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Matching and fair pricing of socially optimal, stable and financially sustainable ride-sharing in congestible networks

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Abstract

The paper deals with matching and fair pricing in urban peer-to-peer ride-sharing schemes where the following desirable properties hold: (i) matchings between passengers and drivers are decided by a social planner to minimize total car-kilometers travelled, (ii) matchings are stable, i.e. no pair of passenger and driver can both increase their fuel cost-related surplus from breaking the current partnership, and (iii) the scheme is financially sustainable, i.e. there is no need of subsidy. The case where travel times are affected by matchings, in the light of the reduced number of cars travelling on the network, is unexplored. The paper fills this gap.

The matching optimization problem is formulated as linear programming problem with nonlinear equilibrium constraints and node-link network representation. Solution to the approximately equivalent mixed-integer linear programming formulation is obtained by available efficient off-the-shelf solvers. Duality theory is used to specify a stability compliant pricing scheme based on fair surplus division: the surplus gained by each traveler is exactly half way between the minimum and the maximum she can obtain from any stable solution. Computation of prices requires solution of two linear programming problems. The price paid by the passenger is received by the driver. Since surplus of each traveler is nonnegative, subsidies are not needed. A toy network and a small network are used to illustrate the theoretical findings, and to appraise the pricing-induced shares of trip cost that accrue to each traveler.

Keywords

Equilibrium, matching, pricing, ride-sharing, stability.

JEL

C78, R40, R48

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1. Introduction

1.1 Setting the scene

Matching and price formation in peer-to-peer ride-sharing in urban areas is the subject of the paper. Prices are the key determinants of how costs are allocated between the travelers, passengers and drivers, who are matched.

Ride-sharing has the potential to reduce congestion, energy consumption, pollutant emissions and the needs of search for parking. The peer-to-peer scheme, where drivers are also travelers, is of interest because of the lower without-passenger travel than ride-hailing services, like Uber, Lyft and Didi Chuxing. In peer-to-peer ride-sharing, without-passenger travel occurs only due to drivers' detours. In contrast, in ride-hailing, without-passenger travel occurs while waiting for the next request: drivers travel to a close-by park location or to a high-demand location, such as an airport or the central business district, or cruise around (Kontou et al., 2020).

One noticeable existing example of peer-to-peer scheme in urban areas is the French 'covoiturage' (Costeseque et al., 2023; Ministère de la Transition écologique et de la Cohésion des territoires, 2023). There are nineteen currently operating platforms. Passengers pay a per-kilometer charge for only the part of the trip between her origin and destination, the driver pays for possible detours. Profits of drivers are excluded. The government recommends setting charges lower than 0.20 EUR/kilometer. Maximum value is 0.60 EUR/kilometer.

Matching and pricing can be evaluated from the social and the individual economic as well as the financial point of view. The following desirable properties are hereafter considered: social optimality, stability and financial sustainability.

Social optimality means that car-kilometers travelled over the network are minimized. Stability has a twofold meaning. The one of coalitional rationality, by which no two travelers would prefer matching together to their current partnership, and no traveler who is matched would prefer travelling alone (Solymosi and Raghavan, 2001). The one of individual rationality, by which no traveler is a loser (Zhao et al., 2014; Yan et al., 2021). Financial sustainability means that subsidies to compensate travelers for losses are excluded. The only need of subsidies arises from platform operation.

Stability requires definition of surplus, also referred to as payoff. This in turn calls for estimation of reservation prices, i.e. of the willingness to pay (WTP, the maximum amount they are willing to spend) of the passenger and of the willingness to accept (WTA, the minimum amount they are willing to receive) of the driver, as well as of prices. The price paid by the passenger can be transferred fully to the driver, a case where side payments are excluded. Alternatively, travelers may pay the platform a charge. A traveler gains if her surplus is strictly positive.

In an attempt to mimic the real operation of the scheme, matching and price formation mechanisms based on bidding processes have been the subject of theoretical investigation (Chen and Valant, 2023). Convergence of the bidding process to a stable matching solution is evaluated by simulation.

Research on ride-sharing mostly addresses matching and price formation from a normative point of view. The aim is the understanding of how the scheme should be operated. Since it is natural to advocate a co-ordination role of the platform manager, decisions about both matchings and prices fall within her remits. Then, if proper incentives are in place, namely if stability holds, travelers accept the decisions. This is certainly the case of peer-to-peer schemes, where, ideally, the objectives of the platform manager should be the ones of a social planner.

Schemes can be classified according to the information that is declared by the travelers when they post their requests and offers. The case of the auctions is the one where passengers express their WTP and drivers their WTA for the matching. Auctions are dealt with, among the others, in Kamar and Horvitz (2009), Zhao et al. (2014), Yan et al. (2021), Schwarzstein and Schouery (2023).

The other case is the one without bids. Only the information on the origin, destination and departure time of the trip is declared. A new strand of literature considers ride-sharing schemes of this type. Schemes which satisfy social optimality, stability and financial sustainability, are analysed in Yan et al. (2021) and Fielbaum et al. (2022).

1.2 Contributions of the paper

The present paper aims to provide a mathematical account of both matching and price formation under the co-ordination of a social planner, in peer-to-peer ride-sharing schemes satisfying social optimality, stability and financial sustainability. Two are the contributions with respect to extant literature. They relate to endogenous travel times and fair prices.

First, travel times are clearly affected by the number of cars travelling on the network. Ride-sharing has an impact on travel times that is larger the higher the number of travelers who participate in the scheme (because the reduction of the number of cars is larger), and the farther the network conditions are from the free-flow state (because the steeper portions of the volume-delay functions are relevant).

The main bulk of research on ride-sharing considers the fixed travel time case. Only few contributions consider matching problems under endogenous congestion. In Zhang and Nie (2022) and Yao and Zhang (2023), the modelling setting is static. In the former, the analysis applies to ride-hailing and restricts to a single origin-destination (OD) pair. In the latter, the analysis applies to multimodal matchings of Mobility-as-a-Service (MaaS) platforms and extends to networks of any size.

In de Palma et al. (2022a and 2022b), the setting is dynamic, because matchings are optimized under equilibrium constraints related to departure time and route choices. Thus, travel times are inherently endogenous. The analysis applies to peer-to-peer ride-sharing. Pricing and stability, however, are not dealt with.

The setting here is the one of static equilibrium. Justification is in order. We consider that trips are booked in advance. Since the exact arrival time of the ride request is not accounted for, travel demand is regarded as independent of time within the time interval modelled. This implies a static setting, where steady state network conditions, in terms of travel times and flows, are considered.

The conditions are the ones of the Wardrop deterministic user equilibrium principle. Perfect information needs to be assumed. This is obtained if cars are equipped with mobile applications providing real-time navigation functionalities, which is increasingly market reality nowadays. Nevertheless, drivers may be well aware of recurrent congestion conditions over the network. This is still true in the presence of ride-sharing, if ride-sharing trips are made on a regular basis. Our deterministic analysis is, in any case, a useful preliminary to stochastic analysis able to account for imperfect information.

Second, prices should at the same time guarantee stability of the socially optimal matchings and be fair. The idea of a fair price is not clearcut. Fairness depends on the criterion that is chosen for its definition.

The most natural solution is the simple equal share (fifty-fifty) allocation, between passenger and driver, of the total profit from the socially optimal matching, the profit being the sum of travelers' surpluses, or of the trip costs. The equal share solution is generally nonstable (see the discussion in Yan et al., 2021, and our toy network example in section 3.1). To deal with this shortcoming, different approaches have been proposed.

One is the best stable ride-matching obtained from the matching optimization problem under stability constraints and given cost allocation. This solution implies a sub-optimal social objective, i.e. a price to pay for stability, which is tantamount to the price of anarchy of network analysis (Roughgarden, 2005). Yan et al. (2021) propose a fair pricing solution that is at the same time socially optimal and stable. They obtain it from the appropriate combination of a socially optimal matching solution with a best stable price solution and an ex-post surplus re-distribution based on the traveler marginal contribution to the total system profit.

A different approach originates from a relaxation of the definition of stability. The requirement is weakened: stability holds if no traveler would prefer travelling alone to current matching. Stable matchings are in this case referred to as hermetic (Fielbaum et al., 2022).

The Vickrey-Clarke-Groves pricing mechanism is extensively investigated (Kamar and Horvitz, 2009; Zhao et al., 2014; Yan et al., 2021). With this mechanism, the passenger pays the amount of the cost increase incurred by the driver because of her participation, and the driver receives the amount of the cost saving incurred by the passenger because of her participation. The mechanism satisfies individual rationality, as well as truthfulness, but can give rise to high deficits. Schwarzstein and Schouery (2023) propose a financially sustainable variant.

The allocation based on the Shapley value that is proposed within game theory is in the direction of fairness, because players gain in dependence of their marginal contribution to the total system profit, but is generally nonstable (Hoffmann and Sudhölter, 2007).

The nucleolous, also proposed within game theory (Schmeidler, 1969), is stable and is based on the maximization of the lowest excess of matchings, the excess being the difference between what the players actually gain minus what they would gain from matching. This fairness criterion looks at the most disadvantaged players only, i.e. those who gain the least, along the lines of the difference principle of justice formulated by Rawls (1999).

The solution of the present paper is based on the fair division of surplus proposed by Thompson (1991). The following two solutions are stable: the one where every passenger takes the maximum she can obtain from any stable solution and every driver takes the corresponding minimum, and the reverse. Then, the fair division solution is the one where the gain of every traveler is exactly half way between the minimum and the maximum each can obtain from any stable solution. This solution is justified by common sense and, as it will be demonstrated in section 2.4, is easily computed.

The paper has the following organisation. The ride-sharing model is outlined in section 2. Section 3 provides two illustrative examples. Section 4 concludes with discussion and future research.

2. Ride-sharing model

2.1 Notation

The following notation is used.

Sets

$\mathcal{A}$	set of links
$\mathcal{C}$	core
$\mathcal{D}$	set of destination nodes
$\mathcal{D}_i$	set of destination nodes having the car flow $\hat{d}_i^k$ strictly positive
$\mathcal{I}$	set of nodes
$\mathcal{N}$	set of excluded matchings
$\mathcal{O}$	set of origin nodes
$\mathcal{O}^k$	set of origin nodes having the car flow $\hat{d}_i^k$ strictly positive

Indexes

i, j	nodes
(ij)	link between node i and node j
k	destination
m	car flow interval for the piecewise approximation of the volume-delay functions
r	OD pair of passengers
s	OD pair of drivers

Input quantities

$\hat{d}_i^k$	car flow from node i to destination k not participating in the ride-sharing scheme
d_r^P	flow of passengers of OD pair r
d_s^D	flow of drivers of OD pair s
B_{ij}	parameter of the volume-delay function of link (i, j)
C_{ij}	capacity of link (i, j)
l_{ij}	length of link (i, j)
M	number of flow intervals used for the piecewise linear approximation of the volume-delay functions
n	number of OD pairs
t_{ij}^0	free-flow travel time of link (i, j)
W_1, W_2	large positive constants
α_m	value of flow identifying the interval of the piecewise linear approximation of the volume-delay functions
$\beta_{ik,rs}$	element of the OD pair-matching incidence matrix

Variables

c_r	fuel cost of passenger of OD pair r in the network conditions of the equilibrium scenario with optimal matching
c_s	fuel cost of driver of OD pair s in the network conditions of the equilibrium scenario with optimal matching
d_i^k	total car flow from node i to destination k
p_r	price paid by passengers of OD pair r
t_{ij}	travel time of link (i, j)
u_r	surplus of passengers of OD pair r
v_s	surplus of drivers of OD pair s
x_{ij}	car flow on link (i, j)

x_{ij}^k	car flow on link (i,j) to destination k
X_{rs}	flow of passengers of OD pair r who are matched with drivers of OD pair s
y_{rs}	dual variable associated with matching between OD pair r and OD pair s
z_{ij}^k	auxiliary binary variable associated with car flow of link (i,j) to destination k
δ_{rs}	fuel cost of the detour of the driver of OD pair s who is matched with passenger of OD pair r in the equilibrium scenario with optimal matching
$\lambda_{ij,m}^L, \lambda_{ij,m}^R$	left and right variable associated with interval m of link (i,j) used for the piecewise linear approximation of the volume-delay functions
μ_i^k	minimum travel time from node i to destination k
π_{rs}	profit associated with matching of passengers of OD pair r and drivers of OD pair s
$\varphi_{ij,m}$	auxiliary binary variable associated with link (i,j) and segment m
χ_{rs}	share of cost of full driver trip for matching rs that is paid by the passenger
ψ_{rs}	share of profit from matching r,s that accrues to the passenger
ω_{rs}^D	reservation price of driver of OD pair s if matched with passenger of od pair r
ω_r^P	reservation price of passenger of OD pair r

2.2 Socially optimal matchings

Consider a road network defined by a link set $\mathcal{A}$ and a node set $\mathcal{J}$. Nodes can be origin and destination of trips, intersections, or both.

Assumption 1: demand.

- On each OD pair we have a fixed flow of cars (cars in the unit time interval),
- a fixed fraction of this flow is represented by travelers who are willing to participate in the ride-sharing scheme as passengers, another fixed fraction as drivers.

The equality between the total number of passengers and the total number of drivers is not imposed. A social planner matches passengers and drivers.

Assumption 2: matching scheme.

- Each driver is matched with one passenger only, and vice-versa.

Definition 1: matchings.

- Matching flows include: (i) flow of passengers who are matched with drivers of the same or different OD pair, (ii) flow of passengers who travel alone, (iii) flow of drivers who travel alone.

The definition is based on the OD pair-by-OD pair matching, in the light of the homogeneity of travelers of each OD pair. The definition includes as a special case the traveler-by-traveler matching.

Only drivers detour, therefore there are four cases as illustrated in Figure 1:

- the origin of passenger is identical to the origin of driver and the destination of passenger is identical to the destination of driver (case 1),
- same origin and different destination (case 2),
- different origin and same destination (case 3),
- the origin of passenger is different from both the origin and the destination of driver, and the destination of passenger is different from both the origin and the destination of driver (case 4).

The following cases:

- the origin of driver is identical to the destination of passenger,
- the destination of driver is identical to the origin of passenger,

are in the set $\mathcal{N}$ of excluded matchings, because the OD pair is travelled by the driver in the outward direction and in the return direction during the same matching and drivers would clearly not accept it.

Definition 2: socially optimal matchings.

- The socially optimal matchings are the ones minimizing total car-kilometers travelled.

If fuel consumption is assumed dependent on distance travelled and independent of speed, then the objective is proportional to total fuel consumption. This is a simplifying assumption: Eisele et al. (2013) find a convex relationship between consumption per kilometer and average speed. Carbon dioxide emissions are produced by the combustion of fossil fuels. Therefore, the emissions of this greenhouse gas are implicitly minimized.

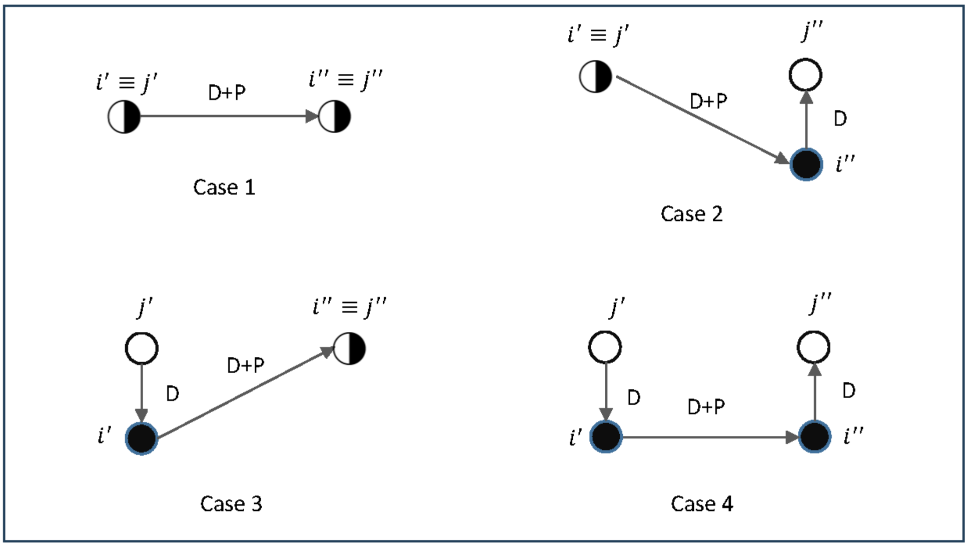


Figure 1. Matchings between OD pair $i'i''$ of passenger (P) and OD pair $j'j''$ of driver (D)

Constraints impose:

- Assignment of passengers of each OD pair to drivers or to travel alone,
- Assignment of drivers of each OD pair to passengers or to travel alone,
- Non negativity of matching flows, while flows of excluded matchings are zero.

Assumption 3: route choice.

- Drivers choose the minimum travel time route.

Assumption 4: travel times.

- Travel times on each link are affected by link flows via the associated continuous volume-delay function.

Then, we have to add equilibrium constraints, in the light of the simultaneity of assumptions 3 and 4. If drivers are perfectly informed, then the network is found in the conditions defined by the Wardrop first principle (Heydecker, 1986): for each driver, the present travel time on any alternative route is at least as great as the travel time on her present route. Clearly, changes in travel times have an indirect impact on total car-kilometer travelled by changing link flows.

Here, the non-linear programming formulation of the equilibrium constraints, with node-link network representation, is adopted (Patriksson, 2015). This amounts to consider only link-related variables (car flow and travel time). The need of route enumeration is obviated. The formulation is based on flow conservation constraints written for each node of the network: the total flow of the forward star, i.e. the flow out, minus the total flow of the backward star, i.e. the flow in, equals the flow that is generated in the node. Flows are disaggregated by destination.

Problem P1: socially optimal matchings.

$$\min_{x,X} \sum_{(ij)\in\mathcal{A}} l_{ij} x_{ij} \tag{1}$$

$$\sum_{s=1}^{n+1} X_{rs} = d_r^P,\ \ r = 1, \dots, n \tag{2}$$

$$\sum_{r=1}^{n+1} X_{rs} = d_s^D,\ \ s = 1, \dots, n \tag{3}$$

$$X_{rs} \geq 0,\ \ r, s = 1, \dots, n;\ X_{r(n+1)} \geq 0,\ \ r = 1, \dots, n;\ X_{(n+1)s} \geq 0,\ \ s = 1, \dots, n;\ X_{rs} = 0,\ \ (r, s) \in \mathcal{N} \tag{4}$$

x is the solution of:

$$\min_{x} \int_0^{x_{ij}} t_{ij}(\theta)\, d\theta$$

$$\sum_{j:(ij)\in\mathcal{A}} x_{ij}^k - \sum_{j:(ji)\in\mathcal{A}} x_{ji}^k = d_i^k,\ \ i \in J, k \in \mathcal{D}\backslash\{i\} \tag{5}$$

$$d_i^k = \hat{d}_i^k + \sum_{r=1}^n \sum_{s=1}^n \beta_{ik,rs} X_{rs} + \sum_{r=1}^n \beta_{ik,r(n+1)} X_{r(n+1)} + \sum_{s=1}^n \beta_{ik,(n+1)s} X_{(n+1)s},\ \ i \in J, k \in \mathcal{D}\backslash\{i\} \tag{6}$$

$$x_{ij} = \sum_{k\in D} x_{ij}^k,\ \ (ij) \in \mathcal{A} \tag{7}$$

$$x_{ij}^k \geq 0,\ \ (ij) \in A, k \in \mathcal{D} \tag{8}$$

where the element of the OD pair-matching incidence matrix $\beta_{ik,rs}$ equals 1 if the OD pair between node i and node k is travelled by the matching flow X_{rs}, equals 0 otherwise. Passengers travelling alone are associated with the dummy driver OD pair indexed as $n + 1$. Drivers travelling alone are associated with the dummy passenger OD pair indexed as $n + 1$.

The OD pair-matching incidence matrix is easily constructed column-wise. Given the matching r, s in the column, the relevant case among the four in Figure 1 is identified. Then, the i, k node pair row elements that are travelled by the matching flow are also identified, and the corresponding cells of the OD pair–matching incidence matrix are assigned value $\beta_{ik,rs} = 1$.

Problem P1 falls into the class of network design problems. Mathematically, it is a bilevel optimization problem, linear at the upper level and nonlinear at the lower level.

One optimal solution to problem P1 always exists based on corollary 3 in Harker and Pang (1988). This is because the lower-level optimization problem is equivalent to a variational inequality according to the conventional route-based and link-based formulations, and the following conditions are satisfied. The objective function of the upper-level problem and the link travel time functions are continuous. The feasible set of matching flows is nonempty (because of the travel-alone possibility) and compact (this follows immediately from defining constraints). The feasible set of link flows is nonempty (because the graph is connected and, therefore, a route exists for every OD pair), compact (this follows immediately from defining constraints) and convex (because the set is defined by linear inequalities).

For computation, Farvaresh and Sepehri (2011) propose the following re-formulation of problem P1. First, BPR volume-delay functions are considered (Bureau of Public Roads, 1964):

$$t_{ij}(x_{ij}) = t_{ij}^0 \left[1 + B_{ij} \left(\frac{x_{ij}}{C_{ij}}\right)^4\right],\ \ (ij) \in \mathcal{A}$$

Second, they provide the Karush-Kuhn-Tucker conditions for the lower-level optimization problem in the following form:

$$x_{ij}^k \leq z_{ij}^k W_1,\ \ (ij) \in \mathcal{A},\ \ k \in \mathcal{D} \tag{9}$$

$$0 \le t_{ij}(x_{ij}) - \mu_i^k + \mu_j^k \le (1 - z_{ij}^k)W_2,\ (ij) \in \mathcal{A},\ k \in \mathcal{D} \tag{10}$$

$$z_{ij}^k \in \{0,1\},\ (ij) \in \mathcal{A},\ k \in \mathcal{D} \tag{11}$$

$$\mu_i^k \ge 0, i \in \mathcal{J}, k \in \mathcal{D} \tag{12}$$

plus Eqs (5), (6), (7) and (8). The Karush-Kuhn-Tucker conditions are necessary and sufficient for the global optimality of car link flows in the lower-level problem. Variables $\mu_i^k, i \in \mathcal{J}, k \in \mathcal{D}$, are OD travel times.

Third, they piecewise linearise the volume-delay functions:

$$x_{ij} = \sum_{m=1,\dots,M}(\alpha_{m-1}\lambda_{ij,m}^L + \alpha_m\lambda_{ij,m}^R),\ (ij) \in \mathcal{A} \tag{13}$$

$$t_{ij}(x_{ij}) \cong t_{ij}^0\left[1 + \left(\frac{B_{ij}}{C_{ij}^4}\right)\sum_{m=1,\dots,M}(\alpha_{m-1}^4\lambda_{ij,m}^L + \alpha_m^4\lambda_{ij,m}^R)\right],\ (ij) \in \mathcal{A} \tag{14}$$

$$\lambda_{ij,m}^L + \lambda_{ij,m}^R = \varphi_{ij,m}, (ij) \in \mathcal{A}, m = 1, \dots, M \tag{15}$$

$$\sum_{m=1,\dots,M}\varphi_{ij,m} = 1,\ (ij) \in \mathcal{A} \tag{16}$$

$$\lambda_{ij,m}^L, \lambda_{ij,m}^R \ge 0,\ (ij) \in \mathcal{A}, m = 1, \dots, M \tag{17}$$

$$\varphi_{ij,m} \in \{0,1\},\ (ij) \in \mathcal{A}, m = 1, \dots, M \tag{18}$$

Constraints (15) to (18) make the model select exactly one segment. If the number of segments is sufficiently large, then the nonlinear volume-delay function can be linearly approximated to any required degree of accuracy. A good approximation is obtained for BPR volume-delay functions with only $M = 20$ segments. Setting a large upper bound α_M on the flow of all links is required.

Using the above, the following single-level optimization problem is obtained:

$$\min_{x,X,z,\lambda,\mu,\varphi} \sum_{(ij)\in\mathcal{A}} l_{ij}x_{ij}$$

subject to constraints of Eqs (2) to (18).

This is a mixed-integer linear programming (MILP) problem in the unknowns represented by the continuous variables x, X, λ and μ, and the binary variables z and φ. Efficient solvers able to provide a good solution exist and are available in commonly used computer programming languages.

The following additional constraints reduce computation times since they reduce the number of binary variables z_{ij}^k.

Let $\mathcal{O}^k$ be the set of the origin nodes i having a strictly positive flow $\hat{d}_i^k > 0$. Let SMT^k the Steiner minimal tree associated with destination k, i.e. the directed tree that connects all nodes in $\mathcal{O}^k$ with destination k having the minimum sum of weights, a unit weight being associated to each link. The following constraints must hold:

$$\sum_{(ij)\in\mathcal{A}} z_{ij}^k \ge |SMT^k|, \qquad k \in \mathcal{D}$$

where $|SMT^k|$ is the number of links of the tree SMT^k.

Let $\mathcal{O}$ be the set of the origin nodes. Let $\mathcal{D}_i$ be the set of the destination nodes k having a strictly positive flow $\hat{d}_i^k > 0$. The following constraints must hold:

$$\sum_{k\in\mathcal{D}^i}\sum_{(ij)\in\mathcal{A}} z_{ij}^k \ge |\mathcal{D}^i|, \qquad i \in \mathcal{O}$$

where $|\mathcal{D}_i|$ is the cardinality of the set $\mathcal{D}_i$.

2.3 Stability and financial sustainability

The theoretical framework of two-sided matching games is followed. In particular the interest is in matching games with transferable utility, also referred to as assignment games, because passengers pay a price which is received by drivers. This is the case of buyers and sellers in auctions and of firms and workers in labour markets. The theory of assignment games is found in Shapley and Shubik (1971), Roth and Sotomayor (1990) and in Galichon (2018). A survey is in Núñez and Rafels (2015).

Here, the passenger acts as the buyer or firm, the driver as the seller or worker. The reservation price of the passenger is driver independent, while, in the classical framework, the reservation price of the buyer is seller dependent because distinct objects are sold by distinct sellers. Also, the reservation price of the driver is passenger dependent because of detours, while, in the classical framework, the reservation price of the seller is buyer independent. Due to symmetry of the framework, these differences are not a barrier to transferability to the setting here.

Assumption 5: fuel cost.

- Fuel cost is directly proportional to distance travelled and independent of speed.

Definition 3: reservation price of travelers.

- The reservation price ω_r^P of a passenger of OD pair r, i.e. her WTP, is, when matched with any driver and when not matched, the fuel cost c_r of the minimum travel time route for her OD pair in the equilibrium scenario with optimal matching.
- The reservation price ω_{rs}^D of a driver of OD pair s, i.e. her WTA, is, when matched with a passenger of OD pair r, the difference between the cost when matched $c_r + \delta_{rs}$ and the cost c_s of the minimum travel time route for her OD pair, all costs being evaluated in the equilibrium scenario with optimal matching. It is zero when not matched.

Assumption 6: absence of side payments.

- There are no side payments, i.e. payments to third parties, typically the platform, because the full price paid by each passenger is transferred to the driver.

Definition 4: surplus of travelers.

- The surplus u_r of a passenger of OD pair r is, when matched with a driver, the difference between her reservation price ω_r^P and the price p_r actually paid.
- The surplus v_s of a driver of OD pair s is, when matched with a passenger of OD pair r, the difference between the price received p_r and her reservation price ω_{rs}^D.
- When the passenger is not matched her surplus is zero, because she pays her reservation price.
- When the driver is not matched her surplus is zero, because she receives zero and her reservation price is also zero.

Based on the above definitions, we implicitly impose that passengers of a given OD pair gain identical surplus irrespective of the driver OD pair they are matched with. Similarly, drivers of a given OD pair gain identical surplus irrespective of the passenger OD pair they are matched with.

Definition 5: profit of matching.

- The profit of a matching is the sum of the surpluses of the passenger and the driver who are involved in the matching. Since prices cancel out of the computation of profit, the profit π_{rs} of a matching of a passenger of OD pair r with a driver of OD pair s is:

$$\pi_{rs} = u_r + v_s = \omega_r^P - \omega_{rs}^D, \;\; r, s = 1, \dots, n$$

- When the traveler is not matched, the profit $\pi_{r(n+1)}$ for the passenger of OD pair r and the profit $\pi_{(n+1)s}$ for the driver of OD pair s are both zero.

Lemma 1: non-negativity of profits. All profits $\pi_{rs}, r, s = 1, \dots, n$, of matchings with strictly positive optimal flow $X^*_{rs} > 0$ are non-negative.

Proof. Consider the optimal matchings. Consider one matching with strictly positive flow $X^*_{rs} > 0$. The profit is $c_s - \delta_{rs}$. The proof is by contradiction. Assume profit is negative, i.e. the cost δ_{rs} of detour of the driver is higher than the cost c_s of the minimum travel time route for the driver on her OD pair. The cost of the optimal matching is $c_r + \delta_{rs}$. Then, we would obtain a lower cost, equal to $c_r + c_s$, if both the driver and the passenger travelled alone on the respective minimum travel time routes. This means that the matching is not optimal, for a contradiction. Therefore, profits of matchings with strictly positive flow may not be negative.

Consider now the following linear programming (LP) problem.

Problem P2: maximization of total profits from matchings.

$$\max_X \sum_{r=1}^{n} \sum_{s=1}^{n} \pi_{rs} X_{rs}$$

$$X_{rs} \le X^*_{rs},\ r, s = 1, \dots, n \tag{19}$$

$$X_{rs} \ge 0,\ r, s = 1, \dots, n$$

$$X_{r(n+1)} = X^*_{r(n+1)}, \qquad r = 1, \dots, n$$

$$X_{(n+1)s} = X^*_{(n+1)s}, \qquad s = 1, \dots, n$$

where $X^*_{rs}, r, s = 1, \dots, n, X^*_{r(n+1)}, r = 1, \dots, n, X^*_{(n+1)s}, s = 1, \dots, n$ are the solutions of problem P1. Notice that for some, but not all, matchings $r, s = 1, \dots, n$ we have $X^*_{rs} = 0$.

Assumption 7: strict positivity of profits.

- All profits $\pi_{rs}, r, s = 1, \dots, n$, of matchings with strictly positive optimal flow $X^*_{rs} > 0$ are strictly positive.

The assumption poses no restriction in applied work. For the assumption to hold, it is sufficient that $c_s \neq \delta_{rs}$ for all r, s matchings.

Clearly, problem P2 has the same and only solutions of problem P1. This is because the inequality constraints (19) are binding, i.e. satisfied as equalities, due to the strict positivity of profits of matchings with strictly positive flow and the search for a maximum.

Also, it is easy to see that the objective function at optimum of problem P2 equals, with minus sign and up to a multiplicative constant (the fuel cost of the travel over a unitary length), the objective function at optimum of problem P1 up to additive constants:

- objective function of P1 = min (car-kilometers of travelers participating in the scheme when matched + car-kilometers of travelers not participating in the scheme)
- objective function of P2 = max (- car-kilometers of travelers participating in the scheme when matched + car-kilometers of all passengers traveling alone + car-kilometers of all drivers traveling alone) x per-kilometer fuel cost.

In mathematical terms, let F_1 be the function which is minimized in problem P1 net of the contribution of travelers not participating in the scheme and multiplied by the per-kilometer fuel cost:

$$F_1 = \sum_{r=1}^{n}\sum_{s=1}^{n}(c_r + \delta_{rs})X_{rs} + \sum_{r=1}^{n} c_r X_{r(s+1)} + \sum_{s=1}^{n} c_s X_{(r+1)s}$$

Let F_2 be the function which is maximized in problem P2:

$$F_2 = -F_1 + \sum_{r=1}^{n} c_r d_r^P + \sum_{s=1}^{n} c_s d_s^D$$

Therefore, application of duality theory of LP to problem P2 makes sense. According to this theory (Eiselt and Sandblom, 2010; table 4.1), the dual of problem P2 is as follows.

Problem P3: dual problem of problem P2.

$$\min_{y} \sum_{r=1}^{n}\sum_{s=1}^{n} X_{rs}^* y_{rs} + \sum_{r=1}^{n} X_{r(n+1)}^* y_{r(n+1)} + \sum_{s=1}^{n} X_{(n+1)s}^* y_{(n+1)s}$$

$$y_{rs} \geq \pi_{rs}, \qquad r,s = 1,\dots,n$$

$$y_{r(n+1)} \geq \pi_{r(n+1)}, \qquad r = 1,\dots,n$$

$$y_{(n+1)s} \geq \pi_{(n+1)s}, \qquad s = 1,\dots,n$$

The following lemma justifies the interpretation of the dual variables $y_{rs}, r,s = 1,\dots,n,\ y_{r(n+1)}, r = 1,\dots,n,\ y_{(n+1)s}, s = 1,\dots,n$, as the value of the corresponding socially optimal matching, i.e. the WTA to forego that matching. Notice that we have extended here the definition of matching to include travel alone, because this is a matching with a dummy traveler.

Lemma 2: dual variables. At optimum of P2 and P3 we have:

$$X_{rs} > 0 \Rightarrow\ y_{rs} = \pi_{rs},\ \ r,s = 1,\dots,n$$

$$X_{r(n+1)} > 0 \Rightarrow\ y_{r(n+1)} = \pi_{r(n+1)},\ \ r = 1,\dots,n$$

$$X_{(n+1)s} > 0 \Rightarrow\ y_{(n+1)s} = \pi_{(n+1)s},\ \ s = 1,\dots,n$$

and

$$y_{rs} > \pi_{rs} \Rightarrow\ X_{rs} = 0,\ \ r,s = 1,\dots,n$$

$$y_{r(n+1)} > \pi_{r(n+1)} \Rightarrow X_{r(n+1)} = 0,\ \ r = 1,\dots,n$$

$$y_{(n+1)s} > \pi_{(n+1)s} \Rightarrow\ X_{(n+1)s} = 0,\ \ s = 1,\dots,n$$

Proof. By the weak complementary slackness conditions that hold at optimum applied to problems P2 and P3 (Eiselt and Sandblom, 2010; theorem 4.9). □

Therefore, problem P3 can be interpreted as the matching valuation problem associated with the matching allocation problem P2.

We set $y_{rs} = u_r + v_s,\ r = 1,\dots,n, y_{r(n+1)} = u_r, r = 1,\dots,n, y_{(n+1)s} = v_s, s = 1,\dots,n,$ and use constraints of Eqs (2) and (3) to obtain the following problem.

Problem P4: re-formulated dual problem of problem P2.

$$\min_{u,v} \sum_{r=1}^{n} u_r\, d_r^P + \sum_{s=1}^{n} v_s d_s^D$$

$$u_r + v_s \geq \pi_{rs}, \qquad r,s = 1,\dots,n$$

$$u_r \geq \pi_{r(n+1)} = 0, \qquad r = 1,\dots,n$$

$$v_s \geq \pi_{(n+1)s} = 0, \qquad s = 1,\dots,n$$

Problems P2 and P4 are key to the proof of the property related to social optimality and stability.

Definition 6: outcome.

- An outcome is the specification of the matching flows $X_{rs}, r, s = 1, \dots, n,$ along with the surplus u_r that each passenger of OD pair r gets as well as the surplus v_s that each driver of OD pair s gets.

Definition 7: feasible outcome.

- An outcome is feasible if the total profit generated from matching is equal to the total profit redistributed to passengers and drivers (remember that the profit of travelling alone is zero):

$$\sum_{r=1}^{n} \sum_{s=1}^{n} \pi_{rs} X_{rs} = \sum_{r=1}^{n} u_r d_r^P + \sum_{s=1}^{n} v_s d_s^D$$

Definition 8: stable outcome.

- A feasible outcome is stable if the following conditions are satisfied:

$$u_r + v_s \geq \pi_{rs}, \; r, s = 1, \dots, n \tag{20}$$

$$u_r \geq 0, \quad r = 1, \dots, n \tag{21}$$

$$v_s \geq 0, \quad s = 1, \dots, n \tag{22}$$

Conditions of Eqs (20) express the exclusion of a blocking pair, i.e. the exclusion of a situation where at least one traveler gains and the other does not lose by leaving the current partner and matching together, and the exclusion of a situation where both the passenger and the driver gain by leaving the current partners and matching together. This is because the profit of the current matching (left-hand side) is not lower than the profit which would be obtained if the partners were matched together (right-hand side). These conditions express Pareto efficiency: no matching is available that makes one traveler better off without making another traveler worse off.

Exclusion of blocking pairs implies that no two travelers have incentive to break up the current partnership and match together. Additionally, since the profit of travelling alone is zero, Eqs (21) and (22) imply that no traveler, passenger or driver, who is matched with another traveler, respectively driver or passenger, would prefer to travel alone. Conditions of Eqs (21) and (22) also express individual rationality, because both passenger and driver do not lose from current matching.

The reservation prices underlying surplus and, hence, the stability definition make reference to the travel times in the equilibrium state of the network with socially optimal matchings. This is akin to the formulation of the Wardrop equilibrium principle which establishes inequalities on travel times with reference to present conditions.

Definition 9: financially sustainable outcome.

- A feasible outcome is financially sustainable if no subsidy is needed to obtain a non-negative surplus for each passenger and driver.

The following proposition establishes an if and only if condition related to social optimality and stability, as well as the financial sustainability of any stable outcome. It is the first main result of the paper.

Proposition 1: social optimality, stability and financial sustainability. Given a feasible outcome, if the matching is socially optimal then the outcome is stable, and if the outcome is stable then the matching is socially optimal. Additionally, every stable outcome is financially sustainable.

Proof. By the strong duality property applied to problem P2 and problems P3 and P4, the objective functions are the same at optimum (Eiselt and Sandblom, 2010; theorem 4.8). Feasibility of the outcome as well as Eqs (20), (21) and (22) follow. Financial sustainability is implied by Eqs (21) and (22).

Stable outcomes represent conditions of competitive equilibrium, in the sense that passengers of each OD pair choose to match with the drivers who provide them with maximum surplus, and drivers of each OD pair choose to match with the passengers who provide them with maximum surplus. Mathematically, for a stable outcome we have:

$$u_r = \max_{s=1,\dots,n+1}(\pi_{rs} - v_s) = (\omega_r^P - p_r), \quad r = 1, \dots, n$$

$$v_s = \max_{r=1,\dots,n+1}(\pi_{rs} - u_r) = \max_{r=1,\dots,n+1}(p_r - \omega_{rs}^D), \quad s = 1, \dots, n$$

To see why it is so, consider that stable outcomes are the optimal solution to the dual problem P4. Then Eqs (20) hold. Eqs (20) imply:

$$v_s \geq \max_{r=1,\dots,n+1}(\pi_{rs} - u_r), \;\; s = 1, \dots, n$$

but the above needs to be satisfied as equality or one could strictly improve on the objective function of the dual problem P4 and contradict optimality of surpluses. Notice that passengers are indifferent to driver OD pair, because each passenger travels on her OD pair and never detours (recall Figure 1).

2.4 Pricing

In section 2.3, we have proved the stability of feasible outcomes associated with socially optimal matchings. Each outcome is the specification of the surplus of each traveler, which, in turn, requires specification of the prices paid by passengers and received by drivers. Specification of prices is the subject of the present section. It is based on the assignment game theory developed in Thompson (1980 and 1991).

Definition 10: core.

- The core is the set of all optimal solutions (u, v) to re-formulated dual problem P4 (Solymosi and Raghavan, 2001).

The core is denoted by $\mathcal{C} = \big(\mathcal{C}(u), \mathcal{C}(v)\big)$, where $\mathcal{C}(u)$ is the passenger core i.e. the set of optimal solutions $u_r, r = 1, \dots, n$, to the dual, and where $\mathcal{C}(v)$ is the driver core i.e. the set of optimal solutions $v_s, s = 1, \dots, n$, to the dual.

The core is non empty, in the light of the strong duality property applied to problems P2 and P4 (Eiselt and Sandblom, 2010; theorem 4.8). Additionally, as discussed in Shapley and Shubik (1971), the core includes usually infinitely many solutions. To see this, consider that if there is a socially optimal and stable matching r, s with surpluses $u_r > 0,\; v_s \geq 0,$ then we can get infinitely many other solutions in the core from it by the transformation $u_r - \gamma,\; v_s + \gamma$, where γ is a small number. This is because the equality $u_r + v_s = \pi_{rs}$ holds in the light of the weak complementary slackness conditions (Eiselt and Sandblom, 2010; theorem 4.9). In words, a small amount can be shifted from u_r to v_s without spoiling any of the conditions for a dual solution.

Definition 11: extremum passenger surpluses.

- The maximum passenger surplus u_r^* for passenger of OD pair r is:

$$u_r^* = \max_{u \in \mathcal{C}(u)} u_r$$

- The minimum passenger surplus u_{*r} for passenger of OD pair r is:

$$u_{*r} = \min_{u \in \mathcal{C}(u)} u_r$$

Using these, we can define points u^* and u_* as:

$u^* = (u_1^*, \dots, u_n^*)$ the maximum passenger surplus point

$u_* = (u_{*1}, \dots, u_{*n})$ the minimum passenger surplus point.

Definition 12: extremum driver surpluses.

- The maximum driver surplus v_s^* for driver of OD pair s is:

$$v_s^* = \max_{v \in \mathcal{C}(v)} v_s$$

- The minimum driver surplus v_{*s} for driver of OD pair s is:

$$v_{*s} = \min_{v \in \mathcal{C}(v)} v_s$$

Using these, we can define points v^* and v_* as:

$v^* = (v_1^*, \dots, v_n^*)$ the maximum driver surplus point

$v_* = (v_{*1}, \dots, v_{*n})$ the minimum driver surplus point.

Definition 13: distinguished surplus points.

- The MaxPassenger-MinDriver surplus point is (u^*, v_*), while the MaxDriver-MinPassenger surplus point is (u_*, v^*).

The following properties characterize the above defined distinguished surplus points: they belong to the core $\mathcal{C}$ (Shapley and Shubik, 1971); are the furthest distance apart of any two points in $\mathcal{C}$ (Shapley and Shubik, 1971); individually and collectively maximize, or minimize, passenger or driver surpluses (Thompson, 1980); there is a traveler who gains zero surplus (Balinski and Gale, 1987).

The first property is well established in the theory of assignment games, and is referred to as polarization of interests (Roth, 1984): there exists a passenger-optimal stable outcome that is the best stable outcome for every passenger and the worst for every driver, and a corresponding driver-optimal stable outcome that is best for every driver and worst for every passenger.

The proposition in the sequel provides the method to compute the two distinguished surplus points of the core. It is the second main result of the paper, because it is essential to price computation. Before, we introduce the following two LP problems.

Problem P5.

$$\min_{u,v} \sum_{s=1,\dots,n} v_s$$

$$u_r + v_s = \pi_{rs},\ r, s: X_{rs} > 0;\ r, s = 1, \dots, n \quad (23)$$

$$u_r = 0,\ r: X_{r(n+1)} > 0;\ r = 1, \dots, n \quad (24)$$

$$v_s = 0,\ s: X_{(n+1)s} > 0;\ s = 1, \dots, n \quad (25)$$

$$u_r + v_s \geq \pi_{rs},\ r, s: X_{rs} = 0;\ r, s = 1, \dots, n \quad (26)$$

$$u_r \geq 0,\ r: X_{r(n+1)} = 0;\ r = 1, \dots, n \quad (27)$$

$$v_s \geq 0,\ s: X_{(n+1)s} = 0;\ s = 1, \dots, n \quad (28)$$

Problem P6.

$$\min_{u,v} \sum_{r=1,\dots,n} u_r$$

subject to Eqs (23) to (28).

Both problems require as inputs the profits and, therefore, the length in kilometers of the minimum travel time route for each OD pair. These can be obtained from the MILP problem optimal solution, which includes link travel times, by the use of shortest path algorithms.

Proposition 2: computation of distinguished surplus points. The optimal solution to problem P5 is the distinguished MaxPassenger-MinDriver surplus point (u^*, v_*), the optimal solution to problem P6 is the distinguished MaxDriver-MinPassenger surplus point (u_*, v^*).

Proof. Consider problem P5. Based on the recalled properties, the distinguished MaxPassenger-MinDriver surplus point (u^*, v_*) is in the core and minimizes the objective function. Therefore, this point is an optimal solution to P5, because the constraints of P5 define all and only the points in the core, in the light of the weak complementary slackness conditions. Thus, to prove the statement of the proposition, it is sufficient to prove that no other point in the core in addition to (u^*, v_*) exist with coordinates $v = v_*$. Such a point cannot exist because if the problem is nondegenerate (i.e. if no matching at all is not an optimal solution and is, therefore, excluded) then there needs to be at least one OD pair r for which at least one equality constraint of Eq. (21) holds: being v_* unchanged, the equality constraint would be violated if we change u_r^*. A similar proof holds for problem P6.

Definition 14: fair division surplus point.

- The fair division surplus point (u^f, v^f) is the midpoint of the line segment connecting the distinguished point (u^*, v_*) and (u_*, v^*):

$$(u^f, v^f) = \frac{1}{2}[(u^*, v_*) + (u_*, v^*)] \tag{29}$$

In words, the surplus gained from the fair division surplus point (u^f, v^f) by each passenger and driver is exactly half way between the minimum and the maximum each can obtain from any solution in the core.

The fair division surplus point is in the core. This is because the core is a convex set, being it defined by linear equalities and inequalities (Arora, 2012), and the fair division surplus point is a convex combination of points in the set.

The price p_r^f, paid by the passenger of OD pair r to the matched driver, that is able to give the fair division surplus point can be obtained as follows. First, problems P5 and P6 associated with an optimal matching are solved. Then the fair division point $u_r^f, r = 1, \dots, n$, and $v_s^f, s = 1, \dots, n,$, is computed by Eqs (29). Finally, prices of surplus fair division are obtained by solving:

$$u_r^f = \omega_r^P - p_r^f, \qquad r = 1, \dots, n$$

Notice that prices are nonnegative. This is because we also have $v_s^f = p_r^f - \omega_{rs}^D \geq 0$ with $\omega_{rs}^D \geq 0$. Moreover, given the matchings, prices associated with the fair division surplus point are unique.

Given a matching r, s, with $X_{rs}^* > 0$ the following two indicators are of interest.

The first indicator is the share ψ_{rs} of the profit from matching that accrues to the passenger. This is the ratio of the passenger surplus to the matching profit:

$$\psi_{rs} = \frac{u_r^f}{\pi_{rs}} = \frac{u_r^f}{u_r^f + v_s^f} = \frac{\omega_r^P - p_r^f}{\omega_r^P - \omega_{rs}^D}$$

We have that $0 \leq \psi_{rs} \leq 1$. To see why, first consider the case where the passenger surplus is at the minimum value which is zero. Then $\psi_{rs} = 0$. Second, consider the case where the passenger surplus is at his maximum value which implies, in the light of proposition 2, that the driver surplus is zero. Then $\psi_{rs} = 1$.

The second indicator is the share χ_{rs} of the cost of the full trip made by the driver that is paid by the passenger. This is the ratio of the price to the cost of the full trip made by the driver:

$$\chi_{rs} = \frac{p_r^f}{c_r + \delta_{rs}}$$

We have that $0 \leq \chi_{rs} \leq 1$. To see why, first consider the case where the price paid by the passenger is at his minimum value which is zero. Then $\chi_{rs} = 0$. In this case, we have the maximum passenger surplus and, in the light of proposition 2, the minimum driver surplus. Second, consider the case where we have the maximum driver surplus which implies, by proposition 2, that the passenger surplus is zero. Then $\delta_{rs} = 0$ and the price equals c_r. Therefore, $\chi_{rs} = 1$.

A final remark relates to travelers participating in the scheme who are not matched. Passengers in particular would clearly be dissatisfied. To cope with this, a subsidisation scheme, with the payment of a flat monetary reward to all travelers posting a ride request or offer, might be implemented neither affecting social optimality nor violating the stability conditions of Eqs (20) to (22).

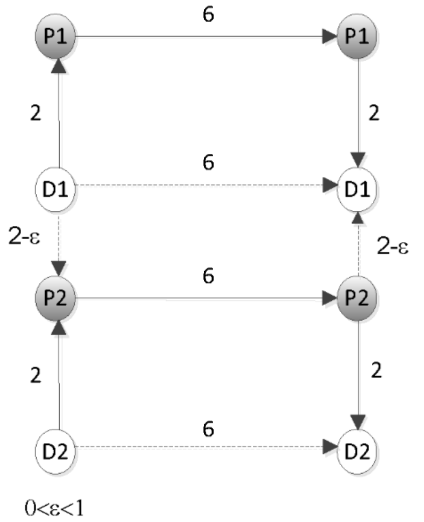
3. Illustrative examples

3.1 Toy network with fixed travel times

In this case, travelers not participating in ride-sharing are irrelevant, because travel times are fixed and, therefore, their route choices are unaffected by matchings. This is also a case where the total fuel cost savings on the all-travel-alone scenario of those participating in the scheme equal their total profits from matchings.

We consider the network in Figure 2 (from Wang et al., 2018). The number near each link is the corresponding cost. The figure includes tables showing the following quantities: cost and cost saving for the individual travelers according to matching, as well as reservation price of individual travelers and profit of matchings.

Matching solution A, where passenger 1 is matched with driver 1 (P1D1) and passenger 2 with driver 2 (P2D2), is socially optimal, because it is associated with the lowest cost and with highest cost saving and profit, in the light of the inequalities $0 < \varepsilon < 1$.



	All-travel-alone scenario	Matching A P1D1, P2D2		Matching B P1, P2D1, D2	
	cost	cost	cost saving	cost	cost saving
P1	6	0	+6	6	0
P2	6	0	+6	0	+6
D1	6	10	-4	10-2ε	-(4-2ε)
D2	6	10	-4	6	0
total	24	20 min	4 max	22-2ε	2+2ε

	Matching A P1D1, P2D2	Matching B P1, P2D1, D2
	reservation price	reservation price
P1	6	6
P2	6	6
D1	4	4-2ε
D2	4	0

Matching A P1D1, P2D2		Matching B P1, P2D1, D2	
	profit		profit
P1D1	2	P1	0
P2D2	2	P2D1	2+2ε
		D2	0
total	4	total	2+2ε

Figure 2. Toy network (P = Passenger; D = Driver)

For this matching solution, the constraints of Eqs (21) to (26) of problems P5 and P6 are as follows:

$$u_1 + v_1 = 2 \tag{30}$$

$$u_2 + v_2 = 2 \tag{31}$$

$$u_2 + v_1 \geq 2 + 2\varepsilon \tag{32}$$

$$u_1, u_2, v_1, v_2 \geq 0 \tag{33}$$

Then, the MaxPassenger-MinDriver surplus point, solution to P5, is:

$$(u_1^*, u_2^*, v_{*1}, v_{*2}) = (2 - 2\varepsilon, 2, 2\varepsilon, 0)$$

and the MinPassenger-MaxDriver surplus point, solution to P6, is:

$$(u_{*1}, u_{*2}, v_1^*, v_2^*) = (0, 2\varepsilon, 2, 2 - 2\varepsilon)$$

Therefore, the fair division surplus point is:

$$\left(u_1^f, u_2^f, v_1^f, v_2^f\right) = (1 - \varepsilon, 1 + \varepsilon, 1 + \varepsilon, 1 - \varepsilon)$$

with the associated price p_1^f paid by passenger 1 to driver 1 equal to $5 + \varepsilon$, and price p_2^f paid by passenger 2 to driver 2 equal to $5 - \varepsilon$.

The share ψ_{11} of the profit from matching that accrues to passenger 1 is $(1 - \varepsilon)/2$, while the corresponding share ψ_{22} that accrues to passenger 2 is $(1 + \varepsilon)/2$. The share χ_{11} of the cost of the full trip made by the driver that is paid by passenger 1 is $(5 + \varepsilon)/10$, while the corresponding share χ_{22} that is paid by passenger 2 is $(5 - \varepsilon)/10$.

It is instructive to see that the equal (between passenger and driver) profit allocation and the equal driver cost allocation are not stable for the socially optimal matching solution. First, consider that the core is provided by Eqs (30) to (33). Therefore, all points in the core satisfy:

$$u_1 = a, u_2 = b$$

$$v_1 = 2 - a, v_2 = 2 - b$$

$$0 \leq a, b \leq 2, b - a \geq 2\varepsilon$$

with associated prices $p_1 = 6 - a$ and $p_2 = 6 - b$. Since the profit is identical for the two optimal matchings P1D1 and P2D2 then the equal cost allocation cannot be in the core (because $a \neq b$ holds) and, as a consequence, it cannot be stable. The same argument holds for the equal driver cost allocation. Prices of the equal profit allocation are equal to 5 for each of the two matchings P1D1 and P2D2. Prices of the equal driver cost allocation also are equal to 5 for each of the two matchings P1D1 and P2D2.

3.2 Small network with endogenous travel times

The illustrative example relates to the network in Figure 3. There are 17 centroids and nodes, 272 OD pairs and 52 directed links. The network is an induced sub-graph of the Sioux Falls network in LeBlanc (1975). The number of travelers is 136 000. Adjustment to OD car flows in LeBlanc (1975) were needed, or link flows would have far exceeded capacity. For each OD pair, the given fraction of travelers who participate in the scheme is divided randomly between passengers and drivers. Equality of total number of passengers with total number of drivers is obtained by final adjustment. Link BPR functions are those in LeBlanc (1975).

A python code is used. The optimal matching with equilibrium constraints MILP problem is solved using Gurobi library. With a 2.50 GHz CPU and 16.00 GB RAM personal computer, the order of magnitude of the computation time is 12 minutes. Results of sensitivity analysis with respect to the fraction of total travelers who participate in the scheme are in Tables 1 and 2.

Table 1 shows a comparison of the results of the model proposed, which endogenizes congestion, with the results of a model where travel times are fixed at all-travel-alone equilibrium. Performance indicators are related to the full traveler population. A matching concordance index is computed as percentage of OD pairs for which the matching is equal in the two models up to a small error (5%). This index is found to be in the range between 82% and 86%.

When the fraction of total travelers who participate in the scheme increases, then the reduction of car-kilometers increases too, because passengers leave their own car and shift to ride-sharing. The percentage reduction is slightly lower than the value of the percentage reduction in car flows which is half that of total travelers. This result is simply explained on the basis of the percentage of matched travelers on total travelers willing to do so: this percentage is lower than 100%, because at optimum a few passengers are not matched and have to use their own car.

The traveler-hours show a different pattern. If travel times are fixed, then traveler-hours are higher than in the all-travel-alone scenario, and increase with increasing fraction of total travelers who participate in the scheme. This result is explained by the higher number of driver detours. By contrast, with endogenous travel times, traveler-hours are lower than in the all-travel-alone scenario, because of the congestion reduction that originates from the reduced number of cars in the network. With increasing fraction of total travelers who participate in the scheme, the traveler-hours increase due to detours.

Table 2 shows performance indicators that are related to only the travelers who participate in the scheme and are matched. Fuel costs are evaluated for a price of 3.5 $/gallon, which is equal to 0.87 $/liter, and an efficiency of 10 kilometers/liter. The order of magnitude of the average distance travelled by a driver is 9.5 kilometers. The following performance indicators are also computed (proposed by Yan et al., 2021):

SP = ratio of total surplus of matched passengers to their total payments,

SD = ratio of total surplus of matched drivers to their total detour costs.

SP is the average net gain of each matched passenger for each car-kilometer of payment. SD is the average net gain each matched driver obtains for each car-kilometer of detour cost.

Figure 4-7 show, for travellers who participate in the scheme and are matched, the distributions of passenger surplus, driver surplus, fair price and share of the driver trip cost that is paid by the passenger (in the case where demand participating in the scheme is 20% of total demand). The distributions show point masses at the zero value of surpluses and price. Consequently, there are point masses at the left and right extremes of the distribution of the share of the driver trip cost that is paid by the passenger: there are optimal matchings for which the trip cost is entirely paid by the driver (left extreme), and optimal matchings for which the trip cost is entirely paid by the passenger (right extreme). On average, passengers pay a higher fraction of the trip cost than the drivers, as confirmed by Table 2 and the right-skewed distribution of Figure 7.

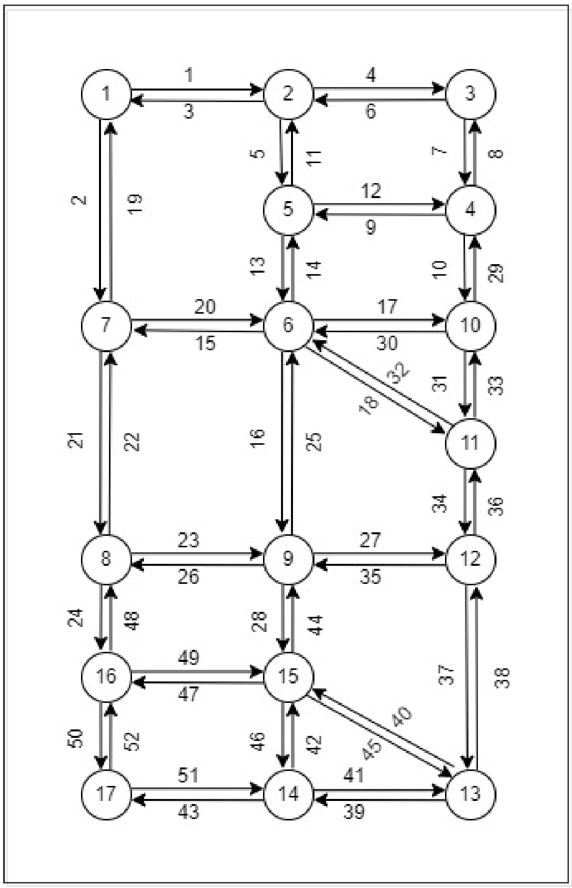


Figure 3. Small network

Table 1. Results – all travelers, and comparison with a model where travel times are fixed at all-travel-alone equilibrium

Travelers participating in the scheme (passengers + drivers) as % of total travelers	Fixed travel times			Endogenous travel times			Matching concordance index (%)
	matched travelers (as % of total travelers)	network car-kilometers (% change on without matching case)	network traveler-hours (% change on without matching case)	matched travelers (as % of total travelers)	network car-kilometers (% change on without matching case)	network traveler-hours (% change on without matching case)	
0%	-	1 169 552	73 395	-	1 169 552	73 395	-
5%	6 242 (4.59%)	1 144 194 (-2.17%)	73 537 (+0.19%)	6 258 (4.6%)	1 144 402 (-2.15%)	55 661 (-24.16%)	86.44%
10%	12 476 (9.17%)	1 118 422 (-4.37%)	73 655 (+0.35%)	12 532 (9.21%)	1 118 424 (-4.37%)	55 706 (-24.1%)	82.63%
15%	18 814 (13.83%)	1 093 904 (-6.47%)	73 845 (+0.61%)	18 818 (13.84%)	1 093 736 (-6.48%)	62 567 (-14.75%)	83.77%
20%	25 166 (18.50%)	1 068 384 (-8.65%)	73 975 (+0.79%)	25 190 (18.52%)	1 067 706 (-8.71%)	65 739 (-10.43%)	85.15%

Table 2. Results – travelers who participate in the scheme and are matched

Travelers participating in the scheme (passengers + drivers) as % of total travelers	Drivers: average increase of car-kilometers per trip on all-travel-alone scenario (%)	Drivers: average increase of travel time per trip on all-travel-alone scenario (%)	Average price paid by the passenger to the driver ($)	Average share of matching profit that accrues to the passenger (%)	Average share of the driver trip cost that is paid by the passenger (%)	SP	SD
5%	8.08%	8.51%	0.51	38.42%	59.17%	0.545	0.51
10%	6.63%	6.99%	0.51	37.60%	58.96%	0.540	0.53
15%	7.62%	8.78%	0.54	33.00%	63.47%	0.437	0.56
20%	7.02%	9.28%	0.50	38.61%	57.72%	0.557	0.51

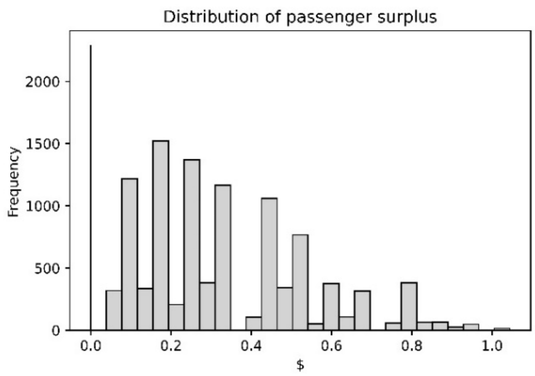


Figure 4. Distribution of passenger surplus u_r^f

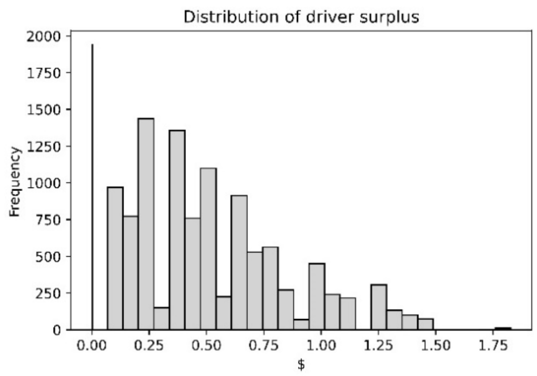


Figure 5. Distribution of driver surplus v_s^f

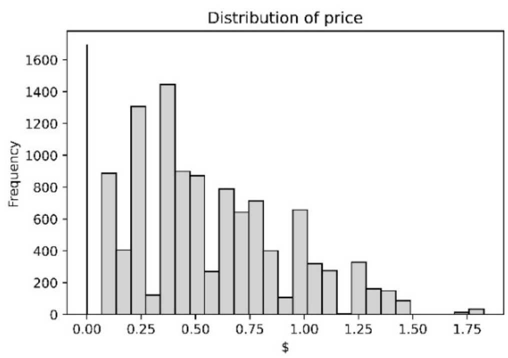


Figure 6. Distribution of fair price p_r^f that is paid by the passenger to the driver

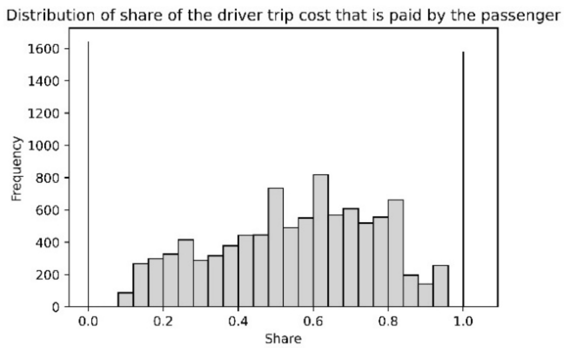


Figure 7. Distribution of share χ_{rs} of the driver trip cost that is paid by the passenger

4. Conclusions

4.1 Discussion

A peer-to-peer ride-sharing model is formulated. The model is intended as a tool for the exploration of the potential of a few innovative policy design ideas. The static model of the paper is a first step towards the development of a dynamic version of the models and algorithms that are needed for the deployment of online peer-to-peer ride-sharing services in cities.

Matching and pricing decisions are co-ordinated by a social planner. Matchings are socially optimal in that they minimize the total car-kilometers traveled over the network. Prices respond to a twofold criterion: stability and fairness. Routing decisions are decentralized, i.e. left to the drivers, who choose the minimum travel time route.

The sequential two-step approach to computation relies on MILP for matching with endogenous congestion, and LP for pricing. In both cases, efficient off-the-shelf solvers are available. These were able to provide the solution for the network of the illustrative example. In larger size networks, the dimensionality of the matching optimization problem may be prohibitive in terms of number of matching flow variables, which, net of the excluded matchings, depends on the square of the number of OD pairs plus one (because of the travelling alone possibility). The simplest approach is to exclude heuristically as many matchings as possible, as an example on the basis of the length of the detour. This may lead to social sub-optima, but is without consequences on stability and method of fair price computation.

The social planner's objective function is in direct proportion to social welfare. Transaction prices cancel out. Resources restrict to fuel consumption expenditure. The framework can be extended to include resources related to travel time expenditure, via the definition of a generalised cost of travel. Consideration of travel time resources implies challenges for the social planner at both stages of matching and price computations. The hurdle is related to the valuation of travel time. Therefore, the implied ride-sharing scheme would be less realistic.

Passengers who sign up to the scheme travel alone by car. No passenger travels alone by public transport, because, if ride-sharing trips substitute for trips by public transport, then the impacts in terms of congestion, energy consumption and pollution would be negative, in the light of the higher car-kilometers due to detours. In practice, registration to the scheme may be permitted depending on car ownership.

Social optimality, stability and financial sustainability properties make the scheme a promising option for real-world implementation. In particular, stability guarantees that travelers have motivations to participate in the scheme. Additionally, a monetary reward to travelers would be able to compensate passengers who are not matched, without impact on social optimality and stability. Thus, proper incentives are in place to encourage travelers increasing participation frequency as well as to attract new travelers.

The numerical illustration makes evident the existence of a number of optimal matchings, around 12% of total matchings, for which the fair price paid by the passenger is zero and, therefore, the cost of the trip is entirely paid by the driver. For these matchings, the driver surplus is zero. Also, there exists a comparable-in-size number of optimal matchings for which the cost of the trip is entirely paid by the passenger. For these matchings, the passenger surplus is zero. Both relate to matched travellers and, therefore, are in addition to un-matched travellers, who have zero surplus by definition. Again, a flat subsidy to all travellers may obviate this shortcoming.

Additional findings from the numerical illustration are the reduction in both total car-kilometers and traveler-hours over the network, as well as the heterogeneity of the share of the trip cost that is paid by the passenger across optimal matchings. On average, the passenger pays more than the driver.

4.2 Future developments

The exclusion of side payments may be usefully relaxed, primarily to cover platform operating costs, or to pay, without the use of subsidies, compensation rewards to passengers who are not matched. A theory of assignment games with side payments is found in Kaneko (1982).

Ride pooling schemes, where more than one passenger shares the ride with one driver, are in favour of sustainability. The stability concept requires extension to set-wise stability with n-tuples of travelers, triplets in the case of two passengers with one driver. For this, literature on multiple partner games is relevant (among the others: Sotomayor, 1992, 2003 and 2007).

Finally, the impacts of real-time matching of ride requests and offers can be analysed in dynamic settings encompassing departure time choices and time-varying congestion. To this aim, the fixed-point within-day macroscopic approach in Bellei et al. (2006), or the mesoscopic approach in de Palma et al. (2022a) can be used. Once matchings are socially optimized under the dynamic multidimensional equilibrium constraints (related to route, departure time and, possibly, mode choices), stability conditions, appropriately defined in terms of reservation prices, hold. To prove this, duality arguments, similar to the one used for the static setting here, can be used. This implication is a new finding of ride-sharing research.

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