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Measuring Differences of Opinion: Axiomatic Foundation, Utility, and Truthtelling

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Understanding how individuals and groups differ in their opinions and preferences is central to analyzing disagreement, measuring polarization, designing institutions, and predicting collective outcomes. Yet comparing preferences requires more than observing how each person ranks alternatives—it requires a method for comparing preference orderings themselves. This paper develops a formal framework to infer how individuals might rank different preference orderings based solely on their observed preferences. We introduce a set of natural and behaviorally plausible axioms—Independence (I), Disagreement Aversion (DA), and Symmetry (S)—and show that they uniquely characterize a class of hyperpreference relations and their associated utility representations. We apply this framework to the study of aggregation mechanisms, deriving necessary and sufficient conditions on utility structures that induce truthful preference reporting in equilibrium and guarantee efficiency. Our results yield new insights into strategyproof mechanism design under deep preference heterogeneity and clarify when differences of opinion can be meaningfully and reliably measured.

KEYWORDS. Preference, Hyperpreference, Hyperutility, Strategy-proofness, Efficiency.

JEL CLASSIFICATION. D01, D04, D71, D78.

Disagreement is a fundamental feature of social and economic life. Individuals and groups routinely differ in their preferences—over risk, fairness, effort, or public policy—and such divergence shapes outcomes within households, firms, and societies. A couple may disagree on how to balance work and caregiving; a manager and an employee may evaluate risk or innovation differently; citizens may prioritize liberty, equality, or security in conflicting ways. Standard economic models accommodate heterogeneity in preferences, typically representing behavior through complete and

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transitive orderings over a set of alternatives. This first-order framework provides a powerful foundation for analyzing choice, welfare, and strategic behavior.

Yet in many important settings, individuals are called upon not only to express their own preferences but also to evaluate the preference orderings of others—whether proposed by peers, institutions, or collective mechanisms. For example, a voter may prefer outcome A to B, but still regard a social ranking that places B above A as more reasonable or fair than one that places B above several other preferred options. Such second-order evaluations—what is generally termed *hyperpreferences*—are central to understanding compromise, polarization, legitimacy, war, and cooperation. As observed by Sen (1977) and Jeffrey (1974), a richer structure is needed to represent how individuals compare preference orderings themselves.

Despite their practical relevance, higher-order preferences remain theoretically underdeveloped and empirically elusive. Existing models lack a general framework for measuring the extent and structure of disagreement, or for inferring higher-order preferences from observed behavior. This limits our ability to evaluate how far apart agents are in their objectives, how such gaps affect collective outcomes, and how institutions can be designed to reconcile divergent views.

This paper develops a formal theory of preference (or opinion) divergence that fills this gap. We ask: How can we represent and measure disagreement in preferences—both in magnitude and in structure—and under what conditions can hyperpreferences be inferred from first-order preferences? Our framework enables meaningful comparisons of disagreement, illuminates the informational requirements for identifying second-order judgments, and offers new foundations for the design of strategyproof aggregation mechanisms and institutions in pluralistic environments. To clarify the motivation and scope of our approach, we begin with two illustrative examples.

Example 1. Suppose a manager (M) must rank three candidates, a, b, and c, for a job opening. M settles on the order $a \succ_M b \succ_M c$. A colleague (E), who has a vested interest in the hiring outcome (e.g., due to task interdependence), believes that $b \succ_E c \succ_E a$ would have been the ideal ranking. How dissatisfied is E with M's decision? Would E be less displeased if M had instead chosen $b \succ_M a \succ_M c$? To answer this, it is not enough to know E's direct preference; we must understand how E evaluates alternative rankings, that is, E's hyperpreference.

Example 2. Consider a public decision where the government must rank several investment projects: the construction of a regional hospital (a), the development of urban transportation (b), subsidies for small businesses transitioning to clean energy (c), and funding for higher education (d). A parent whose child needs specialized medical care might rank these projects as $a \succ_P d \succ_P b \succ_P c$, whereas the government announces the order $b \succ_G a \succ_G d \succ_G c$. Even if the hospital is second, its position may signal a meaningful divergence. Can this individual formally assess how far the government's priorities are from their own? Would they have preferred a different ordering? Suppose the government consults citizens to guide public investments: under what conditions will individuals truthfully reveal their preferences? In each case, evaluating how the collective outcome aligns with personal priorities requires a second-order structure.

These examples illustrate a central insight: first-order preferences alone are insufficient to explain how agents judge other indiviuduals' priorities or evaluate collective outcomes. When individuals reason consequentially—whether selfishly or altruistically—they use their own preferences to assess social decisions. Hyperpreferences capture these second-order evaluations.

A key challenge, however, is that eliciting hyperpreferences directly is cognitively and practically infeasible. While individuals can rank a dozen alternatives, ranking the entire space of possible orderings—whose size grows superexponentially—is overwhelmingly complex. As Sen (1977) observed, it is unrealistic to assume that agents possess fully articulated hyperpreferences.

This motivates the need for a theory that derives hyperpreferences from more tractable primitives. A natural starting point is the Kendall tau (or Kemeny) distance, which counts the number of pairwise disagreements between two rankings (Kendall, 1938, Kemeny, 1959, Kemeny and Snell, 1962). This distance offers a simple metric for how an individual's ranking deviates from another ranking. Laffond et al. (2020) provide a characterization of hyperpreference rules compatible with such distance-based representations.

Yet these approaches typically assume that preferences are transitive and complete, and they do not distinguish between types of disagreement or account for the varying salience of different trade-offs. Our framework generalizes this approach to allow for more nuanced modeling of divergence in both structure and magnitude, thereby enabling a richer analysis of preference heterogeneity and its implications.

This paper proposes a framework for deriving an individual's hyperpreference from their first-order preference over a finite set of alternatives, under natural behavioral assumptions. Formally, let $\mathcal{A}$ be a finite set of alternatives, $\mathcal{B}(\mathcal{A})$ the set of all binary relations over $\mathcal{A}$ (representing all possible first-order preference relations), and $\mathcal{W}(\mathcal{A})$ the set of all weak orders over $\mathcal{A}$. The set of hyperpreference relations over $\mathcal{A}$ is denoted $\mathcal{H}(\mathcal{A})$=$\mathcal{W}(\mathcal{B}(\mathcal{A}))$, which is the set of weak orders over the set $\mathcal{B}(\mathcal{A})$. A hyperpreference rule on $\mathcal{B}(\mathcal{A})$ is a mapping $\mathcal{E}$ that assigns to any first-order preference $R_0 \in \mathcal{B}(\mathcal{A})$ a hyperpreference relation $\mathcal{E}(R_0) \in \mathcal{H}(\mathcal{A})$ over $\mathcal{A}$. Any utility representation of $\mathcal{E}(R_0)$ is called a *hyperutility function*. We introduce a formal theory of hyperpreference based on a class of rules we call *separable hyperpreference rules*, characterized by three behavioral axioms:

- **Independence (I):** the evaluation of disagreements over a given pair of alternatives is independent of other agreements or disagreements elsewhere. That is, a conflict on a vs. b is treated identically regardless of how the two rankings compare on other elements and therefore regardless of the preference an individual holds.[1]
- **Disagreements aversion (DA):** the less disagreement there is with one's ideal preference, the more one is preferred. That is, the hyperpreference $\mathcal{E}(R_0)$ ranks R before

[1] In other words, if the disagreement between two binary relations R and Q over a pair of alternatives $\{a,b\}$ is the same as the disagreement of two other binary relations S and T over $\{a,b\}$, an individual who holds preferences R assesses the disagreement with Q over $\{a,b\}$ the same way as if this individual holds preferences S and assesses the disagreement with T over $\{a,b\}$.

R' whenever the set of disagreements between R_0 and R is a subset of those between R_0 and R'.

- **Symmetry (S):** two disagreements of the same type over the same pair of alternatives are judged equally.

The independence axiom is a well-known property of rational choice. It is also called the cancellation axiom by some authors (Kraft et al., 1959, Fishburn, 1970, Krantz et al., 1971). Our formalization is an extension of this notion to the context of hyperpreference relations. We show that together with the disagreements aversion axiom, it implies an extension of the well-known *independence of irrelevant alternatives* axiom to the hyperpreferences context. The latter concept has been formalized and studied in various contexts related to utility theory (see, e.g., Von Neumann and Morgenstern (1944) for the expected utility theory, and Galanter (1962) and Fishburn (1994) for the subjective utility theory). The Disagreement Aversion axiom can be interpreted as a consequence of a consequentialist perspective: individuals evaluate collective decisions based on their substantive outcomes and thus tend to favor social priority (or aggregation mechanisms that yield outcomes) proximate to those they most prefer. It is also well known in the literature as the *betweenness* axiom when the preferences are linear orders (see, e.g., Lainé et al. (2016), Laffond et al. (2020)). The symmetry axiom seems natural in the sense that it requires to treat the same type of disagreement in the same way.

These axioms, inspired by classical rational choice theory and utility representation frameworks, allow us to construct a unique class of hyperpreference rules (Theorems 1 and 5). They lead to hyperutility functions that assign differential weights to types of disagreements between preferences. For instance, incomparability between two options may be treated differently than indifference. We further provide a condition under which hyperpreferences are preserved under changes in weights (Theorem 6). Additionally, we demonstrate that the three axioms are logically independent, thereby illustrating how relaxing or modifying each of them gives rise to distinct families of hyperpreference rules (Proposition 1). One of the paper's results is that Kendall tau distance arises as a special case of this more general class of separable rules. More importantly, our framework allows for incomplete or even intransitive preferences, which are common in real-world settings involving partial judgments, conflicting values, or uncertainty. It also allows for non-uniform weights across disagreements, capturing differential sensitivity to conflicts.

Beyond its conceptual contributions, our framework provides practical guidance for designing institutions and aggregation mechanisms in contexts marked by deep preference heterogeneity. We apply the theory to characterize when aggregation rules satisfy two core properties: *efficiency*, where no collective outcome can be improved upon (Theorem 7); and *strategy-proofness*, where truth-telling is a dominant strategy, ensuring that reported preferences accurately reflect individual views (Theorem 8).

These results have broad relevance: in domains ranging from organizational decision-making to democratic governance, understanding when individuals are willing to report their preferences truthfully is central to anticipating cooperation, dissent, and reform.

Our analysis contributes to efforts that seek to move beyond classical impossibility theorems in social choice theory (Arrow, 1951, Gibbard, 1973, Satterthwaite, 1975). We show that robust normative outcomes are attainable when second-order reasoning is integrated in a structured and behaviorally grounded way. Specifically, we identify precise conditions on hyperutility parameters under which desirable aggregation properties emerge, offering a new class of possibility results in the spirit of recent work by Maskin (2020), Dasgupta and Maskin (2020), Bahel and Sprumont (2020), Pongou and Tchantcho (2021), Pongou and Sidie (2024).

Taken together, these findings illuminate the design of mechanisms that foster both truth-telling and efficiency in heterogeneous environments. Since accurate measurement of opinion differences depends on truthful reporting, our results also clarify the conditions under which such differences can be meaningfully and reliably assessed.

Our paper can be viewed as a contribution to the literature at the intersection of utility theory and game theory. This line of inquiry has its roots in foundational works by authors such as Bernoulli (1738), Bentham (1890), Von Neumann and Morgenstern (1944), and later developments by Fishburn (1967, 1992) and Krantz et al. (1971), among others. Over time, it has evolved into a broad and diverse field with multiple strands, including informational approaches to welfare economics and decision theory (see, e.g., Fleurbaey (2003), Fleurbaey and Hammond (2004), Fleurbaey (2018), Pivato and Tchouante (2023, 2024), Kleine et al. (2024), Yamazaki (2024), and the references therein).

The remainder of the paper is organized as follows. Section 1 introduces the basic definitions and notational conventions used throughout the paper. Section 2 formalizes the axiomatic foundations of our model. Section 3 presents the main theoretical results. Section 4 explores applications of the framework to the design of strategy-proof and efficient aggregation mechanisms. Section 5 concludes the paper.

1. Generalities

Consider a finite choice set $\mathcal{A}$ containing at least two elements. Elements of $\mathcal{A}$ are called alternatives or options.

A *preference*, *priority*, or *opinion*[2] on $\mathcal{A}$ is a binary relation on $\mathcal{A}$, that is, a subset of the Cartesian product $\mathcal{A} \times \mathcal{A} = \mathcal{A}^2$. A weak order on $\mathcal{A}$ is any complete, reflexive and transitive binary relation on $\mathcal{A}$.[3] If in addition, a weak order is antisymmetric, it is said to be a linear order. We denote by $\mathcal{W}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A})$ ($\mathcal{W}$ and $\mathcal{L}$ if no confusion is possible)

[2]The words *preference*, *priority*, and *opinion* will be used interchangeably throughout the paper.

[3]A binary relation $\succcurlyeq$, where, $a \succcurlyeq b$ (resp. $a \succ b$) means $(a,b) \in \succcurlyeq$ (resp.$(a,b) \in \succcurlyeq$ and $(b,a) \notin \succcurlyeq$) is
reflexive if for all $a \in \mathcal{A}$, $a \succcurlyeq a$,
transitive if for all $a, b, c \in \mathcal{A}$, if $a \succcurlyeq b$ and $b \succcurlyeq c$ then $a \succcurlyeq c$,
complete if for all $a, b \in \mathcal{A}$, $a \succcurlyeq b$ or $b \succcurlyeq a$,
antisymmetry if for all $a, b \in \mathcal{A}$, if $a \succcurlyeq b$ then we can not have $b \succ a$.

the set of weak orders and linear orders on $\mathcal{A}$, respectively. Additionally, we denote by $\mathcal{P}(\mathcal{A})$ (or simply $\mathcal{P}$ if there is no confusion) the set of all the pairs of the elements of $\mathcal{A}$; a pair of elements being a subset of two distinct elements of $\mathcal{A}$:

$$\mathcal{P}(\mathcal{A}) = \{\{a,b\} \subseteq \mathcal{A} : a \neq b\}.$$

For any subset C of $\mathcal{A}$, we denote by $R|_C$ the restriction of the binary relation R on C; that is, $R|_C = R \cap (C \times C)$. For any pair of binary relations (R,R'), we will often write $(R,R')|_{\{a,b\}}$ for $(R|_{\{a,b\}}, R'|_{\{a,b\}})$ if there is no confusion.

For a given binary relation R and a pair $\{a,b\}$ of alternatives in $\mathcal{A}$, we write:

- $a \succ^R b$ (or ab) if $(a,b) \in R$ and $(b,a) \notin R$.
- $a \sim^R b$ (or (ab)) if $(a,b) \in R$ and $(b,a) \in R$.
- $a *^R b$ (or $a * b$) if R does not compare alternatives a and b.

The notations above allow to greatly simply the representation of a binary relation, as shown in the example below.

EXAMPLE. $R = \{(a,b),(b,c),(a,c)\}$, $R_0 = \{(a,b),(a,c)(b,c),(c,b)\}$ and $R' = \{(a,b),(b,a),(a,c)\}$ can be also written as $R = \{ab, ac, bc\} = \{a \succ b, a \succ c, b \succ c\}$, $R' = \{(ab), ac, b * c\} = \{a \sim b,\ a \succ c, b * c\}$ and $R_0 = \{ab, ac, (bc)\} = \{a \succ b, a \succ c, b \sim c\}$. ◊

We assume that all binary relations considered in this study satisfy the reflexivity property, which is a natural property in the preference theories. It follows that if we denote by $|X|$ the cardinality of a given set X, any binary relation can be seen as an element of the set

$$\mathcal{B} = \left\{ R \subset \bigcup_{\{a,b\}\in\mathcal{P}} \mathcal{B}_{ab} : |R \cap \mathcal{B}_{ab}| = 1, \forall \{a,b\} \in \mathcal{P} \right\}, \text{ where } \mathcal{B}_{ab} = \{ab, ba, (ab), a * b\}.$$[4]

We denote by $N = \{c * d : \{c,d\} \in \mathcal{P}\}$ the total uncomparison. For any pair $\{a,b\}$ of alternatives, $X_{ab} = \{ab\} \cup N \setminus \{a * b\}$ and $I_{ab} = I_{ba} = \{(ab)\} \cup N \setminus \{a * b\}$.

Two binary relations (or opinions) are said to be in **disagreement** if they differ on at least one pair of alternatives. A disagreement, in this context, refers to the pair of distinct relational outcomes assigned to a given pair of alternatives. The set of disagreements is thus the collection of all such differing relational evaluations between the two binary relations. We formalize this notion in the following definition.

DEFINITION 1. Let R and R' be two binary relations on $\mathcal{A}$ and $\{a,b\} \in \mathcal{P}$.

(i) The disagreement between R and R' on the pair $\{a,b\}$ is denoted $\mathcal{D}_{ab}(R,R')$ and referred to empty set if $R|_{\{a,b\}} = R'|_{\{a,b\}}$, and $\left\{R|_{\{a,b\}}, R'|_{\{a,b\}}\right\}$ otherwise.

(ii) The disagreement set between R and R' is defined as $\mathcal{D}(R,R') = \{\mathcal{D}_{ab}(R,R') : \{a,b\} \notin A(R,R')$ where $A(R,R')$ is the set of pairs $\{a,b\}$ on which R and R' agree.

[4]It is important to note that $\mathcal{B}$ is not consistent with a classical union symbol $\cup$; that is, we can have $R, R' \in \mathcal{B}$ but $R \cup R' \notin \mathcal{B}$. For example: $R = ab$, $R' = ba$, $R \cup R' = \{ab, ba\} \notin \mathcal{B}$, but its equivalent in $\mathcal{B}$ is $R'' = (ab)$.

In the same vein $\tilde{A}(R,R')$ is the complement of $A(R,R')$, that is the set of pairs on which R an R' disagree.

EXAMPLE. If $R=\{(ab),ac,bc\}$ and $R'=\{a*b,ac,b*c\}$, then $A(R,R')=\{\{a,c\}\}$, $\mathcal{D}_{ab}(R,R')=\{(ab),a*b\}$, $\mathcal{D}_{ac}(R,R')=\emptyset$, $\mathcal{D}_{bc}(R,R')=\{bc,b*c\}$, and $\mathcal{D}(R,R')=\{\{(ab),a*b\},\{bc,b*c\}\}$. ♢

Our objective is to characterize a rule that allows the inference of an individual's hyperpreference from their underlying preference relation. To this end, we begin by introducing the concept of a hyperpreference relation, along with the associated definitions and notions.

DEFINITION 2. (i) A **hyperpreference relation** on $\mathcal{A}$ is defined as a weak order over the set $\mathcal{B}$ of all binary relations on $\mathcal{A}$.

(ii) A **hyperpreference rule** on $\mathcal{B}$ is a mapping that assigns to each binary relation on $\mathcal{A}$ a hyperpreference relation over $\mathcal{A}$.

A weak order here refers to a complete and transitive binary relation over $\mathcal{B}$, meaning that the individual is able to compare any two binary relations and form consistent judgments about their relative desirability. A hyperpreference rule associates to every individual preference (represented by a binary relation) a complete and transitive ranking over the space of all such relations.

DEFINITION 3. Let $\mathcal{E}$ be a hyperpreference rule and U a real value function defined on $\mathcal{B}^2$. U is said to be a **hyperutility function (or representation)** of $\mathcal{E}$ if for all $R_0,R,R'\in\mathcal{B}$,

$$R\succ^{\mathcal{E}(R_0)}R'\Leftrightarrow U(R_0,R)>U(R_0,R').$$

In the same manner, u is the dis-utility function of $\mathcal{E}$ if for all $R_0,R,R'\in\mathcal{B}$,

$$R\succ^{\mathcal{E}(R_0)}R'\Leftrightarrow u(R_0,R)<u(R_0,R').$$

The preceding definition is an extension of the notion of utility representation to hyperpreferences. Assume that R_0 is the preference relation of an individual on the choice set $\mathcal{A}$. Then $U(R_0,R)$ is the utility that this individual derives from the implementation of the ranking R. Indeed, if this individual prefers the ranking R over the ranking R', then the utility that he derives from R should be greater than the utility that he derives from R', and vice versa.
Before moving to the section devoted to our axioms, we need the following definition of concatenation (or accumulation) of subsets which permits to define the accumulation of disagreements and compere them to the accumulation of dis-utilities; this is essential for a hyperpreference rule to be separable.

DEFINITION 4. Let A and B be subsets of a set E. We define the concatenation of A and B, denoted $A\sqcup B$, as the multi-set that contains each element of A and each element of B, preserving their multiplicities.

EXAMPLE. If $A=\{a,b,c,d\}$ and $B=\{a,c,e\}$ then, $A\sqcup B=\{a,b,c,d,a,c,e\}$. ♢

2. Axioms

This section presents three intuitive and normative axioms that a hyperpreference rule is expected to fulfill.

Let $\mathcal{E}$ be a hyperpreference rule on $\mathcal{B}$. We now introduce the aforementioned axioms.

AXIOM 1. ***Independence (I)***
There does not exist a sequence $(Q_1, R_1, S_1, T_1), (Q_2, R_2, S_2, T_2), ..., (Q_p, R_p, S_p, T_p)$ *in* $\mathcal{B}^4$ *such that for all* $k \in \{1, 2, ..., p\}$, $Q_k \succcurlyeq^{\mathcal{E}(R_k)} S_k$ *and* $R_k \succcurlyeq^{\mathcal{E}(S_k)} T_k$, *with at least one of both relations being strict, and* $\bigsqcup_{k=1}^{p} \mathcal{D}(Q_k, R_k) = \bigsqcup_{k=1}^{p} \mathcal{D}(S_k, T_k)$.

AXIOM 2. ***Disagreements aversion (DA)***
For all preferences $R, T, T' \in \mathcal{B}$, *if* $\mathcal{D}(R,T) \subsetneq \mathcal{D}(R,T')$ *then* $T \succ^{\mathcal{E}(R)} T'$.

AXIOM 3. ***Symmetry (S)***
For all preferences $R, T, T' \in \mathcal{B}$, *if for all* $\{a,b\} \in \tilde{A}(T,T')$, $R|_{\{a,b\}} \in \{(ab), a * b\}$ *and* $\mathcal{D}_{ab}(T,T') = \{ab, ba\}$ *then* $T \sim^{\mathcal{E}(R)} T'$.

Often called cancellation, the independence axiom means that each disagreement is assessed in the same way any time that it occurs, independently of the pair of preferences on which it occurs. Indeed, consider $Q_1, R_1, S_1, T_1 \in \mathcal{B}$ such that $Q_1 \succcurlyeq^{\mathcal{E}(R_1)} S_1$ and $R_1 \succ^{\mathcal{E}(S_1)} T_1$. Based solely on disagreements, the behavior of an individual whose hyperpreference respects these conditions can be interpreted as:

- $Q_1 \succcurlyeq^{\mathcal{E}(R_1)} S_1$: disagreement between Q_1 and R_1 is less disappointing than the one between R_1 and S_1.
- $R_1 \succ^{\mathcal{E}(S_1)} T_1$: disagreement between R_1 and S_1 is strictly less disappointing than the one between S_1 and T_1.

Assuming that disappointment is a transitive relation, we can conclude that the disagreement between Q_1 and R_1 is strictly less disappointing than the disagreement between S_1 and T_1. In this case, since disappointment only depends on the level of disagreement, we can not have the same disagreement between Q_1 and R_1 as the one between S_1 and T_1. That is what the Independence axiom generalizes. In other words, the marginal cost or disutility of a disagreement is constant; so, accumulating (or concatenating) the disagreements should be similar to accumulating marginal costs and therefore, if the concatenation of disagreements in two different situations are the same, it will be the same for the sum of marginal costs.

The axiom of Disagreements aversion refers to the comparison of the disagreements between binary relations. Indeed, the disagreements aversion axiom formalizes an aversion to disagreements in the sense that fewer disagreements increase satisfaction. In this sense, it leads to another property of rationality introduced by Laffond et al. (2020) called "self consistency", which states that the hyperpreference derived from R, ranks R before any other else binary relation; in this case any hyperpreference is single peaked and each individual has a consequentialist hyperpreference.

Finally, the symmetry axiom ensures that, for all alternatives a and b, if an individual is indifferent between a and b or cannot compare them, then this individual will be equally indifferent ranking a above b and ranking b above a; that is disagreements $\{ab, a*b\}$ and $\{ba, a*b\}$ are assessed in the same way, $\{ab, (ab)\}$ and $\{ba, (ab)\}$ too. To sum up, since $a \sim b \Leftrightarrow b \sim a$ and $a*b \Leftrightarrow b*a$, the hyperpreference rule respects the natural indifference properties of $\sim$ and $*$; in the sense that each time we are faced with a two-alternative situation $\{a,b\}$, $ab \sim^{\mathcal{E}((ab))} ba$ and $ab \sim^{\mathcal{E}(a*b)} ba$.

3. Main Finding

In this section, we present our main finding, which is that a hyperpreference rule satisfies **I**, **DA** and **S** if and only if its utility representation is **additively separable** with respect to the disagreements between the binary relations that form its arguments. This result provides an easy way to infer an individual's hyperpreference relation from her preferences.

We now define the notion of a separable hyperpreference rule.

Definition 5. Let $\mathcal{E} : \mathcal{B} \longrightarrow \mathcal{W}(\mathcal{B})$ be a hyperpreference rule on $\mathcal{B}$. $\mathcal{E}$ is said to be **separable** if for all pair $\{a,b\} \in \mathcal{P}$, there exists strictly positive real numbers $\alpha_{\{a,b\}}$, $\beta_{\{a,b\}}$, $\lambda_{\{a,b\}}$, $\gamma_{\{a,b\}}$ such that for all $R_0, R, R' \in \mathcal{B}$,

$$R \succcurlyeq^{\mathcal{E}(R_0)} R' \Leftrightarrow \sum_{\{a,b\}\in\mathcal{P}} w_{ab}(R_0,R) \leq \sum_{\{a,b\}\in\mathcal{P}} w_{ab}(R_0,R')$$

$$\text{where } w_{ab}(R_0,R) = \begin{cases} \alpha_{\{a,b\}} \text{ if } \mathcal{D}_{ab}(R_0,R) \in \{\{ab,(ab)\},\{ba,(ab)\}\} \\ \beta_{\{a,b\}} \text{ if } \mathcal{D}_{ab}(R_0,R) \in \{\{ab,a*b\},\{ba,a*b\}\} \\ \lambda_{\{a,b\}} \text{ if } \mathcal{D}_{ab}(R_0,R) = \{(ab),a*b\} \\ \gamma_{\{a,b\}} \text{ if } \mathcal{D}_{ab}(R_0,R) = \{ab,ba\} \\ 0 \text{ if } \mathcal{D}_{ab}(R_0,R) = \emptyset \end{cases}$$

We refer to w_{ab} as a weight function on the pair $\{a,b\}$.

The real numbers $\alpha_{\{a,b\}}$, $\beta_{\{a,b\}}$, $\lambda_{\{a,b\}}$, and $\gamma_{\{a,b\}}$ represent weights or dis-utilities assigned to different types of disagreements between two binary relations on the pair $\{a,b\}$ of alternatives:

- A disagreement of the type $\{a \succ^R b, a \sim^{R'} b\}$ is assigned the weight $\alpha_{\{a,b\}}$.
- A disagreement of the type $\{a \succ^R b, a *^{R'} b\}$ is assigned the weight $\beta_{\{a,b\}}$.
- A disagreement of the type $\{a \sim^R b, a *^{R'} b\}$ is assigned the weight $\lambda_{\{a,b\}}$.
- A disagreement of the type $\{a \succ^R b, b \succ^{R'} a\}$ is assigned the weight $\gamma_{\{a,b\}}$.

The comparison between $\gamma_{\{a,b\}}$, $\alpha_{\{a,b\}}$, $\beta_{\{a,b\}}$, and $\lambda_{\{a,b\}}$ is not predetermined.

If there is no disagreement on the pair $\{a,b\}$, meaning that $\mathcal{D}_{ab}(R_0,R) = \emptyset$, it is natural to set the disagreement weight to zero.

A hyperpreference rule $\mathcal{E}$ is said to be *separable* if, for any given preference R_0, it assigns a weak order $\mathcal{E}(R_0)$ on $\mathcal{B}$ and this weak order compares two binary relations R and R' by simply evaluating the sum of the disagreement weights between R_0 and R, and comparing it to the sum of disagreement weights between R_0 and R'. We refer to a *minimizing weighted rule* as a hyperpreference rule that assigns a non-negative weight to each potential disagreement and aggregates these weights additively to evaluate binary relations. The total weight is interpreted as a disutility to be minimized. Below, we present the necessary and sufficient conditions for a hyperpreference rule to be considered a minimizing weighted rule.

THEOREM 1. *Let $\mathcal{E}$ be a hyperpreference rule on $\mathcal{B}$. $\mathcal{E}$ satisfies (I) and (DA) if and only if it is a minimizing weighted rule.*

The proof in Appendix need the following lemmas whose proofs are also in the appendix.

LEMMA 2. *Let $\mathcal{E}$ be a hyperpreference rule. If $\mathcal{E}$ satisfies (I), then it also satisfies the following property of acyclicity across preferences (AAP)*[5]*: there does not exist a sequence of preferences $R_1, R_2, ..., R_p$ in $\mathcal{B}$ such that $R_1 \succ^{\mathcal{E}(R_2)} R_3$, $R_2 \succ^{\mathcal{E}(R_3)} R_4$,...,$R_{p-1} \succ^{\mathcal{E}(R_p)} R_1$ and $R_p \succ^{\mathcal{E}(R_1)} R_2$.*

Proof in Appendix

LEMMA 3. *Let $\mathcal{E}$ be a hyperpreference rule. If $\mathcal{E}$ satisfies (I), then it also satisfies the following property: Disagreement consistency (DC):*
For all $R, R', T, T' \in \mathcal{B}$, if $\mathcal{D}(R,R') = \mathcal{D}(T,T')$ then , $R' \succsim^{\mathcal{E}(R)} T \Leftrightarrow T' \succsim^{\mathcal{E}(T)} R$.

PROOF. By contradiction, assume that there exist $R, R', T, T' \in \mathcal{B}$ such that $\mathcal{D}(R,R') = \mathcal{D}(T,T')$ and $R' \succsim^{\mathcal{E}(R)} T$ and $R \succ^{\mathcal{E}(T)} T'$.

	left-hand	right-hand
$R' \succsim^{\mathcal{E}(R)} T, R \succ^{\mathcal{E}(T)} T'$	$\mathcal{D}(R,R')$ **and**	$\mathcal{D}(T,T')$

Since $\mathcal{D}(R,R') = \mathcal{D}(T,T')$, the contradiction of the independence axiom is obvious. □

Before moving to the next lemma, it is important to recall that a weak order $\trianglerighteq$ on the set of pairs of binary relations is compatible with $(\mathcal{B}, \{\mathcal{E}(R)\}_{R\in\mathcal{B}})$ if for all $R, R', T \in \mathcal{B}$,

$$\{R,T\} \triangleright \{R,T'\} \Leftrightarrow T \succ^{\mathcal{E}(R)} T'$$

$(\mathcal{B}, \{\mathcal{E}(R)\}_{R\in\mathcal{B}})$ is also said to be $\trianglerighteq$-compatible. If such order $\trianglerighteq$ exists, $(\mathcal{B}, \{\mathcal{E}(R)\}_{R\in\mathcal{B}})$ is said to be order compatible.

[5]The acyclicity was introduced by Laffond et al. (2020) in the context where preferences on the set of alternatives are linear orders, and then it was called "acyclicity across orders (AAO).

LEMMA 4. *Let $\mathcal{E}$ be hyperpreference rule. If $\mathcal{E}$ satisfies (AAP), (DC) and (DA), then there exists a compatible order $\sqsubseteq$ on $\mathcal{P}(\mathcal{B})$ such that for all $R, R', T, T' \in \mathcal{B}$, if $\mathcal{D}(R,R') = \mathcal{D}(T,T')$ then $\{R,R'\} \simeq \{T,T'\}$ and if $\mathcal{D}(R,R') \subsetneq \mathcal{D}(T,T')$ then $\{R,R'\} \sqsubset \{T,T'\}$.*

Proof in Appendix

Another feature of this characterization is its implication for the Independence of Irrelevant Alternative (IIA) axioms; the latter requires that if an alternative a is not involved in any disagreement between preference R and R', then for all preference T,

$$R \succ^{\mathcal{E}(T)} R' \Leftrightarrow R|_{\mathcal{A}\setminus\{a\}} \succ^{\mathcal{E}(T|_{\mathcal{A}\setminus\{a\}})} R'|_{\mathcal{A}\setminus\{a\}}.$$

COROLLARY 1. The independence and disagreements aversion axioms imply the IIA axiom.

This is because the independence and disagreements aversion axioms imply additivity, which implies the IIA. Our main finding is deduced from the above theorem.

THEOREM 5. *Let $\mathcal{E}$ be a hyperpreference rule. $\mathcal{E}$ satisfies (DA), (I) and (S) if and only if it is separable.*

PROOF. Thank to the theorem 1, $\mathcal{E}$ satisfies (DA) and (I) if and only if it is a minimizing weighted rule. In this case, $\alpha_{\{a,b\}} = \alpha'_{\{a,b\}}$ and $\beta_{\{a,b\}} = \beta'_{\{a,b\}}$ for all $\{a,b\} \in \mathcal{P}$ if and only if $\mathcal{E}$ satisfies (S); $\alpha'_{\{a,b\}}$ and $\beta'_{\{a,b\}}$ as defined in the proof of the theorem 1. □

The following remark gives the condition under which the restriction of a separable hyperpreference rule to the set of transitive and complete preference relations yields the Kendall Tau rule or the Kemeny rule.

REMARK 1. When the **Kendall Tau distance** or the **Kemeny distance** between two linear orders is obtained by counting the number of pairwise inversions required to transform one order into another $d_K(R,R') = |R \setminus R'|$, this extends to the set of weak orders as $d_K(R,R') = \frac{1}{2}(|R \setminus R'| + |R' \setminus R|)$. In this case a disagreement of the type $\{ab, (ab)\}$ can be weighted $\frac{1}{2}$ and a disagreement of the type $\{ab, ba\}$ can be weighted 1. In this sense, when $\alpha_{\{a,b\}} = \frac{1}{2}$ and $\gamma_{\{a,b\}} = 1$ for each pair $\{a,b\}$, we obtain the Kemeny rule on the set of complete and transitive preferences on $\mathcal{A}$.

We would like to show that (DA), (I) and (S) form a parsimonious set of axioms in the sense that these axioms are pairwise independent from each other. Indeed, we show that each of these axioms is necessary for this characterization; that is (S) and (I) do not imply (DA); (DA) and (I) do not imply (S) and, (DA) and (S) do not imply (I). We have the following proposition.

PROPOSITION 1. There is logical independence between (I), (DA) and (S).

Proof in Appendix.

To conclude this section, we provide the needed condition on the transformation of weight functions to induce the same hyperpreference rule. We show that given a continuous real-valued function f, to be sure that for any separable hyperpreference rule and any weight function, a transformation of weights via f does not change the hyperpreference rule, it is necessary and sufficient that f be a positive homothetic transformation. This result is stated below.

THEOREM 6. *Let f be a continuous real-valued function. For any weight function $w = (w_{ab})$, w and $f(w) = (f(w_{ab}))$ induce the same separable hyperpreference rule if and only if f is a positive linear transformation on $\mathbb{R}_+$.*

Proof in Appendix.

This means that a transformation of the weights by a function f preserves the hyperpreference rule if and only if f is a positive linear transformation, that is, $f(x) = k \cdot x$ for all $x \geq 0,$ with $k > 0$. We now turn to the application of separable hyperpreference rules in the analysis of strategy-proofness and efficiency in aggregation mechanisms.

4. An Application to Aggregation Mechanisms: Strategy-Proofness and Efficiency

Beyond its conceptual contributions, our framework offers practical guidance for designing institutions and aggregation mechanisms that are compatible with *strategy-proofness*, where truth-telling is a dominant strategy, and *efficiency*, where collective rankings cannot be improved upon.

As an illustration, imagine a scenario in the public decision-making process concerning investments in a community. In this scenario, each individual affected by economic policies must evaluate their utility based on the policies implemented in key sectors such as education (option a), health (option b), and agriculture (option c). Consider three representative members of the community, each possessing distinct interests and requirements:

- Resident 1 (a parent of a student): Values education as they aspire for a brighter future for their children. Their preference order is: $a \succ b \succ c$.
- Resident 2 (a farmer): Values agriculture since their economic livelihood relies on it. Their preference order is: $c \succ b \succ a$.
- Resident 3 (an elderly individual with health concerns): Values health as it represents their pressing need. Their preference order is: $b \succ a \succ c$.

These individual priorities may give rise to the Condorcet paradox when decisions are made using the simple majority rule. As a result, the decision-maker may opt for an alternative rule that reliably yields an outcome maximizing the overall utility of the residents, i.e. an *efficient decision.* To ensure such efficiency, the decision-maker must first verify that each resident (assumed to act rationally) reports their true priorities. This requires the rule to be *strategy-proof*, thereby preventing outcomes where an individual can benefit at the expense of the collective welfare. This naturally raises the following question:

Under what conditions do agents truthfully reveal their preferences?

We will answer this question in a situation where we assume that the different options compete like in a **tournament**. A tournament is a setting where multiple candidates contest in pairs, voters vote over each pair of candidates, and votes are aggregated using a general rule to form a social ranking. We discuss conditions under which certain well-known aggregation rules are strategy-proof and Pareto-efficient when agents (or voters) have separable hyperutility functions. We follow the model developed in Pongou and Tchantcho (2021), but differ from it in that in our model, a voter may have different sensitivities to disagreements between pairs of alternatives.

Consider a finite set N of $n \geq 3$ voters. Each voter i has a preference relation R^i which is a weak order on the set of alternatives. Denote by $R^N = (R^i)_{i \in N}$ the corresponding preference profile. An aggregation rule is a function f which transforms a preference profile R^N into a binary relation $f(R^N)$ on the set of alternatives.

Given a pair $\{a, b\}$ of alternatives, if neither $a \succ^{f(R^N)} b$ nor $b \succ^{f(R^N)} a$, we say that a and b are not comparable by $f(R^N)$. We denote by $\mathcal{R}_{ab} = \{i \in N : R^i|_{\{a,b\}} = ab\}$ the set of voters who prefer a to b, and by $\mathcal{R}_{(ab)} = \{i \in N : R^i|_{\{a,b\}} = (ab)\}$ the set of voters who are indifferent between a and b; we obtain a partition of N into three subsets $\mathcal{R}_{ab}$, $\mathcal{R}_{(ab)}$, and $\mathcal{R}_{ba}$. We refer to $\mathcal{R} = (\mathcal{R}_{ab}, \mathcal{R}_{(ab)}, \mathcal{R}_{ab})$ as a tripartition of N. Such a tripartition $\mathcal{R}$ is said to be winning for a on the pair $\{a, b\}$ if $f(R^N)|_{\{a,b\}} = ab$.

An aggregation rule is said to be *unanimous* if each time that all members of N are unanimous over a comparison between two alternatives a and b, this comparison is adopted. Formally, if $\mathcal{R}_{ab} = \emptyset$, $(\mathcal{R}_{ab}, \mathcal{R}_{(ab)}, \mathcal{R}_{ba})$ cannot be winning for a; $(\mathcal{R}_{ab}, \mathcal{R}_{(ab)}, \mathcal{R}_{ba})$ is winning for a whenever $\mathcal{R}_{ab} = N$; no alternative wins if $\mathcal{R}_{(ab)} = N$.

An aggregation rule is said to be *monotonic* if for all preference profiles R^N, if T^N is obtained by increasing the set of members who prefer a and reducing those who do not prefer a, then, if $\mathcal{R}$ is winning for a, then $\mathcal{T}$ is too. Formally for all profiles R^N and T^N, and all pair $\{a, b\}$, if $\mathcal{R}$ is winning for a and $\mathcal{R}_{ab} \subseteq \mathcal{T}_{ab}$ and $\mathcal{T}_{ba} \subseteq \mathcal{R}_{ba}$, then $\mathcal{T}$ is also winning for a. It also means that if $\mathcal{R}$ is not winning for a and $\mathcal{R}_{ab} \supseteq \mathcal{T}_{ab}$ and $\mathcal{T}_{ba} \supseteq \mathcal{R}_{ba}$, then $\mathcal{T}$ is not winning for a.

We impose the following condition: for any pair $\{a, b\}$, the tripartition $\mathcal{R}$ cannot be winning for both a and b simultaneously. In our model, this condition translates the classical properness condition in voting games, which states that the complement of a winning coalition is a losing coalition.

An aggregation rule is said to be **strategy-proof** (or non-manipulable) if no agent can improve his hyperutility by changing strategically his true first-order preference and consequently the social preference.[6] Formally, f is said to be strategy-proof if there does not exist an individual $i \in N$ and a preference profile R^N and $Q \in \mathcal{W}(\mathcal{A})$ such that $f(R^{N \setminus \{i\}}, Q) \succ^{\mathcal{E}(R^i)} f(R^N)$. It is said to be **Pareto-efficient** if it is not possible to

[6] This is equivalent to truthful reporting of preferences being a pure strategy Nash equilibrium.

change the result in a way that increases the satisfaction (hyperutility) of a voter without decreasing that of another voter.

We assume that each agent has a separable hyperutility function, and that for any given pair $\{a,b\}$ of alternatives $\gamma_{\{a,b\}}$ is the higher level of weight[7], the utility function of an individual on the pair $\{a,b\}$ is such that $U^i_{ab}(R^i, f(R^N)) = \gamma_{\{a,b\}} - w_{ab}(R^i, f(R^N))$; that is,

$$U^i_{ab}(R^i, f(R^N)) = \begin{cases} 0 \text{ if } \mathcal{D}_{ab}(R^i, f(R^N)) = \{ab, ba\} \\ \gamma_{\{a,b\}} - \alpha_{\{a,b\}} \text{ if } \mathcal{D}_{ab}(R^i, f(R^N)) \in \{\{ab, (ab)\}, \{ba, (ab)\}\} \\ \gamma_{\{a,b\}} - \beta_{\{a,b\}} \text{ if } \mathcal{D}_{ab}(R^i, f(R^N)) \in \{\{ab, a*b\}, \{ba, a*b\}\} \\ \gamma_{\{a,b\}} - \lambda_{\{a,b\}} \text{ if } \mathcal{D}_{ab}(R^i, f(R^N)) = \{a*b, (ba)\} \\ \gamma_{\{a,b\}} \text{ if } \mathcal{D}_{ab}(R^i, f(R^N)) = \emptyset \end{cases}$$

and $U^i(R^i, f(R^N)) = \sum_{\{a,b\}\in\mathcal{P}} U^i_{ab}(R^i, f(R^N)) = \sum_{\{a,b\}\in\mathcal{P}}[\gamma_{\{a,b\}} - w_{ab}(R^i, f(R^N))]$.

The idea behind the definition of $U^i_{ab}(R^i, f(R^N))$ is that individual reaches the worst level of utility each time that the social result is the opposition of his preference. Since other else disagreement is less strict than the total opposition, it seems natural to assume that the provided utility is greater than the former which is stated at zero. In the same vein, individual is full satisfied when the social comparison coincides with the individual one and therefore, utility is maximal and equal to what is lost in the worst scenario. This definition of the hyperutility function firstly reflects the symmetry axiom and secondly is consistent with the axiom of disagreements aversion which states that any additional disagreement reduces the hyperutility and the individual first-order preference is the pick of his hyperpreference and so the total agreement leads to full satisfaction and maximizes the hyperutility. The fact that dis-utilities are compute separably is a consequence of the independence axiom. We have the following result:

THEOREM 7. *All unanimous aggregation rules are Pareto-efficient if and only if for all pair $\{a,b\}$ of alternatives, $\alpha_{\{a,b\}} \geq \lambda_{\{a,b\}}$ and $\gamma_{\{a,b\}} > \beta_{\{a,b\}}$.*

Proof in Appendix.

We also provide below a necessary and sufficient condition for all monotonic rules to be strategy-proof.

THEOREM 8. *All monotonic aggregation rules are strategy-proof if and only if for all pair $\{a,b\}$ of alternatives, $\alpha_{\{a,b\}} = \lambda_{\{a,b\}}$.*

The proof in Appendix.

The following results are the generalizations of Theorems 7 and 8 to a situation where agents may have different sensitivities on disagreement over pairs of alternatives.

[7]This seem natural in the context of social decision, since it is the dis-utility associated to the higher level of disagreement.

COROLLARY 2. All unanimous aggregation rules are Pareto-efficient if and only if for all pair $\{a,b\}$ of alternatives and for all individual i, $\alpha^i_{\{a,b\}} \geq \lambda^i_{\{a,b\}}$ and $\gamma^i_{\{a,b\}} > \beta^i_{\{a,b\}}$.

COROLLARY 3. All monotonic aggregation rules are strategy-proof if and only if for all pair $\{a,b\}$ of alternatives and for all individual i, $\alpha^i_{\{a,b\}} = \lambda^i_{\{a,b\}}$.

These findings offer new insights into the design of mechanisms that promote both truth-telling and efficiency. Since truthful preference reporting is essential for accurately assessing differences in opinion, our analysis also clarifies the conditions under which such differences can be meaningfully and reliably measured.

5. CONCLUSION

Disagreement over values, priorities, and trade-offs is a defining feature of economic life. While standard models allow for heterogeneity in first-order preferences, they offer limited tools for analyzing how individuals evaluate alternative preference orderings—especially those proposed by others or produced through collective decision-making. This paper develops a formal theory of preference divergence that introduces and operationalizes the concept of hyperpreferences: second-order judgments about how well one preference ranking diverges from another.

We show how hyperpreferences can be inferred under reasonable axioms, and we provide a systematic method for measuring the structure and magnitude of disagreement between preferences. Our approach extends distance-based methods, such as the Kendall tau and Kemeny distances, by allowing richer distinctions between types of disagreement. This enables a more nuanced account of how individuals assess social decisions—not only in terms of personal gain, but also in terms of fairness, compromise, and alignment with broader priorities.

Beyond its conceptual contributions, the framework developed in this paper has practical implications for the design of institutions and aggregation mechanisms in settings marked by deep preference heterogeneity. Whether in organizational structures or democratic systems, identifying the conditions under which individuals truthfully report their preferences is critical for anticipating cooperation, dissent, and reform. Our analysis provides necessary and sufficient conditions for truth-telling and efficiency under familiar aggregation rules, offering concrete guidance for mechanism design. Since the accurate measurement of opinion differences depends on truthful reporting, the results also clarify when such differences can be meaningfully and reliably assessed.

More broadly, this work opens a new direction for modeling interpersonal and institutional disagreement. By formalizing how individuals compare rankings—not just alternatives—it offers a foundation for future research on legitimacy, deliberation, and the design of systems that can productively navigate pluralism.

APPENDIX

proof of theorem 1

proof of lemma 2

PROOF. We proceed by contradiction. Assume that $\mathcal{E}$ satisfies (I) and there exists a sequence of preferences $R_1, R_2, ..., R_p$ in $\mathcal{B}$ such that $R_1 \succ^{\mathcal{E}(R_2)} R_3$, $R_2 \succ^{\mathcal{E}(R_3)} R_4$,...,$R_{p-1} \succ^{\mathcal{E}(R_p)} R_1$ and $R_p \succ^{\mathcal{E}(R_1)} R_2$. We have

	left-hand		right-hand
$R_1 \succ^{\mathcal{E}(R_2)} R_3, R_2 \succ^{\mathcal{E}(R_3)} R_4$	$\mathcal{D}(R_1, R_2)$	**and**	$\mathcal{D}(R_3, R_4)$
$R_2 \succ^{\mathcal{E}(R_3)} R_4, R_3 \succ^{\mathcal{E}(R_4)} R_5$	$\mathcal{D}(R_2, R_3)$	**and**	$\mathcal{D}(R_4, R_2)$
...	...		...
$R_{p-2} \succ^{\mathcal{E}(R_{p-1})} R_p, R_{p-1} \succ^{\mathcal{E}(R_p)} R_1$	$\mathcal{D}(R_{p-2}, R_{p-1})$	**and**	$\mathcal{D}(R_p, R_1)$
$R_{p-1} \succ^{\mathcal{E}(R_p)} R_1, R_p \succ^{\mathcal{E}(R_1)} R_2$	$\mathcal{D}(R_{p-1}, R_p)$	**and**	$\mathcal{D}(R_1, R_2)$
$R_p \succ^{\mathcal{E}(R_1)} R_2, R_1 \succ^{\mathcal{E}(R_2)} R_3$	$\mathcal{D}(R_p, R_1)$	**and**	$\mathcal{D}(R_2, R_3)$

By concatenating all the left-hand of the disagreements together and all the right-hand together, we obtain the same multi-set; indeed each used disagreement appears one time on the left-hand and one time on the right-hand. That is a contradiction to the Independence axiom. □

proof of lemma 4

PROOF. Assume that $\mathcal{E}$ satisfies (AAP),(DC) and (DA). Because of (AAP) and (DA)[8], thanks to Laffond et al. (2020) in their lemma 3, $(\mathcal{B}, \{\mathcal{E}(R)\}_{R\in\mathcal{B}})$ is order compatible[9]; so there exists a weak order $\trianglerighteq$ on $\mathcal{P}(\mathcal{B})$ such that for all $R, T, T' \in \mathcal{B}$,

$$T \succ^{\mathcal{E}(R)} R' \Leftrightarrow \{R, T\} \triangleright \{R, T'\}$$

Let $\sqsubseteq$ (with the indifference part $\simeq$) be the weak order on $\mathcal{P}(\mathcal{B})$ obtained by ranking all pairs of preferences with the same disagreement equivalently, ensuring that $(\mathcal{B}, \{\mathcal{E}(R)\}_{R\in\mathcal{B}})$ remains $\sqsubseteq$-compatible. This order is constructed using the following procedure.

Let $R, R', T, T' \in \mathcal{B}$ such that $\mathcal{D}(R, R') = \mathcal{D}(T, T')$. For all $P, P' \in \mathcal{B} \setminus \{R, R', T, T'\}$,

- if $\{R, R'\} \trianglerighteq \{P, P'\} \triangleright \{T, T'\}$ (resp. $\{R, R'\} \triangleright \{P, P'\} \trianglerighteq \{T, T'\}$) then, consider $\{R, R'\} \simeq \{T, T'\} \sqsubseteq \{P, P'\}$ (resp. $\{R, R'\} \simeq \{T, T'\} \sqsubset \{P', P\}$).
- if $\{R, R'\} \trianglerighteq \{R, P\} \triangleright \{T, T'\}$ (resp. $\{R, R'\} \triangleright \{R, P\} \trianglerighteq \{T, T'\}$) then, consider $\{R, R'\} \simeq \{T, T'\} \sqsubseteq \{R, P\}$ (resp. $\{R, R'\} \simeq \{T, T'\} \sqsubset \{R, P\}$).
- if $\{R, R'\} \trianglerighteq \{T, P\} \triangleright \{T, T'\}$ (resp. $\{R, R'\} \triangleright \{T, P\} \trianglerighteq \{T, T'\}$) then, consider $\{T, P\} \sqsubset \{R, R'\} \simeq \{T, T'\}$ (resp. $\{T, P\} \sqsubseteq \{R, R'\} \simeq \{T, T'\}$).

[8]Disagreements aversion implies self consistency

[9]Authors shown the compatibility in the context of hyperpreference on the set of linear orders, but the proof doesn't use the fact that preferences are linear orders; therefore it is also true for any form of (first-order) preference relations on the set of alternatives.

- the case $\{R,R'\} \trianglerighteq \{R,T\} \triangleright \{T,T'\}$ (resp. $\{R,R'\} \triangleright \{R,T\} \trianglerighteq \{T,T'\}$) is impossible; else, $R' \succcurlyeq^{\mathcal{E}(R)} T$ and $R \succ^{\mathcal{E}(T)} T'$ (resp. $R' \succ^{\mathcal{E}(R)} T$ and $R \succcurlyeq^{\mathcal{E}(T)} T'$) because $(\mathcal{B}, \{\mathcal{E}(R)\}_{R\in\mathcal{B}})$ is $\trianglerighteq$-compatible; this contradicts the (DC) property.

$$\text{We have for all } R,R',T,T' \in \mathcal{B}, \begin{cases} (R,R') \simeq (T,T') \text{ if } \mathcal{D}(R,R') = \mathcal{D}(T,T') \\ (R,R') \sqsubset (T,T') \Leftrightarrow (R,R') \triangleright (T,T') \text{ else .} \end{cases}$$

We are going to show that $(\mathcal{B}, \{\mathcal{E}(R)\}_{R\in\mathcal{B}})$ is $\sqsubseteq$-compatible; that is for all $R,T,T' \in \mathcal{B}$,

$$T \succ^{\mathcal{E}(R)} T' \Leftrightarrow \{R,T\} \sqsubset \{R,T'\}$$

We proceed by contradiction.
Assume that there exist $R,T,T' \in \mathcal{B}$ such that $T \succ^{\mathcal{E}(R)} T'$ and $\{R,T'\} \sqsubseteq \{R,T\}$.
$T \succ^{\mathcal{E}(R)} R' \Leftrightarrow \{R,T\} \triangleright \{R,T'\}$; so

$$\begin{cases} T \succ^{\mathcal{E}(R)} R' \\ \{R,T'\} \sqsubseteq \{R,T\} \end{cases} \Leftrightarrow \begin{cases} \{R,T\} \triangleright \{R,T'\} \\ \{R,T'\} \sqsubseteq \{R,T\} \end{cases}$$

So there exists $\{P,P'\} \in \mathcal{P}(\mathcal{B})$ such that $\mathcal{D}(P,P') = \mathcal{D}(R,T')$ and $\{P,P'\} \trianglerighteq \{R,T\} \triangleright \{R,T'\}$; but in this case, the obtained order is $\{R,T\} \sqsubset \{P,P'\} \simeq \{R,T'\}$, that contradicts the fact that $\{R,T'\} \sqsubseteq \{R,T\}$; so $(\mathcal{B}, \{\mathcal{E}(R)\}_{R\in\mathcal{B}})$ is $\sqsubseteq$- compatible.
Now, let $R,R',T,T' \in \mathcal{B}$ such that $\mathcal{D}(R,R') \subsetneqq \mathcal{D}(T,T')$. Let $Q \in \mathcal{B}$ such that $\mathcal{D}(R,R') = \mathcal{D}(T,Q)$, so $Q \neq T'$.

$$\mathcal{D}(R,R') = \mathcal{D}(T,Q) \Rightarrow \{R,R'\} \simeq \{T,Q\} \text{ and } \mathcal{D}(T,Q) \subsetneqq \mathcal{D}(T,T') \Rightarrow \{T,Q\} \sqsubset \{T,T'\}$$

By the transitivity of $\sqsubseteq$, we have $\{R,R'\} \sqsubset \{T,T'\}$.

□

Proof of theorem 1

PROOF. Let $\mathcal{E}$ be a hyperpreference rule satisfying (I) and (DA).
$\Rightarrow$) Thank to lemma 2 and 3, $\mathcal{E}$ satisfies (AAP) and (DC). Then, the conditions of lemma 4 are satisfied. Using the weak order $\sqsubseteq$ obtained in the lemma 4, a pair of binary relations can be identify in the relation $\sqsubseteq$ by its set of disagreements.
Let F be the set of feasible disagreements between two given binary relations; that is $D \in F$ if and only if either $D = \emptyset$ or there exists $R,R' \in \mathcal{B}$ such that $R \neq R'$ and $D = \mathcal{D}(R,R')$.

$$F = \left\{ D \subset \bigcup_{\{a,b\}\in\mathcal{P}} F_{ab} : |D \cap F_{ab}| \leq 1, \forall \{a,b\} \in \mathcal{P} \right\}$$

where $F_{ab} = \{\{ab,ba\},\{ab,(ab)\},\{ba,(ab)\},\{ab,a*b\},\{ba,a*b\}\{(ab),a*b\}\}$ is the set of feasible disagreements on the pair $\{a,b\}$.
We define the (opposite) dual weak order $\succcurlyeq$ of $\sqsubseteq$ on F as follows:

$$\mathcal{D}(R,R') \succ \mathcal{D}(T,T') \Leftrightarrow \{T,T'\} \sqsubset \{R,R'\}$$

$\succcurlyeq$ such as defined is also compatible with $(\mathcal{P}(\mathcal{B}), \{\mathcal{E}(R)\}_{R\in\mathcal{B}})$ in the sense that

$$\mathcal{D}(R,T) \prec \mathcal{D}(R,T') \Leftrightarrow T \succ^{\mathcal{E}(R)} T'$$

Recall that $\succcurlyeq$ is a weak order on the finite set F. We are going to show that there exists a real value function w on the set F such that for all $D_1, D_2 \in F$,

$$w(D_1) > w(D_2) \Leftrightarrow D_1 \succ D_2 \text{ and for all } D \in F,\ w(D) = \sum_{d\in D} w(d)$$

For that, we need to show that $\succcurlyeq$ satisfies the independence property of Fishburn (1992). That is, there does not exists a finite sequence $(A_1,B_1),(A_2,B_2),...,(A_p,B_p)$ in $F\times F$ such that for all $k\in\{1,2,...,p\}$, $A_k \succcurlyeq B_k$, with at least one relation being strict, and $\bigsqcup_{k=1}^{p} A_k = \bigsqcup_{k=1}^{p} B_k$.
Our aim is to show that for such a sequence, there exists k such that changing $A_k \succcurlyeq B_k$ to $B_k \succ A_k$ (resp. $A_k \succ B_k$ to $B_k \succcurlyeq A_k$) does not have impact on the compatibility with $(\mathcal{P}(\mathcal{B}), \{\mathcal{E}(R)\}_{R\in\mathcal{B}})$.
If such a sequence exists, since $A_k, B_k \in F$, there exist $Q_k, R_k, S_k, T_k \in \mathcal{B}$ such that $A_k = \mathcal{D}(Q_k,R_k)$ and $B_k = \mathcal{D}(S_k,T_k)$. Reversing the relation of the index k may have impact on the compatibility if and only if there exists $U_k \in \mathcal{B}$ such that $\{Q_k,R_k\} \sqsubset \{R_k,U_k\} \sqsubseteq \{S_k,T_k\}$ or $\{Q_k,R_k\} \sqsubseteq \{R_k,U_k\} \sqsubset \{S_k,T_k\}$ (resp. $\{Q_k,R_k\} \sqsubset \{U_k,S_k\} \sqsubseteq \{S_k,T_k\}$ or $\{Q_k,R_k\} \sqsubseteq \{U_k,S_k\} \sqsubset \{S_k,T_k\}$).
Case 1: If there exists $k\in\{1,2,...,p\}$ such that such U_k doesn't exist, we change $A_k \succcurlyeq B_k$ to $B_k \succ A_k$ (resp. $A_k \succ B_k$ to $B_k \succcurlyeq A_k$) and it doesn't change something on the compatibility with $(\mathcal{B}, \{\mathcal{E}(R)\}_{R\in\mathcal{B}})$.
Case 2: If there exists $k\in\{1,2,...,p\}$ such that U_k exists and $U_k \notin \{Q_k,R_k,S_k,T_k\}$

- If $\{Q_k,R_k\} \sqsubset \{R_k,U_k\} \sqsubseteq \{S_k,T_k\}$(resp. $\{Q_k,R_k\} \sqsubseteq \{R_k,U_k\} \sqsubset \{S_k,T_k\}$), change to $\{S_k,T_k\} \sqsubseteq \{Q_k,R_k\} \sqsubset \{R_k,U_k\}$ (resp. $\{S_k,T_k\} \sqsubset \{Q_k,R_k\} \sqsubseteq \{R_k,U_k\}$).
  If $\{Q_k,R_k\} \sqsubset \{U_k,S_k\} \sqsubseteq \{S_k,T_k\}$(resp. $\{Q_k,R_k\} \sqsubseteq \{U_k,S_k\} \sqsubset \{S_k,T_k\}$), change to
  $\{U_k,S_k\} \sqsubseteq \{S_k,T_k\} \sqsubset \{Q_k,R_k\}$ (resp. $\{U_k,S_k\} \sqsubset \{S_k,T_k\} \sqsubseteq \{Q_k,R_k\}$) and it does not impact on the compatibility.

Case 3: If for all k, U_k exists and $U_k \in \{Q_k,R_k,S_k,T_k\}$, then we obtain a sequence

$(Q_1,R_1,S_1,T_1),(Q_2,R_2,S_2,T_2),...,(Q_p,R_p,S_p,T_p)$ in $\mathcal{B}\times\mathcal{B}\times\mathcal{B}\times\mathcal{B}$ such that for each

$k\in\{1,2,...,p\}$, we have one of the following cases.

$$\{Q_k,R_k\} \sqsubseteq \{R_k,S_k\} \sqsubseteq \{S_k,T_k\} \Leftrightarrow Q_k \succcurlyeq^{\mathcal{E}(R_k)} S_k, R_k \succcurlyeq^{\mathcal{E}(S_k)} T_k$$

$$\{Q_k,R_k\} \sqsubseteq \{Q_k,S_k\} \sqsubseteq \{S_k,T_k\} \Leftrightarrow R_k \succcurlyeq^{\mathcal{E}(Q_k)} S_k, Q_k \succcurlyeq^{\mathcal{E}(S_k)} T_k$$

$$\{Q_k,R_k\} \sqsubseteq \{R_k,T_k\} \sqsubseteq \{S_k,T_k\} \Leftrightarrow Q_k \succcurlyeq^{\mathcal{E}(R_k)} T_k, R_k \succcurlyeq^{\mathcal{E}(T_k)} S_k$$

$$\{Q_k,R_k\} \sqsubseteq \{Q_k,T_k\} \sqsubseteq \{S_k,T_k\} \Leftrightarrow R_k \succcurlyeq^{\mathcal{E}(Q_k)} S_k, Q_k \succcurlyeq^{\mathcal{E}(T_k)} S_k$$

with at least one of both relations being strict and $\bigsqcup_{k=1}^{p}\mathcal{D}(Q_k,R_k)=\bigsqcup_{k=1}^{p}\mathcal{D}(S_k,T_k)$. In this case, we obtain a contradiction of the independence axiom. Thus, each time that there exists such a sequence, we can break it at the way to have $\succcurlyeq$ on F, which is a weak order such that such a sequence $(A_k,B_k)_k$ doesn't exist; that is, $\succcurlyeq$ satisfies the independence axiom as defined in Fishburn (1992). In this case, thanks to Fishburn (1992), the aforementioned function w exists. w being additive, for all $A,B\in F$, if $A\cap B=\emptyset$ and $A\cup B\in F$, then $w(A\cup B)=w(A)+w(B)$.
Firstly, since $\emptyset\in F$ and $\emptyset\cap\emptyset=\emptyset\cup\emptyset=\emptyset$, we have $2w(\emptyset)=w(\emptyset)\Rightarrow w(\emptyset)=0$.
Secondly, for all $R,R'\in\mathcal{B}$,

$$\begin{aligned}R\neq R' &\Rightarrow \mathcal{D}(R,R)=\emptyset\subsetneq\mathcal{D}(R,R')\\&\Rightarrow \mathcal{D}(R,R')\succ\mathcal{D}(R,R)\\&\Rightarrow 0=w(\emptyset)<w(\mathcal{D}(R,R'))\end{aligned}$$

Since,

- any element of F distinct to $\emptyset$ is a collection of elements of the F_{ab} (with $\{a,b\}\in\mathcal{P}$)
- $\succcurlyeq$ is a (opposite) dual of $\sqsubseteq$ and
- $(\mathcal{B},\{\mathcal{E}(R)\}_{R\in\mathcal{B}})$ is $\sqsubseteq$-compatible

let's identify the weights as follows: for all $\{a,b\}\in\mathcal{P}$,

$$\begin{aligned}\alpha_{\{a,b\}}&=w(\{(ab),ab\}) & \alpha'_{\{a,b\}}&=w(\{(ab),ba\})\\ \beta_{\{a,b\}}&=w(\{ab,a*b\}) & \beta'_{\{a,b\}}&=w(\{ba,a*b\})\\ \lambda_{\{a,b\}}&=w(\{(ab),a*b\}) & \gamma_{\{a,b\}}&=w(\{ab,ba\}) & w(\emptyset)=0;\end{aligned}$$

For each $\{a,b\}\in\mathcal{P}(\mathcal{A})$, we define w_{ab} for $R,R'\in\mathcal{B}$ as follows: $w_{ab}(R,R')=w(\mathcal{D}_{ab}(R,R'))$. As consequence we have $w(\mathcal{D}(R,R'))=\sum_{\{a,b\}\in\mathcal{P}}w_{ab}(R,R')$.

$$\begin{aligned}\sum_{\{a,b\}\in\mathcal{P}}w_{ab}(R,T)<\sum_{\{a,b\}\in\mathcal{P}}w_{ab}(R,T') &\Leftrightarrow \mathcal{D}(R,T')\succ\mathcal{D}(R,T)\\&\Leftrightarrow \{R,T\}\sqsubset\{R,T'\}\\&\Leftrightarrow T\succ^{\mathcal{E}(R)}T'\end{aligned}$$

$\Leftarrow$) Conversely, assume that for each pair $\{a,b\}\in\mathcal{P}$, there exists $\alpha_{\{a,b\}},\alpha'_{\{a,b\}},\beta_{\{a,b\}},\beta'_{\{a,b\}},\lambda_{\{a,b\}},\gamma_{\{a,b\}}\in\mathbb{R}_{++}$ such that for all $R,T,T'\in\mathcal{B}$,

$$T\succ^{\mathcal{E}(R)}T'\Leftrightarrow\sum_{\{a,b\}\in\mathcal{P}}w_{ab}(R,T)<\sum_{\{a,b\}\in\mathcal{P}}w_{ab}(R,T')$$

where

$$w_{ab}(R,R') = \begin{cases} 0 \text{ if } \mathcal{D}_{ab}(R,R') = \emptyset \\ \alpha_{\{a,b\}} \text{ if } \mathcal{D}_{ab}(R,R') = \{ab,(ab)\} \\ \alpha'_{\{a,b\}} \text{ if } \mathcal{D}_{ab}(R,R') = \{ba,(ab)\} \\ \beta_{\{a,b\}} \text{ if } \mathcal{D}_{ab}(R,R') = \{ab,a*b\} \\ \beta'_{\{a,b\}} \text{ if } \mathcal{D}_{ab}(R,R') = \{ba,a*b\} \\ \lambda_{\{a,b\}} \text{ if } \mathcal{D}_{ab}(R,R') = \{(ab),a*b\} \\ \gamma_{\{a,b\}} \text{ if } \mathcal{D}_{ab}(R,R') = \{ab,ba\} \end{cases}$$

- Disagreements aversion (DA). Let $R,T,T' \in \mathcal{B}$ such that $\mathcal{D}(R,T) \subsetneqq \mathcal{D}(R,T')$
  $\mathcal{D}(R,T) \subsetneqq \mathcal{D}(R,T') \Rightarrow \tilde{A}(R,T') = \tilde{A}(R,T) \cup D$; with $D \neq \emptyset$ and $\tilde{A}(R,T) \cap D = \emptyset$

  $$\begin{aligned} w(R,T') &= \sum_{\{a,b\} \in \tilde{A}(R,T)} w_{ab}(R,T') + \sum_{\{a,b\} \in D} w_{ab}(R,T') \\ &= \sum_{\{a,b\} \in \tilde{A}(R,T)} w_{ab}(R,T) + \sum_{\{a,b\} \in D} w_{ab}(R,T') \\ &\quad \text{since } T \text{ and } T' \text{ coincide on } \tilde{A}(R,T) \\ &= w(R,T) + \sum_{\{a,b\} \in D} w_{ab}(R,T') > w(R,T), \text{ since } \sum_{\{a,b\} \in D} w_{ab}(R,T') > 0 \end{aligned}$$

  We conclude that $T \succ^{\mathcal{E}(R)} T'$

- Independence (I). By contradiction, assume that there exists a sequence

  $$(Q_1,R_1,S_1,T_1),(Q_2,R_2,S_2,T_2),...,(Q_p,R_p,S_p,T_p) \text{ in } \mathcal{B}^4$$

  such that $Q_k \succcurlyeq^{\mathcal{E}(R_k)} S_k$ and $R_k \succcurlyeq^{\mathcal{E}(S_k)} T_k$ with at least one of both relations being strict, $k, \in \{1,2,...,p\}$ and $\bigsqcup_{k=1}^{p} \mathcal{D}(Q_k,R_k) = \bigsqcup_{k=1}^{p} \mathcal{D}(S_k,T_k)$.
  In this case we have for all $k \in \{1,2,...,p\}$,

  $$\sum_{\{a,b\} \in \mathcal{P}} w_{ab}(Q_k,R_k) \leq \sum_{\{a,b\} \in \mathcal{P}} w_{ab}(S_k,R_k) \leq \sum_{\{a,b\} \in \mathcal{P}} w_{ab}(S_k,T_k)$$

  with at least one of the inequality being strict; and therefore

  $$\sum_{\{a,b\} \in \mathcal{P}} w_{ab}(Q_k,R_k) < \sum_{\{a,b\} \in \mathcal{P}} w_{ab}(S_k,T_k)$$

  Let u be the real value function defined on F for all $R,R' \in \mathcal{B}$ as follows:

  $$u(\mathcal{D}_{ab}(R,R')) = w_{ab}(R,R') \text{ and } u(\mathcal{D}(R,R')) = \sum_{d \in \mathcal{D}(R,R')} u(d)$$

  As defined, we have

  $$\sum_{\{a,b\} \in \mathcal{P}} w_{ab}(R,R') = \sum_{d \in \mathcal{D}(R,R')} u(d)$$

$$\Updownarrow$$

$$\sum_{k=1}^{p}\sum_{\{a,b\}\in\mathcal{P}} w_{ab}(Q_k,R_k) < \sum_{k=1}^{p}\sum_{\{a,b\}\in\mathcal{P}} w_{ab}(S_k,T_k)$$

$$\Updownarrow$$

$$\sum_{k=1}^{p}\sum_{d\in\mathcal{D}(Q_k,R_k)} u(d) < \sum_{k=1}^{p}\sum_{d\in\mathcal{D}(S_k,T_k)} u(d)$$

$$\Updownarrow$$

$$\sum_{d\in\bigsqcup_{k=1}^{p}\mathcal{D}(Q_k,R_k)} u(d) < \sum_{d\in\bigsqcup_{k=1}^{p}\mathcal{D}(S_k,T_k)} u(d)$$

Since $\bigsqcup_{k=1}^{p}\mathcal{D}(Q_k,R_k)=\bigsqcup_{k=1}^{p}\mathcal{D}(S_k,T_k)$, we have a contradiction.

□

proof of proposition 1

PROOF.

- The hyperpreference rule obtaining by using discrete distance satisfies (I), (S) but not (DA).
- The minimizing weighted hyperpreference rule with $\alpha_{\{a,b\}}\neq\alpha'_{\{a,b\}}$ satisfies (DA) and (I) but not (S).
- Consider $R,R',T,T'\in\mathcal{B}$ such that they coincide on any pair $\{c,d\}\in\mathcal{P}(\mathcal{A})\setminus\{\{a,b\},\{a,c\},\{b,c\}\}$ and

$$R|_{\{a,b,c\}}=\{ab,ac,bc\},\qquad R'|_{\{a,b,c\}}=\{(ab),ac,bc\}$$
$$T|_{\{a,b,c\}}=\{ab,a*c,bc\},\qquad T'|_{\{a,b,c\}}=\{(ab),a*c,bc\}$$

Consider a weak order $\succcurlyeq$ on the set of pairs of binary relations such that:

for all $Q,K\in\mathcal{B}$, $Q\neq K\Rightarrow\{Q,Q\}\sim\{K,K\}\succ\{Q,K\}$ and $\ldots\{R,R'\}\succ\{R,T\}\sim\{R',T'\}\ldots\succ\{T,T'\}\ldots$ (1).

Complete the relation $\succcurlyeq$ such that it satisfies the following conditions:

– for all $Q,K,K'\in\mathcal{B}$, $\mathcal{D}(Q,K)\subsetneq\mathcal{D}(Q,K')\Rightarrow\{Q,K\}\succ\{Q,K'\}$ (for Disagreements aversion)

– for all $Q,K,K'\in\mathcal{B}$, if for all $\{a,b\}\in\tilde{A}(K,K')$, $Q|_{\{a,b\}}\in\{(ab),a*b\}$ and $\mathcal{D}_{ab}(K,K')=\{ab,ba\}$ then $\{Q,K\}\sim\{Q,K'\}$ (for Symmetry)

Remember that the part of $\succcurlyeq$ given in (1) doesn't contradict these latter conditions.

Consider a hyperpreference rule $\mathcal{E}$ defined for all $Q,K,K'\in\mathcal{B}$ by:

$$K\succ^{\mathcal{E}(Q)}K'\Leftrightarrow\{Q,K\}\succ\{Q,K'\}$$

This hyperpreference rule satisfies (DA) and (S) but not (DC) because $\mathcal{D}(R,R') = \mathcal{D}\{T,T'\}$ and $R' \succ^{\mathcal{E}(R)} T$ and $R \succ^{\mathcal{E}(T)} T'$. Thank to the lemma 3 it doesn't satisfy (I).

□

Proof of Theorem 6

PROOF. For a separable hyperpreference rule $\mathcal{E}$ with a weight function $w = (w_{ab})_{\{a,b\}\in\mathcal{P}}$. We denote by $w(R_0,R) = \sum_{\{a,b\}\in\mathcal{P}} w_{ab}(R_0,R)$, $\forall R_0, R \in \mathcal{B}$.
Recall that for all $R_0, R, R' \in \mathcal{B}$,

$$R \succ^{\mathcal{E}(R_0)} R' \Leftrightarrow \sum_{\{a,b\}\in\mathcal{P}} w_{ab}(R_0,R) < \sum_{\{a,b\}\in\mathcal{P}} w_{ab}(R_0,R')$$
$$\Leftrightarrow \sum_{\{a,b\}\in\mathcal{P}} [w_{ab}(R_0,R) - w_{ab}(R_0,R')] < 0$$

Remark that for all $\{a,b\} \in \mathcal{P}$, $\pm(\gamma_{\{a,b\}} - \lambda_{\{a,b\}})$ does not appear in the calculation of $w(R_0,R) - w(R_0,R') = \sum_{\{a,b\}\in\mathcal{P}}[w_{ab}(R_0,R) - w_{ab}(R_0,R')]$ because we can not have at the same time $D_{ab}(R_0,R) = \{ab, ba\}$ and $D_{ab}(R_0,R') = \{(ab), a*b\}$.
$\Rightarrow$) Assume that f is a positive linear transformation. We show that w and $w' = f(w)$ induce the same hyperpreference rule.
Because of positive linearity of f, there exists $k > 0$ such that for all $R_0, R, R' \in \mathcal{B}$, $w'(R_0,R) - w'(R_0,R') = k(w(R_0,R) - w(R_0,R'))$. Thus $w(R_0,R) - w(R_0,R') > 0$ if and only if $w'(R_0,R) - w'(R_0,R') > 0$ and obviously w' induce the same hyperpreference with w.
$\Leftarrow$) Assume that for any hyperpreference rule and any weight function w, $w' = f(w)$ is also a weight function for the same hyperpreference rule.
First part: we show that for all positive real values x, y, z, t,

$$x - y = z - t \Rightarrow f(x) - f(y) = f(z) - f(t)$$

We proceed by contradiction by assuming that there exist $x, y, z, t \in \mathbb{R}_+$ such that $x - y = z - t$ and $f(x) - f(y) > f(z) - f(t)$ (with no loss of the generality).
If $x, y, z, t \in \mathbb{R}_{++}$, consider a separable hyperpreference rule $\mathcal{E}$ such that

$$\beta_{\{a,b\}} = x, \alpha_{\{a,b\}} = y, \alpha_{\{a,c\}} = z, \lambda_{\{a,c\}} = t$$

Consider the preferences $R_0 = \{ab, (ac)$ and $e*d$ for all $\{e,d\} \notin \{\{a,b\},\{a,c\}\}\}$, $R = N$ and $R' = \{(ab), ac$ and $e*d$ for all $\{e,d\} \notin \{\{a,b\},\{a,c\}\}\}$. We have
$D(R_0,R) = \{\{ab, a*b\}, \{(ac), a*c\}\} \Rightarrow w(R_0,R) = \beta_{\{a,b\}} + \lambda_{\{a,c\}} = x + t$;
$D(R_0,R') = \{\{ab,(ab)\}, \{(ac), ac\}\} \Rightarrow w(R_0,R') = \alpha_{\{a,b\}} + \alpha_{\{a,c\}} = y + z$;
$w(R_0,R) - w(R_0,R') = (x-y) - (z-t) = 0 \Rightarrow R \sim^{\mathcal{E}(R_0)} R'$.
Since w and $w' = f(w)$ are both weight functions of $\mathcal{E}$, we also have,

$$w'(R_0,R) - w'(R_0,R') = (f(x) - f(y)) - (f(z) - f(t)) > 0 \Rightarrow R \succ^{\mathcal{E}(R_0)} R'$$

which is a contradiction.

If $x=0$ and $y,z,t\in\mathbb{R}_{++}$ then, consider $R_0=\{ab,(ac),e*d;\forall\{e,d\}\notin\{\{a,b\},\{a,c\}\}\}$, $R=X_{ab}$ and $R'=\{(ab),ac$ and $e*d$ for all $\{e,d\}\notin\{\{a,b\},\{a,c\}\}\}$. We have
$D(R_0,R)=\{\{(ac),a*c\}\}\Rightarrow w(R_0,R)=\lambda_{\{a,c\}}=t$;
$D(R_0,R')=\{\{ab,(ab)\},\{(ac),ac\}\}\Rightarrow w(R_0,R')=\alpha_{\{a,b\}}+\alpha_{\{a,c\}}=y+z$;
$w(R_0,R)-w(R_0,R')=(0-y)-(z-t)=0\Rightarrow R\sim^{\mathcal{E}(R_0)}R'$.
Since $f(0)=0$, we also have,

$$w'(R_0,R)-w'(R_0,R')=(-f(y))-(f(z)-f(t))>0\Rightarrow R\succ^{\mathcal{E}(R_0)}R'$$

which is a contradiction. We do the same for $z=0$ and $x,y,t\in\mathbb{R}_{++}$.
If $y=0$ and $x,z,t\in\mathbb{R}_{++}$ then, consider $R_0=\{ab,(ac),e*d;\forall\{e,d\}\notin\{\{a,b\},\{a,c\}\}\}$, $R=N$ and $R'=\{ab,ac$ and $e*d$ for all $\{e,d\}\notin\{\{a,b\},\{a,c\}\}\}$. We have
$D(R_0,R)=\{\{ab,a*b\},\{(ac),a*c\}\}\Rightarrow w(R_0,R)=\beta_{\{a,b\}}+\lambda_{\{a,c\}}=x+t$;
$D(R_0,R')=\{\{(ac),ac\}\}\Rightarrow w(R_0,R')=\alpha_{\{a,c\}}=z$;
$w(R_0,R)-w(R_0,R')=(x)-(z-t)=0\Rightarrow R\sim^{\mathcal{E}(R_0)}R'$.
Since w and $w'=f(w)$ are both weight functions of $\mathcal{E}$, we also have,

$$w'(R_0,R)-w'(R_0,R')=(f(x))-(f(z)-f(t))>0\Rightarrow R\succ^{\mathcal{E}(R_0)}R'$$

which is a contradiction. We do the same for $t=0$ and $x,y,z\in\mathbb{R}_{++}$
If $x=y=0$ or $x=z=0$, we obviously have $x-y=z-t\Rightarrow f(x)-f(y)=f(z)-f(t)$ since $f(0)=0$.
Second part: we show that for all positive rational number r, $f(r)=rf(1)$.
Thank to the first part, for all positive real numbers x and y such that $x\geq y$,

$$x-y\geq 0 \text{ then } x-y=(x-y)-0\Rightarrow f(x)-f(y)=f(x-y), \text{ since } f(0)=0$$

For all $x,y\in\mathbb{R}_+$, $f(x)=f(x+y-y)=f(x+y)-f(y)\Rightarrow f(x+y)=f(x)+f(y)$.
So for all $x\in\mathbb{R}_+$, $p\in\mathbb{N}$,

$$f(px)=f(\underbrace{x+x+...+x}_{p\ times})=\underbrace{f(x)+f(x)+...+f(x)}_{p\ times}=pf(x).$$

Let $p\in\mathbb{N}$ and $p\neq 0$; $f(1)=f(\frac{p}{p})=f(p\times\frac{1}{p})=pf(\frac{1}{p})\Rightarrow f(\frac{1}{p})=\frac{f(1)}{p}$. Let $(p,q)\in\mathbb{N}^2,q\neq 0$, $f(\frac{p}{q})=pf(\frac{1}{q})=\frac{p}{q}f(1)$; thus for all $r\in\mathbb{Q}_+$, $f(r)=rf(1)$.
Third part: we show that for all positive real x, $f(x)=xf(1)$ and $f(1)>0$.
Let $x\in\mathbb{R}_+$; there exists some sequence of positive rationals (x_n) such that $x_n\to x$.
Thank to the second part, for all $n\in\mathbb{N}$, $f(x_n)=x_nf(1)$. Since f is a continuous function, $f(x)=\lim f(x_n)=f(1)\times\lim x_n=xf(1)$. $f(1)>0$ because for all weight function, $f(w)$ is also a weight function and so f transforms a non negative number to another non negative number.
To conclude, f is defined for all $x\in\mathbb{R}_+$ by $f(x)=kx$, with $k=f(1)>0$.

□

Proof of Theorem 7

PROOF. $\Rightarrow$) Assume that for all $\{a,b\} \in \mathcal{P}$, $\alpha_{\{a,b\}} \geq \lambda_{\{a,b\}}$ and $\gamma_{\{a,b\}} > \beta_{\{a,b\}}$.
Let f be a unanimous aggregation rule. Let R^N be a preference profile, $\{a,b\}$ a pair of alternatives and i a voter. Remind that $f(R^N)|_{\{a,b\}} \in \{ab, a*b, ba\}$ and for all $i \in N$, $R^i|_{\{a,b\}} \in \{ab, (ab), ba\}$.

- If $f(R^N)|_{\{a,b\}} = ab$, then $\mathcal{R}_{ab} \neq \emptyset$ since f is unanimous. So changing $f(R^N)|_{\{a,b\}}$ to another value should penalize voters belonging to $\mathcal{R}_{ab}$. Indeed, for all $i \in \mathcal{R}_{ab}$, $U^i_{ab}(R^i, f(R^N)) = \gamma_{\{a,b\}} > \max\{\gamma_{\{a,b\}} - \beta_{\{a,b\}}, 0\}$.
- Symmetrically we obtain the case $f(R^N)|_{\{a,b\}} = ba$.
- If $f(R^N)|_{\{a,b\}} = a*b$.
  - if $\mathcal{R}_{(ab)} = N$ then, changing $f(R^N)|_{\{a,b\}}$ to ab increase utility of a voter if and only if $\alpha_{\{a,b\}} < \lambda_{\{a,b\}}$, which is a contradiction of the assumption. It is the same if $f(R^N)|_{\{a,b\}}$ is changed to ba.
  - if $\mathcal{R}_{(ab)} = \emptyset$ then $\mathcal{R}_{ab} \neq \emptyset \neq \mathcal{R}_{ba}$ and utility of each voter is $\gamma_{\{a,b\}} - \beta_{\{a,b\}}$, so changing value of $f(R^N)|_{\{a,b\}}$ to $f(R^N)|_{\{a,b\}} \in \{ab, ba\}$, some should have $U^i_{ab}(R^i, f(R^N)) = \gamma_{\{a,b\}}$ and other $U^i_{ab}(R^i, f(R^N)) = 0$ and it should not penalize any voter if and only $\gamma_{\{a,b\}} = \beta_{\{a,b\}}$, which is a contradiction of the assumption.
  - if $\mathcal{R}_{(ab)} \neq \emptyset \neq \mathcal{R}_{ab}$ and $\mathcal{R}_{ba} = \emptyset$, changing $f(R^N)|_{\{a,b\}}$ to $f(R^N)|_{\{a,b\}} = ab$ should not penalize voter of $\mathcal{R}_{(ab)}$ if and only if $\lambda_{\{a,b\}} > \alpha_{\{a,b\}}$, which is a contradiction of the assumption. changing $f(R^N)|_{\{a,b\}}$ to ba should penalize each voter in $\mathcal{R}_{ab}$.
  - if $\mathcal{R}_{(ab)} \neq \emptyset$, $\mathcal{R}_{ab} \neq \emptyset$, $\mathcal{R}_{ba} \neq \emptyset$, changing $f(R^N)|_{\{a,b\}}$ to $f(R^N)|_{\{a,b\}} = ab$ should not penalize voters belonging to $\mathcal{R}_{ba}$ if and only if $\gamma_{\{a,b\}} = \beta_{\{a,b\}}$; it should not penalize voters belonging to $\mathcal{R}_{(ba)}$ if and only if $\lambda_{\{a,b\}} > \alpha_{\{a,b\}}$, which is a contradiction of the assumption. By the same reasoning we treat th case where $f(R^N)|_{\{a,b\}}$ is changed to $f(R^N)|_{\{a,b\}} = ba$.

Because of separability of utility function, we have the result.
$\Leftarrow$) Assume that there exists $\{a,b\} \in \mathcal{P}$ such that $\alpha_{\{a,b\}} < \lambda_{\{a,b\}}$ or $\gamma_{\{a,b\}} = \beta_{\{a,b\}}$.

- If $\alpha_{\{a,b\}} < \lambda_{\{a,b\}}$, let f be a simple majority rule, i.e $x \succ^{f(R^N)} y \Leftrightarrow |\mathcal{R}_{xy}| > |\mathcal{R}_{yx}|$.
  Consider a preference profile R^N such that for all $i \in N$, $R^i = (X \setminus \{ab\}) \cup \{(ab)\}$; where $X = abc...$ is a linear order. Then $f(R^N) = (X \setminus \{ab\}) \cup \{a*b\}$ and
  $U^i(R^i, f(R^N)) = \gamma_{\{a,b\}} - \lambda_{\{a,b\}} + \sum_{\{x,y\} \in \mathcal{P} \setminus \{\{a,b\}\}} \gamma_{\{x,y\}}$.
  If $f(R^N)$ is change to $f(R^N) = X$ then the utility of each voter should be
  $U^i(R^i, f(R^N)) = \gamma_{\{a,b\}} - \alpha_{\{a,b\}} + \sum_{\{x,y\} \in \mathcal{P} \setminus \{\{a,b\}\}} \gamma_{\{x,y\}}$. Since $\alpha_{\{a,b\}} < \lambda_{\{a,b\}}$, there is improvement of the utility of players without penalize any other voter.
- if $\gamma_{\{a,b\}} = \beta_{\{a,b\}}$, consider the $\frac{3}{4}$-majority rule, that is $x \succ^{f(R^N)} y \Leftrightarrow |\mathcal{R}_{xy}| \geq \frac{3}{4}n$. Consider the preference profile R^N such that $|\mathcal{R}_{ab}| = |\mathcal{R}_{ba}|$ if n is even and $|\mathcal{R}_{ab}| = |\mathcal{R}_{ba}| + 1$ if n is odd, and $\mathcal{R}_{(ba)} = \emptyset$ and for all $\{x,y\} \in \mathcal{P} \setminus \{\{a,b\}\}$, for all $i,j \in N$, $R^i|_{\{x,y\}} = R^j|_{\{x,y\}} \neq (xy)$. In this case, since $n \geq 3$, $f(R^N)|_{\{a,b\}} = a*b$ and utility of each voter is i, $U^i(R^i, f(R^N)) = \gamma_{\{a,b\}} - \beta_{\{a,b\}} + \sum_{\{x,y\} \in \mathcal{P} \setminus \{\{a,b\}\}} \gamma_{\{x,y\}}$.

Changing $f(R^N)|_{\{a,b\}}$ to ab should improve the utility of all player $i \in \mathcal{R}_{ab}$ to $\sum_{\{x,y\}\in\mathcal{P}} \gamma_{\{x,y\}}$ and does not change the one of any player in $\mathcal{R}_{ba}$ since $\beta_{\{a,b\}} = \gamma_{\{a,b\}}$.

Because of separability, we have the result for all pair of alternatives. □

Proof of Theorem 8

PROOF. $\Rightarrow$) Assume that for all pair $\{a,b\} \in \mathcal{P}$, $\alpha_{\{a,b\}} = \lambda_{\{a,b\}}$. Let f be a monotone aggregation rule. Let R^N be a preference profile.
Let $\{a,b\} \in \mathcal{P}$ and i be a voter. We denote by T^i another binary relation that differs to R^i on the pair $\{a,b\}$.

- If $f(R^N)|_{\{a,b\}} = ab$ then $\{l \in N : R^l|_{\{a,b\}} = ab\}$ is a winning coalition.
  - If $R^i|_{\{a,b\}} = ab$, then $U^i_{ab}(R^i, f(R^N))$ is maximal and there is no possible improvement of utility.
  - If $R^i|_{\{a,b\}} = ba$, then $U^i_{ab}(R^i, f(R^N)) = 0$.
    - if $T^i|_{\{a,b\}} = (ba)$, by monotonicity of f, $f(T^i, R^{N-i})|_{\{a,b\}} = ab$ and
    $$U^i_{ab}(R^i, f(T^i, R^{N-i})) = 0.$$
    - if $T^i|_{\{a,b\}} = ab$, by monotonicity of f, $f(T^i, R^{N-i})|_{\{a,b\}} = ab$ and
    $$U^i_{ab}(R^i, f(T^i, R^{N-i})) = 0.$$
  - If $R^i|_{\{a,b\}} = (ba)$, then $U^i_{ab}(R^i, f(R^N)) = \gamma_{\{a,b\}} - \alpha_{\{a,b\}}$.
    - if $f(T^i, R^{N-i})|_{\{a,b\}} = ab$, then $U^i_{ab}(R^i, f(T^i, R^{N-i})) = \gamma_{\{a,b\}} - \alpha_{\{a,b\}}$.
    - if $f(T^i, R^{N-i})|_{\{a,b\}} = ba$, then $U^i_{ab}(R^i, f(T^i, R^{N-i})) = 0 \le U^i_{ab}(R^i, f(R^N))$.
    - if $f(T^i, R^{N-i})|_{\{a,b\}} = a*b$, then $U^i_{ab}(R^i, f(T^i, R^{N-i})) = \gamma_{\{a,b\}} - \lambda_{\{a,b\}} = \gamma_{\{a,b\}} - \alpha_{\{a,b\}}$.
- If $f(R^N)|_{\{a,b\}} = ba$ we proceed as above.
- If $f(R^N)|_{\{a,b\}} = a*b$
  - If $R^i|_{\{a,b\}} = ab$, then $U^i_{ab}(R^i, f(R^N)) = \gamma_{\{a,b\}} - \beta_{\{a,b\}}$ and $T^i|_{\{a,b\}} \in \{ba, (ab)\}$. Therefore $f(T^i, R^{N-i})|_{\{a,b\}} \in \{ba, a*b\}$ because of monotonicity.
    So $U^i_{ab}(R^i, f(T^i, R^{N-i})) \in \{0, \gamma_{\{a,b\}} - \beta_{\{a,b\}}\}$ and no improvement of utility.
  - If $R^i|_{\{a,b\}} = (ba)$, then $U^i_{ab}(R^i, f(R^N)) = \gamma_{\{a,b\}} - \lambda_{\{a,b\}}$.
    - if $f(T^i, R^{N-i})|_{\{a,b\}} = a*b$ then, no change on the utility.
    - if $f(T^i, R^{N-i})|_{\{a,b\}} \in \{ab, ba\}$, then $U^i_{ab}(R^i, f(T^i, R^{N-i})) = \gamma_{\{a,b\}} - \alpha_{\{a,b\}} = \gamma_{\{a,b\}} - \lambda_{\{a,b\}}$.

In any case, no individual cannot improve his utility by changing his preference.
We obtain that nobody cannot manipulate on the pair $\{a,b\}$. Thank to the separability of hyperutility, we conclude that there is no possible manipulation.

$\Leftarrow$) Assume that there exists $\{a,b\} \in \mathcal{P}$ such that $\alpha_{\{a,b\}} \neq \lambda_{\{a,b\}}$.
Let f be the simple majority rule. Remark that the simple majority is a monotone aggregation rule. $A = \{a,b,c,...\}$, let $X = abc...$ be a linear order on A. If $\alpha_{\{a,b\}} < \lambda_{\{a,b\}}$, consider the preference profile

$$R^i = (X \setminus \{ab\}) \cup \{(ab)\}$$
$$R^j = X$$
$$R^k = (X \setminus \{ab\}) \cup \{ba\}$$
$$R^l = R^1 \text{ for all } l \in N \setminus \{i,j,k\}$$

Consider $T^i = R^j$. Then $f(R^N) = (X \setminus \{ab\}) \cup \{a * b\}$ and

$$U^i(R^i, f(R^N)) = \gamma_{\{a,b\}} - \lambda_{\{a,b\}} + \sum_{\{x,y\} \in \mathcal{P} \setminus \{\{a,b\}\}} \gamma_{\{x,y\}}$$

$f(T^i, R^{N-i}) = X$ and $U^i(R^i, f(T^i, R^{N-i})) = \gamma_{\{a,b\}} - \alpha_{\{a,b\}} + \sum_{\{x,y\} \in \mathcal{P} \setminus \{\{a,b\}\}} \gamma_{\{x,y\}}$
$\alpha_{\{a,b\}} < \lambda_{\{a,b\}} \Rightarrow U^i(R^i, f(R^N)) < U^i(R^i, f(T^i, R^{N-i}))$ and i can manipulate by changing R^i to T^i.
If $\alpha_{\{a,b\}} > \lambda_{\{a,b\}}$, consider the preference profile

$$R^i = (X \setminus \{ab\}) \cup \{(ab)\}$$
$$R^j = X$$
$$R^l = R^i \text{ for all } l \in N \setminus \{i,j\}$$

Consider $T^i = (X \setminus \{ab\}) \cup \{ba\}$. Then $f(R^N) = (X \setminus \{ab\}) \cup \{ab\}$ and

$$U^i(R^i, f(R^N)) = \gamma_{\{a,b\}} - \alpha_{\{a,b\}} + \sum_{\{x,y\} \in \mathcal{P} \setminus \{\{a,b\}\}} \gamma_{\{x,y\}}$$

$f(T^i, R^{N-i}) = (X \setminus \{ab\}) \cup \{a * b\}$ and

$$U^i(R^i, f(T^i, R^{N-i})) = \gamma_{\{a,b\}} - \lambda_{\{a,b\}} + \sum_{\{x,y\} \in \mathcal{P} \setminus \{\{a,b\}\}} \gamma_{\{x,y\}}$$

$\alpha_{\{a,b\}} > \lambda_{\{a,b\}} \Rightarrow U^i(R^i, f(R^N)) < U^i(R^i, f(T^i, R^{N-i}))$ and i can manipulate by changing R^i to T^i. $\square$

REFERENCES

Arrow, Kenneth J (1951), *Social choice and individual values*, volume 12. Yale university press. [5]

Bahel, Eric and Yves Sprumont (2020), "Strategyproof choice of social acts." *American Economic Review*, 110(2), 596–627. [5]

Bentham, Jeremy (1890), *Utilitarianism*. Progressive Publishing Company. [5]

Bernoulli, Danieal (1738), "1738,"exposition of a new theory on the measurement of risk," translated and reprinted in." *Econometrica*. [5]

Dasgupta, Partha and Eric Maskin (2020), "Strategy-proofness, independence of irrelevant alternatives, and majority rule." *American Economic Review: Insights*, 2(4), 459–474. [5]

Fishburn, Peter C (1967), "Methods of estimating additive utilities." *Management science*, 13(7), 435–453. [5]

Fishburn, Peter C (1970), "Utility theory for decision making." Technical report, Research analysis corp McLean VA. [4]

Fishburn, Peter C (1992), "Utility as an additive set function." *Mathematics of Operations Research*, 17(4), 910–920. [5, 18, 19]

Fishburn, Peter C (1994), "Utility and subjective probability." *Handbook of game theory with economic applications*, 2, 1397–1435. [4]

Fleurbaey, Marc (2003), "On the informational basis of social choice." *Social Choice and Welfare*, 21(2), 347–384. [5]

Fleurbaey, Marc (2018), "Welfare economics, risk and uncertainty." *Canadian Journal of Economics/Revue canadienne d'économique*, 51(1), 5–40. [5]

Fleurbaey, Marc and Peter J Hammond (2004), "Interpersonally comparable utility." In *Handbook of Utility Theory: Volume 2 Extensions*, 1179–1285, Springer. [5]

Galanter, Eugene (1962), "The direct measurement of utility and subjective probability." *The American journal of psychology*, 75(2), 208–220. [4]

Gibbard, Allan (1973), "Manipulation of voting schemes: a general result." *Econometrica: journal of the Econometric Society*, 587–601. [5]

Jeffrey, Richard C (1974), "Preference among preferences." *The Journal of Philosophy*, 71(13), 377–391. [2]

Kemeny, John G (1959), "Mathematics without numbers." *Daedalus*, 88(4), 577–591. [3]

Kemeny, John G and James Laurie Snell (1962), *Mathematical models in the social sciences*, volume 9. Blaisdell New York. [3]

Kendall, Maurice G (1938), "A new measure of rank correlation." *Biometrika*, 30(1/2), 81–93. [3]

Kleine, Jens, Thomas Peschke, and Niklas Wagner (2024), "Beyond financial wealth: The experienced utility of collectibles." *The Quarterly Review of Economics and Finance*, 101865. [5]

Kraft, Charles H, John W Pratt, and Abraham Seidenberg (1959), "Intuitive probability on finite sets." *The Annals of Mathematical Statistics*, 30(2), 408–419. [4]

Krantz, David, Duncan Luce, Patrick Suppes, and Amos Tversky, eds. (1971), *Foundations of Measurement, Vol. I: Additive and Polynomial Representations*. New York Academic Press. [4, 5]

Laffond, Gilbert, Jean Lainé, and M Remzi Sanver (2020), "Metrizable preferences over preferences." *Social Choice and Welfare*, 55, 177–191. [3, 4, 8, 10, 16]

Lainé, Jean, Ali Ihsan Ozkes, and Remzi Sanver (2016), "Hyper-stable social welfare functions." *Social Choice and Welfare*, 46, 157–182. [4]

Maskin, Eric (2020), "Arrow's theorem, may's axioms, and borda's rule." [5]

Pivato, Marcus and Élise Flore Tchouante (2023), "Bayesian social aggregation with almost-objective uncertainty." *Theoretical Economics https://econtheory. org/index. php (ISSN (e) 1555-7561 (print) 1933-6837)*. [5]

Pivato, Marcus and Élise Flore Tchouante (2024), "Bayesian social aggregation with non-archimedean utilities and probabilities." *Economic Theory*, 77(3), 561–595. [5]

Pongou, Roland and Ghislain Junior Sidie (2024), "Voting when rankings matter: Truthful equilibria, efficiency, and abstention." *World Bank, Policy Research Working Paper 10837*. [5]

Pongou, Roland and Bertrand Tchantcho (2021), "Round-robin political tournaments: Abstention, truthful equilibria, and effective power." *Games and Economic Behavior*, 130, 331–351. [5, 13]

Satterthwaite, Mark Allen (1975), "Strategy-proofness and arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions." *Journal of economic theory*, 10(2), 187–217. [5]

Sen, Amartya K (1977), "Rational fools: A critique of the behavioral foundations of economic theory." *Philosophy & Public Affairs*, 317–344. [2, 3]

Von Neumann, J and O Morgenstern (1944), "Theory of games and economic behavior.-princeton: Princeton univ. press." [4, 5]

Yamazaki, Kaori (2024), "Order embedding theorems and multi-utility representation of the preorder." *Topology and its Applications*, 109007. [5]