Second Best Efficiency and the English Auction*

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Abstract

We characterize the incentive compatible allocation that maximizes the expected social surplus in a single-unit sale when the efficient allocation is not implementable. We then show that allowing for the possibility that the good remains unsold may increase the expected social surplus even when allocating the good to no bidder generates less social surplus than allocating to any of the bidders. Aside from this option, the English auction implements the second best allocation when there are only two bidders but not always with more than two bidders.

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1 Introduction

Suppose the sale of the right to drill for oil to two wildcatters. The first one, the incumbent, has a high marginal cost and a low fixed cost, whereas the second one, the entrant, has a low marginal cost and a high fixed cost. In this case, it may be efficient to allocate the good to the incumbent if there is little oil and to the entrant if there is much oil. However, Maskin (1992) has shown that this allocation, i.e. the *first best*, is not implementable[1] when the amount of oil is private information of the incumbent. What is then the socially optimal allocation subject to implementability, i.e. the *second best*?

This paper studies this question in a general set-up in which the first best is not implementable. We characterize the set of Bayesian incentive compatible mechanisms that maximize the expected social surplus in the sale of an indivisible unit when buyers have interdependent values. We also show that allowing for the possibility that the good remains unsold may increase the expected social surplus even when allocating the good to no bidder generates less social surplus than allocating to any of the bidders. In our analysis, however, we focus on mechanisms that always allocate the good to one of the bidders.[2]

We use our characterization to study the efficiency properties of the English auction. We show that this auction format has an equilibrium that implements the second best when there are only two bidders. Besides, this is the unique equilibrium outcome[3] when only one bidder has private information about the bidders' common value as in the wildcatter example. In this case, however, there is no equilibrium that implements the second best if there are more than two bidders.

To take a glimpse of the intuition of our results consider the wildcatter example. Since the incumbent knows the amount of oil, she knows her value. Hence, she has a unique weakly dominant strategy as in a private value auction: to stay active until her value is reached. If the incumbent plays this strategy, the entrant's payoff when winning is equal to the difference between her value and the incumbent's, i.e. it is equal to the change in social surplus. It is thus not surprising that the entrant's best response implements the second best.

The above example also illustrates that the strategic analysis of the English auction when the first

[1] In this paper, we mean by implementable that the allocation is the equilibrium outcome to some game. This notion of implementability is also called achievability. Note that it differs from the concept of full implementation. This latter concept requires that the allocation is the unique equilibrium outcome to some game.

[2] This is without loss of generality if the seller's value is sufficiently small relative to buyers's valuations. An extreme case is when not selling is not an option. To some extent this is the case of many public sector auctions.

[3] We restrict to equilibria in which bidders do not use weakly dominated strategies.

best is not implementable is more complex than otherwise. Since the entrant's value is greater than the incumbent's if and only if the latter is large enough, the entrant makes a loss if she wins at a low price but a profit if she wins at a high price. Her best response must trade off these expected losses and gains. As a consequence, the entrant may find it profitable to remain in the auction at prices at which she makes a loss when the incumbent quits, i.e. there may be ex post regret in equilibrium.

The analysis is even more complex with more than one entrant, however it shares the feature than in equilibrium entrants may remain in the auction at prices at which they make a loss if the incumbent quits and they win. This possibility explains our results for more than two bidders. The fact that the incumbent quits may prompt the remaining entrants to quit immediately to avoid losses. Under the standard tie breaking rules, this "rush" implies that the entrant with highest value does not always get the good and the application of our characterization shows that this is incompatible with second best efficiency.

This source of inefficiencies in English auctions is very different from the one already pointed out by Krishna (2003) for the case in which the first best is implementable with other mechanisms. Indeed, we do not expect that the modifications of the English auction used to recover efficiency proposed by Perry and Reny (2002) and Izmalkov (2003) work in our setup. Hence, they should be revised and amended when second best efficiency is a concern.

The wildcatter example is special in that only one bidder has private information about the bidders' common value. Under more general assumptions, the usual problem of multiplicity of equilibria of the English auctions is also a concern here. Consider the following well-known example: two bidders with value functions $v_i(s) = s_i + 2s_j$ where $s_i \in [0, 1]$ is the private type of Bidder i. The symmetric equilibrium of the English auction implements the worst possible allocation: both bidders use the same bid function $b(s_i) = 3s_i$ and hence, the bidder with larger type, and thus lower value, wins the auction. Our analysis shows that focusing on the symmetric equilibrium may be misleading as there are other equilibria that implement the second best and there is no natural refinement to single out an equilibrium.[4]

The rest of the paper is organized as follows. The related literature is in Section 2. We define the formal set-up in Section 3. In Section 4, we study the implementation of the first best allocation.

[4] Chung and Ely (2001) have studied dominance solvability in auctions when the first best is implementable and a single crossing condition is satisfied. With only two bidders their results select the efficient equilibrium of the English auction. This is the symmetric equilibrium when bidders are symmetric. Their results, however, do not extend to our context.

Section 5 includes some motivating examples in which the first best is not implementable. The second best efficient allocation is characterized in Section 6. We discuss the possibility that the good remains unsold in Section 7. Section 8 discusses the implementability of the second best through an English Auction and Section 9 concludes. We include two appendixes: Appendix A with the most technical proofs and Appendix B with an extension of our model to multidimensional types.

2 Related Literature

Most of the papers that study the set of auction mechanisms that maximize the expected social surplus subject to the buyers' incentive compatibility constraints differ from ours in that they assume conditions that guarantee that the incentive compatibility constraints are not binding. This is for instance the case of Vickrey (1961), Krishna and Perry (1998), and Williams (1999), and most of the analysis of Maskin (1992, 2000), and Dasgupta and Maskin (2000).

Maskin (1992, 2000), Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), and Eso and Maskin (2000) also consider the case in which the first best is not implementable. But their results hinge on the assumption that bidders have multidimensional private information. They argue that in this case an implementable allocation cannot depend on the type beyond a particular one-dimensional reduction. The first best is usually not implementable because it requires conditioning on more information than this one-dimensional reduction. Eso and Maskin (2000) define in this setup the constraint efficient allocation. This is the allocation that maximizes expected social surplus when we can only condition the allocation on the former one-dimensional reduction.

Although we assume for our main results a one-dimensional type space, we note in Appendix B that our results may be used in the efficiency analysis based on the one-dimensional reduced types. The reason is that our examples in Section 5 suggests that there are no general arguments that ensure that the one-dimensional reduction verifies the conditions required for the constrained efficient allocation to be implementable.

Another related branch of the literature, in particular Maskin (1992), Krishna (2003), Birulin and Izmalkov (2003), Dubra, Echenique, and Manelli (2008), and Izmalkov (2003), analyzes whether there is an equilibrium of the English auction that allocates the good efficiently when the efficient allocation is implementable. They show that the answer, as in our model, depends on whether the number of bidders is equal to two. The logic, however, differs as our results are due to the particular features of the second best allocation. In fact, the English auction in our setup has an efficient equilibrium when the first best allocation is implementable, regardless of the number of bidders.

On the technical side, our work is related to Mussa and Rosen (1978) and Myerson (1981). They analyze the allocation that maximizes the expected profits using a technique called *ironing*. We use this technique to characterize the allocation that maximizes the expected social surplus. In a recent paper, Boone and Goeree (2008) have used a simplified version of the ironing technique in an environment closely related to our motivating example in Section 5.2. Their focus, as in Myerson (1981), is on the revenue maximizing auction rather than on the maximum expected social surplus.

The problem of second best efficiency has also received attention in the context of two parties that bargain with asymmetric information, see Myerson and Satterthwaite (1983). The difference is that in their setup withdrawing the individual rationality constraints always makes the first best implementable, whereas this is not the case in our setup. In fact, we consider the usual auction environment in which the individual rationality constraints can be trivially met and it is only the incentive compatibility constraints that may be binding.

3 The Model

One unit of an indivisible good is put up for sale to a set $N \equiv \{1, 2, ..., n\}$ of n bidders. Let $s = (s_1, ..., s_n) \in \mathbb{R}^n$ be a vector where s_i corresponds to the realization of an independent random variable with distribution F_i and with a strictly positive density[5] in a bounded support $S_i \subset \mathbb{R}$. Bidder $i \in N$ observes privately s_i and gets a von Neumann-Morgenstern utility $v_i(s) - p$ if she gets the good for sale at price p, and utility $-e_j(s_j) - p$ if Bidder j, $j \neq i$, gets the good and i pays a price p. Thus, e_j denotes a negative externality[6] produced by j on each of the other bidders.

We assume additive separability of the bidders' value functions plus a symmetry assumption on the common value component. Formally,[7] $v_i(s) = t_i(s_i) + \sum_{i \in N} q_j(s_j)$ for any $i \in N$, where $t_i(s_i)$ (t_i stands for taste) is the private value and $\sum_{i \in N} q_j(s_j)$ (q_j stands for quality) is the common value. We also assume that t_i, q_i and e_i are bounded, that $v_i(s)$ is a strictly increasing function of s_i, i.e. that

[5]Monteiro and Svaiter (2009) have recently shown how to extend Myerson's (1981) analysis to general distribution functions. Skreta (2007) also discusses this generalization in some detail.

[6]Note that we also allow for $e_j(s_j) < 0$ and thus for positive externalities.

[7]In the text, we usually give as primitives the v_i's functions for simplicity. A simple way to recover the t_i's and q_i's from the v_i's when $S = [0, 1]^n$ is as follows: $t_i(s_i) = v_i(0, ..., 0, s_i, 0, ..., 0) - v_j(0, ..., 0, s_i, 0, ..., 0) + v_j(0)$, for a $j \neq i$, and $q_i(s_i) = v_j(0, ..., 0, s_i, 0, ..., 0) - v_j(0)$. The functions t_i and q_i deduced in this way correspond to the normalization that $q_i(0) = 0$ for all i.

$\phi_i(s_i) \equiv t_i(s_i) + q_i(s_i)$ is strictly increasing, and that $h_i(s_i) \equiv t_i(s_i) - (n-1)e_i(s_i)$ is measurable and at any point either right or left continuous.

Additive separability and independency of types are usual assumptions in Bayesian implementation. The former assumption is required as otherwise the set of Bayesian implementable allocations does not have a natural-tractable characterization. The role of the latter assumption is to avoid the optimality of sophisticated mechanisms *a la* Cremer and McLean (1985, 1988) that trivialize the implementation problem in an unrealistic way. An alternative approach considered in the literature is the study of ex post implementation. We discuss it in the Conclusions.

Although the independency assumption is usually unrealistic when there are common values, note that this is not the case if only one bidder has private information about the common value. This is the case of our motivating examples, see Section 5. There are other real life examples with common values in which the independency assumption is reasonable, see Bergemann and Välimäki (2002).

Our symmetry assumption simplifies the characterization of the second best allocation. This assumption is without loss of generality with only two bidders. The general case requires a more complex approach. This assumption (together with additive separability) also implies that under the assumptions of Section 8, the English auction implements the first best whenever it is implementable and there are no externalities.[8]

4 Feasible Allocations and First Best Efficiency

Let an *allocation* be a measurable function $p : S \to [0,1]^n$, where $S \equiv \prod_{i=1}^n S_i$, such that $\sum_{i=1}^n p_i(s) = 1$ for any $s \in S$, where $p_i(s)$ denotes the probability that the good is allocated to i when the vector of types is $s \in S$. Note that we do not allow for the possibility that the good remains unsold. This is a common assumption in the papers that study the efficiency of the English auction, for instance Maskin (2000), Krishna (2003), and Birulin and Izmalkov (2003). We show in Section 7 that one reason is that there is very little hope except in very special cases that the English auction is efficient when the efficient allocation requires no selling for some vector of types.

We are interested in the set of allocations that can be implemented. By the revelation principle, there is no loss of generality in restricting to direct mechanisms. A *direct mechanism* is a pair of

[8]This is because under our symmetry and additive separability assumptions the single crossing condition, see below, implies the average crossing condition and the cyclical crossing condition of Krishna (2003), and the generalized single crossing condition of Birulin and Izmalkov (2003).

measurable functions (p, x) where p is an allocation and $x : S \to \mathbb{R}^n$ a payment function. In the direct mechanism (p, x), each bidder announces a type, and $p_i(s)$ denotes the probability that i gets the good and $x_i(s)$ her transfers to the auctioneer when the vector of announced types is $s \in S$.

The expected utility of Bidder i with type s_i who reports s_i' when all the other bidders report truthfully is equal to:

$$U_i(s_i, s_i') \equiv Q_i(s_i', p)\phi_i(s_i) + \Psi_i(s_i', p, x),$$

where[9]

$$Q_i(s_i', p) \equiv \int_{S_{-i}} p_i(s_i', s_{-i}) f_{-i}(s_{-i})\, ds_{-i},$$

and,

$$\Psi_i(s_i', p, x) \equiv \int_{S_{-i}} \left(\sum_{j \neq i} \left(p_i(s_i', s_{-i}) q_j(s_j) - e_j(s_j) p_j(s_i', s_{-i}) \right) - x_i(s_i', s_{-i}) \right) f_{-i}(s_{-i})\, ds_{-i},$$

for $S_{-i} \equiv \prod_{j \neq i} S_j$ and $f_{-i}(s_{-i}) \equiv \prod_{j \neq i} f_j(s_j)$.

Thus, we say that an allocation p is *feasible* if there exists a direct mechanism (p, x) that satisfies the following Bayesian incentive compatibility constraint:[10]

$$U_i(s_i, s_i) = \sup_{s_i' \in S_i} \{U_i(s_i, s_i')\},$$

for all $s_i \in S_i$ and $i \in N$.

The following lemma characterizes the feasible allocation using a standard argument in mechanism design, see for instance Myerson (1981), Rochet (1985) and McAfee and McMillan (1988):

Lemma 1. *An allocation p is feasible if and only if $Q_i(s_i, p)$ is weakly increasing in s_i for all $s_i \in [0, 1]$ and $i \in N$.*

See proof in the Appendix.

We use the following natural definition:

[9] With some abuse of notation, we denote by $p_i(s_i, s_{-i})$ and $p_j(s_i, s_{-i})$ the function p_i and p_j, respectively, evaluated at a vector whose l-th component is equal to the l-th component of s_{-i} if $l < i$, it is equal to s_i if $l = i$ and it is equal to the $l - 1$-th component of s_{-i} if $l > i$. We adopt the same convention for $x_i(s_i, s_{-i})$.

[10] We do not impose individual rationality constraints because they are trivially satisfied in our set-up. For instance, the mechanism proposed in Lemma 1 verifies that all bidders' types get non-negative utility.

Definition: We say that an allocation p is *first best efficient* when $\forall s \in S$, $p_i(s) > 0$ only if:

$$v_i(s) - (n-1)e_i(s_i) = \max\{v_j(s) - (n-1)e_j(s_j)\}_{j=1}^n,$$

or equivalently,

$$h_i(s_i) = \max\{h_j(s_j)\}_{j=1}^n$$

We adapt the following definition to our framework:[11]

Definition: We say that the *single crossing condition* is satisfied for bidder i if,

$$v_i(s) - (n-1)e_i(s_i) > \max\{v_j(s) - (n-1)e_j(s_j)\}_{j\neq i}$$

implies that

$$v_i(s') - (n-1)e_i(s'_i) \geq \max\{v_j(s') - (n-1)e_j(s'_j)\}_{j\neq i},$$

for any $s, s' \in S$ such that $s'_i > s_i$ and $s_j = s'_j$ for $j \neq i$.

The interpretation of the single crossing condition is that if it is (first best) efficient to allocate to Bidder i for some signal profile, it cannot be the case that increasing Bidder i's type (keeping the other types constant) makes it efficient to allocate to Bidder $j \neq i$.

Our additive separability and symmetry assumptions allow for a condition simpler to check in applications.

Lemma 2. *The single crossing condition for Bidder i is satisfied if and only if for any $s_i, s'_i \in S_i$ such that $s'_i > s_i$, the set,*

$$\{s_{-i} \in S_{-i} : \max\{h_j(s_j)\}_{j\neq i} \in (h_i(s'_i), h_i(s_i))\}$$

is empty.

[11]The single crossing condition usually corresponds to the following alternative condition:

$$v_i(s) - (n-1)e_i(s_i) \geq \max\{v_j(s) - (n-1)e_j(s_j)\}_{j\neq i}$$

implies that

$$v_i(s') - (n-1)e_i(s'_i) > \max\{v_j(s') - (n-1)e_j(s'_j)\}_{j\neq i},$$

for any $s'_i > s_i$ and $s'_j = s_j$ for $j \neq i$.

This alternative condition is sufficient for feasibility of the first best. If we add differentiability, it also implies the single crossing condition of Dasgupta and Maskin (2000). We have used instead our definition to get also a necessary condition.

See proof in the Appendix.

Note that the condition in Lemma 2 basically says that $h_i(.)$ must be an increasing function at any point at which a local variation changes the identity of the most efficient bidder.

Proposition 1. *A necessary and sufficient condition for the first best to be feasible is that the single crossing condition is satisfied for all bidders.*

See proof in the Appendix.

Intuitively, if the single crossing condition fails, the first best allocation requires that we move away the allocation of the object from Bidder i to some other bidder as we increase Bidder i's type. This implies that Bidder i's probability of winning conditional on her type must decrease at some point violating the conditions in Lemma 1.

That a version of our single crossing condition is sufficient for feasibility of the first best is well known. The necessary part is a consequence of the additive structure of our model. Dasgupta and Maskin (2000) have also proved that a single crossing condition is necessary for a more demanding definition of feasibility of the first best.

5 Economic Applications

Consider the following examples in which the single crossing condition typically fails.

5.1 An Incumbent's Model

This model formalizes a version of the wildcatters' example mentioned in the Introduction. Suppose the sale of a license to become a monopolist in a market with an inverse demand function $P(Q) = 1 - \frac{Q}{s_1}$. Suppose there is a set N of firms interested in the license. Firm $1 \in N$ is an incumbent that has zero set-up costs to start to operate the license and a constant marginal cost c_1. The other firms are potential entrants. They incur in a set-up cost to start operating the license. We denote by $-s_i$, $i \neq 1$ the set-up cost of Firm i. We assume that all the entrants have the same marginal cost c. We also assume that $c < c_1 < 1$. Thus, as in the example of the Introduction, the incumbent has the lowest fixed cost but the highest marginal cost.

We assume that s_1 is the realization of a random variable with a distribution function F_1 and a density in the support $[\underline{s}, \overline{s}]$, $0 < \underline{s} < \overline{s}$. We also assume that each s_i, $i \neq 1$, is the realization of a random variable with a distribution function F_i and a density in the support $[-\underline{s}\frac{(1-c)^2}{4}, 0]$. The lower bound of the support implies that an entrant always finds it profitable to buy the license at zero price, whereas the upper bound ensures that there is an entrants' type that values the license more than any type of the incumbent. Finally, we assume that all the above random variables are independent, and that s_i is private information of Firm i.

Proposition 2. *In the model of this section, the first best is not feasible.*

See proof in the Appendix.

The intuition is that the single crossing condition is not verified since an increase in the incumbent's type, increases her value less than the entrants'.

5.2 An Insider's Model

Suppose the sale of a painting to a set N of risk neutral bidders. The painting may be an original painting of a well-known (and priced) artist or a fake. Bidder i puts a value on the painting of $\tau_i + \rho$ if the painting is original and otherwise a value of τ_i. We assume that each τ_i is equal to an independent draw of a random variable with a distribution function G_i and a density in the support $[\underline{t}, \overline{t}]$. We assume that τ_i is private information of Bidder i. One of the bidders, Bidder 1, is an expert art dealer and she is the only one knowing whether the painting is original. The other bidders only know that the ex ante probability that the picture is original is equal to $\alpha \in (0, 1)$.

We also assume that $\rho + \underline{t} > \overline{t}$. This assumption means that Bidder 1's multidimensional type can be mapped into a one dimensional type without losing information. Our results can be extended to $\rho + \underline{t} \leq \overline{t}$, but they require the framework of models with multidimensional types of Appendix B.

Proposition 3. *In the model of this section, the first best is not feasible.*

See proof in the Appendix.

The reason is that the single crossing condition for Bidder 1 is not verified. This is because an increase in Bidder 1's value from $\overline{t}$ to $\rho + \underline{t}$ implies that the painting is not a fake but an original.

Hence, the other bidder's values increases by ρ, which is larger than the increase in Bidder 1's value $\rho + \underline{t} - \bar{t}$.

5.3 A Model with Negative Externalities

Suppose n local markets, each with a unit mass of consumers with reservation value 1 for the consumption of the good. Suppose also a set N of n firms and n local markets. Each firm starts with a branch in one of the local markets. Initially, no two firms have a branch at the same local market. Firms can open new branches at a fixed cost $C < 1$ and serve any local market in which they have a branch at a marginal cost $c < 1$.

Suppose that a seller puts up for sale a new technology that reduces the marginal costs of Firm i by an amount s_i. Suppose that s_i is drawn from an independent distribution F_i with support $[0, c]$. If only one firm serves a market, its profits are equal to 1 minus the marginal cost. When more than one firm serves a local market, we assume an outcome consistent with Bertrand competition: the firm with the lowest marginal cost serves the market at a price equal to the second lowest marginal cost.[12]

As a consequence, a firm finds it profitable to open a branch in any of the other markets if and only if she gets the new technology and the reduction in the marginal cost is sufficiently large, in particular, $s_i > C$.

Proposition 4. *In the model of this section, the first best is not feasible.*

See proof in the Appendix.

The reason is that the single crossing condition for Firm i is not verified. To see why, note that Firm i does not create any externality if its cost reduction is less than C, but it causes a negative externality on each of the other firms equal to $1 - c$, otherwise. This means that for cost reductions of Firm i sufficiently close to C but less than C, the first best may allocate to Firm i, but it may allocate to some other firm for a greater cost reduction.

6 Second Best Efficiency

In light of Proposition 1, it is natural to define second best efficiency.

[12] In case of more than one firm with the lowest marginal cost, we assume that they split equally the demand at a price equal to their common marginal cost.

Definition: We say that an allocation p is *second best efficient* if it is feasible and it maximizes:

$$\int_S \sum_{i=1}^{n} \left(v_i(s) - (n-1)e_i(s_i)\right) p_i(s) f(s)\, ds,$$

or equivalently,

$$\int_S \sum_{i=1}^{n} h_i(s_i) p_i(s) f(s)\, ds,$$

where $f(s) \equiv \prod_{i \in N} f_i(s_i)$

Certainly, the set of second best allocations includes the first best allocation when the single crossing condition is satisfied.

To simplify the notation, assume without loss of generality that F_i is uniform on $[0,1]$.[13]

Recall that it is first best efficient to allocate according to $h_i(s_i)$. However, this allocation is not implementable when h_i is not increasing. We next show how to derive from the h_i functions some functions that we denote by g_i that are increasing and that determine the second best allocation like the h_i's determine the first best.

Let $H_i(s_i) \equiv \int_0^{s_i} h_i(\tilde{s}_i)\, d\tilde{s}_i$ for all $i \in n$ and $s_i \in [0,1]$, and let $G_i(s_i) : [0,1] \to \mathbb{R}$ be the convex hull of the function H_i (i.e. the highest convex function on $[0,1]$ such that $G_i(s_i) \leq H_i(s_i)$ for all $s_i \in [0,1]$.) Formally:[14]

$$G_i(s_i) = \min \left\{ wH_i(r_1) + (1-w)H_i(r_2) : w, r_1, r_2 \in [0,1] \text{ and } wr_1 + (1-w)r_2 = s_i \right\}.$$

Next lemma summarizes some properties of G_i, see Section 6 in Myerson (1981).

Lemma 3. *Properties of G_i:*

(a) G_i is convex.

(b) $G_i(0) = H_i(0)$ and $G_i(1) = H_i(1)$.

(c) $G_i(s_i) \leq H_i(s_i)$ for all $s_i \in [0,1]$.

[13] Lehmann (1988) already showed that there is no loss of generality in assuming that signals have a uniform marginal distribution. To see why, suppose that the F_i's were not uniform. Then, we could define a new vector of signals $\tilde{s}_i \equiv F_i(s_i)$ and value functions $\tilde{v}_i(\tilde{s}) \equiv \tilde{t}_i(\tilde{s}_i) + \sum_{i \in N} \tilde{q}_j(\tilde{s}_j)$ and $\tilde{e}_j(\tilde{s}_j)$ where $\tilde{t}_i(\tilde{s}_i) \equiv t_i(F_i^{-1}(\tilde{s}_i))$, $\tilde{q}_j(\tilde{s}_j) \equiv q_j(F_j^{-1}(\tilde{s}_j))$ and $\tilde{e}_j(\tilde{s}_j) \equiv e_j(F_j^{-1}(\tilde{s}_j))$, for $F^{-1}(z) \equiv \min\{s_i \in [\underline{s}, \overline{s}] : F(s) \geq z\}$. To see that each of the new signals $\tilde{s}_i$'s has a uniform distribution on $[0,1]$, note that the probability of $\{\tilde{s}_i \leq z\}$ for $z \in [0,1]$ is equal to the probability of $\{F_i(s_i) \leq z\}$, which is equal to the probability of $\{s_i \leq F_i^{-1}(z)\}$ and thus, it is equal to $F_i(F_i^{-1}(z)) = z$.

[14] See also Rockafellar (1970), Pag. 36.

(d) If $G_i(s_i) < H_i(s_i)$ in an open interval, then G_i is linear in the same open interval.

As a convex function G_i is differentiable except at countably many points, and its derivative is a non-decreasing function. We define $g_i : [0,1] \to \mathbb{R}$ to be the differential of G_i completed by right-continuity.

Note that when h_i is an increasing function then $g_i = h_i$, but this is not the case when h_i is decreasing in some interval. Suppose, for instance, that $h_i(s_i) = \beta - s_i$. Then $H_i(s_i) = \beta s_i - s_i^2/2$, and since it is concave, its convex hull is simply a straight line connecting $(0, H_i(0))$ and $(1, H_i(1))$, i.e. $G_i(s_i) = (\beta - \frac{1}{2})s_1$. Thus, $g_i(s_i) = (\beta - \frac{1}{2})$. Note that in this case g_i is in fact the average value of h_i in $[0,1]$.

More generally, the function g_i is equal to h_i except in some intervals around the points at which h_i is not increasing. In these intervals, g_i takes the average value of h_i in the interval, i.e. the h_i function is "ironed out" in these intervals. The following example illustrates this point: $h_i(s_i) = 2s_i$ if $s_i < 1/2$, and $h_i(s_i) = 2s_i - 1$ otherwise. It can be shown after some algebra that $g_i(s_i) = h_i(s_i) = 2s_i$ if $s_i < 1/4$, $g_i(s_i) = 1/2$ (i.e. the average value of h_i in $[1/4, 3/4)$) if $s_i \in [1/4, 3/4)$ and $g_i(s_i) = h_i(s_i) = 2s_i - 1$ if $s_i \geq 3/4$, see Figure 1.[15]

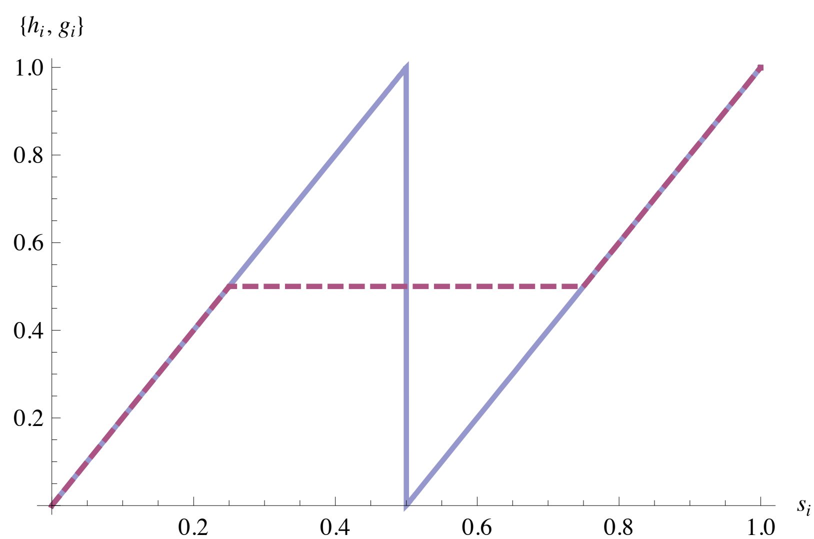

Figure 1: Ironing: the functions h_i and g_i (dashed) when h_i is not increasing.

[15] Mussa and Rosen (1978), pp. 313-314, provide a similar illustration for the case of a price discriminating monopolist that faces a non-monotonic marginal revenue.

Proposition 5. *A feasible allocation p^* is second best efficient if and only if it maximizes:*[16]

$$\int_S \sum_{i=1}^{n} \left(g_i(s_i) + \sum_{j=1}^{n} q_j(s_j) \right) p_i(s)\, ds + \sum_{i=1}^{n} \int_{S_i} \left(G_i(s_i) - H_i(s_i)\right)\, dQ_i(s_i, p). \tag{1}$$

An allocation p^* maximizes the above expression when $\forall i \in N$:

(i) $p_i^*(s) > 0$ only if $g_i(s_i) = \max\{g_j(s_j)\}_{j \in N}$ a.e.

(ii) $Q_i(., p^*)$ is constant in any open interval in which $G_i(s_i) < H_i(s_i)$.

See proof in the Appendix.

Note that an allocation that verifies condition (i) maximizes the first integral in Equation (1), and if it verifies condition (ii), it also maximizes the second integral. To see the latter, recall that by Lemma 3(c), the second integral is non-positive, whereas condition (ii) implies that it is zero.

We next illustrate the proposition with an example with two bidders.[17]

Example 1. *$N = \{1, 2\}$, $v_1(s) = \beta + s_1$, $v_2(s) = \gamma + 2s_1$ and $e_i(s_i) = 0$ for all i, where $\beta, \gamma \geq 0$ and $\beta - \gamma \in (0, 1)$.*

It is straightforward that $h_1(s_1) = \beta - s_1$ and $h_2(s_2) = \gamma$, and hence the first best allocation is to give the good to Bidder 1 if $\beta - s_1 > \gamma$ and otherwise to Bidder 2. This is not feasible since it implies that the probability that Bidder 1 gets the good is decreasing in her type and thus the feasibility condition of Lemma 1 is not met. Note that $g_1(s_1) = \beta - 1/2$, and $g_2(s_2) = \gamma$. Thus, by application of Proposition 5, the second best is to allocate to Bidder 1 if $\beta - 1/2 > \gamma$ and to allocate to Bidder 2 if $\beta - 1/2 < \gamma$.[18]

To understand why this allocation is second best efficient note that there are only two allocations that can potentially be second best: to allocate the good to Bidder 1 for any s_1, or to allocate the

[16] We denote by $\int_E \varphi(x) dF(x)$ the Lebesgue-Stieljes integral of φ with respect to F in E. In particular, for any feasible allocation p, we denote by $\int_{S_i} \varphi(s_i) dQ_i(s_i, p)$ the Lebesgue-Stieljes integral of φ with respect to $Q_i(., p)$ in S_i.

[17] To make it simpler, in our example only Bidder 1 has private information, or equivalently, $v_1(s)$ and $v_2(s)$ are constant with respect to s_2. Although this departs from our general assumptions, the only difference is that the incentive compatibility constraints for Bidder 2 are trivially satisfied and thus the conditions of Lemma 1 only need to hold for Bidder 1.

[18] Note that in the case $\beta - 1/2 = \gamma$ the second best only requires that $Q_1(s_1, p^*)$ is constant in s_1 in the open interval $(0, 1)$.

good to Bidder 2 for any s_1. The reason for this is that the greater s_1 is, the less desirable from an efficient point of view is to allocate to Bidder 1, whereas the feasibility condition requires that if we allocate to Bidder 1 for some type s_1 with some probability we must allocate to Bidder 1 with greater probability for higher types. Between these two candidates, the former is second best if Bidder 1 has greater value than Bidder 2 on average, i.e if $\beta - 1/2 > \gamma$, whereas the latter is second best otherwise. This is precisely what our application of Proposition 5 says.

7 Second Best when the Good may Remain Unsold

In this section, we depart from one assumption that we use in the rest of the paper, that is that the good is always allocated to one of the bidders. We start with an example that illustrates that the English auction is not in general first best efficient when the first best requires allocating to no bidder for some vector of types.

Example 2. *$N = \{1, 2\}$, $v_1(s) = v_2(s) = s_1 + s_2$ and $e_i(s_i) = 0$ for all i, and the social surplus of allocating to no bidder is equal to $1/2$.*

It is easy to see that the efficient allocation of the example requires that the good is allocated to no bidder if $s_1 + s_2 < 1/2$. This cannot be implemented in an English auction with an entry fee and/or a reserve price. The reason is that these instruments induce an entry equilibrium in which bidders use threshold strategies. As a consequence, the good is allocated to no bidder for sets of types $\{(s_1, s_2) \in [0, 1]^2 : s_1 \le \underline{s}_1, s_2 \le \underline{s}_2\}$ for some $\underline{s}_i \in [0, 1]$.

Formally, in this section we assume that an allocation must verify the condition that for any $s \in [0, 1]^n$, $\sum_i p_i(s) \le 1$ rather than the more restrictive condition that we assume in the rest of the paper that $\sum_i p_i(s) = 1$. Note that the probability with which the good remains unsold is equal to $1 - \sum_i p_i(s)$. We only consider the case in which the social surplus of not allocating the good to any bidder is constant and normalize it to be zero.

To distinguish from the analysis in the remaining of the paper, we refer to the allocation that maximizes social surplus when the good may remain unsold, as second best efficient with no selling. Next proposition gives sufficient conditions under which allowing for the good to remain unsold does not improve expected social surplus.

Proposition 6. *If $\max_{i \in N} g_i(s_i) + \sum_{j=1}^{n} q_j(s) \ge 0$, then any second best efficient allocation is also second best efficient with no selling.*

Proof. The proof is direct. The condition of the proposition implies that the proof of Proposition 5 does not require that the good is always allocated to one bidder. ■

The condition ensures that the set of allocations that maximizes the first integral in Equation (1) always allocate the good to one bidder. To interpret it recall that the social surplus of allocating to i is equal to $v_i(s) - (n-1)e_i(s_i) = h_i(s_i) + \sum_{j=1}^{n} q_j(s)$ and that g_i is a version of h_i in which the non-monotone parts of h_i are iron-out by taking mean values. Note that this means that when h_i is weakly increasing for one bidder and the social value of allocating the good to this bidder is greater than the social value of allocating to no bidder, i.e. $v_i(s) - (n-1)e_i(s_i) \geq 0$, $\forall s_i$, the condition in Proposition 6 is verified. This is the case in the examples of Sections 5.1 and 5.2.

We next show by means of an example that the possibility that the good remains unsold may increase the expected social surplus even when allocating the good to no bidder generates less social surplus than allocating to any of the bidders.

Example 3. *$N = \{1,2\}$, $v_i(s) = s_i + 2s_j$ and $e_i(s_i) = 0$ for $i, j \in \{1,2\}$ and $i \neq j$.*

In this example, $h_i(s_i) = -s_i$, $H_i(s_i) = -\frac{s_i^2}{2}$, and thus, $G_i(s_i) = -\frac{s_i}{2}$ and $g_i(s_i) = -\frac{1}{2}$. Hence, the corresponding Equation (1) to Example 3 is:

$$\int_{[0,1]^2} \sum_{i=1,2} \left(-\frac{1}{2} + 2s_1 + 2s_2\right) p_i(s)\, ds + \sum_{i=1,2} \int_0^1 \left(-\frac{s_i}{2} + \frac{s_i^2}{2}\right) dQ_i(s_i, p). \tag{2}$$

Thus, maximizing the first integral requires that a.e. $p_1(s) + p_2(s) = 0$ for $s_1 + s_2 \leq 1/4$ and $p_1(s) + p_2(s) = 1$ otherwise, and maximizing the second integral requires that $Q_i(., p)$ is constant in $(0,1)$ for $i = 1,2$. The allocation described in Figure 2 satisfies both conditions[19] and it is therefore second best efficient with no selling.

In the above example it is (first best) efficient to allocate the good to Bidder 2 when her signal is low and Bidder 1's signal is high. Similarly, it is efficient to allocate the good to Bidder 1 when her signal is low and Bidder 2's signal is high. But this allocation is difficult to implement because both bidders have very little incentives to report truthfully when their signal is high. In fact, if we do not allow for not selling, we cannot do better than ignoring bidders' signals and allocate the object with equal probability between the bidders. This may be easily shown applying Proposition 5.

The allocation in Figure 2 differs from the random allocation described in the paragraph above in that the former does not allocate the good to any bidder in the triangle in the lower-left corner and

[19] Note that for this allocation $Q_1(s_1, p) = Q_2(s_2, p) = \frac{31}{64}$ for any $s_1, s_2 \in [0,1]$.

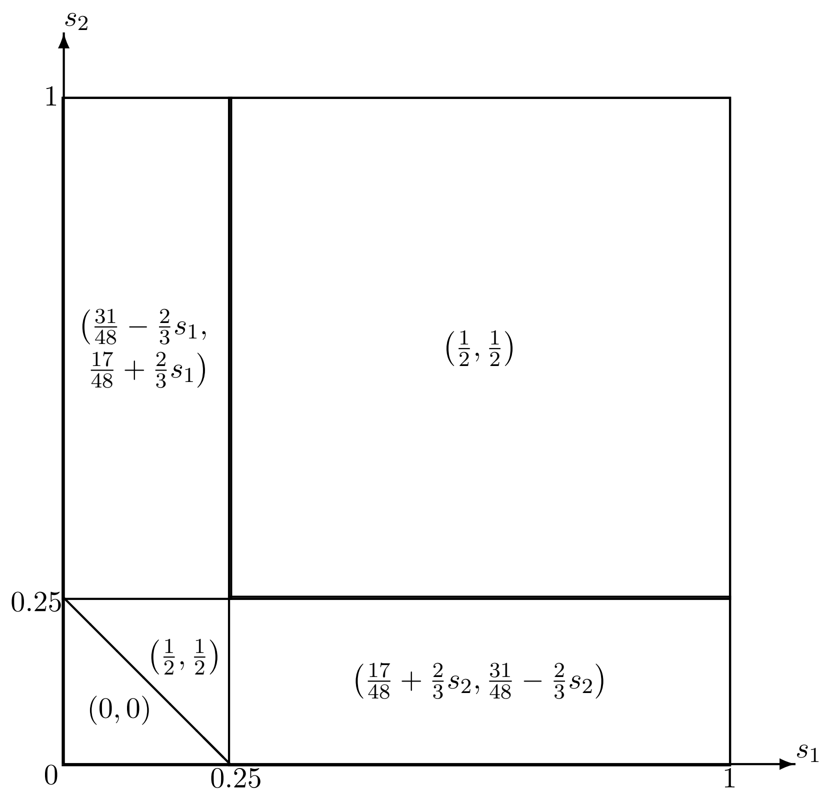

Figure 2: Second best allocation $(p_1(s), p_2(s))$ for Example 3

that it allocates more often to Bidder 2 in the rectangle in the lower-right corner and more often to Bidder 1 in the rectangle in the upper-left corner. The reason why this improves expected surplus is that the efficiency loss of not allocating the good to any bidder in the triangle is small, both values are close to zero, whereas the good is allocated more efficiently in the rectangles.

The above example may give the reader the impression that the second best efficient allocation with no selling is characterized by the allocation that maximizes both integrals in Equation (1) simultaneously. The following slight modification of Example 3 shows that this is not correct:[20]

Example 4. *$N = \{1, 2\}$, $v_1(s_1, s_2) = s_1 + 2s_2 + \epsilon$ and $v_2(s_1, s_2) = s_2 + 2s_1$ with $\epsilon > 0$ and small, and $e_i(s_i) = 0$ for $i, j \in \{1, 2\}$ and $i \neq j$.*

[20]We have chosen an asymmetric counter-example because it makes the argument more transparent. The reader may found in the supplementary material in the authors' web pages that the following is a symmetric counter-example: $N = \{1, 2\}$, $v_i(s) = 20 \cdot \mathbf{1}_{[.9,1]}(s_j) + (s_i + s_j) + \mathbf{1}_{[1/2,1]}(s_i) + \mathbf{1}_{[1/2,1]}(s_j)$ and $e_i(s_i) = 0$.

It may be shown that the maximization of the first integral in Equation (1) requires that p satisfies a.e. that $p(s) = (0, 0)$ if $s_1 + s_2 < \frac{1}{4} - \frac{\epsilon}{2}$, and $p(s) = (1, 0)$, otherwise. Any such allocation verifies that $Q_1(s_1, p)$ is strictly increasing in s_1 for $s_1 \in [0, \frac{1}{4} - \frac{\epsilon}{2}]$. However, the maximization of the second integral of the corresponding Equation (1) requires that p verifies that $Q_1(s_1, p)$ is constant in s_1 in the open interval $(0, 1)$.

Finally, we conclude by showing that the condition in Proposition 6 is not necessary for the second best to be also second best with no selling:[21]

Example 5. *$N = \{1, 2\}$, $v_1(s) = 1/4$ and $v_2(s) = s_1 - 1$, and $e_i(s_i) = 0$ for all i.*

In this example, $h_1(s_1) = 1/4 - s_1$, $q_1(s_1) = s_1$, $g_1(s_1) = -1/4$, $h_2(s_2) = g_2(s_2) = -1$ and $q_2(s_2) = 0$. Thus, $\max_{i \in N} g_i(s_i) + \sum_{j=1}^{n} q_j(s_j) = s_1 - 1/4$, which is negative for $s_1 < 1/4$. However, it is efficient to always allocate to Bidder 1.

8 The English Auction

In this section we analyze whether the second best can be implemented with an English auction. In particular, we assume the model of the English auction described by Krishna (2003). This auction model is a variation of the Japanese auction proposed by Milgrom and Weber (1982) in which the identity of the bidders is observable.

We introduce two additional assumptions. The first one is a simplification, we assume that the functions h_i's are continuous. This assumption implies:

Lemma 4. *The functions g_i's are continuous. Moreover:*

(a) $g_i(s_i) = h_i(s_i)$ if $G_i(s_i) = H_i(s_i)$ and $s_i \in (0, 1)$.

(b) $g_i(0) \leq h_i(0)$ with strict inequality only if $G_i(\epsilon) < H_i(\epsilon)$ for any $\epsilon > 0$ small enough.

(c) $g_i(1) \geq h_i(1)$ with strict inequality only if $G_i(1 - \epsilon) < H_i(1 - \epsilon)$ for any $\epsilon > 0$ small enough.

See proof in the Appendix.

[21] To make the example more transparent, we have violated one of our assumptions, namely that $\phi_1(s_1)$ is strictly increasing. Note, however, that this can be easily fixed changing v_1 to $1/4 + \epsilon s_1$ and v_2 to $s_1 - 1 + \epsilon s_2$. For ϵ sufficiently small, this change generates the appropriate counterexample.

The second assumption is that $\zeta_i(s_i) \equiv q_i(s_i)+e_i(s_i)$ is non-decreasing. All the examples in Section 5 verify this assumption. To understand this assumption consider, first, the case with no externalities, i.e. $e_i(s_i) = 0$. Then our assumption is equivalent to assume that each bidder's value is an increasing function of all bidders' types. This assumption is common to all the papers that study the first best efficiency of English auctions, see Maskin (2000), Krishna (2003) and Birulin and Izmalkov (2003). The reason is that it is well-known that when we relax this assumption the English auction may not be first best efficient even when the first best can be implemented with other mechanisms. We expect this assumption to be satisfied more often in our framework than when the first best is implementable.[22] With externalities, our assumption is verified when the negative externalities induced by a bidder do not decrease too fast as we increase her type.

8.1 An English Auction with Only Two Bidders

Suppose in this subsection that $n = 2$. We shall show that in this case, the English auction implements the second best efficient allocation. We start with an example:

Example 6. *$N = \{1, 2\}$, $v_1(s) = s_1 + 1$, $v_2(s) = s_2 + 2s_1$ and $e_i(s_i) = 0$ for all i.*

The analysis of the second best in this example is very similar to Example 1: $h_1(s_1) = 1 - s_1$, $g_1(s_1) = 1/2$ and $h_2(s_2) = g_2(s_2) = s_2$, the single crossing condition is not satisfied and it is second best to allocate to Bidder 2 if and only if $g_2(s_2) \geq g(s_1)$, i.e. $s_2 \geq 1/2$.

Note that Bidder 1 has no uncertainty about her value. Hence, by the same arguments as in private value auctions, her unique weakly dominant strategy is to bid her value $s_1 + 1$. In this case, Bidder 2's utility when she wins is equal to the difference between Bidder 2's value and Bidder 1's value, i.e. Bidder 2 gets the change in social surplus. Thus, it is not surprising that she finds it optimal to submit a bid that maximizes the expected social surplus. In particular, note that when Bidder 1 with type s_1 bids $p \equiv s_1+1$ and Bidder 2 wins, Bidder 2 gets profits $s_2+2s_1-p = s_2+2(p-1)-p = s_2+p-2$, and thus the greater the price, the more profitable it is for Bidder 2 to win the auction. As a consequence, Bidder 2's best response is either to submit a bid that always loses, e.g. $b = 1$, or a bid that always wins, e.g. $b = 2$. The former option is optimal if and only if the average value of p is larger than $2 - s_2$. Since $p = s_1 + 1$, this is verified if and only if $s_2 \geq 1/2$ as required by the second best.

[22] This is because the fact that the single crossing condition fails implies that there are points at which the function h_i is locally decreasing, see Lemma 2. Since with no externalities $q_i(s_i) = \phi_i(s_i) - h_i(s_i)$ and we normalize types so that $\phi_i(s_i)$ is increasing, this means that q_i is locally increasing, and hence our assumption is locally verified, at these points.

Note that the structure of this equilibrium is more involved than in the more standard model in which the single crossing condition is satisfied. The difference is that Bidder 2 may be active at prices at which she makes a loss if she wins, i.e. there may be ex-post regret. For instance, this occurs in the example if $s_2 = 0.6$ and $s_1 = 0$. The reason why this is profitable for Bidder 2 is that winning at higher prices is sufficiently profitable to offset the losses at lower prices.

The following lemma generalizes this example. Note that this lemma analyzes the case in which one bidder has no uncertainty with respect to her willingness to pay for the object, and it includes, as particular cases, the insider and the incumbent's models of Sections 5.1 and 5.2.

Proposition 7. *If $q_2(s_2) = e_2(s_2) = 0$ for any $s_2 \in [0, 1]$, then any equilibrium of the English auction in non-weakly dominated strategies implements the second best.*

See proof in the Appendix.

Next, we show how this result extends to the general case. As we shall see, there always exists an equilibrium that implements the second best efficient allocation but in some cases there may be other equilibria that are not second best efficient.

We follow three steps. First, we propose a bid function for each bidder; second, we prove that the good is allocated according to the second best allocation when bidders use the proposed bid functions; and finally, we show that the proposed bid functions are an equilibrium of the English auction. Next, we discuss uniqueness.

We start with some auxiliary definitions.

$$\underline{s}_i^j(s_j) \equiv \begin{cases} 0 & \text{if } g_j(s_j) < g_i(0) \\ 1 & \text{if } g_j(s_j) > g_i(1) \\ \min\{s_i : g_i(s_i) = g_j(s_j)\} & \text{o.w.} \end{cases}$$

This is the lowest type of Bidder i that it is consistent with allocating the good to Bidder i in the second best efficient allocation when j's type is equal to s_j. Similarly, we let $\overline{s}_i^j(s_i)$ be defined as $\underline{s}_i^j(s_i)$ above but replacing min by max. Hence, $\overline{s}_i^j(s_i)$ denotes the greatest type of Bidder i that it is consistent with allocating the good to Bidder j in the second best efficient allocation when j's type is equal to s_j.

Let $b_1^*(s_1) \equiv v_1^e(s_1, \overline{s}_2^1(s_1))$, and $b_2^*(s_2) \equiv v_2^e(\underline{s}_1^2(s_2), s_2)$, where $v_i^e(s_i, s_j) \equiv v_i(s_i, s_j) + e_j(s_j)$ (this is $v_i^e(s_i, s_j) = \phi_i(s_i) + \zeta_j(s_j)$). Thus, we propose that Bidder 1 (resp. Bidder 2) bids her willingness to pay for the object when the alternative is that the good goes to the other bidder and conditional on the hypothetical event that the signal of the other bidder is equal to $\overline{s}_2^1(s_1)$ (resp. $\underline{s}_1^2(s_2)$.)

Next lemma and its corollary shows that these bid functions implement a second best efficient allocation. Actually, this is the second best allocation that gives the good to Bidder 1 in as many cases as possible. It is easy to see that there is a symmetric equilibrium that implements the second best allocation that gives the good to Bidder 2 in as many cases as possible.

Lemma 5.

- $b_1^*(s_1) \geq b_2^*(s_2)$ *if and only if* $g_1(s_1) \geq g_2(s_2)$.
- $b_1^*(s_1) < b_2^*(s_2)$ *if and only if* $g_1(s_1) < g_2(s_2)$.

See proof in the Appendix.

Corollary 1. *The allocation induced by* (b_1^*, b_2^*) *is second best efficient.*

See proof in the Appendix.

Note that when Bidder 2 plays b_2^*, Bidder 1 wins if and only if Bidder 2's type is below a certain threshold which depends on Bidder 1's bid. This means that the expected payoffs of Bidder 1 are equal to the expected difference between her value and Bidder 2's bid for all Bidder 2's types below the corresponding threshold. This difference is equal to $v_1^e(s_1, s_2) - v_2^e(\underline{s}_1^2(s_2), s_2)$, which is equal to the variation in social surplus, $v_1^e(s_1, s_2) - v_2^e(s_1, s_2)$, plus the term $v_2^e(s_1, s_2) - v_2^e(\underline{s}_1^2(s_2), s_2)$. The expected value of the variation of the social surplus is maximized when Bidder 1's bid picks a second best efficient allocation, whereas the expected value of the latter term is maximized when Bidder 1's bid picks a threshold $\hat{s}_2$ such that $s_1 = \underline{s}_1^2(\hat{s}_2)$. By application of Lemma 5 and Corollary 1, we know that both maxima are achieved when Bidder 1 uses b_1^*. Hence, b_1^* is a best response to b_2^*. A similar arguments shows that b_2^* is a best response to b_1^*. Consequently:

Proposition 8. *The bid functions* (b_1^*, b_2^*) *form a Bayesian Nash equilibrium of the English auction.*

See proof in the Appendix.

We can thus conclude from Corollary 1 and Proposition 8,

Corollary 2. *The English auction has an equilibrium that implements the second best when there are two bidders.*

The reader may worry that the above bid functions are identity dependent. We shall argue that in general this is not a problem. Let $K_i \equiv \{k \in \mathbb{R} : \exists(\underline{s}, \overline{s}) \neq \emptyset, g_i(s_i) = k, \forall s_i \in (\underline{s}, \overline{s})\}$, this is the set of points in the range of g_i that correspond to a flat in its graph.

Lemma 6. *If $K_1 \cap K_2 = \emptyset$, then $\underline{s}_j^i(s_i) = \overline{s}_j^i(s_i)$ almost everywhere.*

See proof in the Appendix.

Thus, this lemma together with the fact that asymmetries only arise when $\underline{s}_j^i(s_i) \neq \overline{s}_j^i(s_i)$ imply that when $K_1 \cap K_2 = \emptyset$ asymmetries only occur in a set of types with zero measure, and thus, could be removed without upsetting the equilibrium. Moreover, we expect $K_1 \cap K_2 = \emptyset$ to hold true generically since the sets K_i's are countable.[23]

Nevertheless, the issue of asymmetries remains when bidders are symmetric, i.e. when $\phi_1 = \phi_2$, and $\zeta_1 = \zeta_2$. In this case, $K_1 \cap K_2 \neq \emptyset$, and in fact, the symmetric equilibrium is not second best efficient if the single crossing conditions does not hold. To see why, recall Example 2 and note that the only symmetric equilibrium is $b(s_i) = 3s_i$, $i = 1, 2$. Unlike our proposed strategies, $b_1^*(s_1) = s_1 + 2$ and $b_2^*(s_2) = s_2$, this equilibrium allocates the good to the bidder with higher type, which is not second best efficient.

Indeed, the English auction has more problems of multiplicity of equilibria when the single crossing condition fails than when it holds, even when $K_1 \cap K_2 = \emptyset$. The following example provides an illustration of this point.

Example 7. *$N = \{1, 2\}$, $v_1(s_1, s_2) = s_1 + \frac{3}{2}s_2$ and $v_2(s_1, s_2) = s_2 + 2s_1$, and $e_1(s_1) = e_2(s_2) = 0$.*

In this example $h_1(s_1) = -s_1$ and $h_2(s_2) = -\frac{1}{2}s_2$. Thus, $g_1(s_1) = -\frac{1}{2}$ and $g_2(s_2) = -\frac{1}{4}$, and consequently, the second best allocation is to give the good to Bidder 2 for any realization of the bidders' types. This is the allocation that is implemented by our proposed equilibrium applied to this example: $b_1^*(s_1) = s_1$, $b_2^*(s_2) = s_2 + 2$. However, there exist other equilibria that do not implement the second best, for instance,[24] $b_1(s_1) = s_1 + \frac{3}{2}$, $b_2(s_2) = s_2$.

[23] A simple argument is as follows. Let $g_i^{-1}(k) \equiv \min\{s_i \in [0, 1] : g_i(s_i) = k\}$. It is easy to see that if g_i is constant and equal to k in an open interval, then the function g_i^{-1} is discontinuous at k by definition. Finally, note that the set of the discontinuities of g_i^{-1} must be countable since g_i^{-1} is increasing.

[24] Bidder 1 does not have incentives to deviate because she wins with probability one at a price that it is always less

8.2 An English Auction with More than Two Bidders

In this section, we study the case in which there are more than two bidders, i.e. $n > 2$. We shall see that the English auction does not always implement the second best. We start with an example. In this example and in the rest of the section, it is important that we describe the tie-breaking rule. We shall assume the good is allocated with equal probability among the bidders that tie.[25] We conjecture that any other tie-breaking rule that does not condition on the bidders' types would imply similar results.

Example 8. *$N = \{1, 2, 3\}$, $v_1(s) = s_1 + \frac{1}{2}$, $v_2(s) = s_2 + 2s_1$, $v_3(s) = s_3 + 2s_1$ and $e_i(s_i) = 0$ for all i.*

In this example, $h_1(s_1) = \frac{1}{2} - s_1$, $g_1(s_1) = 0$, and $h_i(s_i) = g_i(s_i) = s_i$, $i \in \{2, 3\}$. Hence, it is second best efficient to allocate to Bidder 2 if $s_2 > s_3$ and to Bidder 3, otherwise. Note that, as in Example 6, Bidder 1 has a unique weakly dominant strategy: to remain in the auction until the price reaches her value $s_1 + \frac{1}{2}$. We show in the next lemma that there is no equilibrium of the English auction that implements the second best efficient allocation when Bidder 1 uses her weakly dominant strategy.

Lemma 7. *In Example 8, there is no equilibrium of the English auction in non-weakly dominated*[26] *strategies that implements the second best efficient allocation.*

Proof. To simplify, we refer in the proof to an equilibrium in non-weakly dominated strategies as an equilibrium. The proof has two steps. First, we show by contradiction that in any equilibrium that implements the second best, the strategies of both Bidder 2 and Bidder 3 must specify that types strictly less than 1/2 quit at a price less than Bidder 1's minimum bid (i.e. $\frac{1}{2}$) in information sets in which no bidder has quit yet. Second, we argue that if this is the case Bidder 3 has a profitable deviation.

than her value, whereas Bidder 2 does not have incentives to deviate because any bid $p \in [b_1(0), b_1(1)]$, gives Bidder 2 expected payoffs $\int_0^{b_1^{-1}(p)} \left(s_2 + 2s_1 - (s_1 + \frac{3}{2})\right) ds_1 \leq 0$ for any $s_2 \in [0, 1]$. Moreover, Bidder 2 does not have incentives to bid above $b_1(1)$ because these bids give the same expected payoffs as a bid $b_1(1)$.

[25] In our auction, the price increases continuously until one bidder or more quit. Then, the price is stopped and the following algorithm is repeated: (1) If there are no more active bidders, the good is allocated with equal probability among the bidders that last quitted at the current price. Otherwise, (2) the identity of the bidders that still remain active is announced. (3) After the announcement, bidders that still remain active declare independently and simultaneously whether they quit. If no bidder quits, the price is increased again from the current level. If some bidder quits, we go to (1).

[26] Actually, the lemma (also Proposition 9) only requires that Bidder 1 uses her weakly dominant strategy.

STEP 1: Suppose Bidder 1 plays her weakly dominant strategy and quits at a price p arbitrarily close to 1/2, and both Bidder 2 and 3 are still active and have types s_2 and s_3 strictly less than 1/2. Then Bidder 2 and 3 put a value in getting the good equal to $s_2 + 2(p - 1/2)$ and $s_3 + 2(p - 1/2)$, respectively. These values are less than the price p if p is close to 1/2. Hence, they both quit immediately after Bidder 1 and tie. Our tie breaking rule implies that the induced allocation is not second best efficient.

STEP 2: Suppose now that Bidder 2 quits at a price $p \leq \frac{1}{2}$ when she has a type s_2 in information sets in which no bidder has quit yet. Then, second best efficiency requires that Bidders 3 with a type s_3 in $(0, s_2)$ also quits at a price strictly less than p in the same information sets. In this case, Bidder 3 has a profitable deviation. In this deviation Bidder 3 remains in the auction until either the price reaches $\frac{1}{2}$ or Bidder 2 quits. In the former case, Bidder 3 quits, and in the latter, Bidder 3 remains active until Bidder 1 quits. This deviation lets Bidder 3 win additionally when Bidder 2 has a type in the set (s_3, s_2). In this case, Bidder 3 pays Bidder 1's bid and gets strictly positive expected payoffs:

$$\int_0^1 \left(s_3 + 2s_1 - (s_1 + \frac{1}{2}) \right) ds_1 = s_3 + \int_0^1 (s_1 - \frac{1}{2})\, ds_1 = s_3 > 0.$$

■

Intuitively, if Bidders 2 and 3 have a low type and are still active when Bidder 1 quits at a price close to 1/2 a "rush" occurs: both bidders find out that their values are less than the price. Thus, the need to select the most efficient bidder between Bidder 2 and Bidder 3 before a rush may occur binds Bidders 2 and Bidder 3 drop out prices to a level that is incompatible with their private incentives. Hence the impossibility.

The above result generalizes when Bidder 1 has no uncertainty about the value she puts in winning and under no externalities as follows:

Proposition 9. *Suppose $n > 2$ and $e_i(s_i) = q_i(s_i) = 0$ for $i \neq 1$ and $s_i \in [0, 1]$. There is no equilibrium in non-weakly dominated strategies of the English auction that implements the second best efficient allocation, if:*

$$g_1(\tilde{s}_1) < \max_{j \neq 1, i}\{h_j(\tilde{s}_j)\} = h_i(\tilde{s}_i) < h_1(\tilde{s}_1). \tag{3}$$

for some vector $\tilde{s} \in (0, 1)^n$.

See proof in the Appendix.

In particular, the conditions in the above lemma are the same as in Proposition 7 plus the assumption that $n > 2$ and that when the single crossing condition fails for Bidder 1 there exists a vector of types for which the first best allocates the good to Bidder 1 but the second best allocates the good to either i or j depending on their types. Note that we would expect the conditions of the lemma to hold in general in the models of Section 5.1 and 5.2. However, the above impossibility result does not necessarily hold once we move sufficiently away from the conditions of the above lemma. To see why, note the following generalization of Example 8.

Example 9. *$N = \{1, 2, 3\}$, $v_1(s) = s_1 + \frac{1}{2} + \alpha(s_2 + s_3)$, $v_2(s) = s_2 + 2s_1 + \alpha(s_2 + s_3)$, $v_3(s) = s_3 + 2s_1 + \alpha(s_2 + s_3)$, and $e_i(s_i) = 0$ for all i.*

In this example, $h_1(s_1) = -s_1 + \frac{1}{2}$, $g_1(s_1) = 0$ and $h_i(s_i) = g_i(s_i) = s_i$, $i : 2, 3$. Note that the second best allocation is as in Example 8. Moreover, for values of α close to zero, we expect that a variation of the arguments in Lemma 7 can be used to show that the second best is not implementable with an English auction. However, if α is sufficiently large, there is some multiplicity of equilibria that allows for an equilibrium that avoids the possibility of a "rush" by making bidders bid sufficiently low in information sets in which no bidder has quit yet. As next lemma shows, this is enough to guarantee that the second best can be implemented with an English auction.

Lemma 8. *There exists a perfect Bayesian equilibrium in non-weakly dominated strategies that implements the second best allocation in Example 9 when $\alpha = 1$. In this equilibrium:*

- *Bidder 1 bids $s_1 + \frac{1}{2}$, Bidder $j \neq 1$ bids $3s_j + 2$, in information sets in which no bidder has left the auction yet.*
- *Bidder $i \neq 1$ bids $3s_i + 2p - 1$ in information sets in which Bidder 1 has quit at price p.*
- *Bidder 1 bids $s_1 + \frac{1}{2} + \frac{p-2}{3}$, and Bidder $j \neq 1$ bids $2s_j + \frac{p-2}{3} + 2$ in information sets in which Bidder $i \neq 1, j$ has quit at price p.*

Proof. It is easy to see that the proposed strategies implement the second best: Bidder 1 quits first, followed by the bidder with lower type between Bidder 2 and 3. To show that it is an equilibrium, note that Bidder 1 cannot improve with a downward deviation as it does not change the outcome. In an upward deviation, Bidder 1 pays a price when she wins equal to $2s_2 + s_3 + 2$ if $s_2 \geq s_3$ ($s_2 < s_3$ is symmetric), which is greater than her value $s_1 + s_2 + s_3 + 1/2$, and thus makes the deviation unprofitable. Finally, note that when all bidders but $i \neq 1$ follow the proposed strategy, Bidder i pays a price $3s_j + s_1$, $j \neq 1, i$, when she wins. This price is less that Bidder i's value if and only if $s_i > s_j$. Since our proposed strategy makes Bidder i win in these cases and lose otherwise, she does not have incentives to deviate. ■

Note that the strategies in the lemma are such that in information sets in which no bidder has quit yet Bidder 1 bids as if the types of s_2 and s_3 were zero, whereas Bidder 2 and Bidder 3 bid as if they both had the same type and Bidder 1 had her highest possible type. Note also that Bidder 1 always quits first.

In the previous examples of this section we assumed no externalities. As we shall show next, externalities cause efficiency losses in an English auction that go beyond the problem of feasibility of the first best. To illustrate this point we study two examples in which the single crossing conditions holds and hence the first best is feasible. The first example displays positive externalities and the second one negative externalities.

Example 10. *$N = \{1, 2, 3\}$, $v_1(s) = s_1 + 1$, $v_2(s) = s_2 + 1$, $v_3(s) = s_3 + 1$, $e_2(s_2) = e_3(s_3) = 0$ and $e_1(s_1) = -1/2$.*

Bidder 1 has a weakly dominant strategy, to quit at price $b_1(s_1) = s_1 + 1$ and the first best efficient allocation is to allocate always to Bidder 1. But, this cannot occur in an equilibrium in which Bidder 1 bids $b_1(s_1) = s_1 + 1$ because Bidder 2 with a type $s_2 > 1/2$ finds it strictly profitable to outbid Bidder 1 for prices less than $s_2 + 1/2$. Clearly, Bidder 2 does not internalize the positive externality that allocating the good to Bidder 1 has on Bidder 3.

Example 11. *$N = \{1, 2, 3\}$, $v_1(s) = 2$,*[27] *$v_2(s) = s_2 + 1$, $v_3(s) = s_3 + 1$, $e_2(s_2) = e_3(s_3) = 0$ and $e_1(s_1) = 1$.*

Bidder 1 finds it weakly dominant to bid $b_1(s_1) = 2$ and the first best efficient allocation is that the good is allocated to Bidder 2 if $s_2 \geq s_3$, and to Bidder 3, otherwise. Note that when Bidder 1 bids $b_1(s_1) = 2$, Bidder i, $i \in \{2, 3\}$, with type s_i finds it optimal to outbid Bidder 1 in any continuation game in which only Bidder 1 and i are active. In this case, Bidder i gets a payoff $s_i + 1 - 2$, which is negative if $s_i < 1$. Thus, in any equilibrium in which Bidders 2 and Bidder 3 use this continuation strategy, they both have strict incentives to quit first when all bidders are still active. Consequently, either Bidder 2 or Bidder 3 must quit with positive probability at price 0, which is incompatible with

[27] To make the argument more transparent, we have deviated slightly from the general assumptions of Section 3 and we allow v_1 to be constant on s_1.

the first best efficient allocation. Intuitively, Bidder 2 and 3 do not want Bidder 1 to win, but they both prefer that it is the other bidder who pays the high price necessary to outbid Bidder 1.[28]

9 Conclusions

We study mechanisms that maximize the expected social surplus deriving from the sale of a (single-unit) object subject to Bayesian incentive compatibility constraints. An alternative approach is the equivalent analysis under ex post incentive compatible constraints. Since there is a close connection between ex post implementation and Bayesian implementation, see for instance Chung and Ely (2002), we expect that our characterization of the second best can be easily adapted to the analysis of this alternative framework.

As a matter of fact, the set of second best allocations that we characterize includes an allocation that it is ex post implementable.[29] However, this extension presents an additional difficulty. Second best efficiency requires trading off the cost of implementing inefficiently for different vectors of types. Under Bayesian implementation, the common prior gives natural weights for this comparison, but this is not the case under ex post implementation.

The corresponding analysis of the English auction also presents a major problem. The English auction does not always have an ex post equilibrium when the conditions for the implementation of the first best are violated, at least when we restrict to non-weakly dominated strategies. This is the case in the wildcatter example when it is second best that the entrant wins since all its Bayesian equilibria display ex post regret, see the Introduction.[30]

[28] A related argument was pointed out by Jehiel and Moldovanu (1996) and Hoppe, Jehiel, and Moldovanu (2006) to argue that externalities may induce strategic non participation in auctions.

[29] This may be shown by noting that an allocation such that for any s, $p_i(s) = 1$ if $g_i(s_i) = \max_{j\in N}\{g_j(s_j)\}$ and $i \leq \arg\max_{j\in N}\{g_j(s_j)\}$ satisfies the conditions in Proposition 5 and since the g_i's are increasing, it also satisfies the conditions for ex post implementability provided by Bikhchandani, Chatterji, and Sen (2006).

[30] This is formally shown in Example 6, which has the same qualitative features as the wildcatter example.

Appendix

A Proofs

Proof of Lemma 1

Proof. We first prove the "only if"-part. Suppose a feasible allocation p. Then, there exists a direct mechanism (p, x) for which:

$$
\begin{aligned}
V_i(s_i) \equiv U_i(s_i, s_i) \quad &\geq \quad U_i(s_i, s_i') \\
&= \quad Q_i(s_i', p)\phi_i(s_i) + \Psi_i(s_i', p, x) \\
&= \quad Q_i(s_i', p)\phi_i(s_i') + \Psi_i(s_i', p, x) + Q_i(s_i', p)(\phi_i(s_i) - \phi_i(s_i')) \\
&= \quad V_i(s_i') + Q_i(s_i', p)(\phi_i(s_i) - \phi_i(s_i')),
\end{aligned}
$$

for all $s_i, s_i' \in S_i$, $i \in N$. Hence, we have that for $s_i > s_i'$, $V_i(s_i) \geq V_i(s_i')$, and hence,

$$
Q_i(s_i', p) \leq \frac{V_i(s_i) - V_i(s_i')}{\phi_i(s_i) - \phi_i(s_i')},
$$

and applying the same inequality with the roles of s_i and s_i' interchanged,

$$
Q_i(s_i, p) \geq \frac{V_i(s_i) - V_i(s_i')}{\phi_i(s_i) - \phi_i(s_i')}.
$$

Thus, $Q_i(s_i, p) \geq Q_i(s_i', p)$ as desired.

To prove the "if"-part, suppose an allocation p for which $Q_i(., p)$ is increasing, for all $i \in N$. Note first that by assumption $\phi_i(.)$ is a strictly increasing function, and thus invertible in $[\phi_i(0), \phi_i(1)]$. Let $\tilde{V}_i(y) \equiv \int_{\phi_i(0)}^{y} Q_i(\phi_i^{-1}(\tilde{y}), p)\, d\tilde{y}$ for $y \in [\phi_i(0), \phi_i(1)]$ and,

$$
x_i(s) \equiv Q_i(s_i, p)\phi_i(s_i) + \sum_{j \neq i} (p_i(s_i, s_{-i})q_j(s_j) - e_j(s_j)p_j(s_i, s_{-i})) - \tilde{V}_i(\phi_i(s_i)),
$$

for any $s \in S$. This means that $\Psi_i(s_i, p, x) = \tilde{V}_i(\phi_i(s_i)) - Q_i(s_i, p)\phi_i(s_i)$, for any $s_i \in S_i$, and hence that $\tilde{V}_i(\phi_i(s_i)) = U_i(s_i, s_i)$ for the direct mechanism (p, x). We shall show that this direct mechanism satisfies the Bayesian incentive compatibility constraints. To see why, note that for any $s_i, s_i' \in S_i$

$$
\begin{aligned}
U_i(s_i, s_i) \quad &= \quad \tilde{V}_i(\phi_i(s_i)) \\
&\geq \quad \tilde{V}_i(\phi_i(s_i')) + Q_i(\phi_i^{-1}(\phi_i(s_i')), p)\left(\phi_i(s_i) - \phi_i(s_i')\right) \\
&= \quad Q_i(s_i', p)\phi_i(s_i') + \Psi_i(s_i', p, x) + Q_i(s_i', p)\left(\phi_i(s_i) - \phi_i(s_i')\right) \\
&= \quad Q_i(s_i', p)\phi_i(s_i) + \Psi_i(s_i', p, x) \\
&= \quad U_i(s_i, s_i'),
\end{aligned}
$$

where the inequality is a consequence of $\tilde{V}_i$ being a convex function and $Q_i(\phi_i^{-1}(y), p) \in \partial\tilde{V}_i(y)$ by definition of $\tilde{V}_i$. ■

Proof of Lemma 2

Proof. Substracting $\sum_j q_j(s_j)$ from the two sides of the inequalities that define the single crossing condition, one gets that the single crossing condition is equivalent to say that for any $i \in N$, $s \in S$ and $s'_i \in S_i$ such that $s'_i > s_i$:

$$h_i(s_i) > \max\{h_j(s_j)\}_{j\neq i} \text{ implies } h_i(s'_i) \geq \max\{h_j(s_j)\}_{j\neq i}. \tag{4}$$

This is equivalent to say that for any $i \in N$, $s_i, s'_i \in S_i$ and $s'_i > s_i$, it is verified that $A_i(s_i) \subset B_i(s'_i)$ for $A_i(s_i) \equiv \{s_{-i} \in S_{-i} : h_i(s_i) > \max\{h_j(s_j)\}_{j\neq i}\}$ and $B_i(s'_i) \equiv \{s_{-i} \in S_{-i} : h_i(s'_i) \geq \max\{h_j(s_j)\}_{j\neq i}\}$. This is equivalent to say that for any $i \in N$, $s_i, s'_i \in S_i$ and $s'_i > s_i$, it is verified that $A_i(s_i) \cap [S_{-i} \setminus B_i(s'_i)] = \emptyset$, which corresponds to the condition in the lemma. ■

Proof of Proposition 1

Proof. For the sufficient part, note that an allocation such that $p_i(s) = 0$ if $h_i(s_i) \neq \max_{j\in N} h_j(s_j)$, and $p_i(s) = \frac{1}{m(s)}$, otherwise, where $m(s)$ denotes the cardinality of $\{k \in N : h_k(s_k) = \max_{j\in N} h_j(s_j)\}$ is first best efficient and satisfy the feasibility conditions of Lemma 1 if the condition in the Lemma 2 is satisfied.

Next, let $\overline{J}_i(s_i) \equiv \{s_{-i} : \max\{h_j(s_j)\}_{j\neq i} \leq h_i(s_i)\}$, $\underline{J}_i(s_i) \equiv \{s_{-i} : \max\{h_j(s_j)\}_{j\neq i} < h_i(s_i)\}$ and $\mu_i(A) \equiv \int_A \prod_{j\neq i} f_j(s_j)\, ds_{-i}$ for $A \subset S_{-i}$. Note that for any first best efficient allocation p^*:

$$\mu_i\left(\underline{J}_i(s_i)\right) \leq Q_i(s_i, p^*) \leq \mu_i\left(\overline{J}_i(s_i)\right).$$

We prove the necessary part by contradiction. Suppose that the single crossing condition does not hold. Then, by Lemma 2 there exists a bidder $i \in N$ with types $s'_i > s_i$ and a vector $s_{-i} \in S_{-i}$ such that $\max\{h_j(s_j)\}_{j\neq i} \in (h_i(s'_i), h_i(s_i))$. By either right or left continuity of the $h_j(.)$'s, there exists an open set $O \subset S_{-i}$ such that $\max\{h_j(s'_j)\}_{j\neq i} \in (h_i(s'_i), h_i(s_i))$ for any $s'_{-i} \in O$. By definition, $\overline{J}_i(s'_i) \cap O = \emptyset$ and $\overline{J}_i(s'_i) \cup O \subset \underline{J}_i(s_i)$. Thus, for any first best efficient allocation p^*,

$$Q_i(s'_i, p^*) \leq \mu_i\left(\overline{J}_i(s'_i)\right) < \mu_i\left(\overline{J}_i(s'_i)\right) + \mu_i(O) = \mu_i\left(\overline{J}_i(s'_i) \cup O\right) \leq \mu_i\left(\underline{J}_i(s_i)\right) \leq Q_i(s_i, p^*),$$

which implies a violation of the feasibility conditions of Lemma 1. ■

Proof of Proposition 2

Proof. The model in the section in the notation of Section 3 corresponds to:

$$v_1(s) = s_1 \frac{(1-c_1)^2}{4} \text{ and } e_1(s_1) = 0,$$

$$v_i(s) = s_1 \frac{(1-c)^2}{4} + s_i \text{ and } e_i(s_i) = 0 \text{ for } i \neq 1.$$

Thus, $t_1(s_1) = h_1(s_1) = s_1 \left(\frac{(1-c_1)^2}{4} - \frac{(1-c)^2}{4}\right)$, $q_1(s_1) = s_1 \frac{(1-c)^2}{4}$, $e_1(s_1) = 0$, and $t_j(s_j) = h_j(s_j) = s_j$, $q_j(s_j) = 0$ and $e_j(s_j) = 0$, for $j \neq 1$.

Note that $h_1(\underline{s}) > h_1(\overline{s})$, $h_j(-\underline{s}\frac{(1-c)^2}{4}) < h_1(\underline{s})$ and $h_j(0) > h_1(\overline{s})$, $j \neq 1$. Hence, by continuity of h_j there exists an $s_j \in [-\underline{s}\frac{(1-c)^2}{4}, 0]$ such that $h_j(s_j) \in (h_1(\overline{s}), h_1(\underline{s}))$. Consequently, we can apply Lemma 2 for $s_i = \underline{s}$ and $s_i' = \overline{s}$, and $i = 1$ to show that the single crossing condition for Bidder 1 is not verified, and thus the proof follows by application of Proposition 1. ■

Proof of Proposition 3

Proof. The model in the notation of Section 3 corresponds to: $v_1(s) = s_1$, $e_1(s_1) = 0$, $v_i(s) = s_i + \mathbf{1}_{[\underline{t}+\rho, \overline{t}+\rho]}(s_1)$ and[31] $e_i(s_i) = 0$, $i \neq 1$, where s_1 has a distribution

$$F_1(s_1) = \begin{cases} \alpha G_1(s_1) & \text{if } s_1 < \overline{t} \\ \alpha & \text{if } s_1 \in [\overline{t}, \rho + \underline{t}) \\ \alpha + (1-\alpha) G_1(s_1 - \rho) & \text{otherwise,} \end{cases}$$

with support $[\underline{t}, \overline{t}] \cup [\underline{t} + \rho, \overline{t} + \rho]$ and s_i, $i \neq 1$, is distributed according to $G_i(.)$. Note that according to this convention, $s_1 \in [\underline{t} + \rho, \overline{t} + \rho]$ indicates that the painting is original.

In this application we have that $t_1(s_1) = h_1(s_1) = s_1 - \mathbf{1}_{[\underline{t}+\rho, \overline{t}+\rho]}(s_1)$, $q_1(s_1) = \mathbf{1}_{[\underline{t}+\rho, \overline{t}+\rho]}(s_i)$, $t_i(s_i) = h_i(s_i) = s_i$ and $q_i(s_i) = 0$ for $i \neq 1$, and it is easy to verify that the single crossing condition is violated for Bidder 1. To see why, apply Lemma 2 to $s_1 = \overline{t} - \epsilon$ and $s_1' = \underline{t} + \rho + \epsilon$ for $\epsilon > 0$ and small enough. Thus, the proof follows by application of Proposition 1. ■

Proof of Proposition 4

Proof. This model in terms of the notation of Section 3 corresponds: $v_i(s) = t_i(s_i) = s_i$, and $e_i(s_i) = 0$ if $s_i \leq C$, and $v_i(s) = t_i(s_i) = s_i + (n-1)(s_i - C)$ and $e_i(s_i) = 1 - c$, otherwise (note that $q_i(s_i) = 0$ for any s_i). As consequence, $h_i(s_i) = s_i$ for $s_i \leq C$ and $h_i(s_i) = s_i - (n-1)(C + 1 - c - s_i)$, otherwise.

[31] $\mathbf{1}_X(x)$ is an indicator function that takes value 1 when $x \in X$ and otherwise takes value 0.

In this case, the single crossing condition is violated for any bidder. To see why, apply Lemma 2 to $s_i = C - \epsilon$ and $s_i = C + \epsilon$, and $s_j \in (C - \epsilon, C)$ for all $j \neq i$, and $\epsilon > 0$ and small enough. Thus, the proof follows by application of Proposition 1. ■

Proof of Proposition 5

Proof. The second best maximizes:

$$\int_S \sum_{i=1}^{n} (v_i(t) - (n-1)e_i(s))\, p_i(s)\, ds = \int_S \sum_{i=1}^{n} \left(h_i(s_i) + \sum_{j=1}^{n} q_j(s_j) \right) p_i(s)\, ds. \tag{5}$$

Next note that using integration by parts (see Hewitt (1960)) and Lemma 3 (b) we can show that,

$$\int_S (h_i(s_i) - g_i(s_i))\, p_i(s) ds = \int_{S_i} (h_i(s_i) - g_i(s_i))\, Q_i(s_i, p)\, ds_i =$$
$$\int_{S_i} Q_i(s_i, p)\, dH_i(s_i) - \int_{S_i} Q_i(s_i, p)\, dG_i(s_i) =$$
$$- \int_{S_i} (H_i(s_i) - G_i(s_i))\; dQ_i(s_i, p).$$

Consequently, the expressions in Equation (5) are equal to the expression in Equation (1) as desired.

It is easy to see that an allocation maximizes the first integral in the equation above if and only it satisfies (i). Moreover, since $Q_i(., p)$ is increasing for any p feasible by Lemma 1, and $G_i(s_i) \leq H_i(s_i)$, by Lemma 3 (c), a feasible allocation maximizes the second integral if and only if it satisfies (ii). This completes the proof since the set of feasible allocations that satisfy (i) and (ii) is not empty. For instance, $p_i(s) = 0$ if $g_i(s_i) \neq \max_{j \in N} g_j(s_j)$, and otherwise, $p_i(s) = \frac{1}{m(s)}$, where $m(s)$ denotes the cardinality of $\{k \in N : g_k(s_k) = \max_{j \in N} g_j(s_j)\}$. The monotonicity of $g_i(s_i)$ ensures that the allocation is feasible. ■

Proof of Lemma 4

Proof. The function g_i cannot be discontinuous at points in an open interval in which $G_i(s_i) = H_i(s_i)$ by continuity of h_i, or at points in an open interval in which $G_i(s_i) \neq H_i(s_i)$ by Lemma 3 (d). Take now a point $s_i^* \in (0, 1)$ and an open interval O that includes s_i^* and such that $H_i(s_i) = G_i(s_i)$ if $s_i \in O$ and $s_i < s_i^*$ and $H_i(s_i) > G_i(s_i)$ if $s_i \in O$ and $s_i > s_i^*$ (the other case is symmetric). Then the left derivative of G_i is equal to $h_i(s_i^*)$ and the right derivative is bounded above by $h_i(s_i^*)$. Moreover, by the convexity of G_i the left derivative of G_i must be less than or equal to the right derivative. As a consequence, G_i is differentiable at s_i^* and its differential $g_i(s_i^*)$ is equal to $h_i(s_i^*)$. Continuity at 0 and 1 together with the last two items of the lemma are direct consequences of Lemma 3 (b) and (c) and the boundedness of h_i. ∎

Proof of Proposition 7

Proof. By the same argument as in Example 6, Bidder 1 has a unique weakly dominant strategy, to bid until $b_1(s_1) \equiv t_1(s_1)+q_1(s_1)$. We show next that the resulting allocation when Bidder 1 bids $b_1(s_1)$ and Bidder 2 plays a best response to $b_1(s_1)$ is second best efficient. First, note that b_1 is continuous and strictly increasing, and hence, its inverse b_1^{-1} exists. Bidder 2 wins the auction with a bid b if and only if $s_1 \leq b_1^{-1}(b)$. Thus, Bidder 2's expected payoffs when she bids $b \in [b_1(0), b_1(1)]$ are equal to:

$$\begin{aligned}
&\int_0^{b_1^{-1}(b)} (t_2(s_2) + q_1(\tilde{s}_1) - b_1(\tilde{s}_1))\, d\tilde{s}_1 - \int_{b_1^{-1}(b)}^1 e_1(\tilde{s}_1)\, d\tilde{s}_1 \\
&\qquad = \int_0^{b_1^{-1}(b)} (t_2(s_2) + q_1(\tilde{s}_1) + e_1(\tilde{s}_1) - b_1(\tilde{s}_1))\, d\tilde{s}_1 - \int_0^1 e_1(\tilde{s}_1)\, d\tilde{s}_1 \\
&\qquad = \int_0^{b_1^{-1}(b)} (h_2(s_2) - h_1(\tilde{s}_1))\, d\tilde{s}_1 - \int_0^1 e_1(\tilde{s}_1)\, d\tilde{s}_1 \\
&\qquad = \int_0^{b_1^{-1}(b)} (g_2(s_2) - g_1(\tilde{s}_1))\, d\tilde{s}_1 - \big(H_1(b_1^{-1}(b)) - G_1(b_1^{-1}(b))\big) - \int_0^1 e_1(\tilde{s}_1)\, d\tilde{s}_1,
\end{aligned}$$

where in the third step we have used that h_2 is increasing under the assumptions of the lemma and thus $g_2(s_2) = h_2(s_2)$.

Let $\overline{s}_1^2(s_2)$ be the maximum of the set $\{s_1 \in [0,1] : g_1(s_1) = g_2(s_2)\}$ if non-empty; $\overline{s}_1^2(s_2) = 1$, if $g_2(s_2) > g_1(1)$; and $\overline{s}_1^2(s_2) = 0$, if $g_2(s_2) < g_1(0)$. Note that $b = b_1(\overline{s}_1^2(s_2))$ maximizes the last expression, and in particular the first two terms. That it maximizes the first term is direct from the definition of $\overline{s}_1^2$. To show that it also maximizes the second term, note that by Lemma 3 (c), this second term can only be negative or zero. Thus, it is sufficient to show that for $b = b_1(\overline{s}_1^2(s_2))$ it is equal to zero. This is direct from the definition of $\overline{s}_1^2$ and Lemma 3 (b) and (d). Consequently, any maximum to the above expression must maximize the first two terms. We shall show that this implies that the induced allocation satisfies both conditions in Proposition 5. The maximization of the first term implies directly condition (i). Condition (ii) holds trivially for $i = 2$ since $g_2 = h_2$, i.e. $G_2 = H_2$. Finally, note that the maximization of the second term implies that no type of Bidder 2 bids at points in which $H_1(b_1^{-1}(b)) > G_1(b_1^{-1}(b))$, and thus condition (ii) is also satisfied for $i = 1$. ∎

Proof of Lemma 5

Proof. We only prove the first item. The second one is simply the negation of the first one. We first show the "if" part. Consider first the case $g_1(0) > g_2(1)$. By monotonicity of the bid functions, we only need to show that $g_1(0) > g_2(1)$ implies that $b_1(0) \geq b_2(1)$. Note that it is easy to see that $b_1(0) - b_2(1) = \phi_1(0) + \zeta_2(1) - (\phi_2(1) + \zeta_1(0)) = h_1(0) - h_2(1)$, which is greater than $g_1(0) - g_2(1)$ by Lemma 4, and thus non-negative as desired. Consider now the case $g_1(0) \leq g_2(1)$. In this case, $g_1(s_1) \geq g_2(s_2)$ and continuity of the g_i's, see Lemma 4, implies that there exists a $s_1' \leq s_1$ and a $s_2' \geq s_2$ such that $g_1(s_1') = g_2(s_2')$, $s_1' = \underline{s}_1^2(s_2')$ and $s_2' = \overline{s}_2^1(s_1')$. Thus:

$$
\begin{aligned}
b_1^*(s_1) - b_2^*(s_2) \geq b_1^*(s_1') - b_2^*(s_2') = \\
& \left[\phi_1(s_1') - \zeta_1(\underline{s}_1^2(s_2'))\right] - \left[\phi_2(s_2') - \zeta_2(\overline{s}_2^1(s_1'))\right] = \\
& \left[\phi_1(\underline{s}_1^2(s_2')) - \zeta_1(\underline{s}_1^2(s_2'))\right] - \left[\phi_2(\overline{s}_2^1(s_1')) - \zeta_2(\overline{s}_2^1(s_1'))\right] = \\
& h_1(\underline{s}_1^2(s_2')) - h_2(\overline{s}_2^1(s_1')).
\end{aligned}
$$

We next argue that the last expression is weakly greater than $g_1(\underline{s}_1^2(s_2')) - g_2(\overline{s}_2^1(s_1'))$. To see why, we argue that $h_1(\underline{s}_1^2(s_2')) \geq g_1(\underline{s}_1^2(s_2'))$ and $h_2(\overline{s}_2^1(s_1')) \leq g_2(\overline{s}_2^1(s_1'))$. We only show the former inequality since the latter one has a symmetric proof. Lemma 3 (d), Lemma 4 (a) and the definition of $\underline{s}_1^2$ implies that if $\underline{s}_1^2(s_2') \in (0,1)$ then $h_1(\underline{s}_1^2(s_2')) = g_1(\underline{s}_1^2(s_2'))$. Thus, by Lemma 4, we only need to show that if $\underline{s}_1^2(s_2') = 1$ then it cannot be that $G_1(1-\epsilon) < H_1(1-\epsilon)$ for any ϵ close to zero. By contradiction, suppose that $\underline{s}_1^2(s_2') = 1$ and $G_1(1-\epsilon) < H_1(1-\epsilon)$ for any ϵ close to zero. Then, $g_2(s_2') = g_1(1)$ and $g_1(s_1)$ is flat for any s_1 in $(1-\epsilon, 1]$ by Lemma 3 (d), which contradicts that $\underline{s}_1^2(s_2') = 1$.

Thus, the "if" part follows from the fact that $g_1(\underline{s}_1^2(s_2')) - g_2(\overline{s}_2^1(s_1'))$ is equal to zero since $g_1(s_1') = g_2(s_2')$, $s_1' = \underline{s}_1^2(s_2')$ and $s_2' = \overline{s}_2^1(s_1')$.

We prove the "only if" part by contradiction. We shall show that $g_2(s_2) > g_1(s_1)$ implies that $b_2(s_2) > b_1(s_1)$. The proof is similar to the "if" part. The case $g_2(0) > g_1(1)$ is symmetric to the case $g_1(0) > g_2(1)$ above. In the case $g_2(0) \leq g_1(1)$, $g_2(s_2) > g_1(s_1)$ implies that there exists a strictly decreasing sequence $\{s_{2,m}\}$ starting at s_2 and a strictly increasing sequence $\{s_{1,m}\}$ starting at s_1 with respective limits s_2' and s_1' that satisfy $g_2(s_2') = g_1(s_1')$, $s_2' = \overline{s}_2^1(s_1')$ and $s_1' = \underline{s}_1^2(s_2')$. Note that along the sequence $g_2(s_{2,m}) > g_1(s_{1,m})$ and hence, $s_{2,m} \geq \overline{s}_2^1(s_{1,m})$ and $s_{1,m} \leq \underline{s}_1^2(s_{2,m})$. Using these properties and the monotonicity of the bid functions and ζ_i we have that:

$$b_2^*(s_2) - b_1^*(s_1) > \lim_{m\to\infty} [b_2^*(s_{2,m}) - b_1^*(s_{1,m})] =$$
$$\lim_{m\to\infty} \left[\left(\phi_2(s_{2,m}) - \zeta_2(\overline{s}_2^1(s_{1,m}))\right) - \left(\phi_1(s_{1,m}) - \zeta_1(\underline{s}_1^2(s_{2,m}))\right)\right] =$$
$$\lim_{m\to\infty} \left[\left(h_2(s_{2,m}) + \zeta_2(s_{2,m}) - \zeta_2(\overline{s}_2^1(s_{1,m}))\right) - \left(h_1(s_{1,m}) + \zeta_1(s_{1,m}) - \zeta_1(\underline{s}_1^2(s_{2,m}))\right)\right] \geq$$
$$\lim_{m\to\infty} [h_2(s_{2,m}) - h_1(s_{1,m})] = h_2(\underline{s}_2^1(s_1')) - h_1(\overline{s}_1^2(s_2')),$$

which by the same arguments as in the "if" part is non-negative as desired. ■

Proof of Corollary 1

Proof. That the allocation induced by (b_1^*, b_2^*) satisfies condition (i) in Proposition 5 is direct from Lemma 5. To check condition (ii), note that for the allocation induced by (b_1^*, b_2^*), Lemma 5 implies that:

$$Q_1(s_1, p) = \int_{\{s_2: g_1(s_1) \geq g_2(s_2)\}} p_1(s_1, s_2)\, ds_2,$$

since ties occur with zero probability. Thus, in any open interval in which $H_1(s_1) > G_1(s_1)$ Lemma 3 (d) implies that $Q_1(s_1, p)$ is constant as required by condition (ii). A similar argument shows that Q_2 also satisfies (ii). ■

Proof of Proposition 8

Proof. We only show that Bidder 1 finds it optimal to bid according to b_1^* when Bidder 2 plays b_2^*. The corresponding proof for Bidder 2 is similar.[32] Let $u_1(s_1, b)$ be the expected utility of Bidder 1 when she has a private type s_1, submits a bid b, and Bidder 2 uses the bid function b_2^*. We only show that Bidder 1 does not have incentives to deviate downwards, i.e. $u_1(s_1, b_1^*(s_1)) - u_1(s_1, b) \geq 0$ for $b < b_1^*(s_1)$. The analysis of incentives to deviate upwards, i.e. $b > b_1^*(s_1)$, is symmetric. Downward deviations only affect the payoffs when Bidder 1 wins with $b_1^*(s_1)$ and loses with b, i.e. when[33] $b_2^*(s_2) \in (b, b_1^*(s_1)]$. Recall also from Lemma 5 that $b_1^*(s_1) \geq b_2^*(s_2)$ is equivalent to $g_1(s_1) \geq g_2(s_2)$, which implies: (a) $s_1 \geq \underline{s}_1^2(s_2)$ and (b) $g_1(\underline{s}_1^2(s_2)) \geq g_2(s_2)$. Moreover, by monotonicity of the bid functions we have that $\{s_2 : b_2^*(s_2) \in (b, b_1^*(s_1)]\} = \{s_2 : s_2 \in (\tau_2(b), \overline{s}_2^1(s_1)]\}$ for some $\tau_2(b) \in [0, \overline{s}_2^1(s_1))$. Note also that if

[32] This proof together with the proof that upward deviations are not profitable, see below, is available at the authors' websites.

[33] Note that ties occur with probability zero and thus the conclusions do not change whether considering the boundaries of the interval of bids close or open.

Bidder 1 wins, she gets a good with value $v_1(s_1, s_2)$ and pays Bidder 2's bid and if Bidder 1 loses she suffers a negative externality equal to $e_2(s_2)$. Thus, the change in utility when Bidder 1 wins is equal to $v_1^e(s_1, s_2) - v_2^e(s_2, \underline{s}_1^2(s_2)) = \phi_1(s_1) + \zeta_2(s_2) - \phi_2(s_2) - \zeta_1(\underline{s}_1^2(s_2))$. Thus,

$$
\begin{aligned}
u_1(s_1, b_1^*(s_1)) - u_1(s_1, b) = &\\
&\int_{\tau_2(b)}^{\overline{s}_2^1(s_1)} \big(\phi_1(s_1) + \zeta_2(s_2) - \phi_2(s_2) - \zeta_1(\underline{s}_1^2(s_2))\big)\, ds_2 = \\
&\int_{\tau_2(b)}^{\overline{s}_2^1(s_1)} \big[\phi_1(s_1) - \zeta_1(\underline{s}_1^2(s_2))\big] - [\phi_2(s_2) - \zeta_2(s_2)]\, ds_2 \geq \\
&\int_{\tau_2(b)}^{\overline{s}_2^1(s_1)} \big[\phi_1(\underline{s}_1^2(s_2)) - \zeta_1(\underline{s}_1^2(s_2))\big] - [\phi_2(s_2) - \zeta_2(s_2)]\, ds_2 = \\
&\int_{\tau_2(b)}^{\overline{s}_2^1(s_1)} \big[h_1(\underline{s}_1^2(s_2)) - h_2(s_2)\big]\, ds_2 \geq \\
&\int_{\tau_2(b)}^{\overline{s}_2^1(s_1)} (g_2(s_2) - h_2(s_2))\, ds_2
\end{aligned}
$$

where we use that ϕ_1 is strictly increasing and (a) in the first inequality. As for the second inequality, we use that $h_1(\underline{s}_1^2(s_2)) \geq g_1(\underline{s}_1^2(s_2))$ and (b). To see why $h_1(\underline{s}_1^2(s_2)) \geq g_1(\underline{s}_1^2(s_2))$, we argue by contradiction. Suppose that $h_1(\underline{s}_1^2(s_2)) < g_1(\underline{s}_1^2(s_2))$, then Lemma 4 implies that $\underline{s}_1^2(s_2) > 0$ and thus by continuity there exists an interval $(a, \underline{s}_1^2(s_2)]$, with $a \neq \underline{s}_1^2(s_2)$ such that any s_1 in this interval verifies that $h_1(s_1) < g_1(s_1)$ and hence that $H_1(s_1) \neq G_1(s_1)$. By application of Lemma 3 (d) we have that $g_1(s_1)$ is constant in $(a, \underline{s}_1^2(s_2)]$, which contradicts the definition of $\underline{s}_1^2(s_2)$.

Finally, we argue that $\int_{\tau_2(b)}^{\overline{s}_2^1(s_1)} (g_2(s_2) - h_2(s_2))\, ds_2$ is non-negative. This integral is equal to

$$\big[G_2(\overline{s}_2^1(s_1)) - H_2(\overline{s}_2^1(s_1))\big] + [H_2(\tau_2(b)) - G_2(\tau_2(b))]\,.$$

The second difference is non-negative by Lemma 3 (c). We next argue that the first one is equal to zero. If $\overline{s}_2^1(s_1)$ is either zero or one, this is because of Lemma 3 (b); otherwise, it is because $G_2(\overline{s}_2^1(s_1)) < H_2(\overline{s}_2^1(s_1))$ would imply that g_2 is constant around $\overline{s}_2^1(s_1)$ by Lemma 3 (d), which is a contradiction with the definition of $\overline{s}_2^1(s_1)$. ■

Proof of Lemma 6

Proof. Since any increasing function can be discontinuous in at most countably many points and $\underline{s}_j^i$ is increasing, it is sufficient to show that at any point $s_i \in [0,1]$ for which $\underline{s}_j^i(s_i) < \overline{s}_j^i(s_i)$, the function $\underline{s}_j^i$ is discontinuous. To prove so, suppose a point s_i at which $\delta \equiv \overline{s}_j^i(s_i) - \underline{s}_j^i(s_i) > 0$. Then, by definition of $\underline{s}_j^i$ and $\overline{s}_j^i$, the function g_j is constant and equal to $g_i(s_i)$ in the interval $(\underline{s}_j^i(s_i), \overline{s}_j^i(s_i))$. This means that g_i is strictly increasing at s_i since it cannot be constant because $K_1 \cap K_2 = \emptyset$. Hence, $\overline{s}_j^i(s_i) < \underline{s}_j^i(s_i + \epsilon)$, and consequently $\underline{s}_j^i(s_i) + \delta < \underline{s}_j^i(s_i + \epsilon)$, for any $\epsilon > 0$. This implies that $\underline{s}_j^i$ is discontinuous at s_i. ■

Proof of Proposition 9

Proof. As in the proof of Lemma 7, we refer to an equilibrium in non-weakly dominated strategies simply as an equilibrium. We denote by $b_1(s_1) \equiv t_1(s_1) + q_1(s_1)$ Bidder 1's unique weakly dominant strategy. Finally, note that under the assumptions of the lemma, the functions h_i's, $i \neq 1$, are strictly increasing (since $h_i = \phi_i$) and thus $h_i = g_i$ for $i \neq 1$.

The proof is also sketched in two steps. We first provide three necessary conditions that must be satisfied in an equilibrium that implements the second best:[34]

(i) For any vector of types $s \in [0,1]^n$ that satisfies:

$$g_1(s_1) < \max_{j \neq \{1,i\}} \{h_j(s_j)\} < h_i(s_i) < h_1(s_1),$$

only Bidder i and 1 can be active along the equilibrium path when the price is equal to $b_1(s_1)$. We prove the claim by contradiction. We shall argue that if Bidder 1, Bidder i, and Bidder $l \neq \{1, i\}$ are active at a price $p \equiv b_1(s_1)$ in the equilibrium path induced by the above vector of types, then both Bidder i and l quit immediately if Bidder 1 quits. Hence, there is a tie and thus a contradiction since our tie-breaking rule does not ensure the second best allocation. To understand why Bidder i quits immediately after Bidder 1 in the previous argument, note first that in the equilibrium path Bidder i infers from Bidder 1 quitting at price p that Bidder 1's type is equal to s_1. Thus, Bidder i infers that her value is equal to $t_i(s_i) + q_1(s_1)$, which is strictly less than the price $p = b_1(s_1)$ since

$$t_i(s_i) + q_1(s_1) - b_1(s_1) = t_i(s_i) + q_1(s_1) - t_1(s_1) - q_1(s_1) = h_i(s_i) - h_1(s_1) < 0. \quad (6)$$

Note also that by a similar argument Bidder l also infers that her value is less than the price. This explains why both Bidder i and l must quit immediately after Bidder 1 in equilibrium.

[34] The structure of the proof generalizes the proof of Lemma 7. Basically, the necessary condition (i) corresponds to the first step in the proof of Lemma 7, the necessary conditions (ii) and (iii) only play an auxiliary role, and what follows corresponds to the second step.

(ii) Bidder i, $i \neq 1$, with type s_i does not win at a price $p > b_1(\overline{s}_1^i(s_i))$ in the equilibrium path when Bidder 1 bids p, and thus has a type $s_1 \equiv b_1^{-1}(p) > \overline{s}_1^i(s_i)$. The reason is that the implemented allocation would not be second best because $s_1 > \overline{s}_1^i(s_i)$ implies that $g_1(s_1) > g_i(s_i)$.

(iii) Bidder i, $i \neq 1$, with type s_i does not win in the equilibrium path at a price strictly greater than $t_i(s_i) + q_1(b_1^{-1}(p))$ when Bidder 1 quits at a price p and thus has a type $s_1 = b_1^{-1}(p)$. The reason is that Bidder i does not find it profitable to win at these prices, and hence she would have a profitable deviation, to quit at price $t_i(s_i) + q_1(b_1^{-1}(p))$ if higher than p, or immediately after Bidder 1 otherwise.

We complete the proof by showing that there is a profitable deviation when Bidder 1 uses her unique weakly dominant strategy and all the other bidders a vector of strategies that verifies conditions (i)-(iii) above and that allocates the good according to the second best.

Let $\tilde{s} \in (0,1)^n$ be a vector that verifies the conditions in the statement of the lemma. The profitable deviation exists for a Bidder i, $i \neq 1$, with type $\tilde{s}_i$. To describe it, let $s_{\inf}(s_i)$ denote the infimum of the set $\{s_1 : h_i(s_i) < h_1(s_1)\}$ if not empty and note that $s_{\inf}(s_i)$ is right-continuous since h_1 and h_i are continuous and h_i increasing. Thus, there exists an $\hat{\epsilon} > 0$ small enough such that $h_i(\tilde{s}_i) < h_1(s_1)$ for any $s_1 \in (s_{\inf}(\tilde{s}_i), s_{\inf}(\tilde{s}_i + \hat{\epsilon})]$.

The proposed deviation is that Bidder i with type $\tilde{s}_i$ plays the action prescribed by her strategy but for a type[35] $\tilde{s}_i + \hat{\epsilon}$ (rather than her true type $\tilde{s}_i$) unless either of the following two cases occur: (a) that the price reaches $b_1(\overline{s}_1^i(\tilde{s}_i))$ when Bidder 1 is active; or (b) that Bidder 1 has already quit at price p, and the price is equal or above $t_i(\tilde{s}_i) + q_1(b_1^{-1}(p))$. In either of these two cases, the deviation prescribes that Bidder i quits immediately.

Since the original strategies implemented the second best and satisfy (ii) and (iii) this deviation lets Bidder i win in all the cases in which she was already winning with the original strategy (and at the same price). Moreover, Bidder i's deviation lets her win in some additional cases. To simplify the description of these additional cases, we shall restrict in what follows to the symmetric case in which $h_l = h_k$ for any $l, k \neq 1$ (and thus $g_l = g_k$). The extension to the general case is straightforward but requires a cumbersome notation. Thus, the only additional cases in which i may win with the deviation are when the maximum of the other bidders types but 1 is in $(\tilde{s}_i, \tilde{s}_i + \hat{\epsilon})$, and Bidder 1 has a

[35] The reason why the proof of this lemma is more complex than the proof of Lemma 7 is that we must make sure that the game does not move to an out-of-equilibrium path after the deviation since conditions (i)-(iii) only apply in the equilibrium path.

type less than $\overline{s}_1^i(\tilde{s}_i)$, see a) above. The original strategy did not let i win in these cases because it is not second best efficient. We show next what happens under the deviation in these cases depending on the value of Bidder 1's type s_1:

- If $s_1 \in [0, s_{\text{inf}}(\tilde{s}_i)]$: then, $h_i(\tilde{s}_i) \geq h_1(s_1)$ and thus, $t_i(\tilde{s}_i) + q_1(s_1) \geq b_1(s_1)$ by a similar argument as in Equation (6). As a consequence, condition (b) above ensures that i gets non-negative payoffs with the deviation if she wins.

- If $s_1 \in (s_{\text{inf}}(\tilde{s}_i), s_{\text{inf}}(\tilde{s}_i + \hat{\epsilon})]$: then, $h_i(\tilde{s}_i) < h_1(s_1)$ by definition of $\hat{\epsilon}$, or equivalently $t_i(\tilde{s}_i) + q_1(s_1) - b_1(s_1) < 0$, again by a similar argument as in Equation (6). Thus condition (b) above means that i quits immediately after 1 if i is still active and as a consequence, if i wins, she pays 1's bid.

- If $s_1 \in (s_{\text{inf}}(\tilde{s}_i + \hat{\epsilon}), \overline{s}_1^i(\tilde{s}_i)]$, Bidder i wins with the deviation and pays 1's bid. This is because condition (i) implies that only Bidder 1 and i are active when the price goes above $b_1(s_{\text{inf}}(\tilde{s}_i + \hat{\epsilon}))$. To see why condition (i) applies, note that $s_1 \leq \overline{s}_1^i(\tilde{s}_i)$ means that $g_1(s_1) \leq g_i(\tilde{s}_i)$ and that $g_i(s_i) = h_i(\tilde{s}_i) \leq h_i(\tilde{s}_i + \hat{\epsilon})$ since h_i is increasing. Moreover, for any s_1 arbitrarily close but above $s_{\text{inf}}(\tilde{s}_i + \hat{\epsilon})$, we have $h_i(\tilde{s}_i + \hat{\epsilon}) < h_1(s_1)$. Putting together these facts, we get $g_1(s_1) < h_i(\tilde{s}_i + \hat{\epsilon}) < h_1(s_1)$ as required.

Denote by $\rho(s_1)$ the probability with which i wins conditional on s_1 and on the maximum of $\{s_j\}_{j \neq 1,i}$ being in $(\tilde{s}_i, \tilde{s}_i + \epsilon)$ when i plays her deviation and all the other bidders follow the proposed strategies. Note that our previous arguments imply that $\rho(s_1) = 1$ if $s_1 \in (s_{\text{inf}}(\tilde{s}_i + \hat{\epsilon}), \overline{s}_1^i(\tilde{s}_i)]$. We next use ρ to show that Bidder i gets strictly positive utility with the deviation in the last two cases above:

$$\int_{s_{\inf}(\tilde{s}_i)}^{\overline{s}_1^i(\tilde{s}_i)} (t_i(\tilde{s}_i) + q_1(s_1) - b_1(s_1))\, \rho(s_1)\, ds_1 =$$

$$\int_{s_{\inf}(\tilde{s}_i)}^{\overline{s}_1^i(\tilde{s}_i)} (h_i(\tilde{s}_i) - h_1(s_1))\, \rho(s_1)\, ds_1 \geq$$

$$\int_{s_{\inf}(\tilde{s}_i)}^{\overline{s}_1^i(\tilde{s}_i)} (h_i(\tilde{s}_i) - h_1(s_1))\; ds_1 =$$

$$\int_{s_{\inf}(\tilde{s}_i)}^{\overline{s}_1^i(\tilde{s}_i)} (g_1(\tilde{s}_1) - h_1(s_1))\; ds_1 + \int_{s_{\inf}(\tilde{s}_i)}^{\overline{s}_1^i(\tilde{s}_i)} (h_i(\tilde{s}_i) - g_1(s_1))\; ds_1 =$$

$$\big(G_1(\overline{s}_1^i(\tilde{s}_i)) - H_1(\overline{s}_1^i(\tilde{s}_i))\big) - (G_1(s_{\inf}(\tilde{s}_i)) - H_1(s_{\inf}(\tilde{s}_i))) +$$

$$\int_{s_{\inf}(\tilde{s}_i)}^{\overline{s}_1^i(\tilde{s}_i)} (h_i(\tilde{s}_i) - g_1(s_1))\; ds_1 \geq$$

$$\int_{s_{\inf}(\tilde{s}_i)}^{\overline{s}_1^i(\tilde{s}_i)} (h_i(\tilde{s}_i) - g_1(s_1))\; ds_1 > 0,$$

where in the first inequality we use that $\rho(s_1) = 1$ if $s_1 \in (s_{\inf}(\tilde{s}_i + \hat{\epsilon}), \overline{s}_1^i(\tilde{s}_i)]$ and that $h_i(\tilde{s}_i) < h_1(s_1)$ for $s_1 \in (s_{\inf}(\tilde{s}_i), s_{\inf}(\tilde{s}_i + \hat{\epsilon})]$ by definition of $\hat{\epsilon}$; in the second inequality we use the same arguments as in the last paragraph of the proof of Proposition 8; and in the last inequality we use that $g_i(\tilde{s}_i) \geq g_1(s_1)$ (and that $h_i = g_i$) for $s_1 \leq \overline{s}_1^i(\tilde{s}_i)$, and strictly if $s_1 \in (s_{\inf}(\tilde{s}_i), \underline{s}_1^i(\tilde{s}_i))$ by definition of $\overline{s}_1^i$ and $\underline{s}_1^i$. Note $(s_{\inf}(\tilde{s}_i), \underline{s}_1^i(\tilde{s}_i))$ is non-empty because $s_{\inf}(\tilde{s}_i) < \tilde{s}_1$ and $\tilde{s}_1 < \underline{s}_1^i(\tilde{s}_i)$. The former inequality can be proved using that the vector $\tilde{s}$ verifies Equation (3) and the definition of $s_{\inf}$, and the latter one using also that $\tilde{s}$ verifies Equation (3) and the definition of $\underline{s}_1^i$.

As a consequence, the proposed deviation is profitable as desired. ■

B Multidimensional Type Models

In this appendix we extend our analysis to a family of problems with a multidimensional type space. We shall show that under certain assumptions the analysis of these models can be done with an equivalent model with a one-dimensional type space. This analysis allows the extension of the models in Section 5.

Suppose that Bidder i's private information is a three dimensional vector $\hat{s}_i = (\hat{t}_i, \hat{q}_i, \hat{e}_i)$ that it is drawn according to an independent distribution $\hat{F}_i$ with support in a bounded set $\hat{S}_i \subset \mathbb{R}^3$. We shall assume that this distribution is such that the induced distribution of $\hat{t}_i + \hat{q}_i$, say $\overline{F}_i$, has a strictly positive density $\overline{f}_i$ in all the support $\overline{S}_i \subset \mathbb{R}$. Denote by $\hat{S} = \prod_{i=1}^n \hat{S}_i$ and by $\overline{S} = \prod_{i=1}^n \overline{S}_i$. We assume that Bidder i gets utility $\hat{t}_i + \sum_{j=1}^{n} \hat{q}_j - b$ if she wins and pays b, and utility $-\hat{e}_j - b$ if $j \neq i$ wins and Bidder i pays b.

By the revelation principle, there is no loss of generality in restricting to direct mechanisms. A *direct mechanism* is a pair $(\hat{p}, \hat{x})$, where $\hat{p} : \hat{S} \to [0,1]^n$ and $\hat{x} : \hat{S} \to \mathbb{R}^n$ such that $\sum_{i=1}^{n} \hat{p}_i(\hat{s}) = 1$ and where $\hat{p}_i(\hat{s})$ denotes the probability that Bidder i gets the good and $\hat{x}_i(\hat{s})$ denotes the payments of i to the auctioneer when the announced vector of types is equal to $\hat{s}$. We shall refer to $\hat{p}$ as an allocation*.

The expected utility of Bidder i with type $\hat{s}_i$ that reports $\hat{s}'_i$ when all the other bidders report truthfully is equal to:

$$\hat{U}_i(\hat{s}_i, \hat{s}'_i) \equiv \hat{Q}_i(\hat{s}'_i, \hat{p})(\hat{t}_i + \hat{q}_i) + \hat{\Psi}_i(\hat{s}'_i, \hat{p}, \hat{x}),$$

where

$$\hat{Q}_i(\hat{s}_i, \hat{p}) \equiv \int_{\hat{S}_{-i}} \hat{p}_i(\hat{s}_i, \hat{s}_{-i})\, d\hat{F}_{-i}(\hat{s}_{-i}),$$

and,

$$\hat{\Psi}_i(\hat{s}_i, \hat{p}, \hat{x}) \equiv \int_{\hat{S}_{-i}} \left(\hat{p}_i(\hat{s}_i, \hat{s}_{-i}) \sum_{j \neq i} \hat{q}_j - \hat{x}_i(\hat{s}_i, \hat{s}_{-i}) - \sum_{j \neq i} \hat{p}_j(\hat{s}_i, \hat{s}_{-i}) \hat{e}_j \right) d\hat{F}_{-i}(\hat{s}_{-i}),$$

for $\hat{F}_{-i}(\hat{s}_{-i}) \equiv \prod_{j \neq i} \hat{F}_j(\hat{s}_j)$ and $\hat{S}_{-i} \equiv \prod_{j \neq i} \hat{S}_j$.

Thus, we say that an allocation* $\hat{p} : S \to [0,1]^n$ is *feasible** if there exists a direct mechanism $(\hat{p}, \hat{x})$ that satisfies the following Bayesian incentive compatibility constraint:

$$\hat{U}_i(\hat{s}_i, \hat{s}_i) = \sup_{\hat{s}'_i \in \hat{S}_i} \{\hat{U}_i(\hat{s}_i, \hat{s}'_i)\},$$

for all $\hat{s}_i \in \hat{S}_i$ and $i \in N$.

We shall show that we can study second best efficiency in the model of this section, using the results in the model of Section 3:

Definition: Let the following be the *uni-dimensional equivalent* to a model as in Section 3 in which for all $i \in N$ and $s_i \in \overline{S}_i$:

$$S_i = \overline{S}_i,\; F_i(s_i) = \overline{F}_i(s_i),$$

$$t_i(s_i) = \int_{\hat{\mathcal{S}}_i(s_i)} \hat{t}_i \, \frac{d\hat{F}_i(\hat{s}_i)}{\overline{f}_i(s_i)},\; q_i(s_i) = \int_{\hat{\mathcal{S}}_i(s_i)} \hat{q}_i \, \frac{d\hat{F}_i(\hat{s}_i)}{\overline{f}_i(s_i)},\text{ and } e_i(s_i) = \int_{\hat{\mathcal{S}}_i(s_i)} \hat{e}_i \, \frac{d\hat{F}_i(\hat{s}_i)}{\overline{f}_i(s_i)},$$

where $\hat{\mathcal{S}}_i(s_i) \equiv \{\hat{s}_i \in \hat{S}_i : \hat{t}_i + \hat{q}_i = s_i\}$.

Definition: Let the *uni-dimensional version* of an allocation* $\hat{p}$ be a function $p : \overline{S} \to [0,1]^n$ where

$$p_i(s) = \int_{\hat{\mathcal{S}}_1(s_1)} \ldots \int_{\hat{\mathcal{S}}_n(s_n)} \hat{p}(\hat{s}) \, \frac{d\hat{F}_n(\hat{s}_n)}{\overline{f}_n(s_n)} \ldots \frac{d\hat{F}_1(\hat{s}_1)}{\overline{f}_1(s_1)}.$$

Lemma 9. *The allocation* $\hat{p}$ is feasible* if and only if its uni-dimensional version p is feasible and $\hat{Q}_i(\hat{s}_i, \hat{p}) \in [Q_i(\hat{t}_i + \hat{q}_i, p)^-, Q_i(\hat{t}_i + \hat{q}_i, p)^+]$ for any $\hat{s}_i \in \hat{S}_i$ and $i \in N$.*[36]

Proof. Note that using the definition of the uni-dimensional version, we can show that,

$$Q(s_i, p) = \int_{\hat{\mathcal{S}}_i(s_i)} \hat{Q}_i(\hat{s}_i, \hat{p}) \, \frac{d\hat{F}_i(\hat{s}_i)}{\overline{f}_i(s_i)}$$

where p is the uni-dimensional version of $\hat{p}$. Thus, by application of Lemma 1, p is feasible and $\hat{Q}_i(\hat{s}_i, \hat{p}) \in [Q_i(\hat{t}_i + \hat{q}_i, p)^-, Q_i(\hat{t}_i + \hat{q}_i, p)^+]$ if and only if there exists a vector of increasing functions $\mathcal{Q}_i : \overline{S}_i \to [0, 1]$, $i \in N$, such that $\hat{Q}_i(\hat{s}_i, \hat{p}) \in [\mathcal{Q}_i(t_i + q_i)^-, \mathcal{Q}_i(t_i + q_i)^+]$ for any $\hat{s}_i \in \hat{S}_i$ and $i \in N$, or what is the same, if and only if there exists a set of increasing convex functions $\hat{v}_i : \overline{S}_i \to \mathbb{R}_+$, $i \in N$, such that $\hat{Q}_i(\hat{s}_i, \hat{p}) \in \partial\hat{v}(\hat{t}_i + \hat{q}_i)$ for any $\hat{s}_i \in \hat{S}_i$ and $i \in N$, see Rockafellar (1970).

Thus, to prove the lemma it is sufficient to show the following equivalent statement:

> The allocation* $\hat{p}$ is feasible* if and only if there exists a set of increasing convex functions $\hat{v}_i : \overline{S}_i \to \mathbb{R}_+$, $i \in N$, such that $\hat{Q}_i(\hat{s}_i, \hat{p}) \in \partial\hat{v}(\hat{t}_i + \hat{q}_i)$ for any $\hat{s}_i \in \hat{S}_i$ and $i \in N$.

We first prove the "only if"-part. Suppose a feasible* allocation* $\hat{p} : \hat{S} \to [0, 1]^n$, and let $V_i(\hat{s}_i) \equiv \hat{U}_i(\hat{s}_i, \hat{s}_i)$. Then,

$$\begin{aligned}
V_i(\hat{s}_i) &\geq \hat{U}_i(\hat{s}_i, \hat{s}'_i) \\
&= \hat{Q}_i(\hat{s}'_i, \hat{p})(\hat{t}_i + \hat{q}_i) + \hat{\Psi}_i(\hat{s}'_i, \hat{p}, \hat{x}) \\
&= \hat{Q}_i(\hat{s}'_i, \hat{p})(\hat{t}'_i + \hat{q}'_i) + \hat{\Psi}_i(\hat{s}'_i, \hat{p}, \hat{x}) + \hat{Q}_i(\hat{s}'_i, \hat{p})(\hat{t}_i + \hat{q}_i - \hat{t}'_i - \hat{q}'_i) \\
&= V_i(\hat{s}'_i) + \hat{Q}_i(\hat{s}'_i, \hat{p})(\hat{t}_i + \hat{q}_i - \hat{t}'_i - \hat{q}'_i),
\end{aligned}$$

for all $\hat{s}_i, \hat{s}'_i \in \hat{S}_i$, $i \in N$, and some $\hat{x} : \hat{S} \to \mathbb{R}^n$.

[36] We denote by $Q_i(x_0, p)^-$ and $Q_i(x_0, p)^+$ the limits

$$\lim_{\substack{x \to x_0 \\ x < x_0}} Q_i(x, p) \quad \text{and} \quad \lim_{\substack{x \to x_0 \\ x > x_0}} Q_i(x, p)$$

respectively. To avoid problems at the infimum and supremum of $\overline{S}_i$, we shall adopt the convention that $Q_i(\inf \overline{S}_i, p)^- = Q_i(\inf \overline{S}_i, p)$ and $Q_i(\sup \overline{S}_i, p)^+ = Q_i(\sup \overline{S}_i, p)$. We adopt the same notation and conventions for the functions $\mathcal{Q}_i$ in the proof of the lemma.

The above inequality applied twice, one with the roles of $\hat{s}_i$ and $\hat{s}_i'$ interchanged, to any two vectors $\hat{s}_i, \hat{s}_i' \in \hat{S}_i$ such that $\hat{t}_i + \hat{q}_i = \hat{t}_i' + \hat{q}_i'$, implies that $V_i(\hat{s}_i) = V_i(\hat{s}_i')$. Consequently, there exists a function $v_i : \overline{S}_i \to \mathbb{R}$ such that $V_i(\hat{s}_i) = v_i(\hat{t}_i + \hat{q}_i)$ for any $\hat{s}_i \in \hat{S}_i$. Moreover, v_i is convex because V_i is convex. Note that V_i must be convex because it is equal to the maximum of some linear functions by the incentive compatibility constraint. Finally, note that the above inequality together with the definition of v_i implies that $v_i(y) \geq v_i(\hat{t}_i + \hat{q}_i) + \hat{Q}_i(\hat{s}_i, \hat{p})(y - (\hat{t}_i + \hat{q}_i))$ for any y in $\overline{S}_i$. This means that $\hat{Q}_i(\hat{s}_i, \hat{p}) \in \partial v_i(\hat{t}_i + \hat{q}_i)$ as desired.

To prove the "if"-part, suppose a function $\tilde{v}$ that satisfies the conditions of the lemma for an allocation* $\hat{p}$, and let $\hat{x} : \hat{S} \to \mathbb{R}^n$ be such that $\hat{\Psi}_i(\hat{s}_i, \hat{p}, \hat{x}) = \hat{v}_i(\hat{t}_i + \hat{q}_i) - (\hat{t}_i + \hat{q}_i)\hat{Q}_i(\hat{s}_i, \hat{p})$ for any $i \in N$. We shall show that the direct mechanism $(\hat{p}, \hat{x})$ satisfies the Bayesian incentive compatibility constraints. To see why, note that for any $\hat{s}_i, \hat{s}_i' \in \hat{S}_i$:

$$\begin{aligned}
V_i(\hat{s}_i) = \tilde{v}_i(\hat{t}_i + \hat{q}_i) \quad &\geq \quad \tilde{v}_i(\hat{t}_i' + \hat{q}_i') + \hat{Q}_i(\hat{s}_i', \hat{p})(\hat{t}_i + \hat{q}_i - \hat{t}_i' - \hat{q}_i') \\
&= \quad \hat{Q}_i(\hat{s}_i', \hat{p})(\hat{t}_i' + \hat{q}_i') + \hat{\Psi}_i(\hat{s}_i', \hat{p}, \hat{x}) + \hat{Q}_i(\hat{s}_i', \hat{p})(\hat{t}_i + \hat{q}_i - \hat{t}_i' - \hat{q}_i') \\
&= \quad \hat{Q}_i(\hat{s}_i', \hat{p})(\hat{t}_i + \hat{q}_i) + \hat{\Psi}_i(\hat{s}_i', \hat{p}, \hat{x}) = \hat{U}_i(\hat{s}_i, \hat{s}_i'),
\end{aligned}$$

where the inequality is a consequence of $\hat{Q}_i(\hat{s}_i', \hat{p}) \in \partial \tilde{v}_i(\hat{t}_i' + \hat{q}_i')$. ■

Now, we can state the main result of this appendix:

Proposition 10. *A feasible* allocation* $\hat{p}^*$ is a solution to the problem,*

$$\max \int_{\hat{S}} \sum_{i=1}^{n} \left(\hat{t}_i + \sum_{j=1}^{n} \hat{q}_j - (n-1)\hat{e}_i \right) \hat{p}_i(\hat{s})\, d\hat{F}(\hat{s}),$$

if and only if its uni-dimensional version p^* is second best efficient for the uni-dimensional equivalent model.

Proof. Take any $\hat{p}$ feasible and such that $\sum_{i \in N} \hat{p}_i(\hat{s}) = 1$ for all $\hat{s} \in \hat{S}$, and denote by p its one-dimensional version. Then we can deduce the lemma from the following sequence of algebraic transformations and the fact that by Lemma 9, feasibility* of $\hat{p}$ requires feasibility of p:

$$
\begin{aligned}
&\int_{\hat{S}} \sum_{i=1}^{n} \left(\hat{t}_i + \sum_{j=1}^{n} \hat{q}_j - (n-1)\hat{e}_i \right) \hat{p}_i(\hat{s})\, d\hat{F}(\hat{s}) = \\
&\int_{\hat{S}} \sum_{i=1}^{n} \left(\hat{t}_i - (n-1)\hat{e}_i\right) \hat{p}_i(\hat{s})\, \hat{F}(d\hat{s}) + \int_{\hat{S}} \sum_{i=1}^{n} \hat{p}_i(\hat{s}) \sum_{j=1}^{n} \hat{q}_j\, d\hat{F}(\hat{s}) = \\
&\sum_{i=1}^{n} \int_{\hat{S}_i} \left(\hat{t}_i - (n-1)\hat{e}_i\right) \hat{Q}_i(\hat{s}_i, \hat{p})\, d\hat{F}_i(\hat{s}_i) + \int_{\hat{S}} \sum_{j=1}^{n} \hat{q}_j\, d\hat{F}(\hat{s}) = \\
&\sum_{i=1}^{n} \int_{\overline{S}_i} \int_{\hat{\mathcal{S}}_i(s_i)} \left(\hat{t}_i - (n-1)\hat{e}_i\right) \hat{Q}_i(\hat{s}_i, \hat{p})\, \frac{d\hat{F}_i(\hat{s}_i)}{\overline{f}_i(s_i)} \overline{f}_i(s_i)\, ds_i + \sum_{j=1}^{n} \int_{\hat{S}} \hat{q}_j\, d\hat{F}(\hat{s}) = \\
&\sum_{i=1}^{n} \int_{\overline{S}_i} \int_{\hat{\mathcal{S}}_i(s_i)} \left(\hat{t}_i - (n-1)\hat{e}_i\right) \hat{Q}_i(\hat{s}_i, \hat{p})\, \frac{d\hat{F}_i(\hat{s}_i)}{\overline{f}_i(s_i)} \overline{f}_i(s_i)\, ds_i + \sum_{j=1}^{n} \int_{\hat{S}_j} \hat{q}_j\, d\hat{F}_j(\hat{s}_j) = \\
&\sum_{i=1}^{n} \int_{\overline{S}_i} (t_i(s_i) - (n-1)e_i(s_i))\, Q_i(s_i, p) \overline{f}_i(s_i)\, ds_i + \sum_{j=1}^{n} \int_{\overline{S}_j} \int_{\hat{\mathcal{S}}_j(s_j)} \hat{q}_j\, \frac{d\hat{F}_j(\hat{s}_j)}{\overline{f}_j(s_j)} \overline{f}_j(s_j)\, ds_j = \\
&\sum_{i=1}^{n} \int_{\overline{S}} (t_i(s_i) - (n-1)e_i(s_i))\, p(s) \overline{f}(s) ds + \sum_{j=1}^{n} \int_{\overline{S}_j} q_j(s_j)\, \overline{f}_j(s_j)\, ds_j = \\
&\sum_{i=1}^{n} \int_{\overline{S}} (t_i(s_i) - (n-1)e_i(s_i))\, p(s) \overline{f}(s) ds + \sum_{j=1}^{n} \int_{\overline{S}_j} q_j(s_j) \sum_{i=1}^{n} p_i(s)\, \overline{f}_j(s_j)\, ds_j = \\
&\sum_{i=1}^{n} \int_{\overline{S}} (t_i(s_i) - (n-1)e_i(s_i))\, p(s) \overline{f}(s) ds + \sum_{i=1}^{n} \int_{\overline{S}} \left(\sum_{j=1}^{n} q_j(s_j) \right) p_i(s)\, \overline{f}(s)\, ds = \\
&\sum_{i=1}^{n} \int_{\overline{S}} \left(t_i(s_i) + \sum_{j=1}^{n} q_j(s_j) - (n-1)e_i(s_i) \right) p_i(s) \overline{f}(s) ds,
\end{aligned}
$$

where we have used: in step 2, that $\sum_{j=1}^{n} \hat{p}(\hat{s}) = 1$; in step 4, independency of the $\hat{F}_i$'s; in Step 5, that $\hat{Q}_i(\hat{s}_i, \hat{p}) = Q_i(s_i, p)$ a.e., see below; and in step 7, that $\sum_{i=1}^{n} p_i(s) = 1$.

To see why $\hat{Q}_i(\hat{s}_i, \hat{p}) = Q_i(s_i, p)$ a.e., note that Lemma 9 and $\hat{p}$ feasible* imply that p is feasible, and thus $Q_i(., p)$ increasing by Lemma 1, and hence continuous a.e. As a consequence, applying Lemma 9 again we get that $\hat{Q}_i(\hat{s}_i, \hat{p}) = Q_i(s_i, p)$. ■

Finally, we provide as an application an example that generalizes the model in Section 5.2 to the cases not covered there, i.e. $\rho + \underline{t} \leq \overline{t}$.

Example 12. *Suppose a set $N = \{1, 2, ..., n\}$, and that $\hat{F}_1$ has full support on $\hat{S}_1 = [\underline{t}, \overline{t}] \times \{0, \rho\} \times \{0\}$ and $\hat{F}_i$ on $\hat{S}_i = [\underline{t}, \overline{t}] \times \{0\} \times \{0\}$ for $i \neq 1$. Suppose also that $\hat{t}_1$ and $\hat{q}_1$ are independent and $\hat{t}_1$ has* *a marginal distribution G with strictly positive density g over the support and $\hat{q}_1$ takes value* 0 *with probability $\alpha \in (0,1)$ and ρ with probability $1-\alpha$. Finally, suppose $\rho + \underline{t} \leq \bar{t}$.*

In the above example $q_1(s_1) = \rho \frac{g(s_1-\rho)(1-\alpha)}{g(s_1)\alpha + g(s_1-\rho)(1-\alpha)}$, if $s_1 > \rho$ and zero, otherwise; $t_1(s_1) = s_1 - q_1(s_1)$, $e_1(s_1) = 0$, $t_i(s_i) = s_i$, $q_i(s_i) = e_i(s_1) = 0$ for $i \neq 1$. Thus, its uni-dimensional version violates the single crossing condition. To see why, apply Lemma 2 to $s_1 = \rho + \underline{t} - \epsilon$ and $s_1' = \rho + \underline{t} + \epsilon$ noting that $h_1(s_1) = t_1(s_1)$, and thus that $h_1(\rho + \underline{t} - \epsilon) = \rho + \underline{t} - \epsilon > \rho + \underline{t} + \epsilon - q_1(\rho + \underline{t} + \epsilon) = h_1(\rho + \underline{t} + \epsilon)$ for $\epsilon > 0$ and small enough.

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