Choice by lexicographic semiorders

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Abstract

In Tversky's [30] model of a lexicographic semiorder, preference is generated by the sequential application of numerical criteria, by declaring an alternative x better than an alternative y if the first criterion that distinguishes between x and y ranks x higher than y by an amount exceeding a *fixed* threshold. We generalise this idea to a fully-fledged model of boundedly rational choice. We explore the connection with sequential rationalisability of choice ([1], [20]), and we provide axiomatic characterisations of both models in terms of observable choice data.

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1 Introduction

Lexicographic heuristics have gained much attention in the study of decision making, in several fields: in psychology (e.g. Tversky [30], [31]; Gigerenzer and Todd [9]); in positive economics (e.g. Rubinstein [25]; Leland [13]; Manzini and Mariotti [20]; Apesteguia and Ballester [1]); in normative economics (e.g. Tadenuma [28], [29]; Houy and Tadenuma [11]); in marketing science (e.g. Yee, Dahan, Hauser and Orlin [33]; Kohli and Jedidi [12]). The explanation for this success is obvious: lexicographic procedures look appealingly simple and realistic since they eschew the complex trade-offs between several criteria of classical decision makers. On the other hand, the lack of trade-offs may also seem to constitute a disadvantage (especially among economists). Price may be the most important criterion in the purchase of a house from a set of suitable ones. Yet who would be prevented by a difference of a few bucks from selecting a house in a much more desirable neighbourhood? Arguably, very few people would be so uncompromising as to ignore any significant improvement in one dimension because of an arbitrarily small loss in the most important dimension. When modelling boundedly rational behaviour, the rigid application of simple 'rules of thumb' (such as 'buy the cheapest house among the acceptable ones') may look even less realistic than the trade-offs of textbook utility maximisation.

In other words, it is reasonable that, even in a boundedly rational heuristics, criteria that detect significant differences between the alternatives under consideration should over-ride criteria that do not. In this paper we study a model of choice that formalises this intuition. Note that a number of 'basic criteria' could be aggregated into a single, more complex criterion, to which our observations on the house buyer above would nevertheless still apply: if the agent constructs an index which trades off price and location, that index constitutes a new criterion, for which it may be unwise not to ignore small differences in favour, say, of house size. And so on.[1] Only a fully rational decision maker would be able to pack together all possible trade-offs in a single criterion. However, in a more realistic model of decision making, there is a limit to the number of simultaneous trade-

[1] As another example, in Manzini and Mariotti [19] we have proposed a multi-criterion model of choice over time in which the first criterion is the exponentially discounted value, which trades off the time and size of of a monetary reward.

offs the decision maker is able to carry out. Thus, it seems very plausible to expect the decision maker to rely on a lexicographic list of 'slack' criteria. The choice procedure we propose can explain observed 'anomalies', while at the same time preserving a convincing flexibility.

Considerations of this kind have already led some of the researchers mentioned above[2] to build models of *preference or binary choice* based on the application of numerical criteria where small differences in the values of criteria are ignored.[3] However, such models leave unanswered the issue of *choice* from more complex sets (e.g., budget sets). They do not study choice functions. If binary preferences are derived from a boundedly rational procedure, the issue of associating such preferences with higher order choices is far from trivial: on the one hand it may be impossible to maximise the preference (when it is cyclical); and on the other hand it may be inappropriate to even consider maximisation when the issue is one of bounded rationality.

We focus on Tversky's [30] fruitful notion of *lexicographic semiorder*, in which preference is generated by the sequential application of numerical criteria, by declaring an alternative x better than an alternative y if the first criterion that distinguishes between x and y ranks x higher than y by an amount exceeding a *fixed* threshold. Our first contribution is to define a choice procedure based on Tversky's idea.

Tversky himself considered lexicographic semiorders appealing but restrictive as a model of preference.[4] In fact, this judgement is shown to be somewhat pessimistic. Even when the agent is endowed with very rudimental discriminatory abilities (being only able to classify criteria values in 'good', 'neutral' and 'bad', where just 'good' and 'bad' are rankable), the model can account for a surprisingly rich variety of behaviours.

The proposed model of choice by lexicographic semiorders turns out to be connected with another, much more general-looking, notion of boundedly rational choice, namely 'sequentially rationalisable choice' (Manzini and Mariotti [20]): an arbitrary number of arbitrary asymmetric binary relations ('rationales') is applied sequentially to single out

[2] Tversky [30], Rubinstein [25], Leland [13]

[3] A difference being small is often interpreted as 'similarity'.

[4] See Section 3.

an alternative. On any finite domain[5], bar the restriction that the rationales should be acyclic, the two models have exactly the same reach: they restrict choice data in identical ways (proposition 1).

However, the clause 'on any finite domain' is key. When this clause is relaxed even marginally, by allowing a countably infinite number of *finite* choice sets, the equivalence breaks down: even the use of only two rationales may produce behaviours that cannot be generated by any number of semiorders and any number of discriminations (proposition 2). So, the two models are in general distinguishable by observable choice data.

Next, we characterise choice by lexicographic semiorder (on domains which are not necessarily finite) in terms of a new contraction consistency condition (Reducibility), at the same time providing an algorithm to construct the semiorders (theorem 1).

As a bonus, for the case of finite domains, this result automatically also yields a characterisation of acyclic sequentially rationalisable choice. On the same domain, this leads directly to a relaxation of Reducibility which characterises standard sequential rationalisability, and to an algorithm to construct the rationales (theorem 3). These two results, while tangential to main line of enquiry of this paper, are of independent interest, since the characterisation of sequential rationalisability has proved to be a hard problem which we left open in [20]. Our results in this respect build on and complement those by Apesteguia and Ballester [1], who were the first to draw attention to the restriction of sequential rationalisability to acyclic rationales and to provide a characterisation for it. In the Appendix we work out one of their examples of sequentially rationalisable choices to construct the rationales with our algorithm. Our work can also be seen as an extension of the approach in Mandler, Manzini and Mariotti [18]: we discuss this relation in the concluding section.

2 Lexicographic semiorders: preferences and choice

Fix a nonempty set X. A *semiorder* (Luce [15]) is an irreflexive[6] relation P on X which satisfies two additional properties:

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[5] That is a domain including a finite number of finite sets.

[6] Irreflexivity: for all $x \in X$, $(x, x) \notin P$.

1. $(x, y), (w, z) \in P$ imply $(x, z) \in P$ or $(w, y) \in P$;

2. $(x, y) \in P$ and $(y, z) \in P$ imply $(x, w) \in P$ or $(w, z) \in P$.

Given the irreflexivity of P, each of (1) or (2) imply that P is also transitive.[7] So a semiorder is a very special type of strict partial order. The interest of semiorders is that they can be interpreted as a simple threshold model of (partial) rankings: on suitable domains, P is a semiorder if and only if there exists a real valued function f on X and a number $\sigma \geq 0$ such that $(x, y) \in P$ if and only if $f(x) > f(y) + \sigma$. Here $f(x)$ is the 'value' of the alternative x and σ is the amount by which the value of one alternative x must exceed the value of another alternative y for x to be declared superior to y. The fact that σ is fixed makes this a very parsimonious model of binary preferences.[8]

Tversky [30] essentially proposed a lexicographic procedure, which extends the use of semiorders, to make binary comparisons between alternatives in a set X. There exists an ordered sequence $f = (f_1, ..., f_n)$ of real valued functions on X and a $\sigma > 0$ such that x is declared better than y iff, for the first i for which $|f_i(x) - f_i(y)| > \sigma$, we have $f_i(x) > f_i(y) + \sigma$. The idea is that the agent compares alternatives along several dimensions. As in our opening example, dimensions are ranked in order of importance, and a later dimension is only considered if all previous dimensions failed to discriminate between the two alternatives under consideration. In other words, the agent examines the dimensions *lexicographically*: as soon as a dimension i is found for which one alternative x is superior to another alternative y by an amount exceeding the threshold σ, x is declared better than y. When such an i is found, no dimension j that comes later in the order has any bearing, no matter the size of the differences between the alternatives in these subsequent dimensions. That σ is chosen to be the same for all f_i is not a relevant issue, since even if we had different σ_i, the f_i and σ_i can always be rescaled so as to choose $\sigma_i = 1$. Given f and σ, this procedure can be used to generate a revealed preference

[7] Transitivity: for all $x, y, z \in X$, $(x, y) \in P$, $(y, z) \in P \Rightarrow (x, z) \in P$.

[8] In an *interval order* (Fishburn [7]), characterised by condition 1 alone, the threshold σ is allowed to vary with the alternatives being compared, being a function $\sigma : X \to \mathbb{R}_+$. This makes for a much richer structure. See e.g. Fishburn [8].

relation $\succ_{(f,\sigma)}$ on pairs of alternatives.[9]

Suppose now that the agent wants to apply the procedure to produce a selection out of choice sets S larger than the binary ones. There are several ways to do so, some of which are however problematic. One could for example start from the binary revealed preference relation and use either of the following two plausible methods:

- the choice from S is the set of the maximal elements of $\succ_{(f,\sigma)}$

- the choice from S is the top cycle (or the uncovered set) of $\succ_{(f,\sigma)}$ restricted to each S.[10]

Unfortunately, the preference relation $\succ_{(f,\sigma)}$ may be cyclic - this 'anomalous' feature was indeed the very point of Tversky introducing the procedure. So the first method above may not be well-defined if a nonempty-valued choice function is desired. The second method above borrows the ideas of authors such as Ehlers and Sprumont [5] and Lombardi [14], who use weaker notions of maximization to produce choices out of non-standard preferences formed of asymmetric and complete binary relations (tournaments). These methods would for example select the entire set $S = \{x_1, x_2, ..., x_n\}$ whenever $x_1 \succ_{(f,\sigma)} x_2 \succ_{(f,\sigma)} ... \succ_{(f,\sigma)} x_n \succ_{(f,\sigma)} x_1$.

Here we pursue a different natural way of extending and abstracting Tversky's idea. The method we suggest is, on the one hand, more in line with the procedural (as opposed to maximising) nature of Tversky's approach; and, on the other hand, it can produce a unique selection even from the awkward cycles discussed above. The reason for these two features is that the method, unlike the others suggested, preserves and uses the information on the order in which the dimensions are considered.

We impose no arbitrary uniform bound on the number of dimensions that the agent is allowed to consider. Nevertheless, we insist that the procedure always halts in a finite

[9] Rubinstein [25] proposes a related but distinct procedure. This procedure has recently been studied experimentally by Binmore, Voorhoeve and Wallace [2].

[10] More precisely, let $P|S$ denote the restriction to S of a complete asymmetric binary relation P defined on X. (Completeness: for all $x, y \in X$ either $(x, y) \in P$ or $(y, x) \in P$. Asymmetry: for all $x, y \in X$, $(x, y) \in P \Rightarrow (y, x) \notin P$). Let $(P|S)^t$ denote the transitive closure of $P|S$. The *top cycle* of P in S is the set of maximal elements of $(P|S)^t$ in S. Define the covering relation $C(P, S)$ of P in S by: $(x, y) \in C(P, S)$ iff $x, y \in S$ and either $(x, y) \in P$ or there exists $z \in S$ such that $(x, z) \in P$ and $(z, y) \in P$. The *uncovered set* of P in S is the set of maximal elements of $C(P, S)$ in S.

number of steps in any choice situation.

Our proposed procedure works via a process of sequential elimination. Formally, let Σ be a domain of choice sets, where each S in Σ is a nonempty subset of X. A *choice function* on Σ is a function $c : \Sigma \to X$ such that $c(S) \in S$ for all $S \in \Sigma$. A choice set S which has the form $S = \{x\}$ for some $x \in X$ will be called *trivial.* A collection $\mathcal{C} \subseteq \Sigma$ of choice sets is trivial if each $S \in \mathcal{C}$ is trivial.

An ordered sequence $f = (f_i)_{i \in I}$, where I is either an interval of numbers $\{1, ..., n\}$ or the entire set of natural numbers $\mathbb{N}$, together with a $\sigma > 0$ is a *lexicographic semiorder on* X, denoted $(f_1, f_2, ..., \sigma) = (f_i, \sigma)_{i \in I}$. We abuse terminology and call each f_i directly a semiorder although strictly speaking f_i is a numerical representation of it.

Given a choice set $S \subseteq X$ and a lexicographic semiorder $(f_i, \sigma)_{i \in I}$, define inductively the following 'survivor sets' $M_i(S)$, for all $i > 0$:

$$M_0(S) = S$$

$$M_i(S) = \{s \in M_{i-1}(S) \,|\, f_i(s) + \sigma \geq f_i(s') \;\; \forall s' \in M_{i-1}(S)\}$$

This sequence of sets captures the procedure the agent follows in order to arrive at a final selection from the choice set S: at every round i he looks for alternatives in the current survivor set $M_{i-1}(S)$ which are judged 'worse' than some other alternative in $M_{i-1}(S)$ according to the Tversky procedure described before. He discards all such inferior alternatives (if any), generating the next survivor set $M_i(S)$, and so on.

Definition 1 *A choice function c is a* ***choice by lexicographic semiorder (cles)*** *iff there exists a lexicographic semiorder $(f_i, \sigma)_{i \in I}$ such that, for all $S \in \Sigma$, there is a $j \in I$ for which $\{c(S)\} = M_j(S) = M_k(S)$ for all $k \geq j$.*

In this case we say that $(f_i, \sigma)_{i \in I}$ ***induces*** *c.*

That is, for a cles c, the iterative elimination procedure described before stops on any choice set S after a finite number of steps, yielding precisely the alternative that c picks in S. Note that, in spite of this property of 'finite termination', there might not exist any fixed j that works for all S. When this happens, which means that I can be chosen to be finite, we say that c is a choice by finite lexicographic semiorder.

Basic Semiorders

A semiorder f_i is *basic* if it ranges only in $\{-1, 0, 1\}$ and $\sigma = 1$. A lexicographic semiorder $(f_i, \sigma)_{i \in I}$ is basic if each f_i is basic. So, with a basic lexicographic semiorder the agent has only a very limited power of discrimination. Essentially, on each dimension he can only perform a rough classification of alternatives into 'good' ones (those x for which $f_i(x) = 1$), 'bad' ones ($f_i(x) = -1$), and 'neutral ones' ($f_i(x) = 0$): a good alternative 'beats' a bad one (on the given dimension), and a neutral alternative neither beats a bad one nor is beaten by a good one.

A basic lexicographic semiorder can be denoted simply as $f = (f_i)_{i \in I}$. To emphasise that the survivor sets $M_i(S)$ are obtained from the basic lexicographic semiorder f we write them as $M_i^f(S)$.

Example: Let $X = \{x, y, z\}$ and let $\Sigma = \{\{x, y\}, \{y, z\}, \{z, x\}, X\}$. Let $c(\{x, y\}) = c(X) = x$, $c(\{y, z\}) = y$ and $c(\{x, z\}) = z$. This is a choice function by basic lexicographic semiorder. To see this, let $f_1(x) = 0$, $f_1(y) = 1$, $f_1(z) = -1$, $f_2(x) = 1$, $f_2(y) = -1$, $f_2(z) = 0$, $f_3(x) = -1$, $f_3(y) = 1$, $f_3(z) = 1$. Observe how different (unique) choices from X can be obtained by permuting the order of the f_i.

3 Characterisation

3.1 General remarks

Tversky thought that the model of binary choice by lexicographic semiorder, while useful to explain the anomaly of cyclical preferences, had a narrow scope otherwise. He writes:

> " ... despite its intuitive appeal, it is based on a noncompensatory principle that is likely to be too restrictive in many contexts." ([30], p. 40).

Following this logic, one might conjecture that the version with basic semiorders, with its minimal concession to discriminatory powers, is even more restrictive. We study this issue.

In order to pinpoint the restrictions on behavior implied by the cles model, we begin by recalling a definition from Manzini and Mariotti [20]. For a generic binary relation B

and a set $S \subseteq X$, denote by $\max(S, B)$ the set of $B-$maximal elements in S, $\max(S, B) = \{x : x \in S \text{ and } (y, x) \notin B \text{ for all } y \in S\}$.

Definition 2 *A choice function c is* ***sequentially rationalisable*** *whenever there exists an ordered list* $P_1, ..., P_K$ *of asymmetric relations, with* $P_i \subseteq X \times X$ *for* $i = 1...K$, *such that, defining recursively*

$$M_0^*(S) = S$$
$$M_i^*(S) = \max\left(M_{i-1}^*(S); P_i\right),\ i = 1, ..., K$$

we have

$$\{c(S)\} = M_K^*(S) \text{ for all } S \in \mathcal{P}(X)$$

In that case we say that $(P_1, ..., P_K)$ *sequentially rationalise c. Each* P_i *is a* **rationale**.

Two specialisation of sequential rationalisability are:

Definition 3 *(Manzini and Mariotti [20]) A choice function is a* ***Rational Shortlist Method*** *(RSM) iff it is sequentially rationalisable with two rationales.*

Definition 4 *(Apesteguia and Ballester [1]) A choice function is* ***acyclic sequentially rationalisable*** *iff it is sequentially rationalisable by rationales that are acyclic.*

Both acyclic and standard sequential rationalisability constitute at first sight a much more general model than cles, because the rationales are not required to have any threshold structure and can thus apparently accommodate more sophisticated discriminations. But in fact, for arbitrary finite domains, the behaviours that can be generated by the lexicographic semiorder model and those that can be generated by the acyclic sequential rationalisability model are just the same. And, we need to look no further than *basic* semiorders to yield this equivalence.

On the other side of the coin, the restriction to finite domains is not merely a convenience for the inductive argument used in the proof, but it is necessary for the equivalence to hold. When the restriction is relaxed even marginally (by retaining the finiteness of each choice set but allowing for a countable number of choice sets), the model of acyclic sequential rationalisability suddenly appears far more general than the lexicographic semiorder model: even only two acyclic rationales suffice to produce behaviours that cannot be induced by any basic lexicographic semiorder. And increasing the discriminatory ability of the agent is to no avail: the 'basic' restriction is inessential for this result.

These assertions are made precise in the next two results. The first one can be derived (in the case of Σ being the domain of all nonempty subsets of a finite set X) from theorem B.1 of [1]. We present here a different (inductive) method of proof.

Proposition 1 *Let X be finite. Then a choice function c is acyclic sequentially rationalisable if and only if it is induced by a basic lexicographic semiorder.*

Proof. A semiorder is an acyclic rationale, so it suffices to prove the 'only if' part of the statement. Given acyclic rationales $(P_1, ..., P_K)$, recall the definition 2 of survivor sets $M_i^*(S)$. We will show that, for any domain Σ, there exists a a basic lexicographic semiorder $f = (f_i)_{i \in I}$ such that, for all $S \in \Sigma$, there is a $j \in I$ such that $M_K^*(S) = M_j^f(S) = M_k^f(S)$ for all $k \geq j$. This proves the assertion in the statement.

The proof is by induction on the sum of the cardinalities of the sets S in Σ, which we denote by $n(\Sigma) = \sum_{S \in \Sigma} |S|$. If $n(\Sigma) = 1$ the claim is obviously true. Take now $n(\Sigma) > 1$. If Σ is trivial, then the claim is also obviously true, so assume Σ is not trivial, and w.l.o.g. assume in addition that P_1 is nonempty on some $S \in \Sigma$ (otherwise just exclude P_1 and renumber the remaining P_i). By the acyclicity of P_1 there exist $S \in \Sigma$ and $x, y \in S$ such that $(x, y) \in P_1$ and $(y, z) \notin P_1$ for all $z \in \bigcup_{S \in \Sigma} S$ with $y, z \in T$ for some $T \in \Sigma$ (in words, y is P_1-dominated in some choice set and it does not P_1-dominate any element which appears together with y in any choice set). Fix those x and y, and define

$$\Sigma' = \{S : \{x, y\} \nsubseteq S \in \Sigma\} \cup \{S : S = T \setminus \{y\} \text{ for some } T \in \Sigma \text{ s.t. } \{x, y\} \subseteq T\}$$

Because a T as in the right-hand member of the union above exists by construction, $n(\Sigma') < n(\Sigma)$. So by the inductive hypothesis there exists a basic lexicographic semiorder $f = (f_i)_{i \in I}$ such that, for all $S \in \Sigma'$, there is a $j \in I$ such that $M_K^*(S) = M_j^f(S) = M_k^f(S)$

for all $k \geq j$. Now consider the basic lexicographic semiorder $g = (g_i)_{i \in I'}$ defined by

$$\begin{aligned} g_i &= f_{i-1} \text{ for all } i > 1 \\ g_1(x) &= 1,\ g_1(y) = -1 \text{ and } g_1(z) = 0 \text{ for all } z \neq x, y \end{aligned}$$

Thus, for all $S \in \Sigma$ such that $\{x, y\} \subseteq S$, $M_1^g(S) = S \setminus \{y\} \in \Sigma'$ and consequently $M_K^*(S \setminus \{y\}) = M_{j+1}^g(S) = M_k^g(S)$ for all $k \geq j+1$ (this follows by the second line of the displayed definition of g and the fact that $M_K^*(S \setminus \{y\}) = M_j^f(S \setminus \{y\}) = M_k^f(S \setminus \{y\})$ for all $k \geq j$). Moreover, clearly for all $S \in \Sigma$ such that $\{x, y\} \subseteq S$, $M_K^*(S) = M_K^*(S \setminus \{y\})$. Therefore, for all $S \in \Sigma$, $M_K^*(S) = M_K^*(S \setminus \{y\}) = M_{j+1}^g(S) = M_k^g(S)$ for all $k \geq j+1$.■

Proposition 2 *There exist Rational Shortlist Methods using acyclic rationales which are not induced by any lexicographic semiorder.*

Proof. Let $X = \{1, 2...\}$, let Σ be the collection of finite subsets of X, and let c be uniquely defined as the RSM rationalised by the following two acyclic rationales P_1 and P_2:

$$P_1 = \{(i, i+1) : i \in X\}$$

and

$$P_2 = \{(j, i) : j > i+1\}$$

We show that c is not induced by any lexicographic semiorder. By contradiction, suppose that $(f_\alpha, \sigma)_{\alpha \in I}$ is a lexicographic semiorder which induces c. Let $i, j \in X$ be such that $f_1(j) > f_1(i) + \sigma$. Such an i and j exists w.l.o.g., possibly by renumbering the f_α so that f_1 is the first f_α for which $f_1(k') > f_1(k) + \sigma$ for some $k, k' \in X$. Also, note that $i \neq 1$ since the application of the rationales yields $c(\{1, 2, .., l\}) = 1$ for all $l \in X$. It must be $j = i-1$ (that is, i is eliminated by $i-1$ in the first step in any set that contains both of them). Otherwise suppose first that $j > i$. Then $c(\{i, i+1, i+2, ..., j\}) = i$ would be contradicted by $i \notin M_1(\{i, i+1, i+2, .., j\})$. Alternatively, suppose that $j < i-1$. Then $c(\{j, i\}) = i$ would be contradicted by $i \notin M_1(\{j, i\})$.

Thus, $f_1(i-1) > f_1(i) + \sigma$. Since $c(\{i-1, i+1\}) = i+1$, it must be that, letting n be the first α for which $M_\alpha(\{i-1, i+1\}) \neq \{i-1, i+1\}$, we have $f_n(i+1) > f_n(i-1) + \sigma$.

Applying this fact to $S = \{i-1, i, i+1\}$, we have that either (if $n = 1$) $M_1(S) = \{i+1\}$, or (if $n > 1$) $c(S) = c(M_1(S)) = c(\{i-1, i+1\}) = i+1$. In both cases we have a contradiction with $c(S) = i-1$. ■

Some observations are in order. Apesteguia and Ballester [1] define a *simple rationale* P as a relation of the type $P = \{(x, y)\}$ for some x and y in X. That is, a simple rationale relates only one pair of alternatives. Our notion of 'basic' refers instead to the number of discriminations the agent is able to make, rather than to the number of pairs ranked by the relation (which may be high). In fact, reasonably efficient (that is, short) lists of simple semiorders that induce a cles will 'pack' together several comparisons in each semiorder (so that they will not be simple rationales). It is of course possible to express a simple rationale $P = \{(x, y)\}$ as a basic semiorder (though not vice-versa), by setting $f(x) = 1$, $f(y) = -1$ and $f(z) = 0$ for all other z (this means that, as observed above, proposition 1 could also be derived, in the case of Σ being the domain of all nonempty subsets of a finite set X, with [1]'s reasoning).

However, while the equivalence holds at a formal level, at a conceptual level there is still a distinction, in that there may be circumstances in which the lexicographic semiorder description is more accurate as a decision making procedure (as in the situations conceived byTversky). Using simple rationales instead of basic semiorders may necessitate an unrealistically large number of semiorders in a cles. There is no upper bound to the number of simple rationales needed to express a basic semiorder. For example, the rationale $P = \{(x, y) : y \in X \setminus \{x\}\}$, for a fixed x, is a single basic semiorder for any n, which is nevertheless decomposed into $(n-1)$ distinct simple rationales. In this example, where an agent simply considers that x is better than any other alternative, suppose $n = 1000$. It seems more natural to describe the agent's behaviour by expressing directly (via a semiorder) the agent's discrimination between x and anything else, rather than imagining that he proceeds lexicographically via 1000 steps to recognise that x is better, as a representation by simple rationales would require. In recent work, Mandler [17] has studied in detail the general issue of the minimum number of rationales needed to express a given arbitrary preference relation using the procedure of sequential rationalisability.[11] His main result is that a 'rational agent' (an agent with complete and transitive preferences) never needs more, and sometimes needs fewer, rationales than an 'irrational agent'.

Proposition 2 shows that the domain restriction $|X| < \infty$ of theorem B.1 of [1] is necessary. Their result establishes that, on the domain of all nonempty subsets of a finite set X, the only crucial distinction is between the asymmetry and the acyclicity (a strengthening of asymmetry) of the rationales: further strengthening acyclicity to transitivity, for example, produces no further behavioural restriction. Proposition 2 shows that on larger domains the move from acyclicity to transitivity (semiorders) crosses another important threshold: the transitivity of the agent's discriminatory power alone suffices to rule out behaviours allowed by acyclic rationales. This remains true no matter how limited that power is.

3.2 'Revealed preference' characterisation

Next,we explore directly the restrictions on observable choice data that the procedure we have proposed implies. The following axiom is formulated imagining that the decision maker may find himself choosing from subcollections $\mathcal{C}$ of the entire choice domain Σ: while over the week you express your choice on the set Σ of all restaurants which you patronize, on Sunday you can only express your choice on the collection $\mathcal{C}$ of those restaurants which do not close on that day.

Reducibility: For every $\mathcal{C} \subseteq \Sigma$, there exists $S \in \mathcal{C}$ and $x, y \in S$ such that:

$$(T \setminus \{y\}) \in \mathcal{C},\ x \in T \Rightarrow c(T) = c(T \setminus \{y\}) \text{ for all } T \in \mathcal{C}$$

A choice function which satisfies Reducibility is called *reducible.*

Reducibility refers to the following type of behaviour: independently of which restaurants are open, you simply ignore steak tartare in any restaurant which also offers pizza (though you may or may not choose pizza). Here, pizza is a negative signal about the

[11] To make sense of this assertion, observe that a binary preference relation R can be identified as the base relation of a choice function defined on all pairs of alternatives, i.e. $(x, y) \in R \Leftrightarrow x \in c(\{x, y\})$.

kitchen's sophistication, so that you are induced to ignore sophisticated items on the menu, even if you may end up not choosing the signal item itself.[12]

More abstractly, given a collection of choice sets $\mathcal{C}$, say that x *makes* y $\mathcal{C}-$*irrelevant* if x and y belong to some set in the collection, and whenever this happens, removing y from S has no effect on the final choice from S (so that, in particular y is never chosen if x is available). If x makes y $\mathcal{C}-$irrelevant, then y has no relevance for the purposes of choice whenever x is available. Reducibility requires that the $\mathcal{C}-$irrelevancy relation is nonempty.

One way of satisfying Reducibility is the existence of a 'best' alternative. If c is a choice function that maximizes an ordinary strict preference relation, an alternative which is chosen from an S in $\mathcal{C}$ trivially makes $\mathcal{C}-$irrelevant any alternative which is not chosen from S. In fact in standard theory 'irrelevant' is essentially synonymous with 'unchosen'. Therefore c is reducible in the standard case.

Reducibility relaxes the standard requirement that all rejected alternatives need to be made $\mathcal{C}$-irrelevant on all $\mathcal{C}$ (via the single preference relation) by the 'best' (chosen) alternative, and it does so in two ways. First, some rejected alternatives may not be made $\mathcal{C}$-irrelevant. And, second, an alternative may be made $\mathcal{C}$-irrelevant by some other alternative which is itself not chosen. In other words, Reducibility requires just a bare skeleton of preference to survive.

An example of a reducible non-standard choice function is the three-cycle of choice: $X = \{x, y, z\}$, $c(X) = x$, $c(\{x, y\}) = x$, $c(\{y, z\}) = y$, $c(\{x, z\}) = z$. Here y makes z $\mathcal{C}-$irrelevant when either X or $\{y, z\}$ are in $\mathcal{C}$, and Reducibility is satisfied vacuously otherwise. Observe that the choice from the grand set does not make either y or z $\mathcal{C}-$irrelevant for $\mathcal{C}$ coinciding with the full domain.

On the contrary, the reader can check that the choice function c in the proof of proposition 2 (where c is sequentially rationalisable but not cles) is not reducible. An even

[12] In this example pizza plays a symmetric role that of frog legs in the celebrated example by Luce and Raiffa [16] (a decision maker chooses steak when frog legs are on the menu and chicken when they are not). In Luce and Raiffa's example, frog legs are a positive signal about the quality of the restaurant, so that the decision maker is induced by the presence of frog legs on the menu to choose a high quality item, even if not frog legs themselves.

simpler example of a non-reducible c is given by $X = \{x, y, z\}$, $c(\{x, y\}) = c(\{x, z\}) = x$, $c(\{x, y, z\}) = y$. Letting $\mathcal{C} = \{\{x, y\}, \{x, z\}, X\}$ we have $c(X) \neq c(X \setminus \{y\})$, $c(X) \neq c(X \setminus \{z\})$ so that no alternative makes y or z $\mathcal{C}-$irrelevant. And the choices from binary sets show that no alternative makes x $\mathcal{C}-$irrelevant.

On finite domains Reducibility is easily seen to be a weakening of a standard contraction consistency axiom. Consider the following formulation of Independence of Irrelevant Alternatives:

Independence of Irrelevant Alternatives (IIA): Let $\mathcal{C} \subseteq \Sigma$. Then $c(S) = c(S \setminus \{y\})$ for all $y \in S \setminus \{c(S)\}$ for all $S \in \mathcal{C}$ such that $S \setminus \{y\} \in \mathcal{C}$.

Now consider the following weakening (where we highlight the additional conditions):

Reducibility (restated): Let $\mathcal{C} \subseteq \Sigma$. Then **for some** $x \in X$, $c(S) = c(S \setminus \{y\})$ for **some** $y \in S \setminus \{c(S)\}$ for all $S \in \mathcal{C}$ such that $S \setminus \{y\} \in \mathcal{C}$ **and** $S \ni x$.

While standard IIA requires the choice to be unchanged if *any* unchosen alternative is removed from *any* set, Reducibility requires this to hold only for *some* alternative and for *some* sets (those containing x). Because IIA is so strong, the fact that if it holds, it must hold on the entire domain Σ as well as on any subcollection $\mathcal{C}$, usually needs not to be made explicit.

Below we establish that Reducibility captures all the observable implications of the lexicographic semiorder procedure, and that basic lexicographic semiorders cover exactly the same ground as general lexicographic semiorders. This is true on domains larger than the subsets of a finite set, and therefore also on domains for which the equivalence between the sequential rationalisability and the lexicographic semiorder model fails.

Theorem 1 *Let X be countable. Let c be a choice function defined on the domain Σ of all finite subsets of X. Then the following statements are equivalent:*

(i) c is a choice by lexicographic semiorder;

(ii) c is reducible;

(iii) c is a choice by basic lexicographic semiorder.

Proof. *(i) ⇒ (ii).* Let c be induced by the lexicographic semiorder $(f_i, \sigma)_{i\in I}$, and let $\mathcal{C} \subseteq \Sigma$ be any non-trivial collection of choice sets. Let

$$j = \min\{i : M_i(S) \neq S \text{ for some } S \in \mathcal{C}\}$$

(j is well-defined because of the single valuedness of c).[13]

Let $T \in \mathcal{C}$ be such that $M_j(T) \neq T$. Fix x,$y \in T$ such that $f_j(x) > f_j(y) + \sigma$. For any $S \in \mathcal{C}$ either $\{x, y\} \nsubseteq S$, in which case Reducibility holds vacuously; or $\{x, y\} \subseteq S$. In this latter case (which holds at least for $S = T$), for any $z \in S$, if $f_j(y) > f_j(z) + \sigma$ then also $f_j(x) > f_j(z) + \sigma$. Therefore $M_j(S) = M_j(S\backslash\{y\})$, implying $c(S) = c(S\backslash\{y\})$.

(ii) ⇒ (iii). Let c be a reducible choice function on Σ. We first provide an algorithm to construct a simple lexicographic semiorder for any choice function, then show that this semiorder induces c.

The algorithm proceeds by recursively defining a sequence of collections $\{\mathcal{C}_i\}_{i\in I}$ and an associated sequence of pairs $\{x_i, y_i\}_{i\in I}$, where I is either an interval $\{0, 1, ..., n\}$ or the set of natural numbers. Let $\mathcal{C}_0 = \Sigma$, and let $x_0, y_0 \in X$ be any two alternatives such that, for all $S \in \mathcal{C}_0$, $x_0, y_0 \in S \Rightarrow c(S) = c(S\backslash\{y_0\})$ (alternatives such as x_0 and y_0 exist by Reducibility, and $S\backslash\{y_0\} \in \Sigma$ by assumption). For $0 < i$ define recursively $x_i, y_i \in X$ as any two alternatives such that $(x_i, y_i) \neq (x_j, y_j)$ for all $j < i$, and

$$\text{for all } S \in \bigcap\nolimits_{j<i} \mathcal{C}_j\text{: } x_i, y_i \in S \Rightarrow c(S) = c(S\backslash\{y_i\})$$

and

$$\mathcal{C}_i = \bigcap\nolimits_{j<i} \mathcal{C}_j \backslash \left\{S \in \bigcap\nolimits_{j<i} \mathcal{C}_j : \{x_i, y_i\} \subseteq S\right\}$$

For all i, let $f_i(x_i) = 1$, $f_i(y_i) = -1$, $f_i(z) = 0$ for all $z \in X\backslash\{x_i, y_i\}$, and $\sigma = 1$. Note that, for any i, unless $S \in \mathcal{C}_{i+1} \Rightarrow |S| = 1$ (i.e. unless $\mathcal{C}_i$ is a trivial collection), it is true by Reducibility that $\mathcal{C}_i \neq \mathcal{C}_{i+1}$. Therefore $S \in \bigcap_{i\in I} \mathcal{C}_i \Rightarrow |S| = 1$.

This defines a basic lexicographic semiorder $f = (f_i)_{i\in I}$. As we show below, f induces c. Recall the definition of the survivor sets $M_i(S)$.

Fix $S \in \Sigma$. Suppose by induction that $c(S) \in M_i(S)$. It must be that $M_i(S) \in \mathcal{C}_i$. Otherwise, there would exist $k \leq i$ such that $f_k(x_k) = 1$, $f_k(y_k) = -1$ and

[13] For choice correspondences one would change the qualifier that not all S in $\mathcal{C}$ are singletons with that that not all of them are such that $c(S) = S$.

$\{x_k, y_k\} \subseteq M_i(S) \in \mathcal{C}_k$, contradicting the definition of $M_i(S)$. If also $M_i(S) \in \mathcal{C}_{i+1}$, then $\{x_{i+1}, y_{i+1}\} \nsubseteq M_i(S)$ and so we have immediately $c(S) \in M_{i+1}(S)$. If $M_i(S) \notin \mathcal{C}_{i+1}$, then (since $M_i(S) \in \mathcal{C}_i$) it must be $\{x_{i+1}, y_{i+1}\} \subseteq S$. It cannot be $y_{i+1} = c(S)$ since, by construction of the sequence $\{x_i, y_i\}_{i \in I}$, $c(S) = c(S \setminus \{y_1\}) = ... = c(S \setminus \{y_1, .., y_{i+1}\})$. Therefore $c(S) \in M_{i+1}(S)$.

We now show that for all $s \in S \setminus \{c(S)\}$ there exists a k such that $s \notin M_k(S)$. If not, let $\bigcap_{i \in I} M_i(S) = T$, and let $s \in T$. The definition of T implies that, for all $i \in I$, $\{x_i, y_i\} \nsubseteq T$ (otherwise $x_i, y_i \in M_i(S)$, which is impossible by construction since $f_i(x_i) = 1$ and $f_i(y_i) = -1$). Therefore $T \in \bigcap_{i \in I} \mathcal{C}_i$. But this is a contradiction with $c(S) \neq s \in T$ and $c(S) \in T$, since, as observed before, $T \in \bigcap_{i \in I} \mathcal{C}_i$ implies $|T| = 1$.

(iii) $\Rightarrow$ *(i)*. Trivial. ∎

The countability restriction appearing in theorem 1 is really a product of our insistence that the agent is confined to using a realistic number of dimensions. The techniques we have used in this paper permit relatively easy generalisations of both the model of cles and the proof of theorem 1 to more abstract settings. We could replace the index set I of (a subset of) natural numbers with any well-ordered[14] set $(I, \leq)$. In this way, the definition of survivor sets could be modified using transfinite induction (analogously to what was done in Mandler, Manzini and Mariotti [18]), and the definition of cles would be automatically extended (only noticing that now j might not be finite). The proof would then go through, with obvious adaptations, to the uncountably infinite case.

4 Finite domains

Theorem 1 is in general a characterisation only of choice by lexicographic semiorder and not of sequentially rationalisable choice (a fact which follows from proposition 2 and its proof, where an example with a countable X is used). Yet, the result can naturally be used, together with proposition 1, to provide a characterization of acyclic sequential rationalisability for the special case of a finite X:

[14] A set I is *well-ordered* by $\leq$ if $\leq$ is a linear order (a complete, transitive, and antisymmetric relation) on I such that every nonempty subset of I has a least element $\inf I$ such that $\inf I \leq i$ for all $i \in I$.

Theorem 2 *Let X be finite and let Σ be the set of all nonempty subsets of X. Then a choice function on Σ is acyclic sequentially rationalisable if and only if it is reducible.*

Finally, we study the following natural question: on a finite domain, what types of behaviour can be explained by the sequential rationalisability model but not by the lexicographic semiorder model? To this aim we introduce a weakening of Reducibility:

Weak reducibility: For every $\mathcal{C} \subseteq \Sigma$, there exists $S \in \mathcal{C}$ and a collection of pairs $\{x_i, y_i\}_{i=1,2,...}$, with $x_i, y_i \in S$ for all i, such that:

$$T \backslash \bigcup_{i:x_i \in T} \{y_i\} \in \mathcal{C} \Rightarrow c(T) = c\left(T \backslash \bigcup_{i:x_i \in T} \{y_i\}\right) \text{ for all } T \in \mathcal{C}$$

A choice function that satisfies Weak reducibility is called *weakly reducible.*

The only difference between Reducibility and Weak reducibility is that in the latter the pair (x, y) has been replaced by a *collection* $\{x_i, y_i\}_{i=1,2,...}$ of pairs. In other words, compared to a reducible choice function, a choice function which is only weakly reducible is such that some alternatives which are not individually $\mathcal{C}-$irrelevant (the removal of any one of those alternatives does affect choice) may nevertheless be 'collectively' $\mathcal{C}-$irrelevant (their collective removal from a choice set has no relevance for choice).

We show that the choice functions which are sequentially rationalisable but not cles are exactly those which are only weakly reducible but not reducible.

Theorem 3 *Let X be finite and let Σ be the set of all nonempty subsets of X. Then a choice function on Σ is sequentially rationalisable if and only if it is weakly reducible.*

Proof. *Necessity.* Let c be sequentially rationalisable with rationales $P_1, ..., P_K$, and let $\mathcal{C} \subseteq \Sigma$. Let

$$j = \min\{i : M_i^*(S) \neq S \text{ for some } S \in \mathcal{C}\}$$

Let $A = \{(x, y) : x, y \in S \text{ for some } S \in \mathcal{C} \text{ and } (x, y) \in P_j\}$. A is nonempty by the definition of j. Enumerate the pairs in A to obtain $\{x_i, y_i\}_{i=1,...,n}$. It follows straightforwardly that $M_K^*(S) = M_K^*\left(S \backslash \bigcup_{i:x_i \in S} \{y_i\}\right)$ for all $S \in \mathcal{C}$. The sequential rationalisability of c thus implies that $c(S) = c\left(S \backslash \bigcup_{i:x_i \in S} \{y_i\}\right)$.

Sufficiency. Let c be weakly reducible. We construct the rationales explicitly.[15] Let $\mathcal{C}_0 = \Sigma$, and define recursively

$$\begin{aligned} P_i &= \{(x_{ji}, y_{ji})\}_{j=1,\dots,n(i)}\text{, where } \{x_{ji}, y_{ji}\}_{j=1,\dots,n(i)} \text{ is any collection of pairs such that} \\ c(S) &= c\left(S\backslash \bigcup_{j:x_{ji}\in S} \{y_{ji}\}\right) \ \forall S \in \mathcal{C}_{i-1}; \\ \mathcal{C}_i &= \{S \in \mathcal{C}_{i-1} : S = M_i^*(T) \text{ for some } T \in \mathcal{C}_{i-1}\} \end{aligned}$$

Let $K = \max\{i : P_i \neq \varnothing\}$. The P_i are well-defined for all $i = 1, ...K$ by Weak reducibility. We show that $P_1, ..., P_K$ sequentially rationalize c.

Let $x = c(S)$. Whenever $S \in \mathcal{C}_{i-1}$ for some i, it cannot be $(y, x) = P_i$, since $c(S) \neq c(S\backslash(\{x\} \cup A))$ for any $A \subseteq X$, contradicting the definition of P_i. This implies that $x \in M_i^*(S)$ for all i.

Let $y \in S\backslash\{c(S)\}$. Suppose by contradiction that $y \in M_K^*(S)$. This means that $|M_K^*(S)| \geq 2$, so that, given that $M_K^*(S) \in \mathcal{C}_K$, $\mathcal{C}_K$ is non-trivial. Therefore by Weak reducibility there exists a collection $\{x_{jK+1}, y_{jK+1}\}_{j=1,\dots,n(K+1)}$ such that

$$c(T) = c\left(T\backslash \bigcup_{j:x_{jK+1}\in T} \{y_{jK+1}\}\right) \ \forall T \in \mathcal{C}_K$$

But then $P_{K+1} \neq \varnothing$, contradicting the definition of K. ■

Theorems 2 and 3 are interesting in themselves, as Manzini and Mariotti [20] left the characterization of sequential rationalisability as an open problem.

Apesteguia and Ballester [1] have pioneered a solution to that problem, in so doing offering key insights. Their characterisation of acyclic sequential rationalisability is in terms of a condition called Independence of One Irrelevant Alternative (IOIA). To quickly sketch that condition, we need to define some auxiliary terms. A *binary selector* is a function f which associates to every feasible set S including at least two alternatives a binary feasible set in S. A binary selector f that satisfies certain consistency properties[16] is called *consistent.* Then IOIA requires that $c(S) = c(S\backslash(f(S)\backslash\{c(f(S))\}))$ for some

[15] The algorithm provided below is relatively manageable to execute. We show how in the Appendix.

[16] We refer the reader to Apesteguia and Ballester [1] for a precise statement of the definition, which requires substantial more notation extraneous to the purposes of this paper.

consistent binary selector. While this condition may appear involved, its broad logic is simple, as it essentially imposes a two-stage structure on the choice function c. This is convenient because it reduces the problem of detecting an arbitrarily long sequential structure on c to that of detecting a far simpler construction. Thus, IOIA and Reducibility, which by our results and [1]'s are equivalent conditions in the finite case, highlight different aspects of sequential rationalisability.

Nevertheless, while the results of [1] and ours settle the question for the finite case, the challenge ahead is to provide characterisations in this vein for very general domains, including for example those of standard consumer theory. This remains an open question.

5 Concluding remarks

We have focussed especially on the most minimalist version of the model we are proposing, which attributes to the agent very weak powers of discrimination (basic lexicographic semiorders). On finite domains this version is coextensive with a natural restriction of the seemingly far more general sequentially rationalisable choice model of Manzini and Mariotti [20]. On broader domains the model restricts choice data more narrowly than even a stripped down version of sequential rationalisability (Rational Shortlist Methods).

Our Reducibility condition delimits exactly the restrictions on choice behaviour that our theory implies. While we would argue that this condition (and Weak reducibility) has a more than a whiff of plausibility, we have eschewed defending it as an a priori compelling property of bounded rationality. The appeal of the theory stems from its psychological basis, its tractability and its testability. Our main aim was to work out the observable implication of the theory, in the spirit of the 'revealed preference approach' (see Caplin [3], Gul and Pesendorfer [10], Rubinstein and Salant [26] for methodological discussions of this issue). Reducibility is an easily interpretable and operationally workable concept (as demonstrated by our workouts), and as such we believe it fulfills this role. Our approach is thus in the same spirit as a recent body of work which seeks to characterise models of boundedly rational choice in terms of direct axioms on choice behaviour (e.g. Cherepanov, Feddersen and Sandroni [4]; Eliaz, Richter and Rubinstein [6]; Masatlioglu and Ok [21] and [22]; Masatlioglu and Nakajima [23]; Masatlioglu, Nakajima and Ozbay [24]; Salant and Rubinstein [27]; Tyson [32], beside those already discussed).

The present work is also related to the 'checklist' model of choice in Mandler, Manzini and Mariotti [18]. In that model, an agent goes through an ordered checklist of properties (unary relations), at each step eliminating the alternatives that do not have the specified property. For example, the agent who wishes to buy a house looks first for houses in a certain location, then for those in that location with a minimum square footage, and so on until a final selection is made. A choice by *basic* lexicographic semiorder could be interpreted as a weakening of a choice by checklist, in which the membership of a property is allowed to have three values instead of only two. On this interpretation, $f_i(x) = 1$ (resp., $f_i(x) = -1$) means that x definitely has (resp., does not have) property i, while $f_i(x) = 0$ means that x neither fully has nor fully does not have property i (it falls in a 'grey area' or 'is neutral' with respect to that property). For example, a house's location may neither be entirely convenient (e.g. close to both spouses' workplaces) nor entirely inconvenient (far from both spouses' workplaces). Because (on certain domains) choosing by checklist is exactly equivalent to maximising a utility function (as shown in Mandler, Manzini and Mariotti [18]), a choice by lexicographic semiorder can also be seen as a versatile but minimal departure from the standard model of rational choice.

6 Appendix

It is instructive to see how the algorithm to construct the rationales of theorem 3 works. We use an example provided by Apesteguia and Ballester [1]. The grand set of alternatives is $X = \{\alpha, \beta, \gamma, \delta, \varepsilon, \varphi\}$. The inverse image of the choice function (i.e. the collection of sets from which each alternative is chosen) is given below:

$$
\begin{aligned}
c^{-1}(\alpha) &= \left\{ \begin{array}{l} \{\alpha,\beta,\delta,\gamma,\varepsilon\}, \\ \{\alpha,\beta,\gamma,\varepsilon\}, \{\alpha,\beta,\delta,\gamma\}, \{\alpha,\beta,\delta,\varepsilon\}, \{\alpha,\delta,\gamma,\varepsilon\}, \\ \{\alpha,\beta,\delta\}, \{\alpha,\delta,\varepsilon\}, \{\alpha,\beta,\gamma\}, \{\alpha,\beta,\varepsilon\}, \{\alpha,\gamma,\varepsilon\}, \\ \{\alpha,\beta\}, \{\alpha,\varepsilon\}, \{\alpha,\delta\} \end{array} \right\} \\
c^{-1}(\beta) &= \left\{ \begin{array}{l} \{\beta,\delta,\gamma,\varepsilon,\varphi\}, \\ \{\beta,\delta,\gamma,\varepsilon\}, \{\beta,\delta,\varepsilon,\varphi\}, \{\beta,\gamma,\varepsilon,\varphi\}, \\ \{\beta,\delta,\gamma\}, \{\beta,\delta,\varepsilon\}, \{\beta,\gamma,\varepsilon\}, \{\beta,\varepsilon,\varphi\}, \\ \{\beta,\delta\}, \{\beta,\gamma\}, \{\beta,\varepsilon\} \end{array} \right\} \\
c^{-1}(\gamma) &= \left\{ \begin{array}{l} \{\gamma,\delta,\varepsilon,\varphi\}, \{\alpha,\gamma,\delta,\varphi\}, \\ \{\alpha,\gamma,\varphi\}, \{\alpha,\gamma,\delta\}, \{\gamma,\delta,\varepsilon\}, \{\gamma,\delta,\varphi\}, \\ \{\alpha,\gamma\}, \{\gamma,\delta\}, \{\gamma,\varphi\} \end{array} \right\} \\
c^{-1}(\delta) &= \{\{\beta,\delta,\varphi\}, \{\delta,\varepsilon,\varphi\}, \{\delta,\varepsilon\}, \{\delta,\varphi\}\} \\
c^{-1}(\varepsilon) &= \left\{ \begin{array}{l} X, \{\alpha,\beta,\gamma,\varepsilon,\varphi\}, \{\alpha,\beta,\delta,\varepsilon,\varphi\}, \{\alpha,\delta,\gamma,\varepsilon,\varphi\}, \\ \{\alpha,\beta,\varepsilon,\varphi\}, \{\alpha,\gamma,\varepsilon,\varphi\}, \{\alpha,\delta,\varepsilon,\varphi\}, \\ \{\alpha,\varepsilon,\varphi\}, \{\gamma,\varepsilon,\varphi\}, \\ \{\gamma,\varepsilon\}, \{\varepsilon,\varphi\} \end{array} \right\} \\
c^{-1}(\varphi) &= \left\{ \begin{array}{l} \{\alpha,\beta,\delta,\gamma,\varphi\}, \\ \{\alpha,\beta,\gamma,\varphi\}, \{\beta,\gamma,\delta,\varphi\}, \{\alpha,\beta,\delta,\varphi\}, \\ \{\alpha,\beta,\varphi\}, \{\beta,\gamma,\varphi\}, \{\alpha,\delta,\varphi\}, \\ \{\alpha,\varphi\}, \{\beta,\varphi\} \end{array} \right\}
\end{aligned}
$$

The 'base relation' $P_c = \{(a,b) \in X \times X : a = c(\{a,b\})\}$ is thus:

$$P_c = \left\{ \begin{array}{l} (\alpha,\beta),(\alpha,\varepsilon),(\alpha,\delta),(\delta,\varepsilon),(\delta,\varphi),(\beta,\delta),(\beta,\gamma),(\beta,\varepsilon), \\ (\gamma,\alpha),(\gamma,\delta),(\gamma,\varphi),(\varepsilon,\gamma),(\varepsilon,\varphi),(\varphi,\alpha),(\varphi,\beta) \end{array} \right\}$$

If the rationales P_i and the collections $\mathcal{C}_{i-1}$ are built according to the algorithm in the proof of theorem 3, obviously it can never be $(a,b) \in P_c \cap P_i$ for any a and b such that b is chosen from some $S \in \mathcal{C}_{i-1}$ that also contains a. Consequently we are going to construct the rationales by first ruling out as potential members of P_i all such pairs; then we will verifying whether the residual subcollection of pairs in P_c which have not yet been 'allocated' to any previous rationale P_j, $j < i$, satisfy the requirement in the Weak reducibility axiom, removing more pairs if necessary until we have the largest collection that satisfies the axiom.

Beginning with $\mathcal{C}_0 = \Sigma$, inspection of the inverse images reveals that each alternative is chosen in the presence of any other, with the exception of δ, which is never chosen in the presence of α; moreover, δ is also the only alternative such that, when it is removed from sets that also contain α, leaves choice unchanged. Consequently,

$$P_1 = \{(\alpha,\delta)\}$$

The domain thus reduces from $\mathcal{C}_0$ to $\mathcal{C}_1$ as indicated in the display that follows (simply remove all sets containing α and δ), where observe that the first line is a subcollection of $c^{-1}(\alpha)$, the second line is a subcollection of $c^{-1}(\beta)$, and so on:

$$\mathcal{C}_1 = \left\{ \begin{array}{l} \{\alpha,\beta,\gamma,\varepsilon\},\{\alpha,\beta,\gamma\},\{\alpha,\beta,\varepsilon\},\{\alpha,\gamma,\varepsilon\},\{\boldsymbol{\alpha},\boldsymbol{\beta}\},\{\alpha,\varepsilon\} \\ \{\beta,\gamma,\delta,\varepsilon,\varphi\},\{\beta,\gamma,\delta,\varepsilon\},\{\beta,\delta,\varepsilon,\varphi\},\{\beta,\gamma,\varepsilon,\varphi\}, \\ \qquad\qquad \{\beta,\gamma,\delta\},\{\beta,\delta,\varepsilon\},\{\beta,\gamma,\varepsilon\},\{\beta,\varepsilon,\varphi\},\{\beta,\delta\},\{\boldsymbol{\beta},\boldsymbol{\gamma}\},\{\beta,\varepsilon\} \\ \{\gamma,\delta,\varepsilon,\varphi\},\{\alpha,\gamma,\varphi\},\{\gamma,\delta,\varepsilon\},\{\gamma,\delta,\varphi\},\{\alpha,\gamma\},\{\boldsymbol{\gamma},\boldsymbol{\delta}\},\{\gamma,\varphi\} \\ \{\beta,\delta,\varphi\},\{\delta,\varepsilon,\varphi\},\{\boldsymbol{\delta},\boldsymbol{\varepsilon}\},\{\delta,\varphi\} \\ \{\alpha,\beta,\gamma,\varepsilon,\varphi\},\{\alpha,\beta,\varepsilon,\varphi\},\{\alpha,\gamma,\varepsilon,\varphi\},\{\alpha,\varepsilon,\varphi\},\{\gamma,\varepsilon,\varphi\},\{\gamma,\varepsilon\},\{\boldsymbol{\varepsilon},\boldsymbol{\varphi}\} \\ \{\alpha,\beta,\gamma,\varphi\},\{\beta,\gamma,\delta,\varphi\},\{\alpha,\beta,\varphi\},\{\beta,\gamma,\varphi\},\{\boldsymbol{\alpha},\boldsymbol{\varphi}\},\{\beta,\varphi\} \end{array} \right\}$$

Next, observe that α and φ are chosen in the presence of γ, so that our algorithm prescribes $(\gamma,\alpha) \notin P_2$ and $(\gamma,\varphi) \notin P_2$. Moreover, β is chosen in the presence of φ; γ is chosen in the presence of ε; δ and ε in the presence of β; ε is chosen in the presence of α;

and φ is chosen in the presence of δ. This leaves only (α,β), (β,γ), (γ,δ), (δ,ε), (ε,φ) and (φ,α) as potential members of P_2 (appearing in boldface in the above display), and it is easy to verify that indeed the whole collection of 'candidate pairs'

$$P_2=\{(\alpha,\beta),(\beta,\gamma),(\gamma,\delta),(\delta,\varepsilon),(\varepsilon,\varphi),(\varphi,\alpha)\}$$

is such that $c(S)=c\left(S\setminus\bigcup_{i:x_i\in S}y_i\right)$. Note also that Reducibility fails on the collection $\mathcal{C}_1$: no set contains α and δ, and for the same considerations contained in the previous paragraphs, the only pairs of alternatives that might satisfy Reducibility are $\{\alpha,\beta\}$, $\{\beta,\gamma\}$, $\{\gamma,\delta\}$, $\{\delta,\varepsilon\}$, $\{\varepsilon,\varphi\}$ and $\{\varphi,\alpha\}$. However, none of them does: first of all, because all these binary sets are in $\mathcal{C}_1$, the 'losing' alternative must be the one that is not chosen in pairwise sets; in addition, $x_2,y_2\neq\alpha,\beta$ since e.g. $\alpha=c(\{\alpha,\beta,\gamma\})\neq c(\{\alpha,\gamma\})=\gamma$; $x_2,y_2\neq\beta,\gamma$ since e.g. $\varphi=c(\{\beta,\gamma,\delta,\varphi\})\neq c(\{\beta,\delta,\varphi\})=\delta$; $x_2,y_2\neq\gamma,\delta$ since e.g. $\gamma=c(\{\gamma,\delta,\varepsilon,\varphi\})\neq c(\{\gamma,\varepsilon,\varphi\})=\varepsilon$; $x_2,y_2\neq\delta,\varepsilon$ since e.g. $\beta=c(\{\beta,\gamma,\delta,\varepsilon,\varphi\})\neq c(\{\beta,\gamma,\delta,\varphi\})=\varphi$; and finally $x_2,y_2\neq\varepsilon,\varphi$ since e.g. $\varepsilon=c(\{\alpha,\beta,\gamma,\varepsilon,\varphi\})\neq c(\{\alpha,\beta,\gamma,\varepsilon\})=\alpha$.

Going back to our algorithm, the construction of P_2 yields

$$\mathcal{C}_2=\left\{\begin{array}{c}\{\alpha,\gamma,\varepsilon\},\{\boldsymbol{\alpha},\boldsymbol{\varepsilon}\}\\ \{\beta,\delta\},\{\boldsymbol{\beta},\boldsymbol{\varepsilon}\}\\ \{\alpha,\gamma,\varphi\},\{\alpha,\gamma\},\{\boldsymbol{\gamma},\boldsymbol{\varphi}\}\\ \{\beta,\delta,\varphi\},\{\boldsymbol{\delta},\boldsymbol{\varphi}\}\\ \{\boldsymbol{\gamma},\boldsymbol{\varepsilon}\}\\ \{\boldsymbol{\beta},\boldsymbol{\varphi}\}\end{array}\right\}$$

For the next step, we note that δ is chosen in the presence of β; α is chosen in the presence of γ. So one can verify that all together the remaining candidate pairs provide a suitable P_3, that is:

$$P_3=\{(\alpha,\varepsilon),(\varepsilon,\gamma),(\beta,\varepsilon),(\delta,\varphi),(\varphi,\beta),(\varphi,\gamma)\}$$

As a consequence, the subdomain reduces to:

$$\mathcal{C}_3=\{\{\beta,\delta\},\{\alpha,\gamma\}\}$$

so that we can build the final rationale

$$P_4 = \{(\beta, \delta), (\gamma, \alpha)\}$$

It is straightforward to double check that P_1, P_2, P_3, P_4 so defined sequentially rationalises c.

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