, so is the expectation of the return in period 3 after state of nature g rather than b occurs in period 1. An analogous reasoning shows that if the process is not persistent, the return in period 3 is larger after state of nature b rather than g occurs in period 1.

8 CPO Allocations with Heterogeneous Beliefs

In this section, we highlight the effects that determine whether an economy displays underreaction followed by overreaction.

We consider an homogeneous dogmatic beliefs symmetric economy with constant discount rates that neither display short-term momentum nor long-term reversal and we add a small amount of belief heterogeneity. That is we consider the case in which $\underline{\alpha}_1(1,\mu^\pi) < 1 - \underline{\alpha}_2(2,\mu^\pi)$.

8.1 Underreaction

In this section we show that for any ε one can identify a class of $\varepsilon-$heterogeneous dogmatic beliefs economy where the minimum enforceable weights coincide with those of the correct belief economy and the market belief is less biased towards state of nature 1 at state $\left(1, \alpha'_{cpo}\left(\xi, \underline{\alpha}(1,\mu^\pi)\right)(1)\right)$ than at state $\left(2, \alpha'_{cpo}(\xi, \underline{\alpha}(1,\mu^\pi)\right)(2))$. By the continuity of the conditional expectation in α and the fact that the invariant distribution has finite support, it follows that the conclusion of the Proposition holds also in an open set of welfare weights around $\underline{\alpha}(1,\mu^\pi)$. If one begins with an homogeneous beliefs economy whose invariant distribution is degenerate at $\underline{\alpha}(1,\mu^\pi)$, then a continuity argument shows that the invariant distribution of the $\varepsilon-$heterogeneous dogmatic belief economy puts probability arbitrarily close to 1 in the aforementioned neighbourhood around $\underline{\alpha}(1,\mu^\pi)$. Therefore, Proposition 15 can be used to show that with probability arbitrarily close to 1, the return in period 1 underreacts at the consistent pair of states $\left(1, \alpha'_{cpo}\left(\xi, \alpha\right)(1)\right)$ and $\left(2, \alpha'_{cpo}(\xi, \alpha\right)(2))$. That is, for a set of consistent states with probability close to 1

$$E^{P_{cpo}}\left(R_{k,cpo}\middle| R_{1,cpo} - E\left(R_{1,cpo}\right) > 0\right) > E^{P_e}\left(R_{k,cpo}\middle| R_{1,cpo} - E\left(R_{1,cpo}\right) < 0\right)$$

Proposition 13 shows that when this inequality holds with probability 1 and the distribution of the return in period 1 is symmetric, the asset displays short-term momentum. However, with belief heterogeneity the above inequality holds with probability less than 1 and the distribution becomes asymmetric. Fortunately, continuity of the covariance in the distribution P_e can be used to show that the same conclusion holds if the above inequality holds with probability close enough to 1 and the distribution is slightly asymmetric.

Proposition 18 *Suppose A0 holds, shocks are persistent, agents have dogmatic beliefs satisfying A*1 *and logarithmic preferences. If agent* 1 *satisfies A*2 *, there are* $\varepsilon-$*heterogeneous dogmatic beliefs for agent* 2 *such that the market belief in period* 1 *is less biased towards state of nature* 1 *at state* $\left(1, \alpha'_{cpo}\left(1, \underline{\alpha}(1,\mu^\pi)\right)(1)\right)$ *than at state*

$\left(2, \alpha'_{cpo}(1, \underline{\alpha}(1, \mu^{\pi})\right)(2))$.

The intuition behind Proposition 18 is as follows. Since we assume logarithmic preferences, the market belief at state (ξ, α) is equal to the following average belief

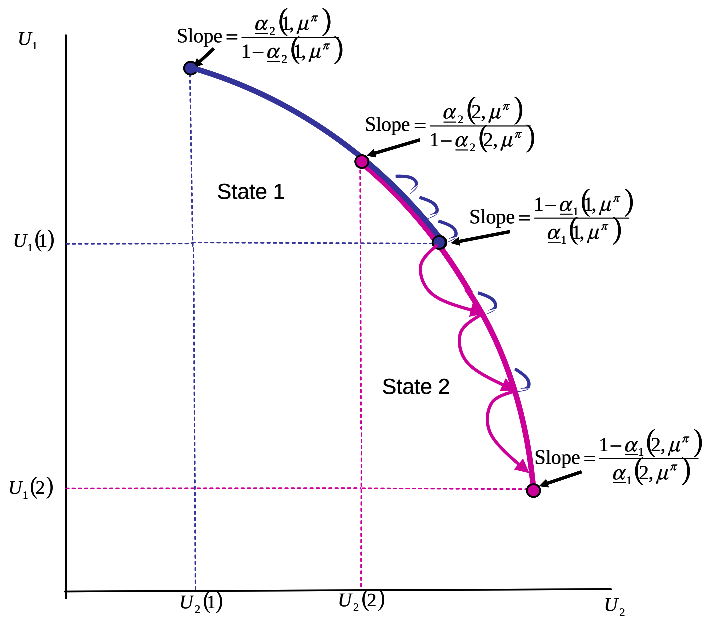
$$m\left(\xi' \left| \mathcal{G}_1 \right.\right)(\xi, \alpha) = \alpha_1 \pi_1 \left(\xi' \left| \xi \right.\right) + \alpha_2 \pi_2 \left(\xi' \left| \xi \right.\right)$$

if no agent is constrained in period 2 at state $\left(\xi', \alpha'_{cpo}(\xi, \alpha)(\xi')\right)$. Since agent 1 has correct beliefs and agent 2 is more pessimistic about state of nature 1, then

$$m\left(1 \left| \mathcal{G}_1 \right.\right)(\xi, \alpha) < \pi^*\left(1 \left| \xi \right.\right)$$

We conclude that agent 2' pessimism creates a bias in the market belief in period 1 when no agent is constrained in period 2.

Suppose one identifies initial welfare weights α such that agent 1 is constrained in period 2 only after history $(1, 2, 1)$. The Figure below.identifies one such example with initial welfare weight $\underline{\alpha}(1, \mu^{\pi})$ We argue in Proposition 18 that the market belief



in period 1 is less biased towards state of nature g at state $\left(1, \alpha'_{cpo}(1, \alpha)(1)\right)$ than at state $\left(2, \alpha'_{cpo}(1, \alpha)(2)\right)$, that is

$$\pi^*\left(1 \left| 1 \right.\right) - m\left(1 \left| \mathcal{G}_1 \right.\right)\left(1, \alpha'_{cpo}(1, \alpha)(1)\right) > \pi^*\left(1 \left| 2 \right.\right) - m\left(1 \left| \mathcal{G}_1 \right.\right)\left(2, \alpha'_{cpo}(1, \alpha)(2)\right)$$

Belief heterogeneity has two effects on the dynamics of the welfare weights and each of them affects the market belief in a different direction. On the one hand, it makes agent 1's welfare weight in period 1 to be larger at state $\left(1, \alpha'_{cpo}(1,\alpha)(1)\right)$ than at state $\left(2, \alpha'_{cpo}(1,\alpha)(2)\right)$. If both consumer are unconstrained in period 2, when the period 1 state is $\left(1, \alpha'_{cpo}(1,\alpha)(1)\right)$ or $\left(2, \alpha'_{cpo}(1,\alpha)(2)\right)$, this effect makes the market belief in period 1 more biased towards state of nature 1 at $\left(1, \alpha'_{cpo}(1,\alpha)(1)\right)$ than at $\left(2, \alpha'_{cpo}(1,\alpha)(2)\right)$. On the other hand, it makes agent 1 to be constrained in period 2 which raises agent 2's welfare weight above its unconstrained (or first best level) at state $\left(2, \alpha'_{cpo}(1,\alpha)(2)\right)$. This effect does not affect the market belief in period 1 at state $\left(1, \alpha'_{cpo}(1,\alpha)(1)\right)$ but it raises the market belief in period 1 at state $\left(2, \alpha'_{cpo}(1,\alpha)(2)\right)$ above the average belief, increasing the biased towards state of nature 1 at state $\left(2, \alpha'_{cpo}(1,\alpha)(2)\right)$. The gist of the proof is to show that the former effect is of second order and so the second effect dominates, making the market belief in period 1 less biased towards state 1 at state $\left(1, \alpha'_{cpo}(1,\alpha)(1)\right)$ than at state $\left(2, \alpha'_{cpo}(1,\alpha)(2)\right)$.

Since the sufficient condition for underreaction requires that such behavior of the market belief occurs, $\psi^{cpo}-$a.s., at every pair consistent with abnormal returns, the result in Proposition 18 is not sufficient to show that underreaction occurs when the degree of belief heterogeneity increases. Indeed, it might be the case that for some other state (ξ, α) in the support of ψ^{cpo} the market belief in period 1 is more biased towards state of nature 1 at state $\left(1, \alpha'_{cpo}(1,\alpha)(1)\right)$ than at state $\left(2, \alpha'_{cpo}(1,\alpha)(2)\right)$. Which behavior of the market belief dominates is a quantitative question that we adress in Section 9.2 in a calibrated version of the model.

8.2 Underreaction followed by Overreaction

We consider the economy identified in Proposition 18 that satisfies the necessary condition for short-term momentum in symmetric economies and show that, unlike in the homogeneous belief economy, the necessary condition for long-term reversal might also holds when belief are heterogeneous. That is, we show that the return in period 3 does not underreact to abnormal returns in period 1 by showing that when the initial state is $(1, \underline{\alpha}(1, \mu^{\pi}))$ and the process is persistent enough, the following inequality holds

$$\pi^*(1|1)\, E^{P_e}\left(R_{3,cpo}|\, 1,1,1\right) + \pi^*(1|2)\, E^{P_e}\left(R_{3,cpo}|\, 1,1,2\right) <$$

$$\pi^*(1|1)\, E^{P_e}\left(R_{3,cpo}|\, 1,2,1\right) + \pi^*(1|2)\, E^{P_e}\left(R_{3,cpo}|\, 1,2,2\right)$$

where the notation $\left(1, \xi, \xi'\right)$ refers to the partial history of states of nature up to date 2 with the understanding that given that history and the initial welfare weight $\underline{\alpha}(1, \mu^{\pi})$ one can recover the history of states (states of nature plus welfare weights) up to period 2. This is by no means obvious since the enforceability constraint binds only in history $(1,2,2,1)$ in period 3, then the market belief in period 2 is less biased towards state of nature 1 at history $(1,1,2)$ than at history $(1,2,2)$ and so

$$E^{P_e}\left(R_{3,cpo}\right|1,1,2) > E^{P_e}\left(R_{3,cpo}\right|1,2,2)$$

So we need to argue that

$$E^{P_e}\left(R_{3,cpo}\right|1,1,1) < E^{P_e}\left(R_{3,cpo}\right|1,2,1) \tag{30}$$

Since no agent is constrained at $(1,1)$ and $(2,1)$, the interest rates are equal to $\frac{1}{\beta}$ and so state of nature 1 is unambiguously good news with constant dispersion for asset 1. Since the welfare weight of agent 1 is larger at history $(1,1,1)$ than at history $(1,2,1)$, the market belief in period 2 is more biased towards state of nature 1 at $(1,1,1)$ than at $(1,2,1)$ and it follows by Proposition 15 that (30)

9 Financial Market Anomalies? Calibrated Examples

In this section we calibrate a more general model to US aggregate as well as idiosyncratic income processes. Following Alvarez and Jermann, our specification the discount factor accomodates to allow for growth in the aggregate endowment when relative risk aversion is constant.

In section 9.1 we consider the same homogeneous belief economy that Alvarez-Jermann and Lustig et al consider and show that it displays neither short-term momentum nor long-term reversal. In section 9.2 we intoduce belief heterogenity and show that the calibrated model displays not only short-term momentum but also long-term reversal.

9.1 Benchmark Model: Homogeneous Beliefs

We specify the endowment process with four values for the share of income of each agent and two values for the discount rate, respecting symmetry across agents. Therefore, $S = 4$ and with symmetry there are 10 parameters to be selected: six for π^*, two for $y_1(\cdot)$ and two for $\beta(\cdot)$. We assume agents have log preferences. As in Lustig et al. we set

$$\pi^* = \begin{bmatrix} 0.1710 & 0.8186 & 0.0040 & 0.0064 \\ 0.3020 & 0.5757 & 0.0172 & 0.1051 \\ 0.0040 & 0.0064 & 0.1710 & 0.8186 \\ 0.0172 & 0.1051 & 0.3020 & 0.5757 \end{bmatrix}$$

$$y = \begin{bmatrix} 0.7952 & 0.6578 & 0.2048 & 0.3422 \\ 0.2048 & 0.3422 & 0.7952 & 0.6578 \end{bmatrix}$$

and so the minimum enforceable weights are given by

$$\begin{bmatrix} \underline{\alpha}_1(1,\mu^\pi) & 1-\underline{\alpha}_2(1,\mu^\pi) \\ \underline{\alpha}_1(2,\mu^\pi) & 1-\underline{\alpha}_2(2,\mu^\pi) \\ \underline{\alpha}_1(3,\mu^\pi) & 1-\underline{\alpha}_2(3,\mu^\pi) \\ \underline{\alpha}_1(4,\mu^\pi) & 1-\underline{\alpha}_2(4,\mu^\pi) \end{bmatrix} = \begin{bmatrix} 0.5308 & 0.7953 \\ 0.4572 & 0.6999 \\ 0.2047 & 0.4692 \\ 0.3001 & 0.5428 \end{bmatrix}$$

It is easy to see that the invariant distribution of the Markov process over states of nature and welfare weights is

$$\psi^{cpo}(\mathcal{S}\times\mathcal{A}) = \begin{cases} 0.1395 & \text{if } \mathcal{S}\times\mathcal{A}\subset\{1\}\times\{\underline{\alpha}_1(1,\mu^\pi)\}\cup\{3\}\times\{1-\underline{\alpha}_2(3,\mu^\pi)\} \\ 0.2873 & \text{if } \mathcal{S}\times\mathcal{A}\subset\{2\}\times\{\underline{\alpha}_1(1,\mu^\pi)\}\cup\{4\}\times\{1-\underline{\alpha}_2(3,\mu^\pi)\} \\ 0.0733 & \text{if } \mathcal{S}\times\mathcal{A}=\{4\}\times\{\underline{\alpha}_1(1,\mu^\pi)\}\cup\{2\}\times\{1-\underline{\alpha}_2(3,\mu^\pi)\} \\ 0 & \quad\text{otherwise} \end{cases}$$

and

$$\begin{aligned} cov^{\psi^e}\left(R^{\xi'}_{1,e}, R^{\xi'}_{2,e}\right) &\approx -0.00085 < 0 \\ cov^{\psi^e}\left(R^{\xi'}_{1,e}, R^{\xi'}_{3,e}\right) &\approx 0.000275 > 0 \end{aligned}$$

One concludes that the calibrated economy with homogeneous beliefs neither displays short-term momentum nor long-term reversal.

9.2 Our Model at Work: Heterogeneous Beliefs

In this section we assume that agent 1 has dogmatic correct beliefs given by $\pi_1 = \pi^*$ while agent 2 has incorrect beliefs given by

$$\pi_2 = \begin{bmatrix} 0.0710 & 0.9186 & 0.0040 & 0.0064 \\ 0.4020 & 0.4757 & 0.0172 & 0.105 \\ 0.0040 & 0.0064 & 0.0710 & 0.9186 \\ 0.0172 & 0.1051 & 0.4020 & 0.4757 \end{bmatrix}$$

Our numerical computations show that

$$\begin{aligned} cov^{\psi^e}\left(R^{\xi'}_{1,e}, R^{\xi'}_{2,e}\right) &\approx 0.0461 > 0 \\ cov^{\psi^e}\left(R^{\xi'}_{1,e}, R^{\xi'}_{3,e}\right) &\approx -0.0346 < 0 \end{aligned}$$

and so the calibrated economy displays short-term momentum and long-term reversal.

10 Appendix A

In this Appendix we prove the results in Section 3.

In what follows, we will use Lemmas 1, 19 and 20. See Beker and Espino [6] for the proofs.

Lemma 19 *$\mathcal{U}(\xi,\mu)$ is compact and convex-valued for all (ξ,μ)*

Lemma 20 *The value function $v^*(\xi,\mu,\alpha)$ is bounded and continuous for all (ξ,μ,α). Moreover, v^* is homogeneous of degree 1 (hereafter HOD 1) and $v^*(\xi,\mu,\alpha)-\alpha\, U(\xi,\mu)$ is increasing in α.*

Given (α,ξ,μ) such that $\alpha\in\Delta(\xi,\mu)$, standard arguments can be used to show that the conditions that characterize the solution to the recursive planner's problem are

$$\alpha_i\ u_i'(c_i(\xi,\alpha))=\lambda(\xi,\mu,\alpha) \tag{31}$$

$$\sum_{i=1}^{I} c_i = y(\xi) \tag{32}$$

$$w_i'(\xi'):\qquad \left[\alpha_i+\eta_i'(\xi')\right]\beta(\xi)\pi_{\mu_i}(\xi'\,|\xi)=\gamma(\xi')\alpha_i'(\xi') \tag{33}$$

$$\left[w_i'(\xi')-U_i(\xi',\mu_i'(\xi,\mu)(\xi'))\right]\eta_i'(\xi')\beta\pi_{\mu_i}(\xi'\,|\xi)=0 \tag{34}$$

$$\alpha_i'(\xi'):\qquad v_{\alpha_i}(\xi',\mu'(\xi,\mu)(\xi'),\alpha'(\xi'))-w_i'(\xi')-\delta(\xi')=0 \tag{35}$$

$$\delta(\xi')\left[1-\sum_i\alpha_i'(\xi')\right]=0$$

where λ, $\eta_i'(\xi')$ and $\delta(\xi')$ denote the Lagrange multipliers for the feasibility constraint, for the agent $i's$ enforceability constraint at ξ', and for the restricted simplex $\Delta(\xi',\mu'(\xi,\mu)(\xi'))$, respectively.

It is useful to draw some conclusions from these conditions. First, note that $\eta_i'(\xi')>0$ implies that $w_i'(\xi')=U_i(\xi',\mu_i'(\xi,\mu)(\xi'))$, i.e. agent i's enforceability constraint is binding at $(\xi',\mu_i'(\xi,\mu)(\xi'))$. Also, it follows from that in this case $\alpha_i'(\xi')\gamma(\xi')>\alpha_i\ \beta\ \pi_{\mu_i}(\xi'\,|\xi)$ (i.e., the evolution derived in Beker and Espino [6] and discussed in (**??**) below) and so, $\alpha_i'(\xi')=\underline{\alpha}_i(\xi',\mu'(\xi,\mu)(\xi'))$. Therefore, we can conclude that

$$\eta_i'(\xi')>0\Rightarrow\alpha_i'(\xi')=\underline{\alpha}_i(\xi',\mu'(\xi,\mu)(\xi')). \tag{36}$$

Secondly, observe that $\eta_i'(\xi')>0$ implies that $w_i'(\xi')=U_i(\xi',\mu_i'(\xi,\mu)(\xi'))$ and thus $w_j'(\xi')>U_j(\xi',\mu_j'(\xi,\mu)(\xi'))$ for some $j\neq i$, i.e., $\eta_j(\xi')=0$ for some $j\neq i$. Therefore, we can conclude that if autarky is not the only feasible enforceable allocation

$$\begin{aligned}\eta_i(\xi') &> 0\Rightarrow\eta_j(\xi')=0\text{ for some }j\neq i, \qquad (37)\\ \text{and }\alpha_j'(\xi') &= \left[1-\sum_{h:\eta_h>0}\underline{\alpha}_h(\xi',\mu'(\xi,\mu))\right]\ \frac{\beta\alpha_j\pi_{\mu_j}(\xi'\,|\xi)}{\sum_{h:\eta_h=0}\beta\alpha_h\pi_{\mu_h}(\xi'\,|\xi)}\text{ for all }j\neq i.\end{aligned}$$

Define $\|f\| = \sup_{(\xi,\mu,\alpha)} \mid f(\xi,\mu,\alpha) : \alpha \in \Delta^{I-1} \mid$ and let

$$\begin{aligned} F &\equiv \{f : S \times \mathbb{R}_+^I \times \mathcal{P}(\Pi) \to \mathbb{R}_+ : \ f \text{ is continuous and } \|f\| < \infty\}. \\ F_H &\equiv \Big\{ f \in F : f(\xi,\mu,\alpha) - \sum_{i=1}^{I} \alpha_i U_i(\xi,\mu_i) \geq 0 \text{ for all } (\xi,\mu,\alpha), \\ &\quad \text{is increasing and HOD 1 in } \alpha \Big\} \end{aligned}$$

F_H is a closed subset of the Banach space F and thus a Banach space itself. Continuity is with respect to the weak topology and thus the metric on F is induced by $\|.\|$.

Given $(\xi,\mu,\alpha) \in S \times \mathbb{R}_+^I \times \mathcal{P}(\Pi)$, define the operator for any $f \in F_H$ as follows-subject to

$$(Tf)(\xi,\mu,\alpha) = \max_{(c,w'(\xi'))} \sum_{i=1}^{I} \alpha_i \left\{ u_i(c_i) + \beta(\xi) \sum_{\xi'} \pi_{\mu_i}(\xi' \,|\xi)\ w_i'(\xi') \right\}, \tag{38}$$

subject to

$$\sum_{i=1}^{I} c_i = y(\xi) \qquad \text{for all } \xi, \qquad c_i \geq 0, \tag{39}$$

$$u_i(c_i) + \beta(\xi) \sum_{\xi'} \pi_{\mu_i}(\xi' \,|\xi)\ w_i'(\xi') \geq U_i(\xi,\mu_i), \tag{40}$$

$$w_i'(\xi') \geq U_i(\xi', \mu_i'(\xi,\mu)(\xi')) \qquad \text{for all } \xi', \tag{41}$$

$$\alpha'(\xi') \equiv \arg\min_{\widetilde{\alpha} \in \Delta^{I-1}} \left[f(\xi', \mu'(\xi,\mu)(\xi'), \alpha') - \sum_{i=1}^{I} \alpha_i'\ w_i'(\xi') \right] \geq 0 \tag{42}$$

We say that $f \in F_H$ is *preserved under* T if $f(\xi,\mu,\alpha) \leq (Tf)(\xi,\mu,\alpha)$ for all (ξ,μ,α). To prove Proposition 2 we need first the two results stated in Proposition 21 and 22.

Proposition 21 *If $f \in F_H$ is preserved under T, then $(Tf)(\xi,\mu,\alpha) \leq v^*(\xi,\mu,\alpha)$ for all (ξ,μ,α).*

Proof of Proposition 21. Let $\mathcal{W}(\xi,\mu)$ denote the constraint correspondence defined by (39)-(42) at any arbitrary (ξ,μ).

Take any arbitrary $\left(\widehat{c}_0, (\widehat{w}_0'(\xi_1))_{\xi_1}\right) \in \mathcal{W}(\xi_0,\mu_0)$ while $\alpha'(\xi_1)$ denotes the corresponding vector of welfare weights such that

$$\widehat{\alpha}'(\xi_1) = \text{argmin}_{\alpha' \in \Delta^{I-1}} \left[f(\xi_1, \mu'(\xi_0,\mu_0)(\xi_1), \alpha') - \sum_{i=1}^{I} \alpha_i'\ \widehat{w}_{i,0}'(\xi') \right]$$

Therefore, since f is preserved under T, we have that

$$\widehat{\alpha}'(\xi_1) \cdot \widehat{w}_0'(\xi') \leq f(\xi_1, \mu'(\xi_0, \mu_0)(\xi_1), \alpha'(\xi_1)) \leq (Tf)(\xi_1, \mu'(\xi_0, \mu_0)(\xi_1), \alpha'(\xi_1))$$

Let $\mu_1(\xi_1) = \mu'(\xi_0, \mu_0)(\xi_1)$. Thus there exists some $\left(\widehat{c}_1, (\widehat{w}_1'(\xi_2))_{\xi_2}\right) \in \mathcal{W}(\xi_1, \mu_1(\xi_1))$ such that

$$\widehat{w}_{i,0}'(\xi_1) = u_i(\widehat{c}_{i,1}) + \beta(\xi_1) \sum_{\xi_2} \pi_{\mu_1(\xi_1)}(\xi_2 \,|\xi_1\,)\ \widehat{w}_{i,1}'(\xi_2)$$

for all i and all ξ_1. Following this strategy repeatedly T times, we can conclude that for any arbitrary $\alpha \in \Delta^{I-1}$

$$\begin{aligned}
& \sum_{i=1}^{I} \alpha_{i,0} u_i(\widehat{c}_{i,0}(\xi_0)) + \beta(\xi_0) \sum_{\xi_1} \pi_{\mu_{i,0}}(\xi_1 \,|\xi_0\,)\ \widehat{w}_{i,0}'(\xi_1) \\
= & \sum_{i=1}^{I} \alpha_{i,0} E^{P_i} \left(\sum_{t=0}^{T} \rho_t\ u_i(\widehat{c}_{i,t}) \right) + \sum_{i=1}^{I} \alpha_{i,0}\ E^{P_i} \left(\rho_{T+1}\ \widehat{w}_{i,T+1}'\right) \\
\leq & \sum_{i=1}^{I} \alpha_{i,0} E^{P_i} \left(\sum_{t=0}^{T} \rho_t\ u_i(\widehat{c}_{i,t}) \right) + \left(\max_{\xi} \beta(\xi) \right)^{T+1} \|f\| \\
\leq & \sum_{i=1}^{I} \alpha_{i,0} E^{P_i} \left(\sum_{t=0}^{\infty} \rho_t\ u_i(\widehat{c}_{i,t}) \right)
\end{aligned}$$

Notice that $(\widehat{c}_i)$ is feasible by construction. Now we argue that it is enforceable as well. To see this, denote recursively $W_{i,t}(s^t) = \left(\widehat{w}_{i,t}'(s_t)(s_0, ..., s_{t-1}, \mu_0)\right)$ and observe that by construction

$$\begin{aligned}
& \left|U_{i,t}(\widehat{c}_i)(s^t) - W_{i,t}(s^t)\right| \\
\leq & \max_{\xi} \beta(\xi) \left| \sum_{s_{t+1}} \pi_{\mu_{i,s^t}}(s_{t+1} \,|s_t\,) \left(U_i(\widehat{c}_i)(s^t, s_{t+1}) - W_{i,t+1}(s^t, s_{t+1})\right) \right| \\
\leq & \max_{\xi} \beta(\xi) \sup_{s'} \left|U_{i,t+1}(\widehat{c}_i)(s^t, s') - W_{i,t+1}(s^t, s')\right| \\
\leq & \left(\max_{\xi} \beta(\xi) \right)^k \sup_{(s_1, ... s_k)} \left|U_{i,t+k}(\widehat{c}_i)(s^t, s_1, ... s_k) - W_{i,t+k}(s^t, s_1, ... s_k)\right|
\end{aligned}$$

Observe that $0 \leq W_{i,t}(s^t) \leq \|f\| < \infty$ for all i and all s^t and $\widehat{c}$ is uniformly bounded by construction. If we take the $\limsup \sup$ as $k \to \infty$ we have that $U_{i,t}(\widehat{c}_i)(s^t) = W_{i,t}(s^t)$ for all i and all s^t. Finally, since $W_{i,t}(s^t) \geq U_i(s_t, \mu_{i,s^{t-1}})$ for all $(s_t, \mu_{i,s^{t-1}})$ and all i, we can conclude that $\widehat{c}$ is enforceable. Thus, since

$\left(\widehat{c}_0, (\widehat{w}'_0(\xi_1))_{\xi_1}\right) \in \mathcal{W}(\xi_0, \mu_0)$ and $\widehat{c}$ are both arbitrary, we have that

$$\begin{aligned} & \sum_{i=1}^{I} \alpha_{i,0} u_i(\widehat{c}_{i,0}(\xi_0)) + \beta(\xi_0) \sum_{\xi_1} \pi_{\mu_0}(\xi_1 \,|\xi_0\,)\ \widehat{w}'_{i,0}(\xi_1) \\ \leq\ & \sum_{i=1}^{I} \alpha_{i,0} E^{P_i} \left(\sum_{t=0}^{\infty} \rho_t\ u_i(\widehat{c}_{i,t}) \right) \\ \leq\ & v^*(\xi, \mu, \alpha) \end{aligned}$$

and therefore since weak inequalities are preserved in the limit

$$\begin{aligned} Tf(\xi, \mu, \alpha) &= \max_{(c,w') \in \mathcal{W}(\xi,\mu)} \sum_{i=1}^{I} \alpha_i u_i(c_i) + \beta(\xi) \sum_{\xi_1} \pi_{\mu_0}(\xi_1 \,|\xi_0\,)\ w'_i(\xi_1) \\ &\leq v^*(\xi, \mu, \alpha). \end{aligned}$$

■

Proposition 22 *$v^* \in F_H$ is preserved under T and thus $v^*(\xi, \mu, \alpha) = (Tv^*)(\xi, \mu, \alpha)$ for all (ξ, μ, α). Moreover, an allocation $(c_i^*)_{i=1}^I$ is PO given (ξ, μ, α) if and only if it is generated by the set of policy functions solving (6) - (10) evaluated at v^*.*

Proof of Proposition 22. Step 1. Given (ξ, μ, α), take any $u \in \mathcal{U}^E(\xi, \mu)$ for which c denotes the corresponding enforceable feasible allocation. Define for each ξ'

$${}_{\xi'}c_i = \{{}_{\xi'}c_i(s^t) = c_i(s^t) \text{ for all } t \geq 1 : (s_0, s_1) = (\xi, \xi')\},$$

as the $\xi'-$continuation of c_i. Also, let

$$P_{i,\xi'}(s^t) = \frac{P_i(C(s^t))}{\pi_{\mu_i}(\xi' \,|\xi\,)},$$

for all s^t such that $t \geq 1$. Note that

$$\sum_{i \in \mathcal{I}} \alpha_i U_i^{P_i}(c_i) = \sum_{i=1}^{I} \alpha_i \left\{ u_i(c_i(\xi)) + \beta(\xi) \sum_{\xi'} \pi_{\mu_i}(\xi' \,|\xi\,)) U_i^{P_{i,\xi'}}({}_{\xi'}c_i) \right\}.$$

Notice that $\left(U_i^{P_{i,\xi'}}({}_{\xi'}c_i^*) \right)_{i=1}^I \in \mathcal{U}(\xi', \mu'(\xi, \mu)(\xi'))$ for all ξ'. It follows by Lemma 1 that

$$v^*(\xi', \mu'(\xi, \mu)(\xi'), \alpha') \geq \sum_{i=1}^{I} \alpha'_i U_i^{P_{i,\xi'}}({}_{\xi'}c_i)$$

for all $\alpha' \in \Delta^{I-1}$ and all ξ'. Therefore, we can conclude that $v^*(\xi, \mu, \alpha) = \sup_{c \in Y^\infty} \sum_{i \in \mathcal{I}} \alpha_i U_i^{P_i}(c_i) \leq (Tv^*)(\xi, \mu, \alpha)$ for all (ξ, μ, α).

Step 2. It is a routine exercise to show that T is a monotone operator (i.e., if $f \geq g$ then $Tf \geq Tg$). Also, observe that $v^* = Tv^* \leq T\widehat{v} \leq \widehat{T}\widehat{v} = \widehat{v}$ by Proposition 2. Monotonicity implies that $T^n(\widehat{v}) \leq T^{n-1}(\widehat{v})$ and thus $v_n \geq v_{n+1} \geq v^*$. Since this implies that $\{v_n\}$ is a monotone decreasing sequence of uniformly bounded functions, then there exists a function $v_\infty \geq v^*$ such that $\lim_{n\to\infty} v_n = v_\infty$. It remains to show that $v_\infty \leq v^*$, for which it is sufficient that v_∞ is preserved under T due to Proposition 21.

Given (ξ, μ, α), $v_\infty(\xi, \mu, \alpha) \leq v_n(\xi, \mu, \alpha)$ implies that for all n there exists $\left(\widehat{c}^n, \left(\widehat{w}'^n(\xi')\right)_{\xi'}\right) \in W^n(\xi, \mu)$ that attains $v_\infty(\xi, \mu, \alpha)$. Observe that $\left(\widehat{c}^n, \left(\widehat{w}'^n(\xi')\right)_{\xi'}\right)$ lies that in a compact set and thus it has a convergent subsequence with limit point $\left(\widehat{c}, \left(\widehat{w}'(\xi')\right)_{\xi'}\right)$. Notice that $\widehat{w}'(\xi') = \lim \widehat{w}'^n(\xi') \geq U_i(\xi, \mu_i)$ and

$$v_n(\xi', \mu', \alpha'(\xi, \mu)) - \sum_{i=1}^{I} \alpha'_i \ \widehat{w}'^n_i(\xi') \geq 0$$

for all n and all ξ' and thus

$$v_\infty(\xi', \mu', \alpha'(\xi, \mu)) - \sum_{i=1}^{I} \alpha'_i \ \widehat{w}'_i(\xi') \geq 0$$

since weak inequalities are preserved in the limit. Therefore, $\left(\widehat{c}, \left(\widehat{w}'(\xi')\right)_{\xi'}\right) \in W^\infty(\xi, \mu)$ and

$$v_\infty(\xi, \mu, \alpha) = \sum_{i=1}^{I} \alpha_i \left(u_i(\widehat{c}_i(\xi)) + \beta(\xi) \sum_{\xi_2} \pi_{\mu_i}(\xi' \,|\xi) \ \widehat{w}'_i(\xi') \right).$$

Finally, since $\left(\widehat{c}, \left(\widehat{w}'(\xi')\right)_{\xi'}\right) \in W^\infty(\xi, \mu)$ is arbitrary, we can conclude that

$$\left(Tv_\infty\right)(\xi, \mu, \alpha) \geq \sum_{i=1}^{I} \alpha_i \left(u_i(\widehat{c}_i(\xi)) + \beta(\xi) \sum_{\xi_2} \pi_{\mu_i}(\xi' \,|\xi) \ \widehat{w}'_i(\xi') \right).$$

Step 3. The rest of the claim follows from the same arguments as in Beker and Espino [6, Theorem 2]. ■

Lemma 23 *Given (ξ, μ), if $\alpha \in \Delta(\xi, \mu)$ then (6) - (10) can be solved ignoring the constraints (8) (i.e. they will not be binding). On the other hand, if $\alpha \notin \Delta(\xi, \mu)$ then there exists some $\widetilde{\alpha} \in \Delta(\xi, \mu)$ such that the solution $c^*(\alpha, \xi, \mu)$ coincides with $c^*(\widetilde{\alpha}, \xi, \mu)$.*

Proof of Lemma 23. If $\alpha \in \Delta(\xi,\mu)$ (i.e., $\alpha_i \geq \underline{\alpha}_i(\xi,\mu)$ for all i), then $u^*(\alpha)$ lies in $\mathcal{FU}(\xi,\mu)$ and thus $u_i^*(\alpha) \geq U_i(\xi,\mu_i)$ for all i by definition of (5). This means that (4) can be solved while the constraints $u_i(\alpha) \geq U_i(\xi,\mu_i)$ can be omitted.

(Only if) If λ_i denotes agent i's Lagrange multiplier corresponding to the constraint (8), then observe that the objective function in (6) - (10) could be re-written

$$\sum_{i=1}^{I} (\alpha_i + \lambda_i) \left\{ u_i(c_i) + \beta(\xi) \sum_{\xi'} \pi_{\mu_i}(\xi' \,|\xi)\; w_i'(\xi') \right\}.$$

Therefore, $\widetilde{\alpha} \in \Delta(\xi,\mu)$ can be defined such that for each i

$$\widetilde{\alpha}_i = \frac{\alpha_i + \lambda_i}{\sum_{h=1}^{I} (\alpha_h + \lambda_h)}.$$

■

Proof of Proposition 7. [14]By Proposition 2 $(c_h^*)_{h=1}^{I}$ is generated by the set of policy functions solving (6) - (10). Then,

$$c_{i,t}^*(s) = c_i(s_t, \alpha_t(s))$$

For any $s \in C\left(s^t, \xi'\right)$, since $\eta_i'\left(s_t, \alpha_t(s), \mu_{s^{t-1}}\right)(\xi') = 0$ (i.e. $\alpha_{i,t+1}\left(s\right) = \frac{\beta \pi_{\mu_{i,s^t}}(\xi')\alpha_{i,t}(s)}{\gamma_t(s)(\xi')}$

[14]Since $U_i\left(\xi', \mu_{i,\left(s^{t-1},\xi'\right)}\right) \leq U_i\left(\xi, \mu_{i,s^{t-1}}\right)$ for all $\xi \in S_t$, it follows that $\underline{\alpha}_i(\xi', \mu_{i,\left(s^{t-1},\xi'\right)}) \leq \underline{\alpha}_i(s_t, \mu_{i,s^{t-1}})$. We first argue that $\eta_i'\left(s_t, \alpha_t(s), \mu_{s^{t-1}}\right)(\xi') = 0$. In order to get a contradiction, suppose $\eta_i'\left(s_t, \alpha_t(s), \mu_{s^{t-1}}\right)(\xi') > 0$ and therefore $\alpha_{i,t+1}\left(s\right) = \underline{\alpha}_i(\xi', \mu_{i,\left(s^{t-1},\xi'\right)})$. It follows by (37) that $\eta_h'\left(s_t, \alpha_t(s), \mu_{s^{t-1}}\right)(\xi') = 0$ for some $h \neq i$ and then $\alpha_{h,t+1}\left(s\right) = \frac{\beta \pi_{\mu_{h,s^t}}(\xi')\alpha_{h,t}(s)}{\gamma\left(s_t,\alpha_t(s),\mu_{s^{t-1}}\right)(\xi')}$. Observe that $\eta_i'\left(s_t, \alpha_t(s), \mu_{s^{t-1}}\right)(\xi') > 0$ implies that

$$\begin{aligned}
\frac{\alpha_{i,t+1}\left(s\right)}{\alpha_{h,t+1}\left(s\right)} &> \frac{\beta \pi_{\mu_{i,s^t}}(\xi')\alpha_{i,t}\left(s\right)}{\beta \pi_{\mu_{h,s^t}}(\xi')\alpha_{h,t}\left(s\right)} \\
&\geq \frac{\alpha_{i,t}\left(s\right)}{\alpha_{h,t}\left(s\right)} \\
&\geq \frac{\underline{\alpha}_i(s_t, \mu_{i,s^{t-1}})}{\overline{\alpha}_h^{-i}(s_t, \mu_{i,s^{t-1}})} \\
&\geq \frac{\underline{\alpha}_i(\xi', \mu_{i,\left(s^{t-1},\xi'\right)})}{\overline{\alpha}_h^{-i}(\xi', \mu_{i,\left(s^{t-1},\xi'\right)})}
\end{aligned}$$

and since the FOC are sufficient it follows that $\eta_i'\left(s_t, \alpha_t(s), \mu_{s^{t-1}}\right)(\xi') \leq 0$, a contradiction.

for any $s \in C\left(s^t, \xi'\right)$), it follows by (31) that

$$\begin{aligned}
\beta(\xi)\, \pi_{\mu_{i,s^t}}(\xi' \,|\xi) \frac{u_i'(c_{i,t+1}^*(s))}{u_i'(c_{i,t}^*(s))} &= \gamma_t(s)\left(\xi'\right) \frac{\beta(\xi)\, \pi_{\mu_{i,s^t}}(\xi' \,|\xi)\alpha_{i,t}(s)}{\gamma_t(s)\left(\xi'\right)} \frac{u_i'(c_{i,t+1}^*(s))}{\alpha_{i,t}(s)\, u_i'(c_{i,t}^*(s))} \\
&= \gamma_t(s)\left(\xi'\right) \frac{\alpha_{i,t+1}(s)\, u_i'(c_{i,t+1}^*(s))}{\alpha_{i,t}(s)\, u_i'(c_{i,t}^*(s))} = \gamma_t(s)\left(\xi'\right) \frac{\lambda_{t+1}(s)}{\lambda_t(s)} \\
&= \gamma_t(s)\left(\xi'\right) \frac{\alpha_{h,t+1}(s)\, u_i'(c_{h,t+1}^*(s))}{\alpha_{h,t}(s)\, u_i'(c_{h,t}^*(s))} \\
&\geq \beta(\xi)\, \gamma_t(s)\left(\xi'\right) \frac{\pi_{\mu_{h,s^t}}(\xi' \,|\xi)\alpha_{h,t}(s)}{\gamma_t(s)\left(\xi'\right)} \frac{u_i'(c_{h,t+1}^*(s))}{\alpha_{i,t}(s)\, u_i'(c_{h,t}^*(s))} \\
&= \beta(\xi)\, \pi_{\mu_{h,s^t}}(\xi' \,|\xi) \frac{u_i'(c_{h,t+1}^*(s))}{u_i'(c_{h,t}^*(s))}
\end{aligned}$$

and for all $h \neq i$ since (33) implies that $\alpha_{h,t+1}(s) \geq \frac{\beta(\xi)\pi_{\mu_{h,s^t}}(\xi'|\xi)\alpha_{h,t}(s)}{\gamma_t(s)(\xi')}$. Consequently, we can conlcude that

$$\beta(\xi)\, \pi_{\mu_{i,s^t}}(\xi' \,|\xi) \frac{u_i'(c_{i,t+1}^*(s'))}{u_i'(c_{i,t}^*(s))} \geq \beta(\xi)\, \pi_{\mu_{h,s^t}}(\xi' \,|\xi) \frac{u_h'(c_{h,t+1}^*(s))}{u_h'(c_{h,t}^*(s))}$$

as desired. ■

Proof of Theorem 8. Define the corresponding (implicit) equivalent martingale measure

$$\pi^{EMM}(\xi' \,|\xi)(\mu, \alpha) = \left(R^{RF}(\xi, \mu, \alpha)\right) \max_h \left\{\beta(\xi)\;\; \pi_{\mu_h}(\xi' \,|\xi)\;\; \frac{\partial u_h(c_h(\xi', \alpha'(\xi, \mu, \alpha)(\xi'))/\partial c_h}{\partial u_h(c_h(\xi, \alpha))/\partial c_h}\right\}$$

Notice that, for all ξ', $\pi^{EMM}(\xi' \,|\xi)(\mu, \alpha) \in (0,1)$ for all (ξ, μ, α). Since (ξ, μ, α) lies in a compact set, it follows by continuity that

$$R_{\min}^{RF} = \min_{(\xi,\mu,\alpha)} R^{RF}(\xi, \mu, \alpha),$$

is well-defined for which $R_{\min}^{RF} > 1$.

Let F be the linear space of functions $f : S \times \mathbb{R}_+^I \times \mathcal{P}(\Delta^{K-1}) \to \mathbb{R}_+$ that are bounded in the norm $\|.\|_\infty$ and continuous with respect to the metric induced by $\|.\|_\infty$. Let $h \in F$ and consider the operator T_i

$$\begin{aligned}
(T_i h)(\xi, \alpha, \mu) &= (c_i(\xi, \alpha) - y_i(\xi)) \\
&\quad + \left(R^{RF}(\xi, \mu, \alpha)\right)^{-1} \sum_{\xi'} \pi^{EMM}(\xi' \,|\xi)(\mu, \alpha)\; h(\xi', \alpha'(\xi, \alpha, \mu)(\xi'), \mu'(\xi, \mu)(\xi')).
\end{aligned}$$

Step 1. First we check that $T_i : F \to F$. Suppose that $h \in F$. Consider first

$$\left(R^{RF}(\xi,\mu,\alpha)\right)^{-1}\sum_{\xi'}\pi^{EMM}(\xi'\,|\xi)(\xi,\mu,\alpha)\ h(\xi',\alpha'(\xi,\alpha,\mu)(\xi'),\mu'(\xi,\mu)), \tag{43}$$

and observe that α' and μ' are both continuous. Thus, the expression (43) is continuous in (ξ,α,μ). (43) is bounded because h and R^{RF} are both bounded. Since $|c_i(\xi,\alpha) - y_i(\xi)|$ is uniformly bounded, we can conclude that $(T_i h) \in F$.

Step 2. Now we check that T_i satisfies Blackwell's sufficient conditions and, thus, it is a contraction mapping.

We start with *discounting*. Consider any $a > 0$ and note that

$$\begin{aligned} T_i(h+a)(\xi,\alpha,\mu) &= (c_i(\xi,\alpha) - y_i(\xi)) \\ &\quad + \left(R^{RF}(\xi,\mu,\alpha)\right)^{-1}\sum_{\xi'}\pi^{EMM}(\xi'\,|\xi)(\xi,\mu,\alpha)\ h(\xi',\alpha'(\xi,\alpha,\mu)(\xi'),\mu'(\xi,\mu)) \\ &\quad + \left(R^{RF}(\xi,\mu,\alpha)\right)^{-1} a. \\ &\leq (T_i h)(\xi,\alpha,\mu) + \left(R^{RF}_{\min}\right)^{-1} a. \end{aligned}$$

where $\left(R^{RF}_{\min}\right)^{-1} \in (0,1)$ since risk-free interest rates are positive.

Monotonicity is obvious. If $h(\xi,\alpha,\mu) \geq z(\xi,\alpha,\mu)$ for all (ξ,α,μ), it is immediate that $(T_i h)(\xi,\alpha,\mu) \geq (T_i z)(\xi,\alpha,\mu)$ for all (ξ,α,μ).

Therefore, we can apply the contraction mapping theorem to conclude that T_i is a contraction with a unique solution $A_i \in F$ that solves (20) for each i.

Finally, the same arguments used in Espino and Hintermaier [15] show that there exists $\alpha_0 = \alpha(s_0,\mu_0) \in \mathbb{R}^I_+$ such that $A_i(s_0,\mu_0,\alpha_0) = 0$ for all i. ■

Proof of Theorem 9. First, notice that $\{\widehat{a}_i\}$ satisfies (19) by construction (22) since $\sum_{i=1}^I A_i(\xi,\mu,\alpha) = 0$ for all (ξ,μ,α). Also, $\widehat{a}_{i,0} = A_i(s_0,\mu_0,\alpha_0) = 0$ for all i by Theorem 8.

We argue that $(\widehat{c}_i,\widehat{a}_i)$ solves agent i's sequential problem for each agent i, (14)-(17). To show that it satisfies the budget set (15) - (17), notice that (15) follows directly from the definition of $(\widehat{c}_i,\widehat{a}_i)$ and the construction of A_i, (20). The solvency constrains (16) are satisfied immediately by construction.

Notice that (21) and (23) imply that

$$\begin{aligned} q_t^{\xi'}(s) &= \beta(\xi)\max_h\left\{\pi_{\mu_{h,s^t}}(\xi'\,|\xi)\frac{u_h'(\widehat{c}_{h,t+1}(s))}{u_h'(\widehat{c}_{h,t}(s))}\right\} \\ &\geq \beta(\xi)\,\pi_{\mu_{i,s^t}}(\xi'\,|\xi)\frac{u_i'(\widehat{c}_{i,t+1}(s))}{u_i'(\widehat{c}_{i,t}(s))} \end{aligned} \tag{44}$$

for all i (with equality if $U_i(\widehat{c}_i)(s^t) > U_i(s_t,\mu_{i,s^t})$).

Consider any alternative plan (c_i, a_i) in the budget set for which $\widehat{B}^{\xi'}_{i,t+1}(s)$ are given by (24). It follows by concavity that

$$u_i(\widehat{c}_{i,t}) - u_i(c_{i,t}) \geq u_i'(\widehat{c}_{i,t})\,(\widehat{c}_{i,t} - c_{i,t})$$

while

$$\begin{aligned}
\widehat{c}_{i,t}(s) - c_{i,t}(s) &= \widehat{a}^{s_t}_{i,t-1}(s) - a^{s_t}_{i,t-1}(s) + \sum_{\xi'} q^{\xi'}_t(s)\left(a^{\xi'}_{i,t}(s) - \widehat{a}^{\xi'}_{i,t}(s)\right) \\
&= -b_{i,t}(s) + \sum_{\widetilde{s}\in C(s^t)} q^{\widetilde{s}_{t+1}}_t(\widetilde{s}) b_{i,t+1}(\widetilde{s}) \\
&= -b_{i,t}(s) + b^*_{i,t}(s),
\end{aligned}$$

where $b_{i,t}(s) = a^{s_t}_{i,t-1}(s) - \widehat{a}^{s_t}_{i,t-1}(s) = a^{s_t}_{i,t-1}(s) - A(s_t, \mu_{s^{t-1}}, \alpha_t(s)) = a^{s_t}_{i,t-1}(s) - \widehat{B}^{s_t}_{i,t-1}(s)$ and $b^*_{i,t}(s) = \sum_{\widetilde{s}\in C(s^t)} q^{\widetilde{s}_{t+1}}_t(\widetilde{s}) b_{i,t+1}(\widetilde{s})$. Therefore, for any $T < \infty$ it holds

$$\begin{aligned}
& E^{P_i}\left(\sum_{t=0}^{T} \rho_t\ \left(u_i(\widehat{c}_{i,t}) - u_i(c_{i,t})\right)\right) \\
\geq\ & E^{P_i}\left(\sum_{t=0}^{T} \rho_t u_i'(\widehat{c}_{i,t})\,(\widehat{c}_{i,t} - c_{i,t})\right) \\
=\ & E^{P_i}\left(\sum_{t=0}^{T} \rho_t u_i'(\widehat{c}_{i,t})\left(-b_{i,t} + b^*_{i,t}\right)\right)
\end{aligned}$$

Since (44) holds, we have that

$$\begin{aligned}
& E^{P_i}\left[\sum_{t=0}^{T} \rho_t\ \left(u_i(\widehat{c}_{i,t}) - u_i(c_{i,t})\right)\right] \\
\geq\ & E^{P_i}\left[\sum_{t=0}^{T} \rho_t u_i'(\widehat{c}_{i,t})\left(-b_{i,t} + b^*_{i,t}\right)\right] \\
=\ & -E^{P_i}\left[\sum_{t=0}^{T} \rho_t u_i'(\widehat{c}_{i,t}) b_{i,t}\right] + E^{P_i}\left[\sum_{t=0}^{T} \rho_t u_i'(\widehat{c}_{i,t}) b^*_{i,t}\right] \\
=\ & -E^{P_i}\left[\sum_{t=0}^{T} \rho_t u_i'(\widehat{c}_{i,t}) b_{i,t}\right] + E^{P_i}\left[\sum_{t=0}^{T} \rho_t E^{P_i}\left[\left. u_i'(\widehat{c}_{i,t}) b^*_{i,t}\right| \mathcal{F}_t\right]\right] \\
\geq\ & -E^{P_i}\left[\sum_{t=0}^{T} \rho_t u_i'(\widehat{c}_{i,t}) b_{i,t}\right] + E^{P_i}\left[\sum_{t=0}^{T} E^{P_i}\left[\left.\rho_{t+1}\ u_i'(\widehat{c}_{i,t+1}) b_{i,t+1}\right| \mathcal{F}_t\right]\right] \\
=\ & -E^{P_i}\left[\sum_{t=0}^{T} \rho_t u_i'(\widehat{c}_{i,t}) b_{i,t}\right] + E^{P_i}\left[\sum_{t=0}^{T} \rho_{t+1} u_i'(\widehat{c}_{i,t+1}) b_{i,t+1}\right]
\end{aligned}$$

where the second and last equalities use the *Law of Iterated Expectations* and the last inequality uses the fact that $u_i'(\widehat{c}_{i,t}(s))b^*_{i,t} \geq \beta E^{P_i}\left[u_i'(\widehat{c}_{i,t+1}(s))b_{i,t+1}|\,\mathcal{F}_t\right]$. Since $a_{i,0}=0$ and $\widehat{a}_{i,0}=A_i(s_0,\mu_0,\alpha_0)=0$ for all i by Theorem **??**, it follows that $b_{i,0}=0$ and thus

$$\begin{aligned} & E^{P_i}\left[\sum_{t=0}^{T}\rho_t\ \left(u_i(\widehat{c}_{i,t})-u_i(c_{i,t})\right)\right] \\ \geq\ & E^{P_i}\left[\rho_{T+1}u_i'(\widehat{c}_{i,T+1}(s))b_{i,T+1}\right] \end{aligned}$$

Notice that $\mathcal{P}(\Pi^K)$ is compact (in the weak topology) and then the continuos function $\underline{\alpha}_i(\xi,\mu)$ is uniformly bounded by some $\underline{\alpha}>0$ for all (ξ,μ) and all i. This fact coupled with (31) and (32) imply that $\widehat{c}_{i,t}(s)$ is uniformly bounded by some $\underline{c}_i>0$ for all (t,s) and all i. The same arguments show that $A_i(\xi,\mu,\alpha)$ is uniformly bounded by some $\underline{A}$ for all (ξ,μ,α) and thus (19) implies that $b_{i,t}(s)$ is uniformly bounded for all (t,s). Therefore, it follows from the Dominated Convergence Theorem that

$$\begin{aligned} & \lim_{T\to\infty} E^{P_i}\left[\sum_{t=0}^{T}\rho_t\ \left(u_i(\widehat{c}_{i,t})-u_i(c_{i,t})\right)\right] \\ & E^{P_i}\left[\sum_{t=0}^{\infty}\rho_t\ \left(u_i(\widehat{c}_{i,t})-u_i(c_{i,t})\right)\right] \\ \geq\ & \lim_{T\to\infty} E^{P_i}\left[\rho_{T+1}u_i'(\widehat{c}_{i,T+1})b_{i,T+1}\right] \\ =\ & E^{P_i}\left[\lim_{T\to\infty}\rho_{T+1}u_i'(\widetilde{c}_{i,T+1})b_{i,T+1}\right]=0 \end{aligned}$$

since $\beta\in(0,1)$. Consequently, since $(\widehat{c},\widehat{a})$ satisfy the market clearing conditions and $(\widehat{c}_i,\widehat{a}_i)$ delivers the highest utility level for every agent i among affordable allocations given prices $\widehat{q}$ and solvency constraints $\widehat{B}_i$,we can conclude that $(\widehat{c},\widehat{a},\widehat{q})$ determine a competitive equilibrium with solvency constraints $\{\widehat{B}_i\}$. ■

11 Appendix B

In this Appendix we prove the results in Section 6

Proof of Theorem 11. TO BE COMPLETED ■

Theorem 24 (Stout [27] and Jensen and Rahbek [16]) *Assume $\{z_t\}_{t=0}^{\infty}$ is a time homogeneous Markov process with transition function F on $(Z,\mathcal{Z})$. If there exists a unique invariant distribution $\psi:\mathcal{Z}\to[0,1]$, then for any $z_0\in Z$, any integer k and any continuous function $f:Z^k\to\Re$,*

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T} f\left(z_t,...,z_{t+k}\right)=E^{P^F(\psi,\cdot)}\left(f\left(\widetilde{z}_0,...,\widetilde{z}_k\right)\right), \qquad P^F\left(z_0,\cdot\right)-a.s.$$

Proof of Proposition 13. Let $\overline{R}_{1,e}(\xi,\alpha)(\xi') \equiv R_{1,e}(\xi,\alpha)(\xi') - E^{P_e}(R_{1,e})$. Note that

$$\begin{aligned}
cov^{P_e}(R_{1,e},R_{k,e}) &= \sum_{\xi,\alpha}\psi^e(\xi,\alpha)\sum_{\xi'}\pi^*(\xi'|\xi)\ \overline{R}_{1,e}(\xi,\alpha)(\xi')\ E^{P_e}(R_{k,e}|R_{1,e}) \\
&= \sum_{\xi,\alpha}\psi^e(\xi,\alpha)\sum_{R\in Supp(\overline{R}_{1,e})}\ \sum_{\xi':\overline{R}_{1,e}(\xi,\alpha)(\xi')=R}\pi^*(\xi'|\xi)\ R\ E^{P_e}(R_{k,e}|R_{1,e}=R) \\
&= \sum_{R\in Supp(\overline{R}_{1,e})} R\ E^{P_e}(R_{k,e}|R_1=R)\sum_{\xi,\alpha,\xi':\overline{R}_{1,e}(\xi,\alpha)(\xi')=R}\psi^e(\xi,\alpha)\pi^*(\xi'|\xi) \\
&= \sum_R \overline{P}(R)\ R\ E^{P_e}(R_{k,e}|\overline{R}_{1,e}=R)
\end{aligned}$$

where $\overline{P}(R) \equiv P^e(\overline{R}_{1,e}(\xi,\alpha)(\xi')=R)$. We want to show that

$$\begin{aligned}
\sum_R \overline{P}(R)\ R\ E^{P_e}(R_{2,e}|\overline{R}_{1,e}=R) &> 0 \\
\sum_R \overline{P}(R)\ R\ E^{P_e}(R_{3,e}|\overline{R}_{1,e}=R) &< 0
\end{aligned}$$

Note that $\sum_R \overline{P}(R)\ R\ E^{P_e}(R_{k,e}|\overline{R}_{1,e}=R)$ is

$$\begin{aligned}
&= \sum_{R>0}\overline{P}(R)\ R\ E^{P_e}(R_{k,e}|\overline{R}_{1,e}=R)+\sum_{R<0}\overline{P}(R)\ R\ E^{P_e}(R_{k,e}|\overline{R}_{1,e}=R) \\
&= \sum_{R>0}\overline{P}(R)\ R\ E^{P_e}(R_{k,e}|\overline{R}_{1,e}=R)-\sum_{R>0}\overline{P}(-R)\ R\ E^{P_e}(R_{k,e}|\overline{R}_{1,e}=-R) \\
&= \sum_{R>0}\overline{P}(R)\ R\ \left(E^{P_e}(R_{k,e}|\overline{R}_{1,e}=R)-E^{P_e}(R_{k,e}|\overline{R}_{1,e}=-R)\right)
\end{aligned}$$

where the last equality uses the simmetry of $\overline{P}$.

Our assumptions that $R_{2,e}$ underreacts to abnormal returns and $R_{3,e}$ overreacts to abnormal returns imply that for any $R>0$ such that $\overline{P}(R)>0$,

$$\begin{aligned}
R\ \left(E^{P_e}(R_{2,e}|\overline{R}_{1,e}=R)-E^{P_e}(R_{2,e}|\overline{R}_{1,e}=-R)\right) &> 0 \\
R\ \left(E^{P_e}(R_{3,e}|\overline{R}_{1,e}=R)-E^{P_e}(R_{3,e}|\overline{R}_{1,e}=-R)\right) &< 0
\end{aligned}$$

as desired. ■

Proof of Proposition 15. Let states $(g, \widetilde{\alpha})$ *and* $(b, \widehat{\alpha})$ be consistent with $R^*_{k,e} = \widetilde{R} > 0$ and $R^*_{k,e} = \widehat{R} < 0$, respectively. Since state of nature g is constant good news for $R_{k+1,e}$ *at* $(\xi, \alpha) \in \{(g, \widetilde{\alpha}), (b, \widehat{\alpha})\}$, then

$$R_{k+1,e}(g, \widetilde{\alpha}, d)(g) - R_{k+1,e}(g, \widetilde{\alpha}, d)(b) = R_{k+1,e}(b, \widehat{\alpha}, d)(g) - R_{k+1,e}(b, \widehat{\alpha}, d)(b) > 0 \tag{45}$$

Since

$$0 = E^m(R_{k+1,e} | \mathcal{G}_k) = E^m(R_{k+1,e} | \xi, \alpha, d), \tag{46}$$

then

$$\begin{aligned} E^{\pi^*}(R_{k+1,e} | \xi, \alpha, d) &= E^{\pi^*}(R_{k+1,e} | \xi, \alpha, d) - E^m(R_{k+1,e} | \xi, \alpha, d) \\ &= (\pi^*(g | \xi) - m(g | \xi, \alpha))(R_{k+1,e}(\cdot)(g) - R_{k+1,e}(\cdot)(b)) \end{aligned}$$

Finally, note that (45) implies

$$E^{\pi^*}(R_{k+1,e} | g, \widetilde{\alpha}) > E^{\pi^*}(R_{k+1,e} | b, \widehat{\alpha}) \Leftrightarrow \pi^*(g | g) - m(g | g, \widetilde{\alpha}) > \pi^*(b | b) - m(b | b, \widehat{\alpha}). \tag{47}$$

Therefore,

$$\begin{aligned} E^{P_e}\left(R_{k+1,e} | R^*_{k,e} = \widetilde{R}\right) &= \sum_{(\xi,\alpha,\xi'):R^*_{k,e}(\xi,\alpha)(\xi')=\widetilde{R}} \frac{\psi^e(\xi,\alpha)\pi^*(\xi'|\xi)}{P^e\left(R^*_{k,e}(\xi,\alpha)(\xi')=R\right)} E^{P_e}(R_{k+1,e} | \mathcal{G}_\tau)\left(\xi', \alpha'(\xi,\alpha)\left(\xi'\right)\right) \\ &> \sum_{(\xi,\alpha,\xi'):R^*_{k,e}(\xi,\alpha)(\xi')=\widehat{R}} \frac{\psi^e(\xi,\alpha)\pi^*(\xi'|\xi)}{P^e\left(R^*_{k,e}(\xi,\alpha)(\xi')=-R\right)} E^{P_e}(R_{k+1,e} | \mathcal{G}_\tau)\left(\xi', \alpha'(\xi,\alpha)\left(\xi'\right)\right) \\ &= E^{P_e}\left(R_{k+1,e} | R^*_{k,e} = \widehat{R}\right) \end{aligned}$$

where the second inequality follows from (47). ■

Proof of Theorem 16 . Let agent h be some agent whose prior satisfies A2. A straightforward extension of Beker and Espino [6] can be used to show that the welfare weights associated with a PO allocation satisfy that for every agent i and every path $s \in S^\infty$

$$\begin{aligned} \alpha_{i,t}(s) &= \frac{\alpha_{i,0}\ P^{\pi^*}_{i,t}(s)}{\sum_{j=1}^I \alpha_{j,0}\ P^{\pi^*}_{j,t}(s)} \\ &= \frac{\dfrac{\alpha_{i,0}\ P^{\pi^*}_{i,t}(s)}{\alpha_{h,0}\ P^{\pi^*}_{h,t}(s)}}{\sum_{i=1}^I \dfrac{\alpha_{i,0}\ P^{\pi^*}_{i,t}(s)}{\alpha_{h,0}\ P^{\pi^*}_{h,t}(s)}} \end{aligned}$$

and so the limit behavior of the welfare weights depends on the limit behavior of the likelihood ratio

$$\frac{\alpha_{i,0}\ P_{i,t}^{\pi^*}(s)}{\alpha_{h,0}\ P_{h,t}^{\pi^*}(s)}$$

If h's prior satisfies A1.a then one can use Phillip and Polberger's [23, Theorem 4.1] results to show that, $P^{\pi^*}-a.s.$,

$$\frac{\alpha_{i,0}\ P_{i,t}^{\pi^*}(s)}{\alpha_{h,0}\ P_{h,t}^{\pi^*}(s)} \to \frac{\alpha_{i,0}}{\alpha_{h,0}}\frac{\mu_i(\pi^*)}{\mu_h(\pi^*)}$$

while if h's prior satisfies A1.b then one can use Sandroni's results to show that, $P^{\pi^*}-a.s.$,

$$\frac{\alpha_{i,0}\ P_{i,t}^{\pi^*}(s)}{\alpha_{h,0}\ P_{h,t}^{\pi^*}(s)} \to \frac{\alpha_{i,0}}{\alpha_{h,0}}\frac{f_i(\pi^*)}{f_h(\pi^*)}$$

It follows that $\alpha_{i,t}(s) \to \alpha_\infty$, $P^{\pi^*}-a.s.$

Since every agent prior satisfies A1, it is well known that there exists some $\pi = (\pi_1, ..., \pi_I)$ where $\pi_i \in \Pi^K$ such that μ_{i,s^t} converges weakly to μ^{π_i} for $P^{\pi_i}-$almost all $s \in S^\infty$ and π_i is the element of i's support which is closer to π^* in terms of entropy. By assumption A.2, $\pi_h = \pi^*$.

Since convergence almost surely implies convergence in distribution, we conclude that, $P^{\pi^*}-a.s.$, the marginal distribution over welfare weights and beliefs converges to a mass point on (α_∞, μ^π).

Now notice that for any $k \geq 1$

$$E^{P_{po}}(R_{k,PO}|\mathcal{G}_1)(\xi, \alpha_\infty, \mu^\pi) = E^{P^{\pi^*}}(R_{k,PO}|\xi)$$

and since we assume there is at least one agent i who satisfies $A2$, the argument above implies that

$$m\left(\xi'\,|\mathcal{G}_1\right)(\xi, \alpha_\infty, \mu^\pi) = \pi^*\left(\xi'\,|\xi\right)$$

Therefore, for any $k \geq 1$

$$E^{P_{po}}(R_{k,PO}|\mathcal{G}_1) = E^{P^{\pi^*}}(R_{k,PO}|\xi) = E^{P^m}(R_{k,PO}|\mathcal{G}_1) = 0$$

where the last equality follows by the no-arbitrage condition. The desired result follows from Proposition 12. ■

Proof of Proposition 3. Let f_i be defined as

$$f_i(\xi,\mu)(u_{-i}) = \max_{u_i} \quad \{u_i \in \Re : (u_i, u_{-i}) \in \mathcal{FU}(\xi,\mu)\},$$

Then f_i is decreasing with respect to u_j for $j \neq i$. Notice also that

$$\underline{\alpha}_1(1,\mu) \geq \underline{\alpha}_1(2,\mu) \quad \text{and} \quad \underline{\alpha}_2(2,\mu) \geq \underline{\alpha}_2(1,\mu) \tag{48}$$

Suppose first that $u \in \mathcal{U}^{FB}(\mu) \cap \mathcal{U}(\mu)$. Since $u \in \mathcal{U}^{FB}(\mu)$, there exists $\alpha(\xi,\mu)$ such that $\alpha(\xi,\mu) = \alpha_{\mathcal{FU}^{FB}(\xi,\mu)}(u_1(\xi,\mu), u_2(\xi,\mu)) \in \Delta^1$ and $\alpha(1,\mu) = \alpha(2,\mu)$. Since $u \in \mathcal{U}(\mu)$,

$$u_i(\xi) \geq U_i(\xi,\mu) \qquad \text{for all } \xi \text{ and } i.$$

It follows that

$$\begin{aligned} \alpha_1(\xi,\mu) &= \alpha_{1,\mathcal{FU}^{FB}(\xi,\mu)}(u_1(\xi,\mu), u_2(\xi,\mu)) = \alpha_{1,\mathcal{FU}(\xi,\mu)}(u_1(\xi,\mu), u_2(\xi,\mu)) \\ &\geq \alpha_{1,\mathcal{FU}(\xi,\mu)}(U_1(\xi,\mu), f_2(\xi,\mu)(U_1(\xi,\mu))) = \underline{\alpha}_1(\xi,\mu) \end{aligned}$$

$$\begin{aligned} 1-\alpha_1(\xi,\mu) &= 1-\alpha_{1,\mathcal{FU}^{FB}(\xi,\mu)}(u_1(\xi,\mu), u_2(\xi,\mu)) \\ &\geq 1-\alpha_{1,\mathcal{FU}^{FB}(\xi,\mu)}(f_2(\xi,\mu)(U_2(\xi,\mu)), U_2(\xi,\mu)) = \underline{\alpha}_2(\xi,\mu) \end{aligned}$$

then by (48)

$$\underline{\alpha}_1(1,\mu) \leq \alpha_1(1,\mu) = \alpha_1(2,\mu) \leq 1-\underline{\alpha}_2(2,\mu)$$

Suppose now that

$$\underline{\alpha}_1(1,\mu) < 1-\underline{\alpha}_2(2,\mu).$$

Then there is α such that

$$\underline{\alpha}_1(2,\mu) < \alpha < 1-\underline{\alpha}_2(1,\mu)$$

and so if one sets $\alpha_1(1,\mu) = \alpha_1(2,\mu) = \alpha$, it follows by (48) that

$$\underline{\alpha}_1(1,\mu) < \alpha_1(1,\mu) < 1-\underline{\alpha}_2(1,\mu) \text{ and } \underline{\alpha}_1(2,\mu) < \alpha_1(2,\mu) < 1-\underline{\alpha}_2(2,\mu)$$

Then there exists $u \in \mathcal{U}(\mu)$ such that $\alpha(\xi,\mu) = \alpha_{\mathcal{FU}(\xi,\mu)}(u_1(\xi,\mu), u_2(\xi,\mu))$ for all ξ. Therefore

$$U_1(\xi,\mu) < u_1(\xi,\mu) \quad \text{and} \quad U_2(\xi,\mu) < u_2(\xi,\mu)$$

and therefore $u \in \mathcal{U}^{FB}(\mu)$. ■

The following Lemma will be used in the proof of Proposition 4

Lemma 25 *There exists $L > 0$ such that $L < \frac{\underline{\alpha}_i(\xi,\mu)}{\underline{\alpha}_j(\xi,\mu)} < \frac{1}{L}$ for all (ξ,μ) and for all i,j.*

Proof of Lemma 25. Let $y_i^{\min} = \min_{s_t} y_i(s_t) > 0$ and observe that for all s^t

$$\begin{aligned} U_i(y_i)(s^t) &\geq u_i(y_i^{\min}) + \beta(s_t) \sum_{s_{t+1}} \pi_{\mu_{i,s^t}}(s_{t+1} | s_t) U_i(y_i)(s^t, s_{t+1}) \\ &= U_i(s_i^{\min}, \mu^{\delta^{s_i^{\min}}}) > 0 \end{aligned}$$

for which $s_i^{\min}$ is defined such that $y_i^{\min} = y_i(s_i^{\min})$. Define

$$\underline{\alpha}_i = \min_{\mu_{-i}} \left(\max_{u_{-i}} \quad \alpha_{\mathcal{FU}(\xi, \mu^{\delta^{s_i^{\min}}}, \mu_{-i})} (U_i(s_i^{\min}, \mu^{\delta^{s_i^{\min}}}), u_{-i}) \right),$$

Notice that $\underline{\alpha}_i > 0$ for all i (otherwise, $U_i(s_i^{\min}, \mu^{\delta^{s_i^{\min}}})$ can be attained) and therefore it is easy to that there exists $L > 0$ such that $L < \frac{\underline{\alpha}_i(\xi,\mu)}{\underline{\alpha}_j(\xi,\mu)} < \frac{1}{L}$ for all (ξ, μ) and for all i, j. ■

Proof of Proposition 4. Suppose there exists $u \in \mathcal{U}^{FB}(\mu) \cap \mathcal{U}(\mu)$. Since $u \in \mathcal{U}^{FB}(\mu)$, there exists $\alpha_0 \in \Delta^{I-1}$ such that

$$\frac{\alpha_{i,t}(s)}{\alpha_{j,t}(s)} = \frac{P_{i,t}(s)}{P_{j,t}(s)} \frac{\alpha_{i,0}}{\alpha_{j,0}} \quad \text{for all } s \in S^\infty \text{ and } t \geq 0$$

Since $u \in \mathcal{U}(\mu)$,

$$\min_\xi \frac{\underline{\alpha}_i(\xi, \mu_{s^t})}{\underline{\alpha}_j(\xi, \mu_{s^t})} \leq \frac{\alpha_{i,t}(s)}{\alpha_{j,t}(s)} \leq \max_\xi \frac{\underline{\alpha}_i(\xi, \mu_{s^t})}{\underline{\alpha}_j(\xi, \mu_{s^t})} \quad \text{for all } s \in S^\infty \text{ and } t \geq 0$$

Thus, it follows by Lemma 25 that

$$L \leq \frac{\alpha_{i,t}(s)}{\alpha_{j,t}(s)} \leq \frac{1}{L} \quad \text{for all } s \in S^\infty \text{ and } t \geq 0$$

Therefore, it suffices to show that there exists a path s and a date t where

$$\frac{P_{i,t}(s)}{P_{j,t}(s)} < \frac{\alpha_{j,0}}{\alpha_{i,0}} L \tag{49}$$

Notice that for each path s and date t

$$\begin{aligned} \frac{P_{i,t}(s)}{P_{j,t}(s)} &= \frac{\int_{\Delta^{K-1}} P_t^\pi(s) \mu_i(d\pi)}{\int_{\Delta^{K-1}} P_t^\pi(s) \mu_j(d\pi)} \qquad (50) \\ &= \frac{\int_{\Delta^{K-1}} \Pi_{\xi=1}^K \pi(\xi)^{n_{\xi,t}(s)} \mu_i(d\pi)}{\int_{\Delta^{K-1}} \Pi_{\xi=1}^K \pi(\xi)^{n_{\xi,t}(s)} \mu_j(d\pi)} = \frac{\int_{\Delta^{K-1}} \left(\Pi_{\xi=1}^K \pi(\xi)^{\frac{n_{\xi,t}(s)}{t}} \right)^t \mu_i(d\pi)}{\int_{\Delta^{K-1}} \left(\Pi_{\xi=1}^K \pi(\xi)^{\frac{n_{\xi,t}(s)}{t}} \right)^t \mu_j(d\pi)} \end{aligned}$$

For a given t, apply the Dominated Convergence Theorem to argue that

$$\lim_{x\to\overline{\pi}} \frac{\int_{\Delta^{K-1}} \left(\Pi_{\xi=1}^{K}\pi\left(\xi\right)^{x(\xi)}\right)^{t}\mu_i\left(d\pi\right)}{\int_{\Delta^{K-1}} \left(\Pi_{\xi=1}^{K}\pi\left(\xi\right)^{x(\xi)}\right)^{t}\mu_j\left(d\pi\right)} = \frac{\int_{\Delta^{K-1}} \left(\Pi_{\xi=1}^{K}\pi\left(\xi\right)^{\overline{\pi}(\xi)}\right)^{t}\mu_i\left(d\pi\right)}{\int_{\Delta^{K-1}} \left(\Pi_{\xi=1}^{K}\pi\left(\xi\right)^{\overline{\pi}(\xi)}\right)^{t}\mu_j\left(d\pi\right)} \tag{51}$$

So, if one shows that there exists τ such that

$$\frac{\int_{\Delta^{K-1}} \left(\Pi_{\xi=1}^{K}\pi\left(\xi\right)^{\overline{\pi}(\xi)}\right)^{\tau}\mu_i\left(d\pi\right)}{\int_{\Delta^{K-1}} \left(\Pi_{\xi=1}^{K}\pi\left(\xi\right)^{\overline{\pi}(\xi)}\right)^{\tau}\mu_j\left(d\pi\right)} < \frac{\alpha_{j,0}}{\alpha_{i,0}}L, \tag{52}$$

it follows by (51) that there is ε' such that $\left\|\frac{n_\tau(s)}{\tau}-\overline{\pi}\right\| \leq \varepsilon'$ implies

$$\frac{\int_{\Delta^{K-1}} \left(\Pi_{\xi=1}^{K}\pi\left(\xi\right)^{\frac{n_{\xi,\tau}(s)}{\tau}}\right)^{\tau}\mu_i\left(d\pi\right)}{\int_{\Delta^{K-1}} \left(\Pi_{\xi=1}^{K}\pi\left(\xi\right)^{\frac{n_{\xi,\tau}(s)}{\tau}}\right)^{\tau}\mu_j\left(d\pi\right)} < \frac{\alpha_{j,0}}{\alpha_{i,0}}L$$

Since $\left\{s\in S^\infty : \left\|\frac{n_\tau(s)}{\tau}-\overline{\pi}\right\| \leq \varepsilon'\right\}$ is non-empty, it follows that (49) holds.

To complete the proof, we shall show that (52) holds. Choose $\varepsilon \leq \frac{1}{2}\frac{\alpha_{j,0}}{\alpha_{i,0}}L$. By A.1 there exists n such that $\pi\in\Theta_n\left(\overline{\pi}\right)$ implies $\frac{d\mu_i}{d\mu_j}\left(\pi\right)<\varepsilon$. Let $\pi_n = \underset{\pi\in\Delta^{K-1}\setminus\Theta_n}{\arg\max}\ \Pi_{\xi=1}^{K}\pi\left(\xi\right)^{\overline{\pi}(\xi)}$ and $\overline{\Theta}_n = \left\{\pi\in\Theta_n\left(\overline{\pi}\right) : \Pi_{\xi=1}^{K}\pi\left(\xi\right)^{\overline{\pi}(\xi)} > \Pi_{\xi=1}^{K}\pi_n\left(\xi\right)^{\overline{\pi}(\xi)}\right\}$. Since $\overline{\pi} = \underset{\pi\in\Delta^{K-1}}{\arg\max}\Pi_{\xi=1}^{K}\pi\left(\xi\right)^{\overline{\pi}(\xi)}$, it follows that $\overline{\Theta}_n \neq \varnothing$. Then,

$$\frac{\Pi_{\xi=1}^{K}\pi\left(\xi\right)^{\overline{\pi}(\xi)}}{\Pi_{\xi=1}^{K}\pi_n\left(\xi\right)^{\overline{\pi}(\xi)}}\begin{cases} <1 & \text{if } \pi\notin\overline{\Theta}_n \text{ and } \pi\neq\pi_n \\ >1 & \text{if } \pi\in\overline{\Theta}_n \end{cases} \tag{53}$$

Since $\overline{\Theta}_n \subset \Theta_n\left(\overline{\pi}\right)$, it follows by A.1 that

$$\frac{d\mu_i}{d\mu_j}\left(\pi\right) < \varepsilon \quad \forall\pi\in\Theta_n\left(\overline{\pi}\right) \tag{54}$$

and thus (53) implies that there is some T such that for any $\tau \geq T$,

$$\frac{1}{\varepsilon} < \int_{\overline{\Theta}_n}\left(\frac{\Pi_{\xi=1}^{K}\pi\left(\xi\right)^{\overline{\pi}(\xi)}}{\Pi_{\xi=1}^{K}\pi_n\left(\xi\right)^{\overline{\pi}(\xi)}}\right)^{\tau}\mu_j\left(d\pi\right) \tag{55}$$

Moreover, for $\tau \geq T$

$$
\begin{aligned}
& \frac{\int_{\Delta^{K-1}} \left(\Pi_{\xi=1}^{K} \pi\left(\xi\right)^{\overline{\pi}(\xi)}\right)^{\tau} \mu_i\left(d\pi\right)}{\int_{\Delta^{K-1}} \left(\Pi_{\xi=1}^{K} \pi\left(\xi\right)^{\overline{\pi}(\xi)}\right)^{\tau} \mu_j\left(d\pi\right)} \leq \frac{\int_{\Delta^{K-1}} \left(\Pi_{\xi=1}^{K} \pi\left(\xi\right)^{\overline{\pi}(\xi)}\right)^{\tau} \mu_i\left(d\pi\right)}{\int_{\overline{\Theta}_n} \left(\Pi_{\xi=1}^{K} \pi\left(\xi\right)^{\overline{\pi}(\xi)}\right)^{\tau} \mu_j\left(d\pi\right)} \\
= & \frac{\int_{\Delta^{K-1}\setminus\overline{\Theta}_n} \left(\frac{\Pi_{\xi=1}^{K} \pi(\xi)^{\overline{\pi}(\xi)}}{\Pi_{\xi=1}^{K} \pi_n(\xi)^{\overline{\pi}(\xi)}}\right)^{\tau} \mu_i\left(d\pi\right) + \int_{\overline{\Theta}_n} \left(\frac{\Pi_{\xi=1}^{K} \pi(\xi)^{\overline{\pi}(\xi)}}{\Pi_{\xi=1}^{K} \pi_n(\xi)^{\overline{\pi}(\xi)}}\right)^{\tau} \mu_i\left(d\pi\right)}{\int_{\overline{\Theta}_n} \left(\frac{\Pi_{\xi=1}^{K} \pi(\xi)^{\overline{\pi}(\xi)}}{\Pi_{\xi=1}^{K} \pi_n(\xi)^{\overline{\pi}(\xi)}}\right)^{\tau} \mu_j\left(d\pi\right)} \\
\leq & \frac{1 + \int_{\overline{\Theta}_n} \left(\frac{\Pi_{\xi=1}^{K} \pi(\xi)^{\overline{\pi}(\xi)}}{\Pi_{\xi=1}^{K} \pi_n(\xi)^{\overline{\pi}(\xi)}}\right)^{\tau} \mu_i\left(d\pi\right)}{\int_{\overline{\Theta}_n} \left(\frac{\Pi_{\xi=1}^{K} \pi(\xi)^{\overline{\pi}(\xi)}}{\Pi_{\xi=1}^{K} \pi_n(\xi)^{\overline{\pi}(\xi)}}\right)^{\tau} \mu_j\left(d\pi\right)} \leq \frac{1 + \varepsilon \int_{\overline{\Theta}_n} \left(\frac{\Pi_{\xi=1}^{K} \pi(\xi)^{\overline{\pi}(\xi)}}{\Pi_{\xi=1}^{K} \pi_n(\xi)^{\overline{\pi}(\xi)}}\right)^{\tau} \mu_j\left(d\pi\right)}{\int_{\overline{\Theta}_n} \left(\frac{\Pi_{\xi=1}^{K} \pi(\xi)^{\overline{\pi}(\xi)}}{\Pi_{\xi=1}^{K} \pi_n(\xi)^{\overline{\pi}(\xi)}}\right)^{\tau} \mu_j\left(d\pi\right)} \\
= & \frac{1}{\int_{\overline{\Theta}_n} \left(\frac{\Pi_{\xi=1}^{K} \pi(\xi)^{\overline{\pi}(\xi)}}{\Pi_{\xi=1}^{K} \pi_n(\xi)^{\overline{\pi}(\xi)}}\right)^{\tau} \mu_j\left(d\pi\right)} + \varepsilon < \frac{\alpha_{j,0}}{\alpha_{i,0}} L
\end{aligned}
$$

where the second inequality follows by the definition of $\overline{\Theta}_n$, the third inequality follows by (54) and the last one follows from (55) and our assumption on ε. ■

12 Appendix C

In this section we offer the proofs corresponding to Section 7

Proof of Theorem 17. First we show that asset g displays short-term momentum.

We begin showing that the return in period 1 is symmetrically distributed. Let $\widehat{p}_{k,cpo}^{\xi'}(\xi, d)$ be the ex-dividend price of asset ξ' in period k in state $(\xi, \underline{\alpha}(\xi))$ with cumulated dividend $d \in \{1, .., k\}$. Then, $\widehat{p}_{3,cpo}^{\xi'}(\xi, d) = d$ and for $k \in \{0, 1, 2\}$

$$\widehat{p}_{k,cpo}^{g}(\xi, d) = Q\left(\xi, \underline{\alpha}(\xi)\right)(g)\, \widehat{p}_{k+1,cpo}^{g}(g, d+1) + Q\left(\xi, \underline{\alpha}(\xi)\right)(b)\, \widehat{p}_{k+1,cpo}^{b}(b, d)$$

For $\widehat{r}(\xi) \equiv RF\left(\xi, \underline{\alpha}(\xi)\right)$, its return in period k is

$$\widehat{R}_{k,cpo}^{g}(\xi, d)\left(\xi'\right) \equiv \frac{\widehat{p}_{k,cpo}^{g}\left(\xi, \delta\left(\xi', d\right)\right)}{\widehat{r}(\xi)} - \widehat{p}_{k,cpo}^{g}(\xi, d)$$

Note that

$$
\begin{aligned}
\widehat{R}_{1,cpo}^{\xi'}(\xi, 0)(g) &= \frac{\widehat{p}_1^{\xi'}(g, 0)}{\widehat{r}(\xi)} - \left(m^*(g\,|\xi)\, \frac{\widehat{p}_{1,cpo}^{\xi'}(g, 0)}{\widehat{r}(\xi)} + \left(1 - m^*(g\,|\xi)\right) \frac{\widehat{p}_{1,cpo}^{\xi'}(b, 0)}{\widehat{r}(\xi)}\right) \\
&= \left(1 - m^*(g\,|\xi)\right) \left(\frac{\widehat{p}_{1,cpo}^{\xi'}(g, 0)}{\widehat{r}(\xi)} - \frac{\widehat{p}_{1,cpo}^{\xi'}(b, 0)}{\widehat{r}(\xi)}\right) \\
&= \beta(\xi)\, \pi^*(b\,|\xi)\, \theta^*(b\,|\xi) \left(\widehat{p}_{1,cpo}^{\xi'}(g, 0) - \widehat{p}_{1,cpo}^{\xi'}(b, 0)\right)
\end{aligned}
$$

Since $\pi^*(b|g) = \pi^*(g|b)$ and $\pi^*(g|g) = \pi^*(b|b)$, it follows that

$$\widehat{R}^g_{1,cpo}(g,0)(g) = \beta\pi^*(b|g)\,\theta^*(b|g)\left(\widehat{p}^g_{1,cpo}(g,0) - \widehat{p}^g_{1,cpo}(b,0)\right) = -\widehat{R}^g_{1,cpo}(b,0)(b)$$

$$\widehat{R}^g_{1,cpo}(b,0)(g) = \beta\pi^*(b|b)\,\theta^*(b|b)\left(\widehat{p}^g_{1,cpo}(g,0) - \widehat{p}^g_{1,cpo}(b,0)\right) = -\widehat{R}^g_{1,cpo}(g,0)(b)$$

and we conclude that the return in period 1 is symmetrically distributed.

Since the return in period 1 is symmetrically distributed, then by Proposition 13 it suffices to show that the return in period 2 underreacts to abnormal returns in period 1. Note that $(g, \underline{\alpha}(g))$ is consistent with $\widehat{R}^g_{1,cpo}(g,0)(g)$ and $\widehat{R}^g_{1,cpo}(b,0)(g)$ while $(b, \underline{\alpha}(b))$ is consistent with $\widehat{R}^g_{1,cpo}(b,0)(b)$ and $\widehat{R}^g_{1,cpo}(g,0)(b)$. Therefore, $\psi_{cpo} - a.s.$, $(g, \underline{\alpha}(g))$ and $(b, \underline{\alpha}(b))$ is the only pair consistent with abnormal returns in period 1. Moreover, for every $\xi \in \{g, b\}$ we have that

$$\widehat{R}^g_{1,cpo}(\xi,0)(g) - \widehat{R}^g_{1,cpo}(\xi,0)(b) = \frac{\widehat{p}^g_1(g,0) - \widehat{p}^g_1(b,0)}{\widehat{r}(\xi)}$$

is independent of ξ and so state of nature g is unambiguously good news with constant dispersion. Since the market belief in period 1 is less biased towards state of nature g at state $(g, \underline{\alpha}(g))$ than at state $(b, \underline{\alpha}(b))$, the desired result follows by Proposition 15.

Now we show the asset return does not display long-term reversal. By Corollary 14 it suffices to show that the return in period 3 underreacts to abnormal returns in period 1. Note that by the symmetry of the distribution of the return in period 1, $\sum_R \overline{P}(R)\; R\; E^{P_e}\left(R_{3,e}|\,\overline{R}_{1,e} = R\right)$ is

$$\begin{aligned} &= \sum_{R>0} \overline{P}(R)\;\; R\; E^{P_e}\left(R_{3,e}|\,\overline{R}_{1,e} = R\right) + \sum_{R<0} \overline{P}(R)\;\; R\; E^{P_e}\left(R_{3,e}|\,\overline{R}_{1,e} = R\right) \\ &= \sum_{R>0} \overline{P}(R)\;\; R\;\; \left[E^{P_e}\left(R_{3,e}|\,\overline{R}_{1,e} = R\right) - E^{P_e}\left(R_{3,e}|\,\overline{R}_{1,e} = -R\right)\right] \end{aligned}$$

and so it suffices to show that for $R \in \left\{\widehat{R}^g_{1,cpo}(g,0)(g), \widehat{R}^g_{1,cpo}(b,0)(g)\right\}$,

$$E^{P_e}\left(R_{3,e}|\,\overline{R}_{1,e} = R\right) \geq E^{P_e}\left(R_{3,e}|\,\overline{R}_{1,e} = -R\right)$$

Notice that by the law of iterated expectations for any $R > 0$,

$$E^{P_e}\left(R_{3,e}|\,\overline{R}_{1,e} = R\right) = E^{P_e}\left(E^{P_e}\left(R_{3,e}|\,\mathcal{G}_2\right)\big|\,\overline{R}_{1,e} = R\right)$$

Note that

$$\widehat{R}^g_{3,cpo}(\xi,d)(g) - \widehat{R}^g_{3,cpo}(\xi,d)(b) = \frac{\widehat{p}^g_3(g,d+1) - \widehat{p}^g_3(b,d)}{\widehat{r}(\xi)} = \frac{1}{\widehat{r}(\xi)}$$

and so g is unambiguously good news with constant dispersion for the return in period 3. Therefore, $E^{P_e}\left(R_{3,e}\right|\mathcal{G}_2)(\omega)$ depends only on the state in period 2 (ξ,α) and we write $E^{P_e}\left(R_{3,e}\right|\mathcal{G}_2)(\xi,\alpha)$. Then,

$$E^{P_e}\left(R_{3,e}\right|\overline{R}_{1,e}=R\Big) = \pi^*(g|g)\,E^{P_{cpo}}\left(R^g_{3,cpo}\Big|\,\mathcal{G}_2\right)(g,\underline{\alpha}(g)) + \pi^*(b|g)\,E^{P_{cpo}}\left(R^g_{3,cpo}\Big|\,\mathcal{G}_2\right)(b,\underline{\alpha}(b))$$

$$E^{P_e}\left(R_{3,e}\right|\overline{R}_{1,e}=-R\Big) = \pi^*(g|b)\,E^{P_{cpo}}\left(R^g_{3,cpo}\Big|\,\mathcal{G}_2\right)(g,\underline{\alpha}(g)) + \pi^*(b|b)\,E^{P_{cpo}}\left(R^g_{3,cpo}\Big|\,\mathcal{G}_2\right)(b,\underline{\alpha}(b))$$

Once again, $\psi_{cpo} - a.s.$, $(g,\underline{\alpha}(g))$ and $(b,\underline{\alpha}(b))$ is the only pair consistent with the return in period 2. Since the market belief in period 2 is less biased towards state of nature g at state $(g,\underline{\alpha}(g))$ that at state $(b,\underline{\alpha}(b))$, it follows by Proposition 15 that $E^{P_{cpo}}\left(R^g_{3,cpo}\Big|\,\mathcal{G}_2\right)(g,\underline{\alpha}(g)) \geq E^{P_{cpo}}\left(R^g_{3,cpo}\Big|\,\mathcal{G}_2\right)(b,\underline{\alpha}(b))$. Therefore,

$$E^{P_e}\left(R_{3,e}\right|\overline{R}_{1,e}=R\Big) \geq E^{P_e}\left(R_{3,e}\right|\overline{R}_{1,e}=-R\Big) \Leftrightarrow \pi^*(g|g) \geq \pi^*(g|b) \Leftrightarrow \pi^*(g|g) \geq \frac{1}{2}$$

as desired. We conclude that if the states of nature are persistent, the return in period 3 underreacts to abnormal returns in period 1 and so the economy does not display long-term reversal. ■

Proof of Proposition 18. TO BE COMPLETED ■

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