Household behavior and individual autonomy

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Abstract

The paper proposes a model of household behavior with both private and public consumption where the spouses independently maximize their utilities, but taking into account, together with their own individual budget constraints, the collective household budget constraint with public goods evaluated at Lindahl prices. The Lagrange multipliers associated with these constraints are used to parameterize the set of equilibria, in addition to the usual parameterization by income shares. The proposed game generalizes both the 'collective' model of household behavior and the non-cooperative game with voluntary contributions to public goods.

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1 Introduction

This paper addresses the issue of group behavior for a set of individuals consuming both privately and jointly. It presents a model specifically formulated for

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simplicity in terms of two-person household behavior, but which can be straightforwardly extended to larger groups. The natural starting point of such analysis is to discard the so-called unitary approach, which assumes that the household acts as if it were maximizing a single utility function, possibly a well-defined social welfare function.

Two alternative non-unitary approaches have been used in the literature on household behavior:[1] the fully cooperative, which entails Pareto-efficiency of household decisions, and the fully non-cooperative, with household decisions resulting from a Nash equilibrium of some game where each individual maximizes utility under a personal budget constraint. The first approach started with models based on axiomatic bargaining theory (Manser and Brown, 1980, McElroy and Horney, 1981), which result in Pareto-efficient outcomes varying according to the specified threat point, itself possibly determined by the solution of a non-cooperative game (Lundberg and Pollak, 1993, Chen and Woolley, 2001). Subsequent papers proposed 'collective' models in order to explore the restrictions on observable household behavior implied by the assumption of Pareto efficiency, without explicitly referring to a specific bargaining or other decision making process (Chiappori, 1988, 1992, Browning and Chiappori, 1998). The second approach is based on two types of non-cooperative games, generally leading to inefficient equilibrium outcomes. In the first type each individual is supposed to be responsible for a 'separate sphere' of joint consumption (Lundberg and Pollak, 1993). In the second type, each individual voluntarily contributes to any public good (Ulph, 1988, Chen and Woolley, 2001, Lechene and Preston, 2005, Browning, Chiappori and Lechene, 2006b).

In this paper, we want to propose a more general strategic approach, which includes as sub-cases fully cooperative solutions and fully non-cooperative equilibrium outcomes, together with intermediate cases. This more general approach will provide a double parameterization of the set of equilibria, in terms of the income distribution between the two spouses (allowing to move along the utility

[1] A synthesis of the field is provided by Donni (2008b). See also Donni (2008a) for a general presentation of the so-called 'collective' models of household behavior.

possibility frontier) and in terms of their autonomy in spending decisions (implying downward movements below that frontier, as autonomy increases). The two extreme cases, the one where both spouses have full autonomy and the one where they have none, correspond to a fully non-cooperative and a fully cooperative outcome, respectively. By filling the gap between these two extreme cases, our approach provides a theoretical development that has already been hoped for in the literature.[2]

The autonomy we are referring to is related to the way in which the household organizes its finances. An important distinction appearing in empirical sociological studies (for instance two surveys of the International Social Survey Programme of 1994 and 2002, analyzing representative samples of 38 countries) is the one between money management "systems in which couples operate more or less as single economic units" and "individualized or privatized systems in which couples operate largely as two separate, autonomous economic units" (Vogler, Brockmann and Wiggins, 2006, Pahl, 2008). The former comprehend systems in which one of the two spouses manages all the household money, except possibly a fraction left to the other spouse for his/her personal expenses, but also systems (used by more than half of the couples surveyed by the ISSP) in which all the household money is pooled in a common bank account and managed jointly by the two spouses, not necessarily on a 50-50 basis. These systems afford a good illustration of the economic household models of both the unitary and the fully cooperative approaches. In contrast with them, we

[2] For instance, in a recent paper focusing on the household decisions concerning labor supply, Del Boca and Flinn (2006) write: "We view labor supply outcomes as either being associated with a particular utility outcome on the Pareto frontier (the one chosen under symmetric Nash bargaining) or to be associated with the noncooperative equilibrium point. In reality there are a continuum of points that dominate the noncooperative equilibrium point and that do not lie on the Pareto frontier, however developing a model that allows such outcomes to enter the choice set of the household seems beyond our means" (pp. 1-2). Cf. also Lechene and Preston (2005): "neither the assumption of fully efficient cooperation nor of complete absence of collaboration is likely to be an entirely accurate description of typical household spending behaviour and analysis of such extreme cases can be seen as a first step towards understanding of a more adequate model" (p. 19).

find two kinds of individualized systems. The first one is the 'independent management system' in which each spouse keeps his/her own income separate and has responsibility for different items of household expenditure. This system may be easily approached by fully non-cooperative economic household models displaying 'separate spheres', either exogenously or endogenously. The other individualized system (used by 13% of the couples in the ISSP 1994 survey, 17% in the 2002 survey) is "the *partial pool* in which couples pool some of their income to pay for collective expenditure and keep the rest separate to spend as they choose" (Vogler, Brockmann and Wiggins, 2006).[3]

In our approach, the autonomy of each spouse can be evaluated in terms of the proportion of his/her contribution to the expenditure on public goods which is directly effected through individual purchases in the market, hence properly 'spent as he/she chooses'. The rest of the expenditure on public goods is supposed to be paid from a 'pool' of financial contributions of the spouses computed according to Lindahl prices corresponding to their relative incomes. The degree of autonomy of each partner may be preliminarily agreed upon within the household or else be determined by social norms. But it may also emerge spontaneously as one of the characteristics of a specific equilibrium of a non-cooperative game played by the spouses. We assume that in such game they both take into account not only their own personal budget constraint, but also the collective budget constraint computed at Lindahl prices. At equilibrium, the weight of each constraint is evaluated by a Lagrange multiplier, and the relative weight of the personal constraint may be taken as an index of the degree of autonomy attained by the corresponding spouse.[4]

[3] The terminology 'partial pooling' or else 'joint pooling', applied by sociologists to specific systems of financial management within the household, should not be confused with the terminology 'income pooling' used by economists to designate situations in which households behave *as if* their income was pooled, so that it does not matter which member receives the income (see Bradbury, 2004, p.504).

[4] A similar procedure has been used to parameterize the set of equilibria of oligopolistic games by d'Aspremont, Dos Santos Ferreira and Gérard-Varet (2007) and by d'Aspremont and Dos Santos Ferreira (2009).

In Section 2, we will briefly present the household decision model, in both its cooperative and non-cooperative versions, and develop our own general non-cooperative approach. In Section 3, we will analyze local and observable properties of the household demand function which may be used to discriminate among the different regimes of household behavior. In Section 4, we will exploit an example already used by Browning, Chiappori and Lechene (2006), in order to illustrate the implications of varying degrees of autonomy. We conclude in Section 5.

2 The household decision model

We study a two-adult household, consuming goods that are either private or public (within the household). Denote by A and B the two household members, and let $\left(q^A, q^B\right) \in \mathbb{R}_+^{2n}$ be the vector of consumption by the two members of n private goods and $Q \in \mathbb{R}_+^m$ the consumption vector of m public goods. The preferences of each individual J ($J = A, B$) are represented by a utility function $U^J\left(q^J, Q\right)$, which is defined on $\mathbb{R}_+^n \times \mathbb{R}_+^m$, increasing and strongly quasi-concave.[5] Each member J of the household is supposed to receive an initial income $Y^J \geq 0$. The total income of the household is $Y = Y^A + Y^B$. We want to study how the household decides on its total consumption given the vector of private good prices $p \in \mathbb{R}_{++}^n$ and the vector of public good prices $P \in \mathbb{R}_{++}^m$. The first private good, assumed to be desired in any household environment, is taken as numéraire ($p_1 = 1$).

2.1 The efficient intra-household decision approach

If a *collective point of view* is adopted inside the household, and a *Pareto-optimal decision* is looked for, the usual approach is to fix a parameter $\mu \in [0, 1]$ and

[5] For simplicity, we shall stick to the egoistic case where the utility of each spouse only depends upon his/her own consumption, either private or public.

solve a programme of the type:

$$\begin{aligned} &\max_{(q^A,q^B,Q)\in\mathbb{R}_+^{2n+m}} \ \mu U^A\left(q^A,Q\right)+(1-\mu)\,U^B\left(q^B,Q\right) \\ &\text{s.t. } p\left(q^A+q^B\right)+PQ\leq Y. \end{aligned} \tag{1}$$

All the Pareto-optimal decisions can be characterized by varying the Pareto weight μ. According to the specific collective decision process (*e.g.* Nash bargaining with a Nash non-cooperative equilibrium as the threat point), the Pareto weight may depend upon the environmental variables (p,P,Y) as well as on distributional factors, either environmental or not, but not affecting the individual preferences (*e.g.* parameters determining the threat point). As well discussed in Browning, Chiappori and Lechene (2006a), if the Pareto weight is independent of (p,P,Y), while possibly depending on distributional factors, then the efficient intra-household decision approach reduces to the unitary model, in the sense that the household decides as a single decision unit, maximizing under the common budget constraint $pq+PQ\leq Y$ the utility function

$$\widetilde{U}\left(q,Q\right)\equiv\max_{\left\{(q^A,q^B)\in\mathbb{R}_+^{2n}\,|\,q^A+q^B=q\right\}}\mu U^A\left(q^A,Q\right)+(1-\mu)\,U^B\left(q^B,Q\right), \tag{2}$$

where the function $\widetilde{U}$ may be affected at most by the distributional factors. However, as soon as the Pareto weights do depend on the environmental variables (p,P,Y) (*e.g.* through the determination of the threat point), the function $\widetilde{U}$ becomes a 'generalized' utility function, depending through μ on prices and household income, so that the collective model must indeed be distinguished from the unitary model.

For every J, let $\tau^J\left(q^J,Q\right)$ denote the marginal-willingness-to-pay vector for the public goods in terms of the numéraire:

$$\tau^J\left(q^J,Q\right)\equiv\frac{1}{\partial_{q_1}U^J\left(q^J,Q\right)}\partial_Q U^J\left(q^J,Q\right). \tag{3}$$

Under usual regularity conditions, the Pareto-optimal decisions (corresponding to all values of μ in $[0,1]$) are characterized by the Bowen-Lindahl-Samuelson conditions:

$$\tau^A\left(q^A,Q\right)+\tau^B\left(q^B,Q\right)=P \tag{4}$$

together with the budget condition $p\left(q^A+q^B\right)+PQ=Y$. With each such solution (corresponding to some value of μ), one can thus associate *Lindahl* (or *personalized*) *prices*

$$P^J \equiv \tau^J\left(q^J,Q\right),\ J=A,B \tag{5}$$

(such that $P^A+P^B=P$) and individual expenditures

$$\rho^J Y \equiv pq^J+P^JQ \tag{6}$$

(such that $\rho^A+\rho^B=1$). Both the Lindahl prices and the expenditure shares ρ^J $(J=A,B)$ are functions of $(\mu,p,P,Y)\in[0,1]\times\mathbb{R}^{n+m}_{++}\times\mathbb{R}_+$ (where μ may itself depend upon (p,P,Y)). In particular, we may take $\rho^A=\rho\left(\mu,p,P,Y\right)$ as the *sharing rule* applying to the household, and interpret P^JQ as J's tribute to the household expenditure in public goods.

In the following, we are going to reverse this procedure and start, for a given environment (p,P,Y), by fixing the income distribution $\left(Y^A,Y^B\right)=(\rho,1-\rho)\,Y$ (with $\rho\in[0,1]$).[6] With each such distribution, one can associate a *Lindahl equilibrium* $\left(P^A,P^B,Q^A,Q^B,q^A,q^B\right)\in\mathbb{R}^{4m+2n}_+$ (where Q^J denotes the vector of public consumptions desired by J), such that

(i) for $J=A,B$,

$$\left(q^J,Q^J\right)\in\arg\max_{\left(\widetilde{q}^J,\widetilde{Q}^J\right)\in\mathbb{R}^{n+m}_+} U^J\left(\widetilde{q}^J,\widetilde{Q}^J\right) \quad \text{s.t. } p\widetilde{q}^J+P^J\widetilde{Q}^J\leq Y^J; \tag{7}$$

(ii)

$$P^A+P^B=P \text{ and } Q^A=Q^B=Q. \tag{8}$$

By varying $\rho=Y^A/Y$ on $[0,1]$ all the Pareto-optimal decisions $\left(q^A,q^B,Q\right)$ may be obtained, according to the second welfare theorem, in a 'decentralized' way, together with the corresponding Pareto weights (which we will leave implicit).

[6] Notice that this income distribution depends upon environmental factors but may itself be determined by the distribution of power within the household.

For the sake of later comparisons, recall the first order conditions for a Lindahl equilibrium (for $J = A, B$):

$$\begin{aligned} \frac{1}{\partial_{q_1} U^J\left(q^J, Q^J\right)} \partial_q U^J\left(q^J, Q^J\right) &\leq p \\ \tau^J\left(q^J, Q^J\right) &\leq P^J \\ pq^J + P^J Q^J &= Y^J, \end{aligned} \tag{9}$$

with an equality for any private good i s.t. $q_i^J > 0$ or any public good k s.t. $Q_k^J > 0$. Together with condition (ii) in the definition of a Lindahl equilibrium, they entail the Bowen-Lindahl-Samuelson conditions for any interior solution.

2.2 The fully non-cooperative approach

An alternative non-unitary model of household decisions is non-cooperative,[7] with each spouse having full autonomy in allocating income to public consumption. More precisely, we may define a *game with voluntary contributions to public goods* where each spouse J chooses a strategy $\left(q^J, g^J\right) \in \mathbb{R}_+^{n+m}$ (q^J denoting J's private consumptions and g^J his/her contributions to public goods) in order to solve the programme:

$$\begin{aligned} &\max_{\left(q^J, g^J\right) \in \mathbb{R}_+^{n+m}} U^J\left(q^J, g^J + g^{-J}\right) \\ &\text{s.t. } pq^J + Pg^J \leq Y^J. \end{aligned} \tag{10}$$

A Nash equilibrium of this game can be characterized by the first order conditions (for $J = A, B$):

$$\begin{aligned} \frac{1}{\partial_{q_1} U^J\left(q^J, g^J + g^{-J}\right)} \partial_q U^J\left(q^J, g^J + g^{-J}\right) &\leq p \\ \tau^J\left(q^J, g^J + g^{-J}\right) &\leq P \\ pq^J + Pg^J &= Y^J, \end{aligned} \tag{11}$$

with an equality for any private good i s.t. $q_i^J > 0$ or any public good k s.t. $g_k^J > 0$.

[7] See Ulph (1988), Chen and Woolley (2001), Lechene and Preston (2005), Browning, Chiappori and Lechene (2006b).

This approach is in sharp contrast with the cooperative approach where each individual has to choose the same *aggregate* quantity of each public good (in the centralised version), so as to maximise a *collective* objective, determined by the Pareto weights. In that collective approach, the spouses lose their autonomy in allocating income to public consumption. Even in the decentralised version of the efficient household decision approach, the spouses contribute to the collective acquisition of public goods by paying a tribute computed according to Lindahl prices imposed upon them.

Both household decision processes, the Pareto efficient behavior (whether centralised or decentralised) and the fully non-cooperative behavior, appear as extreme in terms of the autonomy left to the individuals. In reality, intermediate forms of household behavior, with for instance incomes partially transferred to a common bank account for collective decisions, can often be observed. In order to cover such intermediate forms, we will combine the two approaches in the next subsection.

2.3 The non-cooperative approach with Lindahl prices

We let each spouse decide under two different budget constraints, one personal, the other collective, together with a feasibility vector constraint stating that the desired public consumptions cannot be higher than the sum of the spouses' planned contributions. To be explicit, let us define a *household game with Lindahl prices*, associated with the environment $(p, P, Y) \in \mathbb{R}^{n+m}_{++} \times \mathbb{R}_+$, a given income distribution $\left(Y^A, Y^B\right) = (\rho, 1-\rho)\, Y$ (with $\rho \in [0,1]$) and corresponding Lindahl prices $\left(P^A, P^B\right) \in \mathbb{R}^{2m}_+$. Each spouse $J \in \{A, B\}$ is supposed to choose a vector of private consumptions $q^J \in \mathbb{R}^n_+$, a vector of voluntary contributions to public goods $g^J \in \mathbb{R}^m_+$ and a vector of desired public consumptions $Q^J \in \mathbb{R}^m_+$,

solving the programme:

$$\begin{aligned} &\max_{(q^J,g^J,Q^J)\in\mathbb{R}_+^{n+2m}} U^J\left(q^J,Q^J\right) \\ &pq^J+Pg^J\leq Y^J \\ &p\left(q^A+q^B\right)+P^AQ^A+P^BQ^B\leq Y \\ &Q^J\leq g^A+g^B. \end{aligned} \tag{12}$$

The first constraint is the personal budget constraint, stating that the individual income of spouse J should be enough to finance his/her private consumption plus the value, at market prices P, of his/her contribution to public consumption. The second constraint is the collective budget constraint, stating that the household income should be enough to cover the sum of the two private expenditures plus the value of the desired public consumption. This value is computed for each spouse J and each public good k by applying the Lindahl price P_k^J to the desired public consumption Q_k^J. The third constraint ensures the consistency of the two budget constraints, by restraining the desired public consumptions to be at most equal to the sum of the voluntary contributions of the two spouses.

A non-cooperative equilibrium of this game, consisting in vectors $\left(q^A,g^A,Q^A\right)$ and $\left(q^B,g^B,Q^B\right)$ that solve both individual programmes simultaneously and satisfy the feasibility constraints of the two spouses as equalities ($Q^A=Q^B=g^A+g^B$), is called a *household behavioral equilibrium.*

Proposition 1 *At a household behavioral equilibrium* $\left(q^A,g^A,Q,q^B,g^B,Q\right)$ *associated with Lindahl prices* $\left(P^A,P^B\right)$*, all the constraints in (12), for* $J=A,B$*, are satisfied as equalities.*

Proof. The feasibility constraint is satisfied as an equality by definition of a household behavioral equilibrium. By adding the two personal budget constraints and using $P^A+P^B=P$, we see that the collective budget constraint can be satisfied as a strict inequality only if one at least of the personal constraints, say the one of spouse A, is a strict inequality at equilibrium. But, in

this case, A could increase her utility by increasing simultaneously Q^A and g^A. Finally, if the collective budget constraint is satisfied as an equality, so are both personal budget constraints. ■

The first order conditions characterizing for agent J a household behaviorial equilibrium can be written as follows:

$$\begin{bmatrix} \partial_q U^J\left(q^J, Q\right) \\ \partial_Q U^J\left(q^J, Q\right) \end{bmatrix} \leq \lambda^J \begin{bmatrix} p \\ 0 \end{bmatrix} + \nu^J \begin{bmatrix} p \\ P^J \end{bmatrix} + \begin{bmatrix} 0 \\ \kappa^J \end{bmatrix}, \tag{13}$$

with equality for any coordinate i (resp. k) s.t. $q_i^J > 0$ (resp. $Q_k > 0$), and

$$\kappa^J \leq \lambda^J P, \tag{14}$$

with equality for any coordinate k s.t. $g_k^J > 0$. We thus obtain, for private good $i = 1, ..., n$, the condition $\partial_{q_i} U^J\left(q^J, Q\right) \leq \left(\lambda^J + \nu^J\right) p_i$ or, in terms of marginal rates of substitution,

$$\frac{\partial_{q_i} U^J\left(q^J, Q\right)}{\partial_{q_1} U^J\left(q^J, Q\right)} \leq p_i \tag{15}$$

with equality if $q_i^J > 0$. Also, the marginal willingness to pay public good $k = 1, ..., m$ with the numéraire, at equilibrium, is

$$\tau_k^J\left(q^J, Q\right) \equiv \frac{\partial_{Q_k} U^J\left(q^J, Q\right)}{\partial_{q_1} U^J\left(q^J, Q\right)} \leq \frac{\lambda^J P_k + \nu^J P_k^J}{\lambda^J + \nu^J} \equiv \theta^J P_k + \left(1 - \theta^J\right) P_k^J, \tag{16}$$

with equality if $g_k^J > 0$ (implying $Q_k > 0$). The parameter $\theta^J \equiv \lambda^J / \left(\lambda^J + \nu^J\right)$ is simply a normalized Lagrange multiplier associated with the personal budget constraint.

The parameter pair $\theta = \left(\theta^A, \theta^B\right) \in [0, 1]^2$ can be used to parameterize the set of household behavioral equilibria. A Lindahl equilibrium outcome and the outcome of a Nash equilibrium of the game with voluntary contributions to public goods are also outcomes of two extreme elements of this set, as formally stated in the following proposition.

Proposition 2 *Take environment (p, P, Y), income distribution $\left(Y^A, Y^B\right)$ and associated Lindahl equilibrium $\left(P^A, P^B, Q^A, Q^B, q^A, q^B\right)$. Outcome $\left(q^A, q^B, Q\right)$ with $Q = Q^A = Q^B$ is the outcome of a household behavioral equilibrium characterized by $\theta^A = \theta^B = 0$. Also, the outcome $\left(q^A, g^A, q^B, g^B\right)$ of a household*

behavioral equilibrium characterized by $\theta^A = \theta^B = 1$ *is a Nash equilibrium of the game with voluntary contributions to public goods.*

Proof. The collective constraint in programme (12) can be expressed as $pq^J + P^J Q^J \leq Y^J + \left(Y^{-J} - pq^{-J} - P^{-J}Q^{-J}\right)$, the expression in parentheses being nil if we take the Lindahl equilibrium values P^{-J}, Q^{-J} and q^{-J}. Since the personal budget constraints are not binding if $\theta^A = \theta^B = 0$, we can thus make programmes (12) coincide with programmes (7) characterizing the Lindahl equilibrium. Consequently, the tuple $\left(q^A, g^A, Q, q^B, g^B, Q\right)$ is a household behavioral equilibrium if the vectors g^A and g^B satisfy $g^A + g^B = Q$ and $Pg^A = P^A Q$ ($m^* + 1$ equations with $2m^*$ unknowns, where m^* is the number of actually consumed public goods). As to the second statement of the proposition, if $\left(q^A, g^A, Q, q^B, g^B, Q\right)$ is a household behavioral equilibrium characterized by $\theta^A = \theta^B = 1$, the collective budget constraint is not binding for both spouses, so that the pairs $\left(q^A, g^A\right)$ and $\left(q^B, g^B\right)$ solve the two programmes (10) of the game with voluntary contributions to public goods. ■

At a specific equilibrium, the parameter $\theta^J \equiv \lambda^J / \left(\lambda^J + \nu^J\right)$, the normalized shadow price associated with the personal budget constraint, may be seen as representing the *degree of autonomy* of spouse J. Indeed, equilibria with larger and larger values of θ^J imply that J is more and more concerned with his/her own personal budget, relative to the collective budget.

In this analysis, up to now, the degrees of autonomy of both spouses are fixed endogenously, *ex post*, as characteristics of a specific equilibrium. We may however invert the approach, and take the parameters as preliminarily and conventionally fixed, *ex ante,* within the household. To define the new corresponding game, one can assume that each spouse J contributes to the funding of public goods in two ways. On the one hand, J spends autonomously in the market place a share $\theta^J \in [0, 1]$ of the market value of his/her contribution to public consumption: $\theta^J Pg^J$. On the other hand, J remits to a common account, designed to finance public expenses in which both spouses concur, the complementary share of the value, now at Lindahl prices, of his/her contribution to

public consumption. This value is the sum of spouse J's Lindahl tax on his/her planned contribution ($P^J g^J$) plus the Lindahl tax the other spouse would have to pay on what she/he would like J's contribution to be ($P^{-J}\left(Q^{-J} - g^{-J}\right)$). Notice that by aggregating these values over the two spouses, we obtain, as expected, the sum of the two Lindahl taxes:

$$P^A g^A + P^B\left(Q^B - g^B\right) + P^B g^B + P^A\left(Q^A - g^A\right) = P^A Q^A + P^B Q^B. \quad (17)$$

The new game, which we shall call in the following the *θ-household game*, has the same strategies as the household game with Lindahl prices $\left(P^A, P^B\right)$, namely $\left(q^J, g^J, Q^J\right) \in \mathbb{R}_+^{n+2m}$ for $J = A, B$, and the programme of spouse J can be formulated as follows:

$$\begin{aligned} &\max_{(q^J, g^J, Q^J) \in \mathbb{R}_+^{n+2m}} U^J\left(q^J, Q^J\right) \quad (18)\\ &pq^J + \theta^J P g^J + \left(1 - \theta^J\right)\left[P^J g^J + P^{-J}\left(Q^{-J} - g^{-J}\right)\right] \leq Y^J,\\ &Q^J \leq g^A + g^B. \end{aligned}$$

As will become clear in the following proposition, this programme differs from programme (12) only in so far as the personal and the collective budget constraints are now merged into a single personal budget constraint. We show that the equilibria of the two games coincide for the same value of θ.

Proposition 3 *The tuple $\left(q^A, g^A, Q, q^B, g^B, Q\right)$ is a household behavioral equilibrium with associated parameters $\theta = \left(\theta^A, \theta^B\right)$ if and only if it is an equilibrium of the θ-household game.*

Proof. First notice that, since U^J is increasing, any solution to programme (18) must verify both constraints as equalities. We thus obtain in particular that an equilibrium $\left(q^A, g^A, Q^A, q^B, g^B, Q^B\right)$ of the θ-household game necessarily satisfies $Q^A = Q^B = Q = g^A + g^B$. Using in addition the equality $P = P^A + P^B$, we then see that the budget equation of the θ-household game coincides at equilibrium with the personal budget equation of the household game with Lindahl prices $\left(P^A, P^B\right)$, since

$$\theta^J P g^J + \left(1 - \theta^J\right)\left[P^J g^J + P^{-J}\left(Q^{-J} - g^{-J}\right)\right] = P g^J.$$

Also, the aggregation of the two budget equations naturally results in the collective budget equation, since

$$P\left(g^A + g^B\right) = P^A Q^A + P^B Q^B.$$

Finally, by simply reproducing the argument applied to programme (12), we see that the equilibrium of the θ-household game must satisfy the following first order conditions for the solutions of the two programmes (18) with $J = A, B$:

$$\begin{aligned} \frac{1}{\partial_{q_1} U^J\left(q^J, Q\right)} \partial_q U^J\left(q^J, Q\right) &\leq p \\ \tau^J\left(q^J, Q\right) &\leq \theta^J P + \left(1 - \theta^J\right) P^J, \end{aligned}$$

with an equality for any private good i s.t. $q_i^J > 0$ or for any public good k s.t. $g_k^J > 0$ (implying $Q_k > 0$). These are exactly the first order conditions of a household behavioral equilibrium (see (15) and (16)). Consequently, there is a complete coincidence of all the conditions that must be satisfied for $\left(q^A, g^A, Q^A, q^B, g^B, Q^B\right)$ to be a household behavioral equilibrium associated with the parameter values θ^A and θ^B, and to be a θ-household game with the same parameter values now exogenously fixed. ■

2.4 Separate spheres and local income pooling

Two properties of the Nash equilibrium of the game with voluntary contributions to public goods, proved by Browning, Chiappori and Lechene (2006b), are that there is generically at most one public good to which both spouses contribute (*separate spheres* up to one) and that, in the case they both contribute to one public good, income redistributions have locally no effect on household expenditures (*local income pooling*). The first property (Proposition 1 in Browning, Chiappori and Lechene, 2006b)[8] can be extended to household behavioral equilibria (as well as to equilibria of the θ-household game), except in the extreme Lindahl equilibrium case where the degree of autonomy vanishes for both members of the household ($\theta^A = \theta^B = 0$):

[8] The result is only 'generic', and does not apply in specific cases, for instance when individual preferences over the public goods are identical, as in Lechene and Preston (2005).

Proposition 4 *Take any household behavioral equilibrium characterized by (endogenous) degrees of autonomy* $\left(\theta^A, \theta^B\right) \neq (0,0)$*, and let* m^* *(*$m^* \leq m$*) be the number of public goods actually consumed by the household, of which* m^J *are contributed by spouse* J *(*$J = A, B$*). Then, generically, either* $m^A + m^B = m^*$ *or* $m^A + m^B = m^* + 1$*: there is at most one public good to which both spouses contribute.*

Proof. Consider a household behavioral equilibrium $\left(q^A, g^A, Q, q^B, g^B, Q\right)$ characterized by degrees of autonomy $\left(\theta^A, \theta^B\right)$, with environment (p, P, Y), income distribution $\left(Y^A, Y^B\right)$ and corresponding Lindahl prices $\left(P^A, P^B\right)$. As all constraints in the agents' programmes (12) are satisfied as equalities at equilibrium, each vector q^J must maximise J's utility $U^J(\cdot, Q)$ under the constraint $pq^J = y^J$ (with $y^J = Y^J - Pg^J$), and hence be uniquely determined by $\left(Q, y^J\right)$ in addition to the prices p of private goods. The m^* positive coordinates of Q and the two scalars y^A and y^B are then determined by the $m^A + m^B$ first order conditions

$$\tau_k^J\left(q^J\left(Q, y^J\right), Q\right) = \theta^J P_k + \left(1 - \theta^J\right) P_k^J$$

corresponding to any good k s.t. $g_k^J > 0$ (for $J = A, B$), together with the household budget constraint $y^A + y^B = Y - PQ$. Hence, we have $m^A + m^B + 1$ equations in $m^* + 2$ unknowns. Except in the case of a Lindahl equilibrium (corresponding to $\theta^A = \theta^B = 0$), where $\tau_k^J\left(q^J\left(Q, y^J\right), Q\right) = P_k^J$ implies $\tau_k^{-J}\left(q^{-J}\left(Q, y^{-J}\right), Q\right) = P_k^{-J}$ for any k s.t. $Q_k > 0$, there is overdetermination if $m^A + m^B > m^* + 1$: a solution can then only exist for singular parameter values. It is moreover clear that $m^* \leq m^A + m^B$, which completes the proof. ■

By contrast, the second property, that of local income pooling (Proposition 2 in Browning, Chiappori and Lechene, 2006b) does not generalize, and can only be obtained in the extreme case of full autonomy of the two spouses ($\theta^A = \theta^B = 1$). A household behavioral equilibrium outcome $\left(q^A, q^B, Q\right)$ with $n^A + n^B + m^*$ positive consumptions is fully determined by $n^A + n^B - 2 + m^A + m^B$ FOC equations (15) and (16), plus the collective budget equation, provided $m^A + m^B = m^* + 1$ (joint contribution to some public good k). Personal budget

equations can then be used to determine the two individual contributions g_k^A and g_k^B in Q_k. However, contrary to what happens in the full autonomy case, a change in the income distribution ρ influences through the Lindahl prices P_k^A and P_k^B each spouse's willingness to pay the public goods, and eventually the equilibrium outcome. Hence, the local income pooling property appears in our context as a symptom of fully non-cooperative behavior, lost as soon as we inject some dose of cooperation.

3 Local properties of household demand

The purpose of this section is to characterize the properties of demand functions that could be observable and used for discriminating among the different types of household behavior: full cooperation (Browning and Chiappori, 1998), full autonomy (Lechene and Preston, 2008) and intermediate types, involving partial autonomy of the two spouses.

3.1 Foundations of the demand functions

Take the game with *exogenous* degrees of autonomy $\left(\theta^A, \theta^B\right) \in [0,1]^2$ of the two spouses, associated with a given environment $(\pi, Y) \equiv (p, P, Y) \in \mathbb{R}_+^{n+m+1}$, a given income distribution $\left(Y^A, Y^B\right) \equiv \left(\rho^A, \rho^B\right) Y \equiv (\rho, 1-\rho) Y$, and corresponding Lindahl prices P^A and P^B. For a given choice $\left(g^{-J}, Q^{-J}\right) \in \mathbb{R}_+^{2m}$ of the other spouse, the Marshallian *conditional* demand function of spouse $J \in \{A, B\}$ can be straightforwardly derived, from his/her utility maximisation programme (18), with $Q^J = g^A + g^B$:

$$\begin{aligned} & x^J\left(p, \mathcal{P}^J, \mathcal{Y}^J, g^{-J}\right) = \arg \max_{\left(q^J, g^J\right) \in \mathbb{R}_+^{n+m}} U^J\left(q^J, g^J + g^{-J}\right) \\ & pq^J + \mathcal{P}^J g^J \leq \mathcal{Y}^J, \end{aligned} \tag{19}$$

with $\mathcal{P}^J = P - \left(1-\theta^J\right) P^{-J}$ and $\mathcal{Y}^J = \rho^J Y - \left(1-\theta^J\right) P^{-J}\left(Q^{-J} - g^{-J}\right)$.

Now, fix both ρ and $\left(\theta^A, \theta^B\right)$ (which will in general be omitted, for simplicity of notation, as arguments of the functions to be introduced in the fol-

lowing), and consider environment perturbations. More precisely, take an open set $\Omega \subset \mathbb{R}_+^{n+m+1}$ of environment values, with associated Lindahl price functions $P^J : \Omega \to \mathbb{R}_+^m$, such that the public goods *actually* purchased by each spouse (corresponding to the non-zero elements of equilibrium vectors g^A and g^B) are the same for each element of this set (that is, such that there is no regime switching over Ω). For $\left(\theta^A, \theta^B\right) \neq (0,0)$, we assume equilibrium *uniqueness* for any element of Ω, so that we can refer to the functions $G^J : \Omega \to \mathbb{R}_+^m$ $(J = A, B)$, associating with each environment the individual contributions to public consumption at equilibrium.[9] In the fully cooperative case ($\left(\theta^A, \theta^B\right) = (0,0)$), when more than one public good is consumed, equilibrium uniqueness does not prevail as concerns $\left(g^A, g^B\right)$, so that we must introduce a selection $\left(G^A, G^B\right) : \Omega \to \mathbb{R}_+^{2m}$ such that, according to the proof of Proposition 2, the two functions add up to the household consumption and are such that, for any $(\pi, Y) \in \Omega$, $PG^A(\pi, Y) = P^A(\pi, Y)\left(G^A(\pi, Y) + G^B(\pi, Y)\right)$.

In the Marshallian conditional demand function $x^J : \mathbb{R}_+^{n+m+1+m} \to \mathbb{R}_+^{n+m}$ as defined above, we thus take the arguments $\mathcal{P}^J$, $\mathcal{Y}^J$ and g^{-J} as functions of the environment, differentiable by assumption: $\mathcal{P}^J = P - \left(1 - \theta^J\right) P^{-J}(\pi, Y)$, $\mathcal{Y}^J = \rho^J Y - \left(1 - \theta^J\right) P^{-J}(\pi, Y) G^J(\pi, Y)$ and $g^{-J} = G^{-J}(\pi, Y)$. We are thus assuming that U^J has the usual properties required to ensure differentiability

[9] The uniqueness assumption is incompatible with the non-generic case of joint contribution to more than one public good. Indeed, take equilibrium values $\left(\widehat{g}^A, \widehat{g}^B\right)$ such that $\widehat{g}_k^A$ and $\widehat{g}_k^B$ are both positive for any k in some set K of public goods. Clearly, we see by simple inspection of the spouses' programmes, that a replacement of elements $\widehat{g}_k^A$ and $\widehat{g}_k^B$ by other positive elements g_k^A and g_k^B satisfying, for any $k \in K$, $g_k^A + g_k^B = \widehat{g}_k^A + \widehat{g}_k^B$ and, for $J \in \{A, B\}$,

$$\begin{aligned} & \sum_{k \in K} \left(\theta^J P_k + \left(1 - \theta^J\right) P_k^J\right)\left(g_k^J - \widehat{g}_k^J\right) - \sum_{k \in K} \left(1 - \theta^J\right) P_k^{-J}\left(g_k^{-J} - \widehat{g}_k^{-J}\right) \\ = & \sum_{k \in K} P_k \left(g_k^J - \widehat{g}_k^J\right) = 0 \end{aligned}$$

will lead to another equilibrium with the same outcome. We thus obtain a system of $\#K + 2$ equations (of which only $\#K + 1$ are independent) in $2(\#K)$ unknowns. Uniqueness consequently requires $\#K \leq 1$. Lechene and Preston (2008) also rely on the uniqueness assumption, but with a slightly different game where each spouse J chooses, rather than his/her own contribution g^J, his/her preferred household consumption Q^J (which should not be less than g^{-J}), with $Q^A = Q^B = Q$ at equilibrium.

of x^J.

3.2 Full cooperation ($\theta^A = \theta^B = 0$)

Consider individual demands ξ^A and ξ^B directly expressed as functions of the environment, namely

$$\xi^J\left(\pi, Y, \rho^J\right) \equiv x^J\left(p, P^J\left(\pi, Y\right), \rho^J Y - P^J\left(\pi, Y\right) G^{-J}\left(\pi, Y\right), G^{-J}\left(\pi, Y\right)\right) \tag{20}$$

and, recalling that $\rho^A = \rho = 1 - \rho^B$, the corresponding household demand

$$\xi\left(\pi, Y, \rho\right) \equiv \xi^A\left(\pi, Y, \rho\right) + \xi^B\left(\pi, Y, 1-\rho\right).$$

In the case of the *unitary model* with a fixed Pareto weight μ, the Lindahl prices P^A and P^B allowing for decentralization of the spouses' decisions are indeed functions of the environment (π, Y) alone, and so are the selections G^A and G^B. Of course, expenditure shares $\rho^J = \pi\xi^J/Y$ must then be adjusted to changes in the environment by lump sum transfers between the two spouses, so that we are in fact referring to the individual demand functions

$$\xi^J_\mu\left(\pi, Y\right) \equiv \xi^J\left(\pi, Y, \rho^J\left(\pi, Y\right)\right) \tag{21}$$

and to the corresponding household demand function

$$\xi_\mu\left(\pi, Y\right) \equiv \xi^A_\mu\left(\pi, Y\right) + \xi^B_\mu\left(\pi, Y\right). \tag{22}$$

The household demand has of course the usual properties of Marshallian demand functions, in particular a Slutsky matrix $\Sigma_\mu = \left[\partial_\pi \xi_\mu\right] + \left[\partial_Y \xi_\mu\right]\left[{}^T\xi_\mu\right]$. As to individual demands, they have the *household* income Y as an argument and income effects correspondingly work through *household* income. This means that a compensation of those effects at the household level so as to leave the household utility constant also ensures constancy of individual utilities, which are just fixed shares μ^A and μ^B of household utility. As a consequence, the *individual* demand ξ^J_μ also has the usual properties of Marshallian demand, with a Slutsky matrix $\Sigma^J_\mu = \left[\partial_\pi \xi^J_\mu\right] + \left[\partial_Y \xi^J_\mu\right]\left[{}^T\xi_\mu\right]$, where $\partial_Y \xi^J_\mu$ is a partial

derivative with respect to *household* income, the income effects being evaluated relative to *household* expenditure ξ_μ. Thus, from the definition of household demand ξ_μ as the sum of the two individual demands ξ_μ^A and ξ_μ^B we immediately obtain: $\Sigma_\mu = \Sigma_\mu^A + \Sigma_\mu^B$.

By using the definition (21) of these individual demand functions, we may also make explicit in the expression of the Slutsky matrix Σ_μ the adjustment of expenditure shares which is required to keep μ constant:

$$\Sigma_\mu = \overbrace{[\partial_\pi \xi] + [\partial_Y \xi] \left[{}^T \xi\right]}^{\Psi} + \overbrace{\left(\left[\partial_\rho \xi^A\right] - \left[\partial_\rho \xi^B\right]\right) \left([\partial_\pi \rho] + (\partial_Y \rho) \left[{}^T \xi_\mu\right]\right)}^{\Delta}. \quad (23)$$

In the *collective* model, with fixed income distribution given by the parameter ρ and an implicit Pareto weight varying with the environment, the effects described by the matrix Δ of the adjustment in the income distribution required to keep μ fixed are absent. As a consequence, the household demand $\xi(\pi, Y, \rho)$ has only a *pseudo-Slutsky matrix* $\Psi = [\partial_\pi \xi] + [\partial_Y \xi] \left[{}^T \xi\right]$ differing from the genuine Slutsky matrix Σ_μ of $\xi_\mu(\pi, Y)$ by the *deviation matrix* Δ, an outer product, hence with rank at most equal to 1, an observation that reproduces Browning and Chiappori (1998) main result. We may be more precise about the expression of Δ in terms of individual demands x^A and x^B.

Proposition 5 *Under* full cooperation *($\theta^A = \theta^B = 0$), the household demand function $\xi(\pi, Y, \rho) = \xi^A(\pi, Y, \rho) + \xi^B(\pi, Y, 1-\rho)$ has a pseudo-Slutsky matrix Ψ which deviates from a Slutsky matrix Σ_μ by an outer product, which can be expressed in terms of the individual demands x^A and x^B as*

$$\Delta = \left(\left[\partial_y x^A\right] - \left[\partial_y x^B\right]\right) \left[{}^T \left(\rho^B x^A - \rho^A x^B\right)\right]. \quad (24)$$

Proof. *It remains to show that the two expressions of Δ, in (23) in terms of ξ^A and ξ^B and in (24) in terms of x^A and x^B, are equivalent. By using $\rho(\pi, Y) = (1/Y)\, \pi \xi_\mu^A(\pi, Y)$, we can compute:*

$$\begin{aligned} & [\partial_\pi \rho] + (\partial_Y \rho) \left[{}^T \xi_\mu\right] \\ = \; & (1/Y) \left(\left[{}^T \xi_\mu^A\right] + \left[{}^T \pi\right] \left[\partial_\pi \xi_\mu^A\right]\right) + (1/Y) \left(-\rho + \left[{}^T \pi\right] \left[\partial_Y \xi_\mu^A\right]\right) \left[{}^T \xi_\mu\right]. \end{aligned}$$

By symmetry of the Slutsky matrix $\Sigma_\mu^A = \left[\partial_\pi \xi_\mu^A\right] + \left[\partial_Y \xi_\mu^A\right] \left[{}^T\xi_\mu\right]$

$$\begin{aligned}\left[{}^T\pi\right]\left(\left[\partial_\pi \xi_\mu^A\right] + \left[\partial_Y \xi_\mu^A\right]\left[{}^T\xi_\mu\right]\right) &= \left[{}^T\pi\right]\left({}^T\left[\partial_\pi \xi_\mu^A\right] + \left[\xi_\mu\right]\left({}^T\left[\partial_Y \xi_\mu^A\right]\right)\right) \\ &= \left[{}^T\pi\right]\left({}^T\left[\partial_\pi \xi_\mu^A\right]\right) + \overbrace{\left[{}^T\pi\right]\left[\xi_\mu\right]}^{Y}\left({}^T\left[\partial_Y \xi_\mu^A\right]\right).\end{aligned}$$

By Euler's identity applied to ξ_μ^A*, a homogeneous function of degree* 0*, we see that this expression is nil, so that we are left with*

$$\begin{aligned}\left[\partial_\pi \rho\right] + \left(\partial_Y \rho\right)\left[{}^T\xi_\mu\right] &= (1/Y)\left(\left[{}^T\xi_\mu^A\right] - \rho\left(\left[{}^T\xi_\mu^A\right] + \left[{}^T\xi_\mu^B\right]\right)\right) \\ &= (1/Y)\left(\rho^B\left[{}^T\xi_\mu^A\right] - \rho^A\left[{}^T\xi_\mu^B\right]\right).\end{aligned}$$

Finally, by referring to the definition of ξ^J *in terms of* x^J*, we have*

$$\left[\partial_\rho \xi^J\right] = \left[\partial_{\mathcal{Y}} x^J\right] Y,$$

leading to

$$\Delta = \left(\left[\partial_{\mathcal{Y}} x^A\right] - \left[\partial_{\mathcal{Y}} x^B\right]\right)\left[{}^T\left(\rho^B x^A - \rho^A x^B\right)\right],$$

by just making the values of ξ_μ^J *and* x^J *coincide.* ■

It should be emphasized that the deviation $\Delta = \Sigma_\mu - \Psi$ is independent of the existence of any public consumption. It results from an *aggregation effect*, working in the general case where there is no representative consumer.

3.3 Full autonomy ($\theta^A = \theta^B = 1$)

The pseudo-Slutsky matrix of the household demand function $\xi \equiv \xi^A + \xi^B$ can be easily decomposed by detailing the different effects of the environment through the arguments of the sum $x^A + x^B$. This pseudo-Slutsky matrix $\Psi = \left[\partial_\pi \xi\right] + \left[\partial_Y \xi\right]\left[{}^T\xi\right]$ can be computed from the Jacobian

$$\begin{aligned}\left[\partial_{(\pi,Y)}\xi\right] &= \begin{bmatrix} \partial_\pi x^A & \rho^A \partial_{\mathcal{Y}} x^A \end{bmatrix} + \begin{bmatrix} \partial_\pi x^B & \rho^B \partial_{\mathcal{Y}} x^B \end{bmatrix} \\ &\quad + \left[\partial_g x^A\right]\left[\partial_{(\pi,Y)} G^B\right] + \left[\partial_g x^B\right]\left[\partial_{(\pi,Y)} G^A\right]\end{aligned} \tag{25}$$

giving, with the notation $\Gamma^J \equiv \left[\partial_\pi G^J\right] + \left[\partial_Y G^J\right]\left[^T\left(x^A + x^B\right)\right]$:

$$\Psi = \overbrace{\left[\partial_\pi x^A\right] + \left[\partial_\mathcal{Y} x^A\right]\left[^T x^A\right]}^{\Sigma^A} + \overbrace{\left[\partial_\pi x^B\right] + \left[\partial_\mathcal{Y} x^B\right]\left[^T x^B\right]}^{\Sigma^B} \tag{26}$$
$$-\overbrace{\left(\left[\partial_\mathcal{Y} x^A\right] - \left[\partial_\mathcal{Y} x^B\right]\right)\left[^T\left(\rho^B x^A - \rho^A x^B\right)\right]}^{\Delta} + \overbrace{\left[\partial_g x^A\right]\Gamma^B + \left[\partial_g x^B\right]\Gamma^A}^{\Xi}.$$

The Slutsky matrices Σ^A and Σ^B of the individual demand functions x^A and x^B express the direct effects of a change in the environment on individual optimizing decisions. Their sum Σ has also the properties of a Slutsky matrix. The matrix Δ is an outer product, with a rank at most equal to $r_\Delta = 1$. It was already present in the fully cooperative case, resulting as already stated from an *aggregation effect*.

The matrix Ξ is new. It expresses an *externality effect*, when this effect ceases to be compensated by the response of Lindahl taxation to changes in the environment, as it was in the fully cooperative case. Notice that, because of the assumption of no regime switching over Ω, if $g_k^J = 0$ for some k, then $\partial_g x_{n+k}^J = 0$ and $\partial_{(\pi,Y)} G_k^J = 0$, so that the matrix $\left[\partial_g x^J\right]$ (resp. Γ^J) has at most $n + m^J$ (resp. m^J) non-zero rows, m^J being the number of public goods contributed by spouse J. In the absence of public consumption or, more generally, under preference separability, when the utility derived from each spouse's private and public consumption is unaffected by the other spouse's exclusive contributions to public goods, the matrix $\left[\partial_g x^J\right]$ vanishes (at least in the regime of separate spheres), so that $\Xi = 0$, bringing us back to the result of the fully cooperative case: the deviation matrix $\Sigma - \Psi$ has a rank at most equal to 1. But this result due to inoperative externality effects is of course lost as soon as we abandon separability. The generic result requires the rank of the deviation matrix to be equal to an upper bound introduced in the following proposition (a result first formulated by Lechene and Preston, 2008, for the case $m^* = m^A + m^B - \delta = m$, with $\delta = 0$ under separate spheres and $\delta = 1, ..., m$ under joint contribution to δ public goods).[10]

[10] The upper bound established by Lechene and Preston (2008) is in fact independent of

Proposition 6 *Under* full autonomy *($\theta^A = \theta^B = 1$), the household demand function $\xi(\pi, Y) = x^A\left(\pi, \rho^A Y, G^B(\pi, Y)\right) + x^B\left(\pi, \rho^B Y, G^A(\pi, Y)\right)$ has a pseudo-Slutsky matrix Ψ which deviates from a Slutsky matrix $\Sigma = \Sigma^A + \Sigma^B$ by a matrix $\Delta - \Xi$ of rank at most equal to*

$$r_{\Delta-\Xi} = 1 + m^* + \min\left\{n - 1 - \max\left\{m^A - m^B, 1\right\}, 0\right\}.$$

The upper bound $r_{\Delta-\Xi}$ can be neither higher than $1 + m^*$ nor lower than 1 (for $n = 1$ and either $m^A = 1$ or $m^B = 0$).

Proof. *(**Separate spheres**) This is the simpler case. We first determine the maximum possible rank of Ξ. The matrix $\left[\partial_g x^J\right]$ has at most $n + m^J$ non-zero rows, which however cannot be linearly independent since $\left[{}^T\pi\right]\left[\partial_g x^J\right] = 0$ (consumption changes induced by the sole externality effect should not modify the expenditure πx^J). Hence, the rank of $\left[\partial_g x^J\right]$ is at most equal to $n + m^J - 1$. The matrix Γ^J has at most m^J non-zero rows so that, assuming WLOG that $m^A \geq m^B$, the rank of the matrix $\Xi = \left[\partial_g x^A\right]\Gamma^B + \left[\partial_g x^B\right]\Gamma^A$ cannot be higher than*

$$r_\Xi = m^B + \min\left\{n + m^B - 1, m^A\right\} = m^* + \min\left\{n - 1 - \left(m^A - m^B\right), 0\right\}.$$

Now, by applying Euler's identity to the functions ξ and x^J, which are homogeneous of degree 0, we see that $\Psi[\pi] = \Sigma[\pi] = 0$, implying $(\Delta - \Xi)[\pi] = 0$, so that the columns of the matrix $\Delta - \Xi$ are not linearly independent. Hence, the rank of this matrix is at most equal to $n + m^* - 1$, since it has only $n + m^*$ non-zero columns (variations in the prices of the $m - m^*$ public goods which are actually not consumed by the household cannot induce changes in the spouses' contributions). Taking into account this upper bound and simply adding r_Δ and r_Ξ completes the proof:

$$\begin{aligned} r_{\Delta-\Xi} &= \min\left\{n + m^* - 1, 1 + m^* + \min\left\{n - 1 - \left(m^A - m^B\right), 0\right\}\right\} \\ &= 1 + m^* + \min\left\{n - 1 - \max\left\{m^A - m^B, 1\right\}, 0\right\}. \end{aligned}$$

the value of δ. For simplicity, we have limited our analysis to the generic case $\delta \in \{0, 1\}$. If $m^* = m$ (the case contemplated by Lechene and Preston), their result coincides with ours.

*(**Joint contribution to one public good**) Now suppose that both spouses contribute to the k-th public good. Because of local income pooling, the equilibrium outcome (except as concerns the way Q_k is decomposed into g_k^A and g_k^B) will be the same at given prices and household income if we let spouse A make the whole purchase of public good k, compensating her by an income transfer from B equal to $P_k G_k^B$. This transfer triggers the appearance of a new component of the pseudo-Slutsky matrix of household demand, namely*

$$\left(\left[\partial_{\mathcal{Y}} x^A\right] - \left[\partial_{\mathcal{Y}} x^B\right]\right)\left(P_k \Gamma_k^B + e_{n+k} G_k^B\right),$$

where $e_{n+k} = [\partial_\pi P_k] + (\partial_Y P_k)\left[^T\left(x^A + x^B\right)\right]$ is the $n+k$-th row of the identity matrix I_{n+m}. Clearly, this component does not increase the rank of the deviation matrix, since it can be added to Δ without changing its nature of outer product. Otherwise, the income transfer brings us back to a regime of separate spheres with m^A and $m^B - 1$ public goods contributed by spouses A and B, respectively. Hence, the maximum rank of Γ^B is now $m^B - 1$. However, the relevant upper bound for the rank of $\left[\partial_g x^B\right]$ remains $n + m^B - 1$, since we cannot apply in this context the implication $x_k^B = 0 \Longrightarrow \partial_g x_k^B = 0$ imposed by the assumption of no regime switching over Ω. Indeed, B's marginal willingness to pay for the k-th public good remains equal to P_k (whereas it is generically smaller than its price for any non contributed public good), making it eligible for a contribution by B in response to any perturbation of his environment. By simply reproducing the argument developed for the case of separate spheres, we thus obtain for the maximum rank of the deviation matrix:

$$\begin{aligned} r_{\Delta - \Xi} &= \min\left\{n + m^* - 1, 1 + \left(m^B - 1\right) + \min\left\{n + m^B - 1, m^A\right\}\right\} \\ &= 1 + m^* + \min\left\{n - 1 - \max\left\{m^A - m^B, 1\right\}, 0\right\}. \end{aligned}$$

■

3.4 Intermediate cases ($0 < \theta^J < 1$, $J = A, B$)

The analysis of the intermediate cases where both spouses have some degree of autonomy, but also cooperate through Lindahl taxation, can be seen as a

generalization of the previous cases. The Jacobian of the household demand function $\xi = x^A + x^B$ is now:

$$\begin{aligned}
\left[\partial_{(\pi,Y)}\xi\right] &= \left[\begin{array}{cc} \partial_{(p,\mathcal{P})}x^A & \rho^A\partial_{\mathcal{Y}}x^A \end{array}\right] + \left[\begin{array}{cc} \partial_{(p,\mathcal{P})}x^B & \rho^B\partial_{\mathcal{Y}}x^B \end{array}\right] \\
&+ \left[\partial_g x^A\right]\left[\partial_{(\pi,Y)}G^B\right] + \left[\partial_g x^B\right]\left[\partial_{(\pi,Y)}G^A\right] \\
&- \left(1-\theta^A\right)\left[\partial_{\mathcal{P}}x^A\right]\left[\partial_{(\pi,Y)}P^B\right] - \left(1-\theta^B\right)\left[\partial_{\mathcal{P}}x^B\right]\left[\partial_{(\pi,Y)}P^A\right] \\
&- \left(1-\theta^A\right)\left[\partial_{\mathcal{Y}}x^A\right]\left(\left[{}^TG^A\right]\left[\partial_{(\pi,Y)}P^B\right] + \left[{}^TP^B\right]\left[\partial_{(\pi,Y)}G^A\right]\right) \\
&- \left(1-\theta^B\right)\left[\partial_{\mathcal{Y}}x^B\right]\left(\left[{}^TG^B\right]\left[\partial_{(\pi,Y)}P^A\right] + \left[{}^TP^A\right]\left[\partial_{(\pi,Y)}G^B\right]\right).
\end{aligned} \tag{27}$$

By introducing the additional notation $\Pi^J = \left[\partial_\pi P^J\right] + \left[\partial_Y P^J\right]\left[{}^T\left(x^A + x^B\right)\right]$, we can express the *pseudo-Slutsky matrix* Ψ of the household demand as follows:

$$\begin{aligned}
\Psi = &\overbrace{\left[\partial_{(p,\mathcal{P})}x^A\right] + \left[\partial_{\mathcal{Y}}x^A\right]\left[{}^Tx^A\right]}^{\Sigma^A} + \overbrace{\left[\partial_{(p,\mathcal{P})}x^B\right] + \left[\partial_{\mathcal{Y}}x^B\right]\left[{}^Tx^B\right]}^{\Sigma^B} \\
&- \overbrace{\left(\left[\partial_{\mathcal{Y}}x^A\right] - \left[\partial_{\mathcal{Y}}x^B\right]\right)\left[{}^T\left(\rho^Bx^A - \rho^Ax^B\right)\right]}^{\Delta} \\
&+ \overbrace{\left(\begin{array}{c} \left(\left[\partial_g x^A\right] - \left(1-\theta^B\right)\left[\partial_{\mathcal{Y}}x^B\right]\left[{}^TP^A\right]\right)\Gamma^B \\ + \left(\left[\partial_g x^B\right] - \left(1-\theta^A\right)\left[\partial_{\mathcal{Y}}x^A\right]\left[{}^TP^B\right]\right)\Gamma^A \end{array}\right)}^{\Xi} \\
&- \overbrace{\left(1-\theta^A\right)\left(\left[\partial_{\mathcal{P}}x^A\right] + \left[\partial_{\mathcal{Y}}x^A\right]\left[{}^TG^A\right]\right)\Pi^B}^{\Theta^A} \\
&- \overbrace{\left(1-\theta^B\right)\left(\left[\partial_{\mathcal{P}}x^B\right] + \left[\partial_{\mathcal{Y}}x^B\right]\left[{}^TG^B\right]\right)\Pi^A}^{\Theta^B}.
\end{aligned} \tag{28}$$

The matrices Σ^A and Σ^B are again the Slutsky matrices of the individual demand functions, and their sum Σ has the same properties. The matrix Δ, an outer product, expresses the aggregation effect. The matrix Ξ expresses the externality effects, now including income effects through Lindahl taxation, which may increase its maximum rank

$$r_\Xi = m^A + m^B + \min\left\{n - \left(m^A - m^B\right), 0\right\}. \tag{29}$$

But the new element in the decomposition of Ψ is the sum of the two matrices Θ^A and Θ^B, expressing the substitution effects of price changes *through the*

Lindahl prices. The matrix $\left[\partial_{\mathcal{P}} x^J\right]$ has at most m^J non-zero columns, since variations in the prices of the $m - m^J$ public goods to which spouse J does not contribute cannot induce changes in the demand for any good. So has the matrix $\left[\partial_{\mathcal{P}} x^J\right] + \left[\partial_{\mathcal{Y}} x^J\right] \left[{}^T G^J\right]$, since $\left[{}^T G^J\right]$ has the *same* $m - m^J$ zero columns. Hence, the rank of $\Theta = \Theta^A + \Theta^B$ is upper bounded by $m^A + m^B = m^* + \delta$.

Proposition 7 *In the intermediate cases where* $\left(\theta^A, \theta^B\right) \in (0,1)^2$*, the household demand function* $\xi(\pi, Y)$ *has a pseudo-Slutsky matrix* Ψ *which deviates from a Slutsky matrix* $\Sigma = \Sigma^A + \Sigma^B$ *by the matrix* $\Delta - \Xi + \Theta$*, the rank of which is at most equal to*

$$r_{\Delta - \Xi + \Theta} = 1 + 2m^* + \min\left\{n - (m^* + 2), 2\delta\right\}.$$

The upper bound $r_{\Delta - \Xi + \Theta}$ *can neither be higher than* $1 + 2(m^* + \delta)$ *nor lower than* 1 *(for* $n = m^* = 1$*).*

Proof. *Just add the maximum ranks of* Δ*,* $-\Xi$ *and* Θ*, as previously established, and take into account the upper bound of the rank of the deviation matrix, which has at most* $n + m^* - 1$ *linearly independent non-zero columns (since* $\Psi[\pi] = \Sigma[\pi] = 0$*, implying* $(\Delta - \Xi + \Theta)[\pi] = 0$*), to obtain:*

$$\begin{aligned} & \min\left\{n + m^* - 1, 1 + 2\left(m^A + m^B\right) + \min\left\{n - \left(m^A - m^B\right), 0\right\}\right\} \\ = \quad & 1 + 2m^* + \min\left\{n - (m^* + 2), 2\delta\right\}. \end{aligned}$$

■

The strategy of making the results for the two regimes of separate spheres and of joint contribution to one public good coincide, by exploiting local income pooling in the latter regime, does not work here. Indeed, the presence of Lindahl prices depending upon income distribution makes income pooling incomplete. Hence, our result on the maximum rank of the deviation matrix is now regime dependent.

The upper bound imposed upon the rank of the deviation matrix can be used to test the different models of household behavior. Browning and Chiappori (1998) have used this upper bound to discriminate between the unitary

model (which predicts that the matrix $\Psi - \left({}^{T}\Psi\right)$ has rank 0, as $\Psi = \Sigma$, hence symmetric) and the collective model (which predicts that $\Psi - \left({}^{T}\Psi\right)$ has rank at most 2), and have shown that this test requires at least 5 goods. This requirement stems from the fact that the rank of $\Psi - \left({}^{T}\Psi\right)$ cannot be higher than 2 if the number $n+m$ of goods is not larger than 4 (given the linear dependence of the columns of Ψ introduced by homogeneity of degree zero of the demand functions). Lechene and Preston (2008) have used the properties of Propositions 5 and 6 to discriminate between the cooperative and the non-cooperative models, and have shown that this test requires $n \geq m+5$. Their Lemma 1 shows indeed that, if $\Psi - \left({}^{T}\Psi\right)$ has rank at most $n+m-1$, then Ψ can always be expressed as the sum of a symmetric matrix and a matrix of rank not higher than r such that $2r+1 \geq n+m-1$ (with $r = 1+m$ according to Proposition 6). If we apply this lemma to our own Proposition 7, we see that $n \geq 3m^{*}+4\delta+5$ is needed to discriminate between full and partial autonomy. If, for instance, there is only one public good and a single contributor, at least 8 private goods are required. The maximum possible rank of $\Psi - \left({}^{T}\Psi\right)$, given homogeneity of degree 0 of the demand functions, is then 8. As the observed rank increases from 0 to 8, the test successively rejects the unitary model (at 2), full cooperation (at 4), full autonomy (at 6) and the collective model as a whole (at 8).

4 Household decisions under varying degrees of autonomy: an example

In order to study household decisions when we vary not only the income shares but also the degrees of autonomy, and to make comparisons with previous results on the game with voluntary contributions to public goods obtained by Browning, Chiappori & Lechene (2006b), we use their example, with Cobb-Douglas preferences over one private good and two public goods. We denote by c and d the private consumptions of spouses A (the wife) and B (the husband), respectively, and by G and H the quantities of the two public goods. The utility

functions are given by:

$$U^A(c,G,H) = cG^{5/3}H^{8/9} \text{ and } U^B(d,G,H) = dG^{15/32}H^{1/2}, \tag{30}$$

so that A cares more about the first public good, and B about the second ($(5/3)(9/8) > (15/32)2$). We further use the following normalization:

$$p = P_G = P_H = Y = 1.$$

The environment in this example is thus described by the vector $(1,1,1,1)$ and the income distribution by the pair $(\rho, 1-\rho)$.

4.1 Income distribution and public consumptions

We first need the Lindahl prices in our example. After some computations, we get:

$$\left(P_G^A, P_H^A\right) = \left(\frac{63\rho}{31\rho+32}, \frac{63\rho}{64-\rho}\right), \ \left(P_G^B, P_H^B\right) = \left(\frac{32(1-\rho)}{31\rho+32}, \frac{64(1-\rho)}{64-\rho}\right). \tag{31}$$

Straightforward application of first order conditions (16) for optimal public consumptions, as desired by agents A and B in the game with Lindahl prices, leads to:

$$\begin{aligned} c\left(\frac{5/3}{G}, \frac{8/9}{H}\right) &\leq \left(\frac{32(1-\rho)\theta^A + 63\rho}{31\rho+32}, \frac{64(1-\rho)\theta^A + 63\rho}{64-\rho}\right) \text{ and} \\ d\left(\frac{15/32}{G}, \frac{1/2}{H}\right) &\leq \left(\frac{32(1-\rho) + 63\rho\theta^B}{31\rho+32}, \frac{64(1-\rho) + 63\rho\theta^B}{64-\rho}\right), \end{aligned} \tag{32}$$

with an equality when the corresponding public good receives a positive voluntary contribution. Hence, a positive contribution by both spouses to both public goods would imply that the four inequalities above hold as equalities. Since an equal ratio G/H should obtain for A and B, we then get

$$\frac{64(1-\rho)\theta^A + 63\rho}{32(1-\rho)\theta^A + 63\rho} = \frac{1}{2}\frac{64(1-\rho) + 63\rho\theta^B}{32(1-\rho) + 63\rho\theta^B},$$

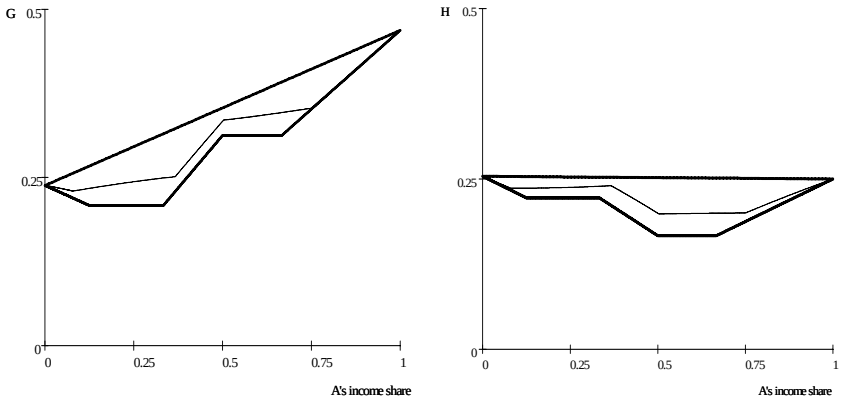

Figure 1: Household public consumptions as ρ varies

an equality that can be solved for $\theta^J \in [0,1]$ and $J = A, B$ only if $\theta^A = \theta^B = 0$, that is, in the case of the Lindahl equilibrium outcome.[11] This illustrates Proposition 4, implying in this case that the spouses will jointly contribute to at most one public good (separate spheres up to one). Consequently, for any given pair $\left(\theta^A, \theta^B\right)$ of degrees of autonomy, we should expect different equilibrium regimes concerning the contributions for public goods as we increase A's income share ρ from 0 to 1: (I) where B is the only spouse to contribute to (both) public goods, (II) where A contributes to her preferred public good and B still contributes to both, (III) where each spouse specializes on his/her preferred public good, (IV) and (V) symmetric to (II) and (I) respectively (with inverted roles of A and B). Figure 1 illustrates these regimes, for the household consumptions of the first and second public goods, when the degrees of autonomy are $(0,0)$ (the upper thick curves), $(1,1)$ (the lower thick curves) and $(1/2, 1/2)$ (the thin curves). The upper curves, corresponding to the Lindahl equilibrium, are straight lines, increasing in ρ for the first public good (the preferred one), (slightly) decreasing for the second. The other curves are broken lines, each kink corresponding to a change of regime. Each lower broken line, portraying the

[11] The corresponding equality would trivially hold, for any values of ρ, θ^A and θ^B, in the singular case where both spouses would equally care for the two public goods. See footnote 8.

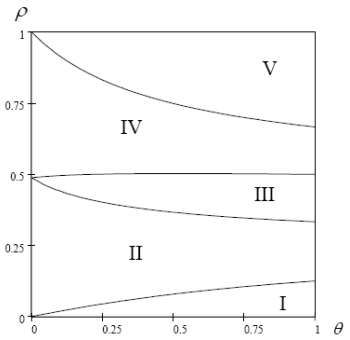

Figure 2: Regime switching values of ρ as θ varies

Nash equilibrium outcome of the game with voluntary contributions, exhibits two horizontal segments (corresponding to regimes II and IV). These segments illustrate the local income pooling phenomenon formulated in Proposition 2 of Browning, Chiappori and Lechene (2006b). Such horizontal segments do not appear in the other broken lines, a feature that illustrates our claim in Subsection 2.3.

We have represented, as the sole intermediate case between the Lindahl equilibrium and the fully non-cooperative equilibrium, the case where both degrees of autonomy are equal to 1/2, but all other intermediate cases would be represented by broken lines similarly located between the thick curves. The non-monotonicity of the household public consumptions as functions of ρ is a phenomenon which is not limited to the fully non-cooperative case, but it eventually disappears as the degrees of autonomy tend to vanish. Finally, observe that regime switches occur at different values of ρ for different configurations of the degrees of autonomy. In Figure 2, we have represented the regime switching values of ρ as functions of $\theta = \theta^A = \theta^B$ in the symmetric case of equal degrees of autonomy. We may notice that the regimes where both spouses contribute to the same public good tend to expand as individual autonomy tends to vanish.

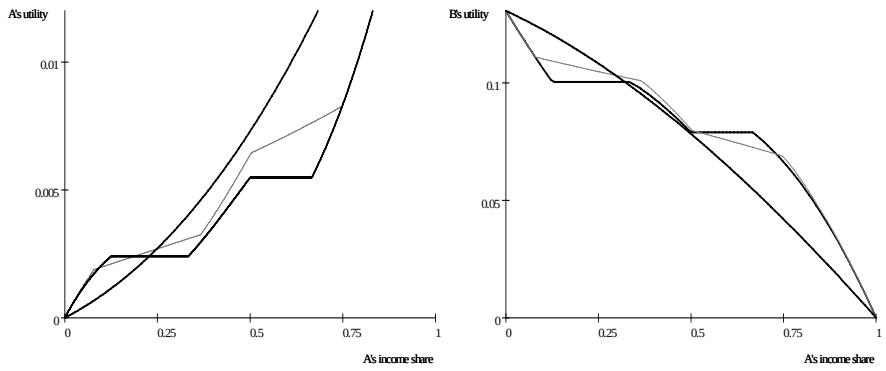

Figure 3: Agents' utilities under symmetric autonomy

4.2 Degrees of autonomy and spouse welfare

Another interesting comparative statics issue consists in looking at the way the welfare of each spouse varies as the wife's income share increases for the same three configurations of the degrees of autonomy. We represent in Figure 3 A's and B's utilities as functions of A's income share ρ, when the degrees of autonomy are $(0, 0)$ (the smooth thick curves), $(1, 1)$ (the broken thick curves) and $(1/2, 1/2)$ (the thin broken curves).

As expected, A's (resp. B's) utility is increasing (resp. decreasing) in the three cases, but the three curves cannot be monotonically ranked in terms of the (common) degree of autonomy. When a spouse's income share is low, his/her utility is higher for the fully non-cooperative equilibrium than for the Lindahl equilibrium. This relationship is reversed as soon as the spouse's income share is moderately high. In the intermediate case where the degrees of autonomy are $(1/2, 1/2)$, the relationship of the spouse's utility with that obtained at the Lindahl equilibrium follows the same pattern (the two curves cross only once), but its relationship with the utility obtained at the fully non-cooperative equilibrium is more complex (the two curves cross several times). Besides, because of the absence of local income pooling, the curve corresponding to the case $\theta^A = \theta^B = 1/2$ does not exhibit the same horizontal segments as the curve

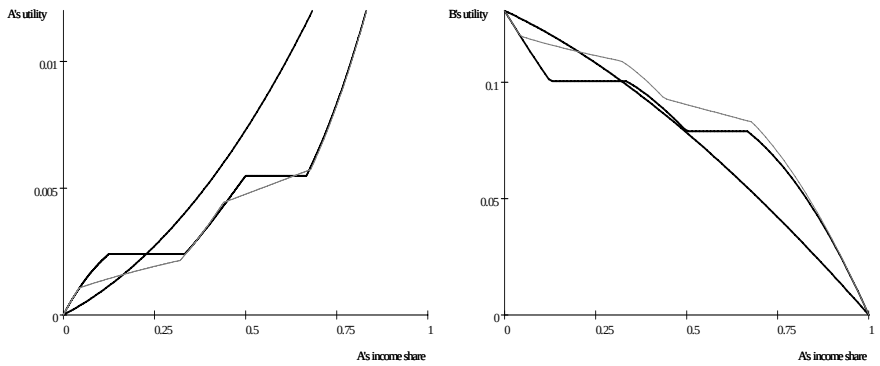

Figure 4: Agents' utilities under asymmetric autonomy

corresponding to the case $\theta^A = \theta^B = 1$.

Figure 4 reproduces the same extreme cases, but the thin broken curves represent now the asymmetric case $\left(\theta^A, \theta^B\right) = (1/4, 3/4)$. We see that the husband, whose degree of autonomy is higher, tends to attain the highest utility level in this case. A higher degree of autonomy has a negative effect on utility through a loss of efficiency, but this effect is more than compensated by a positive redistributive effect.

5 Conclusion

We have proposed a model of household behavior with both private and public consumption, where the spouses independently maximize their utilities, but taking into account, together with their own personal budget constraints, the collective budget constraint with public goods evaluated at Lindahl prices. This model generalizes both the collective, fully cooperative, model of household behavior and the non-cooperative model with voluntary contributions to public goods. This is achieved in two ways. The first is through a generalized game of voluntary contributions where the set of equilibria can be parameterized according to the degree of autonomy of each spouse, measured by the relative pressure of the two budget constraints at equilibrium, personal vs. collective. The second

is by fixing as an exogenous parameter the share of public consumption that is autonomously taken care of by each spouse. The degrees of autonomy introduce a complementary dimension to the income shares, allowing to consider variations of household behavior not only along the Pareto frontier, but also inside the utility possibility set.

Our analysis has shown that the three types of household behavior (full cooperation, full autonomy and partial autonomy) impose sufficient restrictions on observed household demand to allow for testability under some conditions on the number of goods (typically, a number of private goods much higher than the number of public goods). Further work is needed. In particular, the estimation of the autonomy parameters would be welcome. This has been done in other fields. The New Empirical Industrial Organization has been estimating the so-called conduct parameters which measure the relative weight of competitive toughness and play in the analysis of firm behavior a role similar to the one of our degrees of autonomy in the analysis of household behavior.

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