THEMA

théorie économique, modélisation et applications

THEMA Working Paper n°2023-22
CY Cergy Paris Université, France

Risk management of margin based portfolio strategies for dynamic portfolio insurance with minimum market exposure

Killian Pluzanski, Jean-Luc Prigent

CERGY PARIS UNIVERSITÉ

cnrs

January 2023

Risk management of margin based portfolio strategies for dynamic portfolio insurance with minimum market exposure

K. Pluzanski*and J.L. Prigent†

January 25, 2023

Abstract

We extend the standard Constant Proportion Portfolio Insurance (CPPI) by introducing simultaneously margin based dynamic strategies and constraints on minimum market exposure. This leads us to introduce specific conditional floors, allowing the portfolio of not being monetized (to avoid the cash-lock risk) while ensuring better participation in potential market increases. To control the risk of such strategies, we introduce risk measures based both on quantile conditions. Our empirical analysis is mainly conducted on S&P 500 and Euro Stoxx 50, by using Monte-Carlo experiments based on circular block boostrap method. This allows us to analyze the impact of the different parameters that define our CPPI strategies (i.e. CPPI multiple, successive margins, level of the minimum market exposure). We estimate and compare the cumulative distribution functions of the portfolio returns corresponding to the various insurance strategies that we investigate. We provide also their first four moments and measure their respective performances using both the Sharpe and the Omega ratios. Our results highlight the benefits of introducing time-varying floors associated to a decreasing sequence of margins while keeping the market exposure above a minimum level.

Key words: Portfolio insurance; CPPI strategy; time vaying floor; margin based strategy; market exposure.

JEL classification: C 22, C 61, G 11.

*Amundi, 91 Boulevard Pasteur, 75015, Paris, France Email: k.pluzanski@gmail.com

†Thema and Labex MME-DII, CY Cergy Paris University, 33 Bd du Port, 95011, Cergy-Pontoise, France. Email: jean-luc.prigent@cyu.fr

1 Introduction

The recent events[1] and the history of the financial markets[2] point out both the plausibility and the severity of the potential losses that an investor can experience. For risk adverse investors, such as insurers subject to regulatory constraints or pension funds with defined contributions, the control of the downside risk plays a key role in their investment process. However, downside risk control requires to address both statistical and practical concerns. First, estimating the downside risk represents an important challenge due to the structure of asset prices ([1]). For instance, the non-homogeneous behavior of asset returns implies that the risk is time-varying and need to be dynamically assessed. Second, from a practical point of view, downside risk control is not as simple as withdrawing the capital from a position at risk. Indeed, due to the market structure, a portfolio manager cannot reallocate his entire portfolio at once due to liquidity issue or without suffering from an execution risk. In addition, a significant reduction in exposure at the source of the risk significantly limits the potential benefit of a future market recovery..

In this framework, the concept of portfolio insurance has been developed to limit the portfolio downside risk while maintaining a certain upside participation. There are two main approaches to portfolio insurance: (i) the option based strategies usually known as the *option based portfolio insurance* (OBPI) and the (ii) floor-based strategies covering the well known, *constant proportion portfolio insurance* (CPPI) and *time invariant portfolio insurance* (TIPP) strategies.

Option-based strategies use option instruments to target a desired payoff profile at a given time. First introduced by Leland and Rubinstein ([2]), with the use of European put options to guarantee a minimal portfolio value in the future, these strategies have evolved considerably over time, mainly with the use of hedging strategies to mitigate the insurance cost or to benefit from more exotic option payoff profile. For instance, Föllmer et al. ([3], [4]) introduce a quantile hedging framework for investor facing budget constraints but requiring to achieve a specific goal. In a similar manner, Strassberger ([5]) implements a dynamic risk budgeting strategy based on the replication of a synthetic put to hedge the value at risk and the expected shortfall. Alternatively, Carr et al. ([6]) focus on hedging the maximum drawdown using double barrier options. Additionally, since double barrier options are not liquid, they provide alternative hedging strategies based on more vanilla options. Other different approaches rely on optimization procedures to select the structure of the option strategies. Capinski ([7]) proposes to find the optimal allocation of put options that minimize the conditional value at risk of a portfolio subject to a cost constraints.

Although they offer a wide range of solutions, option based strategies are not always easily implementable. On one hand, options are not necessarily liquid instruments and even the most liquid option markets are limited in the choice of the strike or maturity. On the other hand, option pricing and hedging requires advanced statistical methods and computational resources that are not available to all investors.

The second approach to portfolio insurance provides a much simpler and less restricted implementation. Indeed, floor-based strategies consist of directly adjusting the portfolio exposure over time to maintain a minimal guarantee value, referring to the floor level. These strategies are based on the *constant proportion portfolio insurance* (CPPI) allocation mechanism introduced by Black and Perold ([8]) and only differ in the design of the floor process. The CPPI strategy allocates dynamically the portfolio value between two assets: a risky asset and a risk-free asset. The floor is set at the inception of strategy and is assumed to evolve at the risk free rate. This design allows an investor to insure a proportion of his initial capital. The exposition into these assets is based on the distance of the portfolio value with its floor level, corresponding to the guarantee, and a given parameter, the so-called multiple, which measures the market exposition and can be related for example to risk aversion. The weight of the risky asset decreases or increases as the portfolio value converges towards or moves away from the floor level. Then, the initial level and adjustment speed of the exposure in the risky asset is amplified by the value of the multiple. Higher multiple value results in an higher initial exposure into the risky asset with a higher variability over time.

[1] The COVID-19 crisis and the Russia-Ukraine war.
[2] Subprime crisis, sovereign debt.

This parameter drives the investor ability to benefit from a rise in asset prices but, on the opposite side, increase the risk of reaching faster the minimal desired portfolio value. Note that it must be upper bounded to control the gap risk (i.e. the portfolio value becomes smaller than the floor).

Although the CPPI provides a flexible and easy to implement insurance strategy, it comes at the cost of two major drawbacks. Initially developed in continuous time, this strategy ensures that the portfolio value to never be lower than the floor level (i.e. there is no gap risk). However, under real market conditions, time is discrete. Thus asset prices exhibit a jump risk making this strategy subject to the gap risk ([9]). The main issues with gap risk is twofold: (i) the solution cannot guarantee a minimal portfolio value with probability one and (ii) once the portfolio breaches the floor, the portfolio is monetized (equivalently cash-lock). The exposure to the risky asset is set to zero and the portfolio can no longer benefit from any rise in asset prices. In general, the second issue does not necessarily require the portfolio to breach the floor level. The cash-lock risk occurs implicitly since, before breaching the floor, the level of the exposure to the risky asset is already significantly reduced. In most cases, the exposure mechanically becomes close to zero before the gap risk materializes. Then the portfolio becomes almost fully concentrated on risk free asset and thus misses a large part of a potential market increase.

The other major concern with the CPPI framework comes from the drawdown risk. By definition, this strategy only focuses on one aspect of the downside risk, the initial capital loss. The allocation scheme does not take into account the current gain of the portfolio and thus is subject to an high drawdown risk. Let us consider an investor with an initial capital of \$100 and floor level of \$70. If the portfolio reaches a net asset value of \$300 then the maximum possible loss for the investor is about 76.6% [3]. It is unlikely that investors will tolerate such loss level. Indeed, as suggested by Cheklov et al. ([10],[11]), investors usually withdraws their funds after a drawdown of about 20% on a one year time period.

These issues lead to several modifications of the initial CPPI framework from the choice of the multiple to the choice of the floor process. For instance, Ben Ameur and Prigent ([12]) address the gap risk and thus indirectly the cash-lock risk by allowing the multiple to vary over time. They use a risk control approach based on quantile and expected shortfall criteria to select the multiple conditionally to the market environment. They find that using conditional multiple provides significant different performance than the standard CPPI formulation due to the greater reactivity to the local market configuration. Thus the strategy benefits from low risk environment to be more inclined to increase its exposure to the risky asset and reciprocally to be more conservative in high risk environment.

Other extensions focus on the change of the floor process. One of the most known alternative to the CPPI strategy is the *time varying portfolio protection* (TIPP) strategy of Grossman and Zhou ([13]) which focus specifically on controlling the maximum drawdown. The TIPP considers the floor level as a step function increasing every time the portfolio reaches a new maximum value. The floor dynamic caps the exposure to the risky asset and thus ensures the portfolio drawdown to not exceed a predefined level. However, every time the portfolio reaches a new maximum the exposure is reset to a lower level limiting potential future gains. In a less restrictive approach, Kanniganti and Boulier ([14]) propose a more flexible framework based on two different floor process: (i) the margin and (ii) the ratchet effects. The margin effect consists in setting the initial floor higher than the target floor and use the difference, namely the margin, as a reserve to differ in time the investment mechanism. This reserve is partially or fully consumed to increase the strategy's exposure to the risky asset when it becomes too low. This reduces the risk of cash-lock. Conversely, ratchet effects increase the floor level when the strategy value increases above a predefined level. This mechanism is used to lock a proportion of the strategy's current gain and then limit the drawdown risk. However, the authors limit their work to arbitrary choices of decrease and increase of the floor level. Based on this framework, Ben Ameur and Prigent ([15]) propose for the two effects to adjust the floor level according to the same risk control they use to find conditional multiples ([12]). As a result, the floor is adjusted according to the expectation of the risk of the underlying asset. They provide, for both margin and ratchet effects, a set of rules to update the floor level while maintaining a risk control

[3] Assuming there is no gap-risk.

over the strategy.

However, in their formulation, the use of the risk control implies strong conditions. In what follows, we consider the margin based CPPI strategies. Floor adjustments subject to the risk control are triggered if the portfolio value becomes smaller than the conditional floor (equal to the target floor plus the margin) of if the cushion level is (conditionally) expected at the next period to become negative. In order to be active, this latter rule, requires either that the underlying asset must be subject to substantial losses for low to moderate multiple level or to consider very high multiple level to compensate for lower loss magnitude. Additionally, previous conditions can lead to too conservative strategy (small market exposure) since the strategy can end up in a cash-lock situation with a remaining margin since the magnitude of the expected price variation might never be enough to expect a negative cushion.

The purpose of this article is to combine the approach of Ben Ameur and Prigent ([15]) with an additional control of the minimum market exposition. Indeed, our approach is twofold : first we want to introduce a more versatile way to trigger margin effects while maintaining a local risk control over the strategy; second we search to better benefit from market rises by keeping the marker exposure above a given minimum level. in this respect, we propose an ex-post version of the triggering mechanism which relies directly on the exposure level of the strategy as in Boulier and Kanniganti ([14]). Second, we determine the floor adjustments based on a risk control which focus on the variability of the cushion instead of its level.

The paper is organized as follows. Section 2 presents the standard CPPI framework in discrete-time and impact of the parameter selection on the strategy. Section 3 reviews the time-varying floor approach of Ben Ameur and Prigent applied to the CPPI with margin effects. The third section introduces the use of a different triggering event and an alternative gap risk control. Then the fourth section provides the comparative analysis of our contribution using simulated and empirical data.

2 The CPPI strategy in discrete-time

In the discrete-time framework, at a set of trading dates t_k, the CPPI strategy allocates the portfolio value V_{t_k} between two assets: the risky asset S_{t_k} and the risk-free or reserve asset B_{t_k} over a given investment horizon T. The allocation mechanism consists of investing an amount called the exposure, $e_{t_k} = m * C_{t_k} = m * (V_{t_k} - P_{t_k})$ into the risky asset S_{t_k} and the remaining amount $V_{t_k} - e_{t_k}$ into the risk free asset B_{t_k}. The exposure is a function of the distance between the portfolio value V_{t_k} and the floor level P_{t_k}, namely the cushion C_{t_k} and of the multiple, $m \in \mathbb{R}^{+,*}$. The multiple can be usually related to the investor risk aversion.

In the standard formulation, the floor level is determined at inception and evolves at the same rate as the reserve asset, namely with returns, $r^B_{t_k}$, over the period $[t_{k-1}, t_k]$. For instance, an investor requiring a capital insurance of 70% at one year horizon with a risk-free rate equal to 3% per year sets at inception his floor level to $P_{t_0} = 0.7 * V_{t_0} * \exp[-0.03]$. Finally, in the case of a floor breach, i.e. $C_{t_k} \leq 0$, the exposure is immediately set to zero and the portfolio becomes fully concentrated in the risk free asset. Therefore, due to the discrete-time setting, there is a non-negligible probability that the targeted guarantee is not meet and the actual portfolio value is lower than the desired one.

The strategy dynamic is obtained through a two-step process: (i) the implementation step and (ii) the evaluation step. The first stage allocates at time t_k the portfolio value into the risky and risk-less asset while the second assesses at time t_{k+1} the results of the allocation. Therefore, we get the following representation of the CPPI strategy:

$$\textbf{Implementation} \qquad V_{t_k} = \frac{e_{t_k}}{S_{t_k}} * S_{t_k} + \frac{V_{t_k} - e_{t_k}}{B_{t_k}} * B_{t_k} \tag{1}$$

$$\textbf{Evaluation} \qquad V_{t_{k+1}} = \frac{e_{t_k}}{S_{t_k}} * S_{t_{k+1}} + \frac{V_{t_k} - e_{t_k}}{B_{t_k}} * B_{t_{k+1}} \tag{2}$$

From these two steps, we deduce the portfolio value and cushion dynamics over one period of time $[t_k, t_{k+1}]$. The portfolio dynamics is given by:

$$\Delta V_{t_{k+1}} = V_{t_{k+1}} - V_{t_k} = e_{t_k} * \frac{\Delta S_{t_{k+1}}}{S_{t_k}} + (V_{t_k} - e_{t_k}) * \frac{\Delta B_{t_{k+1}}}{B_{t_k}} \tag{3}$$

We deduce the cushion dynamics from the previous equation. Indeed, by definition, the cushion satisfies $C_{t_k} = V_{t_k} - P_{t_k}$. Thus, we have:

$$\begin{aligned}\Delta C_{t_{k+1}} = C_{t_{k+1}} - C_{t_k} &= \Delta V_{t_{k+1}} - \Delta P_{t_{k+1}} \\ &= e_{t_k} * \frac{\Delta S_{t_{k+1}}}{S_{t_k}} + (V_{t_k} - e_{t_k}) * \frac{\Delta B_{t_{k+1}}}{B_{t_k}} - \Delta P_{t_{k+1}}\end{aligned}$$

Due to the fact that $e_{t_k} = m * C_{t_k}$ and $V_{t_k} = C_{t_k} + P_{t_k}$, the previous expression becomes:

$$\begin{aligned}\Delta C_{t_{k+1}} &= m * C_{t_k} * \frac{\Delta S_{t_{k+1}}}{S_{t_k}} + (C_{t_k} + P_{t_k} - m * C_{t_k}) * \frac{\Delta B_{t_{k+1}}}{B_{t_k}} - \Delta P_{t_{k+1}} \\ &= C_{t_k} * \left(m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}} + (1-m) * \frac{\Delta B_{t_{k+1}}}{B_{t_k}} \right) + P_{t_k} * \frac{\Delta B_{t_{k+1}}}{B_{t_k}} - \Delta P_{t_{k+1}}\end{aligned}$$

Since the floor P_{t_k} evolves at the same rate, $r^B_{t_{k+1}}$, as the reserve asset B_{t_k} we deduce that:

$$P_{t_k} * \frac{\Delta B_{t_{k+1}}}{B_{t_k}} - \Delta P_{t_{k+1}} = 0$$

Finally, the dynamics of the cushion is given by:

$$\Delta C_{t_{k+1}} = C_{t_k} * \left(m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}} + (1-m) * \frac{\Delta B_{t_{k+1}}}{B_{t_k}} \right) \tag{4}$$

$$C_{t_{k+1}} = C_{t_k} * \left(1 + m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}} + (1-m) * \frac{\Delta B_{t_{k+1}}}{B_{t_k}} \right) \tag{5}$$

If we consider that $r^B_{t_k}$ is very small (usually due to the small time period $[t_k, t_{k+1}]$), the previous equation simplifies to:

$$C_{t_{k+1}} \approx C_{t_k} * \left(1 + m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}}\right) = (V_{t_k} - P_{t_k}) * \left(1 + m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}}\right) \tag{6}$$

This equation fully describes the behavior of the strategy and the role of the parameters. Indeed, it appears that the multiple drives the variability of the cushion while, in some sense, the floor controls its level. Moreover, this expression provides additional useful information over the relationship between the parameters choice and the strategy risks. For instance, one way to escape rapidly from the cash-lock risk (i.e. C_{t_k} close to zero) is to use a high enough multiple value. However, such value increases the risk of breaching the floor level (i.e. increases the gap risk). Alternatively, the floor can be adjusted, downwards or upwards, to either mitigate the cash-lock risk or the drawdown risk, respectively. A lower floor level mechanically results in an higher cushion and thus exposure, while a higher floor level reduces the exposure and set a lower tolerance for losses.

In what follows, we are going to consider various time varying and conditional floors. However, we note the following property of "independence" w.r.t. the floor.

Remark 1 *(Cushion positivity and floor) According to Equation 6, the positivity of the cushion after the variations of the asset prices does not depend on the floor value.*

3 Time varying floor framework

Due to these relationships, Ben Ameur and Prigent ([12],[15]) use the previous equation (5) as a starting point to provide a risk based framework to the selection of the parameters. In a first instance, they show in ([12]) that the gap risk can be controlled when considering multiples satisfying the following quantile rule:

$$\mathbb{P}\left(\forall\ t_k \in [0,T], C_{t_k} > 0\right) \geq 1-\epsilon \underset{r_{t_k}<<}{\Leftrightarrow} \mathbb{P}\left(\forall\ t_k \in [0,T], (1 + m * \frac{\Delta S_{t_k}}{S_{t_{k-1}}}) > 0\right) \geq 1-\epsilon$$

with $\epsilon \in (0,1)$. Equivalently, considering $M_T = \max_{1\leq l\leq n}\left(-\frac{\Delta S_{t_l}}{S_{t_l-1}}\right)$ the maximum loss over one period of time, we get:

$$\begin{aligned}
\mathbb{P}\left(\forall\ t_k \in [0,T], -\frac{\Delta S_{t_l}}{S_{t_{l-1}}} < \frac{1}{m}\right) \geq 1-\epsilon &\Leftrightarrow \mathbb{P}\left(M_T < \frac{1}{m}\right) \geq 1-\epsilon \\
&\Leftrightarrow F_{M_T}\left(\frac{1}{m}\right) \geq 1-\epsilon \\
&\Leftrightarrow \frac{1}{m} \geq F^{-1}_{M_T}(1-\epsilon) \\
&\Leftrightarrow m < \frac{1}{F^{-1}_{M_T}(1-\epsilon)}
\end{aligned}$$

where $F^{-1}_{M_T}$ is the inverse of the cumulative distribution function of M_T. This approach allows investors to target multiple value depending on their choice of the probability threshold ϵ. Indeed, the upper bound is an increasing function of this latter one. For example, in some sense, a risk averse investor will choose a small ϵ implying that he will select a low multiple. However, this rule considers the asset returns distribution in its globality and do not account for its temporal properties. In this way, the multiple is constant over the entire investment period and thus the strategy cannot adapt to the different risk environments.

In this context, the authors address the lack of adaptability by considering a more general framework. First the multiple is no longer constant. Second it evolves over time in such a way that the gap risk is controlled over two consecutive trading dates. This local feature yields from the use of the current state of the cushion in the risk control selection of the parameter. Since the cushion is mainly driven by the asset returns, this approach allows to account for the asset price dynamic and thus its different risk environments.

This framework is not only limited to the selection of the multiple under a gap risk control. Ben Ameur and Prigent ([15]), extended the previous approach to the floor process. Based on the previous work of Kanniganti and Boulier ([14]), they show that the floor can be adjusted to reduce both the cash-lock risk and the drawdown risk, using respectively margin and ratchet effects, while maintaining a gap risk control. The margin effect consists of reducing the floor level to regain in exposure into the risky asset. Reciprocally, the ratchet effect increases the floor level to lock in the current gains of the strategy. Both of these effects are triggered based on predefined events corresponding to specific states of the strategy. For example, in the case of the margin effect the floor can be reduced when the exposure decreases below a specific level.

The time-varying floor mechanism is common to both effects. First it assumes a target floor, denoted $\hat{P}_{t_k}$, referring to the usual floor of the standard strategy. This floor allows to control the global loss risk of the portfolio over time and allows to recover a predefined percentage of the initial investment amount at the terminal horizon. If at any trading date, t_k, the portfolio breached the target floor (i.e. $\hat{C}_{t_k} = V_{t_k} - \hat{P}_{t_k} \leq 0$) then the portfolio becomes monetized. Second this mechanism allows the investor to modify his floor at any time during the management period. Thus it defines an effective dynamic and conditional floor as follows:

$$\nabla P_{t_k} = P^+_{t_k} - P^-_{t_k}, \tag{7}$$

which means that ∇P_{t_k} represents the variation of the floor at time t_k due to the specific choice of the new floor $P^+_{t_k}$.

- The value $P^-_{t_k}$ is equal to the previous floor value chosen at time t_{k-1} for the period $[t_{k-1}, t_k[$ and invested in the riskless asset with rate $r^B_{t_k}$ during this time period. Thus it evolves according to:

$$P^-_{t_k} = P^+_{t_{k-1}} * \exp(r^B_{t_k}).$$

- The value $P^+_{t_k}$ is chosen at time t_k in order to satisfy the portfolio management objectives at that time. This can be based on a triggering event modeled by a Bernoulli random variable X_{t_k} depending on the considered effects.

In the same way, we define the variations of the cushion at time t_k, resulting from the choice of the portfolio strategy at time t_k:

$$\nabla C_{t_k} = C^+_{t_k} - C^-_{t_k}.$$

Note that we have $P^+_{t_k} \geq \hat{P}_{t_k}$. Therefore, we get the following general form for the dynamic floor $P^+_{t_k}$:

Proposition 2 *(Choice of the new floor) At any time t_k, the floor $P^+_{t_k}$ is chosen in the following manner:*

$$P^+_{t_k} = \begin{cases} h(t_k, \Gamma_{t_k}) & \text{if } X_{t_k} = 1, \\ \\ P^-_{t_k} = P^+_{t_{k-1}} * (1 + r^B_{t_k}) & \text{if } X_{t_k} = 0, \end{cases} \tag{8}$$

where $h(t_k, \Gamma_{t_k})$ denotes a generic function and Γ a set of parameters fully determined from the considered effects.

Based on this new floor process, we deduced the following dynamics for the strategy value and the cushion level:

$$V_{t_{k+1}} = V_{t_k} + e^+_{t_k} * \frac{\Delta S_{t_{k+1}}}{S_{t_k}} + (V_{t_k} - e^+_{t_k}) * \frac{\Delta B_{t_{k+1}}}{B_{t_k}} \tag{9}$$

with $e^+_{t_k} = m * C^+_{t_k} = m * (V_{t_k} - P^+_{t_k})$
and the dynamics of the cushion is defined by:

$$C^-_{t_{k+1}} \approx C^+_{t_k} * \left(1 + m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}}\right) = \left(V_{t_k} - P^+_{t_k}\right) * \left(1 + m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}}\right) \tag{10}$$

3.1 Value-at-Risk constraints on the cushion value

The floor can be adjusted up or down based on the following risk control on the cushion value:

$$\forall\ k \in \mathbb{N},\ \mathbb{P}^{\mathcal{G}_{t_k}}(C^-_{t_{k+1}} < -L_{t_k}) < \epsilon \tag{11}$$

where $\forall\ k \in \mathbb{N}, L_{t_k} > 0$ is a predefined threshold and $\mathcal{G}_{t_k}$ corresponds to a set of information such that $\mathcal{F}_{t_{k-1}} \subset \mathcal{G}_{t_k}$ with $\mathcal{F}_{t_k} = \sigma\left(\frac{\Delta S_{t_1}}{S_{t_0}}, \dots, \frac{\Delta S_{t_k}}{S_{t_{k-1}}}\right)$ the σ-algebra generated by the asset returns. This quantile condition is considered as "local" since the control at time t_k concerns only the variation on the time period $]t_k, t_{k+1}]$.

Developing this risk control results to the following restriction over the floor level:

$$\begin{aligned} C^-_{t_{k+1}} < -L_{t_k} &\Leftrightarrow C^+_{t_k} * \left(1 + m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}}\right) < -L_{t_k} \\ &\Leftrightarrow \frac{\Delta S_{t_{k+1}}}{S_{t_k}} < -\frac{1}{m} * \left(1 + \frac{L_{t_k}}{C^+_{t_k}}\right) \end{aligned}$$

Let $F_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\cdot)$ be the conditional cumulative distribution function of the asset returns w.r.t. the information $\mathcal{G}_{t_k}$. We assume that it is invertible.[4]

$$\begin{aligned} \forall\ k \in \mathbb{N},\ \mathbb{P}^{\mathcal{G}_{t_{k-1}}}(C^-_{t_{k+1}} < -L_{t_k}) < \epsilon &\Leftrightarrow F_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}\left(-\frac{1}{m}\left(1 + \frac{L_{t_k}}{C^+_{t_k}}\right)\right) < \epsilon \\ &\Leftrightarrow -\frac{1}{m}\left(1 + \frac{L_{t_k}}{C^+_{t_k}}\right) < F^{-1}_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\epsilon) \\ &\Leftrightarrow -\frac{L_{t_k}}{C^+_{t_k}} < \left(1 + m * F^{-1}_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\epsilon)\right) \end{aligned}$$

Denote by $\theta^m_{t_k}(\epsilon)$ the term $\left(1 + m * F^{-1}_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\epsilon)\right)$ which is the quantile of $\left(1 + m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}}\right)$ at the probability level ϵ. Therefore we deduce: If $\theta^m_{t_k}(\epsilon) < 0$ then

$$\begin{aligned} -\frac{L_{t_k}}{\theta^m_{t_k}(\epsilon)} > C^+_{t_k} &\Leftrightarrow -\frac{L_{t_k}}{\theta^m_{t_k}(\epsilon)} > (V_{t_k} - P_{t_k^+}) \\ &\Leftrightarrow V_{t_k} + \frac{L_{t_k}}{\theta^m_{t_k}(\epsilon)} < P_{t_k^+} \end{aligned}$$

and since $V_{t_k} - P^+_{t_k} > 0$ we have the following restriction over the choice of the new floor level:

$$V_{t_k} + \frac{L_{t_k}}{\theta^m_{t_k}(\epsilon)} < P_{t_k^+} < V_{t_k} \tag{12}$$

Finally if $\theta^m_{t_k}(\epsilon) > 0$ we have

$$P_{t_k^+} < \min\left(V_{t_k} + \frac{L_{t_k}}{\theta^m_{t_k}(\epsilon)}, V_{t_k}\right) \tag{13}$$

Since $\frac{L_{t_k}}{\theta^m_{t_k}(\epsilon)} > 0$ then $P_{t_k^+} < V_{t_k}$. In this configuration the underlying risk is low enough to not impose any particular restriction on the choice of the floor level.

[4] Otherwise, we consider its left inverse, as it is a monotononic function.

Proposition 3 *(VaR constraints on the new floor due to the risk control on the cushion value) The risk control adjustment of the floor is completely determined by the sign of the quantity $\theta^m_{t_k}(\epsilon)$:*

1. *If $\theta^m_{t_k}(\epsilon) < 0$ then*

$$V_{t_k} + \frac{L_{t_k}}{\theta^m_{t_k}(\epsilon)} < P_{t_k^+} < V_{t_k} \tag{14}$$

2. *If $\theta^m_{t_k}(\epsilon) > 0$, then*

$$P_{t_k^+} < V_{t_k} \tag{15}$$

As emphasized in previous proposition, the sign of the quantile $\theta^m_{t_k}(\epsilon)$ plays a key role when controlling locally the cushion value.

Remark 4 *The quantile $\theta^m_{t_k}(\epsilon)$ depends on the conditional distribution of the asset returns as follows:*

$$\theta^m_{t_k}(\epsilon) < 0 \Leftrightarrow F^{-1}_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\epsilon) < -\frac{1}{m}$$

$$\theta^m_{t_k}(\epsilon) > 0 \Leftrightarrow F^{-1}_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\epsilon) > -\frac{1}{m}$$

For example, if $m = 3$ then the expected asset return over $[t_k, t_{k+1}]$ at a given probability level $F^{-1}_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\epsilon)$ must be lower than -33.33%. This anticipated loss threshold is very significant and suggests that the underlying instrument must be in very bad configuration to reach such loss level. Likewise, if the expected asset returns is above this level then the instrument is considered to be in a good enough configuration to require any risk control. The multiple plays an important role in the classification of the underlying risk environment. Indeed, the multiple determines the sensitivity of the strategy to the risky asset. Thus a low multiple implies a lower sensitivity and results in an higher loss threshold. Reciprocally, strategies with higher multiples are more prone to require a risk control since there more sensitive to the risky asset and thus considered riskier. Therefore, the multiple determines the definition of the risk environment.

To summarize, the choice of the new floor $P^+_{t_k}$ at time t_k, based on the risk control of the cushion value, can be expressed as follows.

Proposition 5 *(Choice of the new floor with risk control of the cushion value) At any time t_k, we follow the following process:*

1. *If $\widehat{P}_{t_k} \geq V_{t_k}$, then the exposure is set to 0 until maturity.*
2. *If $\widehat{P}_{t_k} < V_{t_k}$, then the floor $P^+_{t_k}$ is chosen in the following manner:*

$$P^+_{t_k} = \begin{cases} h(t_k, \Gamma_{t_k}) & \text{if } X_{t_k} = 1, \\ P^-_{t_k} = P^+_{t_{k-1}} * (1 + r^B_{t_k}) & \text{if } X_{t_k} = 0, \end{cases} \tag{16}$$

with

$$X_{t_k} = \begin{cases} 1 & \text{if } P^-_{t_k} > V_{t_k} \text{ or } P^-_{t_k} < V_{t_k} \text{ but } \theta^m_{t_k}(\epsilon) < 0 \\ 0 & \text{if } P^-_{t_k} < V_{t_k} \text{ and } \theta^m_{t_k}(\epsilon) > 0 \end{cases} \tag{17}$$

and

$$h(t_k, \Gamma_{t_k}) = \begin{cases} \widehat{P}_{t_k} + q_{t_k}\widehat{C}_{t_k} \text{ with } 0 < q_{t_k} < \frac{(V_{t_k} - \widehat{P}_{t_k})}{\widehat{C}_{t_k}} & \text{if } P^-_{t_k} > V_{t_k} \\ \left(V_{t_k} + \frac{L_{t_k}}{\theta^m_{t_k}(\epsilon)}\right) & \text{if } P^-_{t_k} < V_{t_k} \text{ and } \theta^m_{t_k}(\epsilon) < 0 \end{cases} \tag{18}$$

Note that this latter choice corresponds to the choice of the maximal possible exposure when $P^-_{t_k} < V_{t_k}$ and $\theta^m_{t_k}(\epsilon) < 0$, under the quantile constraint.

3.2 Alternative risk control

As discussed previously, the risk control is not easy to apply since it requires very strong conditions to be active. In this context, we can control for example the downside variation of the cushion induced by the choice of a new floor level instead of its direct level. Such approach allows to remove the dependence of the risk control activation to the multiple. Thus, the application of the risk control is now fully driven by the distribution of the asset returns. Let $\Delta C^-_{t_{k+1}} = C^-_{t_{k+1}} - C^+_{t_k}$ be the variation of the cushion right on the time period $]t_k, t_{k+1}]$. The risk control is design as follows:

$$\forall\, k \in \mathbb{N},\ \mathbb{P}^{\mathcal{G}_{t_k}}\left(\Delta C^-_{t_{k+1}} < -\tilde{L}_{t_k}\right) < \epsilon \tag{19}$$

Since we have:

$$\Delta C^-_{t_{k+1}} = C^-_{t_{k+1}} - C^+_{t_k} \approx C^+_{t_k} * m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}},$$

we deduce that

$$\begin{aligned}\forall\, k \in \mathbb{N},\ \mathbb{P}^{\mathcal{G}_{t_k}}\left(\Delta C^-_{t_{k+1}} < -\tilde{L}_{t_k}\right) < \epsilon &\Leftrightarrow \mathbb{P}^{\mathcal{G}_{t_k}}\left(m * C^+_{t_k} * \frac{\Delta S_{t_{k+1}}}{S_{t_k}} < -\tilde{L}_{t_k}\right) < \epsilon \\ &\Leftrightarrow \mathbb{P}^{\mathcal{G}_{t_k}}\left(\frac{\Delta S_{t_{k+1}}}{S_{t_k}} < \frac{-\tilde{L}_{t_k}}{m * C^+_{t_k}}\right) < \epsilon\end{aligned}$$

Assuming that the cumulative conditional distribution function of the asset returns, $F_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\cdot)$ is invertible and $\tilde{\theta}^m_{t_k}(\epsilon) = m * F^{-1}_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\epsilon)$, we obtain:

$$\begin{aligned}\forall\, k \in \mathbb{N},\ \mathbb{P}^{\mathcal{G}_{\sqcup_\parallel}}\left(\Delta C^-_{t_{k+1}} < -\tilde{L}_{t_k}\right) < \epsilon &\Leftrightarrow \frac{-L_{t_k}}{C^+_{t_k}} < \tilde{\theta}^m_{t_k}(\epsilon) \\ &\Leftrightarrow -\tilde{L}_{t_k} < C^+_{t_k} * \tilde{\theta}^m_{t_k}(\epsilon) \\ &\Leftrightarrow -\tilde{L}_{t_k} < \left(V_{t_k} - P^+_{t_k}\right) * \tilde{\theta}^m_{t_k}(\epsilon)\end{aligned}$$

Finally based on the sign of $\tilde{\theta}^m_{t_k}(\epsilon)$ we have the following relationships:

$$\begin{cases}\tilde{\theta}^m_{t_k}(\epsilon) < 0 \implies V_{t_k} + \dfrac{\tilde{L}_{t_k}}{\tilde{\theta}^m_{t_k}(\epsilon)} \leq P^+_{t_k} \\ \\ \tilde{\theta}^m_{t_k}(\epsilon) > 0 \implies P^+_{t_k} \leq V_{t_k} + \dfrac{\tilde{L}_{t_k}}{\tilde{\theta}^m_{t_k}(\epsilon)},\end{cases} \tag{20}$$

the latter condition being always satisfied by construction of $P^+_{t_k}$.

Remark 6 *These inequalities only differ from the ones (12, 13) obtained with the previous gap risk control on how the risk environment is determined.*

$$\textbf{\textit{Level}} \qquad \theta^m_{t_k}(\epsilon) = 1 + m * F^{-1}_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\epsilon) \tag{21}$$

$$\textbf{\textit{Variation}} \qquad \tilde{\theta}^m_{t_k}(\epsilon) = m * F^{-1}_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\epsilon) \tag{22}$$

Looking at these measures, we note that the use of the new risk control provides two additional features: (i) the risk environment (i.e. the sign of θ) is independent from the multiple m and (ii) the risk measure is no longer restricted to extreme risk scenario. For the special cases $L_{t_k} = 0$ and $\tilde{L}_{t_k} = 0$, the first risk is to get a negative floor (gap risk of the conditional floor) while the second risk corresponds to a (simple) decrease of the cushion.

4 Floor adjustments with margin effects

In the case of margin effects, the effective floor process is composed as the sum of two elements: (i) the target floor, $\hat{P}_{t_k}$, and (ii) the margin, $M_{t_k} > 0$. The margin works as a buffer that decreases every time a triggering event is reached. By construction, the floor is initially higher than the target floor and converges toward it gradually.

To detail the margin strategy, we first consider the case of the risk control according to previous quantile condition.

4.1 Choice of the new margin with risk control of the cushion value

A) If the portfolio value V_{t_k} satisfies $P^-_{t_k} < V_{t_k}$ then we reduce possibly the margin according to the following rule using the VaR condition, namely:

- 1) If $\theta_{t_k} < 0$, then the new value $P^+_{t_k}$ of the floor is equal to the usual floor plus the previous margin evaluated at time t_k, which is reduced by factor γ_{t_k}. We have:

$$P^+_{t_k} = \hat{P}_{t_k} + M^+_{t_k} \text{ and } M^+_{t_k} = M^-_{t_k}\gamma_{t_k}.$$

The new cushion is equal to:

$$C^+_{t_k} = V_{t_k} - P^+_{t_k}$$

Thus, to satisfy the general condition determining the lower bound on the floor if $\theta_{t_k} < 0$, we must set:

$$\frac{V_{t_k} + \frac{L_{t_k}}{\theta_{t_k}} - P^+_{t_k}}{M^-_{t_k}} \leq \gamma_{t_k} \leq \frac{V_{t_{k1}} - P^+_{t_k}}{M^-_{t_k}}. \tag{23}$$

In our illustrations, we set:

$$\gamma_{t_k} = \gamma^{VaR*}_{t_k} = Max[\frac{V_{t_k} + \frac{L_{t_k}}{\theta_{t_k}} - P^+_{t_k}}{M^-_{t_k}}, 0] \tag{24}$$

- 2) If $\theta_{t_{k-1}} > 0$, then the VaR condition is not stringent. We keep the same floor (i.e. $P^+_{t_k} = P^+_{t_k}$).

B) If the portfolio value V_{t_k} satisfies $P^-_{t_k} > V_{t_k}$ then we reduce the floor as follows: We define a new margin $M^+_{t_k}$ equal to a given proportion q_{t_k} $(0 < q_{t_k} < 1)$ of the target cushion $\widehat{C}_{t_k} = V_{t_k} - \widehat{P}_{t_{k1}}$. Thus we set:

$$P^+_{t_k} = \widehat{P}_{t_k} + M^+_{t_k} \text{ with } M^+_{t_k} = q_{t_k}\left(V_{t_k} - \widehat{P}_{t_k}\right).$$

Recall that, if $\widehat{P}_{t_{k1}} \geq V_{t_k}$, then the exposure is set to 0 until maturity.

Remark 7 *In the original case of the margin introduced by Boulier and Kanniganti (2005), the proportion $\gamma_{t_k} = q_{t_{k-1}}$ is assumed to be constant. Additionally, there is no explicit risk control. In our framework, this proportion is variable and based on the quantile condition depending at each time on the values of several parameters such as m, L_t, and $\widehat{P}_0$ together with the current portfolio value V_{t_k}.*

To summarize, for the margin based strategy based on risk control, the floor process $(P^+_{t_k})_k$ is defined through a sequence of margins $(M_{t_l})_l$ as follows.

Proposition 8 *(Choice of the new margin with risk control of the cushion value) At any time t_k, the process $P^+_{t_k}$ is defined as follows:*

1. *If $\widehat{P}_{t_k} \geq V_{t_k}$, then the exposure is set to 0 until maturity.*
2. *If $\widehat{P}_{t_k} < V_{t_k}$, then the floor $P^+_{t_k}$ satisfies:*

$$P^+_{t_k} = \begin{cases} h_m(t_k, \Gamma_{t_k}) & \text{if } X_{t_k} = 1, \\ P^-_{t_k} = P^+_{t_{k-1}} * (1 + r^B_{t_k}) & \text{if } X_{t_k} = 0, \end{cases} \tag{25}$$

with

$$X_{t_k} = \begin{cases} 1 & \textit{if } P^-_{t_k} > V_{t_k} \textit{ or } P^-_{t_k} < V_{t_k} \textit{ but } \theta^m_{t_k}(\epsilon) < 0 \\ \\ 0 & \textit{if } P^-_{t_k} < V_{t_k} \textit{ and } \theta^m_{t_k}(\epsilon) > 0 \end{cases} \tag{26}$$

and

$$h_m(t_k, \Gamma_{t_k}) = \begin{cases} \widehat{P}_{t_k} + M^+_{t_k} \textit{ with } M^+_{t_k} = q_{t_k}\widehat{C}_{t_k} \textit{ and } 0 < q_{t_k} < 1 & \textit{if } P^-_{t_k} > V_{t_k} \\ \widehat{P}_{t_k} + M^+_{t_k} \textit{ with } M^+_{t_k} = M^-_{t_k} * \gamma^{VaR*}_{t_k} & \textit{if } P^-_{t_k} < V_{t_k} \textit{ and } \theta^m_{t_k}(\epsilon) < 0 \end{cases} \tag{27}$$

Finally from these conditions, the portfolio manager is able to reduce the cash lock risk while controlling the gap risk. For instance, when $\theta^m_{t_k}(\epsilon) < 0$ selecting the lower bound minimizes the cash-lock risk since it represents the greatest increase in exposure while maintaining the gap risk under control. Reciprocally, when $\theta^m_{t_k}(\epsilon) > 0$ the risk of the underlying asset is considered low enough to be subject to the risk control.

4.2 Floor adjustments with margin effects and minimal exposure

As seen previously, the dependence of the triggering event to $\theta^m_{t_k}(\epsilon)$ limits drastically its reachability, except for very small values of the probability threshold φ. As a result, the strategy based on the previous risk control is almost identical to the one introduced by Boulier and Kanniganti ([14]). Our approach to mitigate this dependency is to change the triggering event for a simpler one only based on the observed exposure of the strategy. In what follows, instead of monitoring an expected breach of the running cushion, we focus on the proportion invested in the risky asset defined as:

$$w_{t_k} = \frac{e_{t_k}}{V_{t_k}} = \frac{m * C_{t_k}}{V_{t_k}}.$$

This choice is equivalent to monitor the cushion due to their proportional relationship[5] (note also that sgn $w^-_{t_k}$ = sgn $C^-_{t_k}$) but provides an easier interpretation. Let $\overline{w}_{t_k} \in \mathbb{R}^+$ be the triggering threshold such that a triggering event occurs every time $w_{t_k} \leq \overline{w}_{t_k}$. This event is considered as ex-post compared to the previous one since it depends only on a realized observation and not on any expectation. Thus the

$$X_{t_k} = 1_{\{w^-_{t_k} \leq \ \overline{w}_{t_k}\}}$$

The use of this type of triggers requires to distinguish two cases depending on the value of the multiple m and the trigger threshold $\overline{w}_{t_k}$. In the case of $m \leq \overline{w}_{t_k}$ margin call will constantly occurs since the maximal exposure is limited to a lower level than the trigger threshold:

$$\max_{1\leq k\leq n}\left(w^-_{t_k}\right) = \max_{1\leq k\leq n}\left(m * \left(1 - \frac{P^-_{t_k}}{V_{t_k}}\right)\right) < \max_{1\leq k\leq n}\left(\overline{w}_{t_k} * \left(1 - \frac{P^-_{t_k}}{V_{t_k}}\right)\right) < \max_{1\leq k\leq n}\left(\overline{w}_{t_k}\right)$$

However this case is rarely implemented in practice since in most cases portfolio managers do not have the ability to use large leverages. For instance, if $m = 3$ the portfolio exposure into the risky asset must reached at least 300% to trigger a margin effect. Moreover to benefit from a convex payoff the multiple tends to be quite high compared to the allowed level of leverage. In the case $m > \overline{w}_{t_k}$, the use of soft triggers implies that the available margin at the time of the event is implicitly higher than the one initially defined. Indeed, at a triggering time we have:

$$w_{t_k} \leq \overline{w}_{t_k} \Leftrightarrow m * \left(V_{t_k} - P^-_{t_k}\right) \leq \overline{w}_{t_k} * V_{t_k} \Leftrightarrow V_{t_k} \leq P^-_{t_k} * \left(1 - \frac{\overline{w}_{t_k}}{m}\right)^{-1} = P^{(I,-)}_{t_k},$$

where $P^{(I,-)}_{t_k}$ denotes an implied effective floor from which we deduce the following implied margin:

[5] When using the exposition level, there is no need to screen the cushion and reciprocally.

$$M^{(I,-)}_{t_k} = P^{(I,-)}_{t_k} - \hat{P}_{t_k} = P^-_{t_k} * \left(1 - \frac{\overline{w}_{t_k}}{m}\right)^{-1} - \hat{P}_{t_k} = M^-_{t_k} + P^-_{t_k} * \frac{\overline{w}_{t_k}}{m - \overline{w}_{t_k}}$$

Thus, to set the weight above its minimal value, we must choose the new floor such as:

$$P^+_{t_k} \leq \frac{m - \overline{w}_{t_k}}{m} V_{t_k}.$$

However, since we must choose the new floor above the target floor, we consider finally:

$$P^+_{t_k} = Max\left(\frac{m - \overline{w}_{t_k}}{m} V_{t_k}, \hat{P}_{t_k}\right)$$

When $\frac{m-\overline{w}_{t_k}}{m} V_{t_k} \geq \hat{P}_{t_k}$, we note that:

$$P^+_{t_k} = \hat{P}_{t_k} + M^+_{t_k}$$

$$\text{with } M^+_{t_k} = \left(1 - \frac{\overline{w}_{t_k}}{m}\right) V_{t_k} - \hat{P}_{t_k} = \left(\hat{C}_{t_k} - \frac{\overline{w}_{t_k}}{m} V_{t_k}\right).$$

To summarize, for the margin based strategy with minimal exposure, the floor process $(P^+_{t_k})_k$ is defined through a sequence of margins $(M_{t_l})_l$ as follows.

Proposition 9 *(Choice of the new margin with minimal exposure) At any time t_k, the process $P^+_{t_k}$ is defined as follows:*

1. *If $\widehat{P}_{t_k} \geq V_{t_k}$, then the exposure is set to 0 until maturity.*
2. *If $\widehat{P}_{t_k} < V_{t_k}$, then the floor $P^+_{t_k}$ satisfies:*

$$P^+_{t_k} = \begin{cases} h_e(t_k, \Gamma_{t_k}) & \text{if } X_{t_k} = 1, \\ P^-_{t_k} = P^+_{t_{k-1}} * (1 + r^B_{t_k}) & \text{if } X_{t_k} = 0, \end{cases} \tag{28}$$

with

$$X_{t_k} = \begin{cases} 1 & \text{if } P^-_{t_k} > V_{t_k} \text{ or } P^-_{t_k} < V_{t_k} \text{ but } w_{t_k} \leq \overline{w}_{t_k} \\ 0 & \text{if } P^-_{t_k} < V_{t_k} \text{ and } w_{t_k} \geq \overline{w}_{t_k} \end{cases} \tag{29}$$

and

$$h_e(t_k, \Gamma_{t_k}) = \begin{cases} \widehat{P}_{t_k} + M^+_{t_k} \text{ with } M^+_{t_k} = q_{t_k} \widehat{C}_{t_k} \text{ and } 0 \leq q_{t_k} \leq 1 & \text{if } P^-_{t_k} > V_{t_k} \\ \widehat{P}_{t_k} + M^+_{t_k} \text{ with } M^+_{t_k} = Max\left[\left(\hat{C}_{t_k} - \frac{\overline{w}_{t_k}}{m} V_{t_k}\right), 0\right] & \text{if } P^-_{t_k} < V_{t_k} \text{ and } w_{t_k} \leq \overline{w}_{t_k} \end{cases} \tag{30}$$

If we impose also the constraint of minimal exposure when $P^-_{t_k} > V_{t_k}$, then we choose q_{t_k} such that $q_{t_k} = Max\left[\left(1 - \frac{\overline{w}_{t_k}}{m}\frac{V_{t_k}}{\hat{C}_{t_k}}\right), 0\right]$. Thus, the margin $M^+_{t_k}$ satisfies:

$$M^+_{t_k} = \gamma_{e,t_k} M^-_{t_k} \text{with } \gamma_{e,t_k} = \begin{cases} Max\left[\frac{\left(\hat{C}_{t_k} - \frac{\overline{w}_{t_k}}{m} V_{t_k}\right)}{M^-_{t_k}}, 0\right] & \text{if } X_{t_k} = 1, \\ 1 & \text{if } X_{t_k} = 0, \end{cases} \tag{31}$$

5 Numerical analysis

In this section, we illustrate numerically the different versions of the previous floor adjustments. For the margin effects, we review the following strategies: the ex-ante triggering mechanism with a level risk control, the ex-post mechanism based on the strategy's exposure associated with the level and the variation risk control. Additional we introduce a naive strategy to compare with the ex-post triggering mechanism. This strategy is similar to the later but does not account for a risk control. The floor is adjusted such that the exposure is set to the minimal level that does not trigger a floor adjustment. When the margin becomes too low to reach this level then all the remaining cash is invested into the risky asset.

Our analysis is conducted in two parts: (i) the first part uses numerical simulations to analyze the payoff profile of the strategies with respect to their parameters and (ii) the second part considers empirical observations to assess their performances from a practical point of view.

In both cases, we use end of week observations[6] of the S&P 500 Index and the Eurostoxx 50 Index over two specific time periods. The first period lies from the 31st December 2003 to the 31st January 2012 which is associated to the subprimes crisis and the second period spans from the 1st January 2018 to the 1st January 2022 which refers to the COVID-19 crisis. These two periods are selected for their different crisis behaviors (see table 1 and figure 1). The subprimes crisis defines a significant drop in prices occuring at a relatively slow pace followed by a slow recovery period for the S&P 500 Index and a no recovery for the Eurostoxx 50 Index. Conversely, the COVID-19 crisis characterizes a very fast sell-off lasting one month followed by a fast recovery period of approximately six months for the S&P 500 Index and one year for the Eurostoxx 50 Index. These scenario are particularly important for insurance strategies since they allow to access the efficiency of the protection and the associated opportunity cost when the underlyer asset recover. For instance, the COVID-19 period is one of the worst scenario in term of opportunity cost for floor based strategies due to the speed and the strength of the drawdown and the recovery. Such strategies tends to be cash-lock during the market recovery resulting in an important opportunity cost. The management of this cost is crucial for portfolio manager since it allows to show their abilities to adjust the protection when needed.

Index	**Start date**	**Drawdown date**	**Recovery date**	**Time to drawdown (days)**	**Time to recover (days)**	**Drawdown**	**Recover**
SPX Index	09-Oct-2007	09-Mar-2009	28-Mar-2013	355	1021	-56.78%	131.95%
SPX Index	19-Feb-2020	23-Mar-2020	18-Aug-2020	23	106	-33.92%	51.51%
SX5E Index	16-Jul-2007	09-Mar-2009	none	430	No recovery	-60.29%	none
SX5E Index	19-Feb-2020	18-Mar-2020	18-Mar-2021	20	261	-38.27%	62.11%

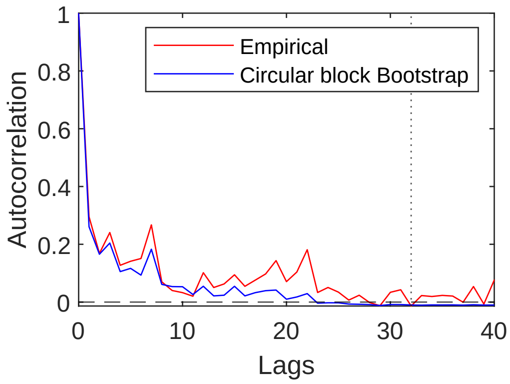
Table 1: Features of the subprimes and COVID-19 drawdown for both the S&P 500 Index and the Eurostoxx 50 Index.

[6] End of week = friday.

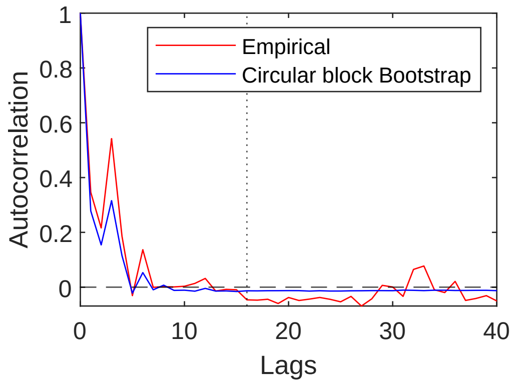
[POINTBREAK] I stop here since i need to re-run the strategies on both simulated and empirical price series.

5.1 Simulation based approach

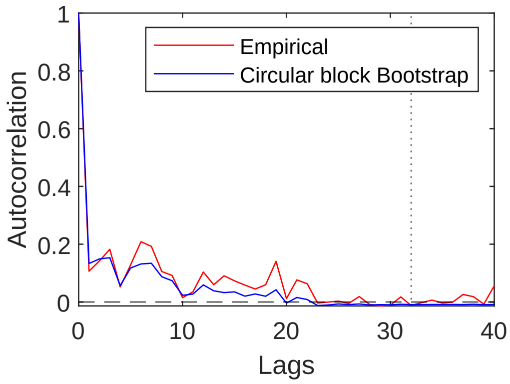
The simulation approach consists of analyzing the impact of the parameters over the payoff and the distribution of the different strategy configurations. We consider Monte-Carlo experiments based on the Circular Block Bootstrap method. This method is purely data driven and aims to the preserve the empirical properties of the asset returns. The data generation process uses the permutation of block of data as describe by the illustration 7 in the appendix. This technique provides an easy and efficient approach to simulate time series while preserving their dependencies. The choice of an adequate block size is crucial since only the temporal dependencies within a block are conserved. In our case, the block size is choosen proportionally to the size of initial sample to reproduce the observed hetoroscedasticity[7] as shown in the figure 2 below.



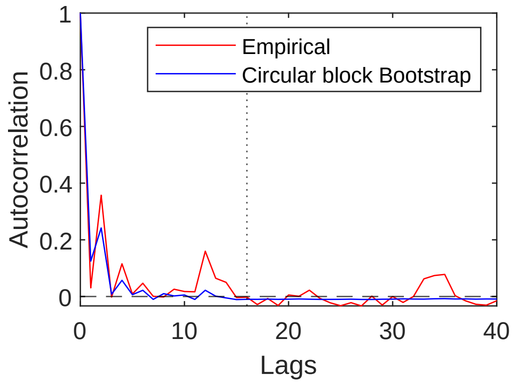
(a) SPX Index from 31/12/2003 to 31/01/2012



(b) SPX Index from 01/01/2018 to 01/01/2022



(c) SX5E Index from 31/12/2003 to 31/01/2012



(d) SX5E Index from 31/12/2018 to 01/01/2022

Figure 2: Representation of the estimated square returns auto-correlation for both simulations methods. The black dash-line corresponds to the size of a block for the circular block Bootstrap.

[7]Replication of the slow decay in the auto correlation of the square returns.

For Monte-Carlo experiments we generate 5000 returns trajectories that are transformed into price series using compound returns. The length of the simulated price series corresponds to the size of the observed sample. Then we use as risk free rates the annualized average returns of cash indexes[8], in the table below (2), associated to the same geographical area as the equity indexes.

Period		**SPX Index**	**SX5E Index**
31/12/2003	31/01/2012	2,05%	2,03%
01/01/2018	01/01/2022	1,09%	-0,42%

Table 2: Annualized average of the cash index daily returns over the specified time period. The annualization coefficient corresponds to 52 weeks.

Finally, the application of the strategy risk controls requires to estimate the conditional quantile of asset returns $F^{-1}_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\cdot)$ for each sample path. For simplicity purposes we use an empirical quantile estimated on a moving window corresponding of the size of the bootstrap block.

In the following part we show the results of the different strategy configurations, the hard and soft triggering event respectively associated with the level and variation risk controls as defined previously. The hard triggering event represents a breach of the current floor (every time the current cushion becomes negative) while the soft triggering is associated with a breach of the exposure level (every time the exposure goes below a given level). The tables 3 and 4 below, presents the different parameters used in the simulation process and the empirical application. The green row corresponds to the baseline parameters while the others are used to analyze the sensitivity of the strategies to these parameters. In other words, we obtain the sensitivity of the strategies to their parameters by varying the value of one parameter (see rows of the table) and keeping the other constant with the value in the green row. The second part of table corresponds to the standard floor configuration of the CPPI with margin and ratchet effects. For the margin strategies we consider that 7.5% of the global floor is dynamically adjusted and in case of a margin effect which does not fall under the risk control conditions the margin is reduced by 20%. Then for the ratchet effect, the floor is revisited upward every time the portfolio value exceeds the floor by a proportion based on the sign of $\theta^m_{t_k}(\cdot)$ and $\tilde{\theta}^m_{t_k}(\cdot)$.

Margin strategy parameters

Trigger x_{t_k}	**Multiple** m	**Quantile** ϵ	**Threshold** L_{t_k}/V_{t_k}	**Max exposure**	**Target floor**	**Margin** M_0/V_0	**Reduction** p_{t_k}
1.0%	3	0.25%	1.0%				
5.0%	6	1.0%	2.5%	100%	20%	7.5%	80%
10%	8	5.0%	5.0%				

Table 3: Parameters used in the simulation based analysis and the empirical application for both indexes and time period.

5.1.1 Margin effects

The overall results[9] show that soft triggering events produce more volatile strategies with a significantly lower skewness and kurtosis compare to the hard triggering events (see table 5, 6, 7 and 8). These results are expected since the soft triggering mechanism induces relatively more trades as shown by the margin consumption in every market configuration (cf last row of the tables below).

[8] The Bloomberg tickers of the indexes are BXIIBUS0 Index and BXIIBEU0 Index.
[9] Independent from the considered index or time period.

Ratchet strategy parameters

Upward revision p_{t_k}	Multiple m	Quantile ϵ	Target floor
90%	3	0.25%	
85%	6	1.0%	20%
80%	8	5.0%	

Table 4: Parameters used in the simulation based analysis and the empirical application for both indexes and time period.

This higher trading activity results on one hand in an higher volatility and on the other hand in a smoother transition between pre and post monetization or cash lock events. Particularly the change in the strategy returns amplitude is less sudden reducing the skewness and the kurtosis compare to the hard triggering mechanism.

The analysis of the different strategy configurations yields in mixed results based on the price behavior following a drawdown event: (i) a fast recovery represented by the period 2018-2022 for the S&P 500 index (see table 7), (ii) a slower recovery associated to the S&P 500 index (see table 5) and the Eurostoxx 50 (see table 8), respectively, over the period 2004-2012 and 2018-2022 and (iii) a no recovery defined by the Eurostoxx 50 index (see table 6) over the period 2004-2012.

First, the difference in performance between triggering types suggests that soft triggering events provide on average better performances compare to hard triggering events for simulated scenarios based on a recovery period. However, the performance improvement is significantly greater for the period with a fast recovery (cf. table 7) except for low multiple values ($m = 3$)[10]. Indeed, for the simulations based on this period, the total return is about 5% better, the annualized average return is at least 1.5% higher and the omega ratio is about 0.1 point higher than the ones of the strategy using hard triggering events. These improvements are followed by a small increase of volatility of about 1% while the skewness and the kurtosis are significantly lower. Conversely, for the period associated to a slower recovery the performance increase is less notable. For instance, based on the simulation obtained from the Eurostoxx 50 index over the period 2018-2022, soft triggering events slightly improve the total returns by about more than 1%, the average returns by 0.3% and the omega ratio by at least 0.04 for less than 1 point of volatility increase.

Then in the case of simulations generated from a non recovery scenario (cf. table 6) the performance of the strategies using soft triggering mechanism are degraded. Such performance loss is due to the greater trading activity of these strategies. By construction these strategies increase more often their exposure into the risky asset in period of downward markets, characterized by an important consumption of the margin, to benefit as soon as possible from a potential recovery. Thus when the recovery is not materializing this extra risk is not rewarded and yields to greater losses.

Second, when focusing on the risk control style we note that the variation risk control is more aggressive than the level risk control. The consumption of the margin appears to be significantly greater in all market configuration. Indeed based on the equation (21) and (22), the variation risk control is solicited more easily since the condition $\tilde{\theta}^m_{t_k}(\epsilon) < 0$ is less restrictive than the condition $\theta^m_{t_k}(\epsilon) < 0$ of the level risk control. From a performance point of view, this increase in reactivity yields to the same results as above for simulations generated from period with a fast and without a recovery while the results are very mixed for the simulations associated to periods with a slower recovery.

As previously indicated, in the case of fast and non recovery, the higher activity allows either to

[10]For lower multiple the risk of breaching the floor can be very low resulting in no performance differences between the hard and soft triggering mechanisms.

drastically increase or severely deteriorated the global performance of the strategies. For the fast recovery scenario (cf. table 7), the variation risk control provides better performance increase than the level risk control, independently from the strategy configuration. On the other hand, for the non recovery scenario (cf. table 6) the variation risk control also generates the worst performance regarding the strategy configuration.

For scenario based on a slower recovery, the results are mixed regarding the index. For instance, based on the S&P 500 index over the period 2004-2012, there is no obvious gains in performance between the level and the variation risk control. First, we note that when considering riskier or less conservative parameters (right part of the table 5) the risk variation degrades the performance in every configuration. Second when parameters are more conservative the performance differences tend to disappear for the strategies using soft triggering events with the variation or the level risk control. However for the strategies using the hard triggering mechanism the variation risk control reduces globally the performance compares to the level risk control. Conversely, for the Eurostoxx 50 index on the period 2018-2022 (see table 8) the use of the variation risk control provides better performance in every configuration. Both period even if labeled under a slow recovery are very different. The drawdown and recovery behavior over the period 2018-2022 for the Eurostoxx 50 index are indeed much faster than the ones of the S&P 500 index over the period 2004-2012. The variation risk control is more reactive than the level risk control thus for a period with a stronger recovery it allows to benefit more from it but when the recovery is less pronounced this reactivity tends to add supplementary risks which are not compensated by the realized gains.

Third, the sensitivity of the strategies to their parameters are difficult to interpret overall from the tables. There is no clear evidence of a relationship between the performance of the strategies and the parameter values. However, in general considering less conservative parameters tend to increase the global risk of the strategies since it implies an higher trading activity, i.e. the margin is decreased. The gain in performance for the extra risk depends essentially on the considered case. As suggested previously, an higher trading activity accentuate the behavior of the sample used for the simulations. For instance, for the simulations based on the Eurostoxx 50 index over the period 2004-2012 the performance is decreased as parameter values increase while for the S&P 500 index we obtain the opposite results.

Since by construction the distribution of these strategies are truncated the use of averaged statistic provide only a limited interpretation of their distributions. Thus, we analyze the estimated cumulative distribution function of the strategies total return (see appendix B.1) to assess the impact of the parameters on the distribution of the payoff. This approach allows to obtain more detailed information of the strategies performance based on the value of a parameter. From this analysis, we see that the quantile level and threshold parameters have a very little impact on the total return distributions while the trigger level and the multiple change significantly these distributions. However, we note that strategies using the variation risk control are more sensitive to a change in the threshold parameter. This result comes from the fact that the variation risk control is more often used than the level risk control. Thus the threshold which regulates the magnitude of the margin decrease plays a more active role.

Overall the change in the distribution due to an increase of a parameter is twofold: it augments the probability of ending with a negative total returns but also the probability of having a larger positive returns. However, this increase is not linear and depends mostly on the value of the parameter. For instance, in most cases going from the mid to the highest parameter value yields to increase more significantly the negative side than the positive one. In other words, the gain in probability of having a larger positive returns is marginal compare to the gain in probability of ending with a negative returns.

Threshold	1,00%				2,50%				5,00%			
Trigger	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**
Risk control	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**
Total return	17,13%	18,13%	18,70%	19,78%	16,42%	17,33%	18,09%	19,80%	16,42%	17,33%	17,30%	19,58%
Avg. return (an.)	2,01%	2,14%	2,22%	2,35%	1,92%	2,03%	2,14%	2,35%	1,92%	2,03%	2,03%	2,31%
Volatility (an.)	8,82%	9,41%	8,46%	9,38%	9,14%	9,41%	8,56%	9,89%	9,14%	9,41%	8,77%	10,12%
Skewness	-1,83	-1,58	-2,29	-1,57	-1,69	-1,59	-2,12	-1,57	-1,69	-1,59	-1,87	-1,76
Kurtosis	20,15	16,74	26,52	16,72	19,10	17,42	24,05	16,75	19,10	17,42	20,69	19,78
Omega ratio	1,06	1,06	1,09	1,08	1,05	1,05	1,07	1,07	1,05	1,05	1,06	1,06
Margin	5,27%	3,90%	8,16%	4,77%	4,17%	3,56%	7,04%	3,52%	4,17%	3,56%	5,56%	2,89%

(a) Sensitivity to the threshold parameter.

Multiple	3				6				8			
Trigger	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**
Risk control	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**
Total return	18,27%	18,48%	18,27%	18,73%	16,42%	17,33%	18,09%	19,80%	16,89%	18,75%	18,06%	20,53%
Avg. return (an.)	2,19%	2,21%	2,19%	2,24%	1,92%	2,03%	2,14%	2,35%	1,98%	2,21%	2,14%	2,42%
Volatility (an.)	6,23%	6,47%	6,23%	6,74%	9,14%	9,41%	8,56%	9,89%	9,15%	10,28%	8,91%	10,71%
Skewness	-2,04	-1,53	-2,04	-1,39	-1,69	-1,59	-2,12	-1,57	-2,28	-1,74	-2,55	-1,80
Kurtosis	21,57	14,75	21,57	13,16	19,10	17,42	24,05	16,75	27,91	20,84	32,62	20,65
Omega ratio	1,14	1,12	1,14	1,12	1,05	1,05	1,07	1,07	1,05	1,05	1,07	1,05
Margin	9,37%	7,36%	9,37%	6,13%	4,17%	3,56%	7,04%	3,52%	5,58%	3,38%	7,32%	2,97%

(b) Sensitivity to the multiple parameter

Trigger level	1,00%				5,00%				10,0%			
Trigger	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**
Risk control	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**
Total return	16,42%	16,67%	18,09%	19,94%	16,42%	17,33%	18,09%	19,80%	16,42%	18,07%	18,09%	20,26%
Avg. return (an.)	1,92%	1,95%	2,14%	2,37%	1,92%	2,03%	2,14%	2,35%	1,92%	2,12%	2,14%	2,39%
Volatility (an.)	9,14%	9,20%	8,56%	9,40%	9,14%	9,41%	8,56%	9,89%	9,14%	9,71%	8,56%	10,23%
Skewness	-1,69	-1,65	-2,12	-1,62	-1,69	-1,59	-2,12	-1,57	-1,69	-1,58	-2,12	-1,57
Kurtosis	19,10	18,49	24,05	17,40	19,10	17,42	24,05	16,75	19,10	17,13	24,05	16,65
Omega ratio	1,05	1,05	1,07	1,08	1,05	1,05	1,07	1,07	1,05	1,05	1,07	1,06
Margin	4,17%	4,00%	7,04%	5,06%	4,17%	3,56%	7,04%	3,52%	4,17%	3,10%	7,04%	2,83%

(c) Sensitivity to the trigger parameter.

Quantile level	0,25%				1,00%				5,00%			
Trigger	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**
Risk control	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**
Total return	16,42%	17,33%	18,09%	19,80%	16,42%	17,33%	18,09%	19,80%	18,40%	19,26%	17,22%	19,09%
Avg. return (an.)	1,92%	2,03%	2,14%	2,35%	1,92%	2,03%	2,14%	2,35%	2,18%	2,29%	2,02%	2,25%
Volatility (an.)	9,14%	9,41%	8,56%	9,89%	9,14%	9,41%	8,56%	9,89%	8,51%	9,32%	8,79%	9,97%
Skewness	-1,69	-1,59	-2,12	-1,57	-1,69	-1,59	-2,12	-1,57	-2,21	-1,59	-1,85	-1,76
Kurtosis	19,10	17,42	24,05	16,75	19,10	17,42	24,05	16,75	25,45	16,90	20,40	19,73
Omega ratio	1,05	1,05	1,07	1,07	1,05	1,05	1,07	1,07	1,08	1,07	1,06	1,05
Margin	4,17%	3,56%	7,04%	3,52%	4,17%	3,56%	7,04%	3,52%	7,64%	4,70%	5,43%	3,08%

(d) Sensitivity to the quantile parameter.

Table 5: Bootstrap statistics based on the SPX index for the period 2004-2012.

Threshold	1,00%				2,50%				5,00%			
Trigger	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**
Risk control	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**
Total return	5,96%	4,70%	6,83%	5,20%	4,67%	4,20%	6,16%	4,46%	3,81%	4,00%	5,25%	4,13%
Avg. return (an.)	0,94%	0,79%	1,05%	0,85%	0,77%	0,72%	0,96%	0,77%	0,66%	0,70%	0,84%	0,74%
Volatility (an.)	7,65%	8,69%	7,57%	8,76%	7,87%	8,72%	7,62%	9,25%	8,14%	8,74%	7,75%	9,56%
Skewness	-3,73	-2,91	-3,91	-2,89	-3,45	-2,95	-3,77	-2,99	-3,43	-2,98	-3,56	-3,00
Kurtosis	50,16	37,03	53,47	36,25	44,72	37,07	51,00	38,08	44,03	37,35	47,00	38,14
Omega ratio	1,02	0,99	1,04	1,00	1,00	0,99	1,03	0,99	0,99	0,98	1,01	0,98
Margin	6,63%	2,50%	7,76%	2,60%	4,98%	2,27%	6,89%	1,76%	3,97%	2,20%	5,71%	1,39%

(a) Sensitivity to the threshold parameter.

Multiple	3				6				8			
Trigger	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**
Risk control	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**
Total return	8,65%	7,71%	8,65%	7,21%	4,67%	4,20%	6,16%	4,46%	4,60%	3,95%	5,71%	3,78%
Avg. return (an.)	1,19%	1,07%	1,19%	1,01%	0,77%	0,72%	0,96%	0,77%	0,79%	0,73%	0,93%	0,71%
Volatility (an.)	5,63%	6,02%	5,63%	6,45%	7,87%	8,72%	7,62%	9,25%	8,52%	9,65%	8,19%	9,84%
Skewness	-3,27	-2,34	-3,27	-2,21	-3,45	-2,95	-3,77	-2,99	-3,78	-3,16	-4,13	-3,26
Kurtosis	41,98	25,64	41,98	23,37	44,72	37,07	51,00	38,08	54,29	42,55	59,51	42,94
Omega ratio	1,09	1,05	1,09	1,04	1,00	0,99	1,03	0,99	1,01	0,98	1,02	0,97
Margin	9,37%	5,44%	9,37%	3,72%	4,98%	2,27%	6,89%	1,76%	5,39%	2,23%	6,75%	1,95%

(b) Sensitivity to the multiple parameter

Trigger level	1,00%				5,00%				10,0%			
Trigger	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**
Risk control	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**
Total return	4,67%	4,54%	6,16%	4,98%	4,67%	4,20%	6,16%	4,46%	4,67%	4,12%	6,16%	3,98%
Avg. return (an.)	0,77%	0,75%	0,96%	0,82%	0,77%	0,72%	0,96%	0,77%	0,77%	0,72%	0,96%	0,72%
Volatility (an.)	7,87%	8,26%	7,62%	8,64%	7,87%	8,72%	7,62%	9,25%	7,87%	9,10%	7,62%	9,45%
Skewness	-3,45	-3,14	-3,77	-3,10	-3,45	-2,95	-3,77	-2,99	-3,45	-2,89	-3,77	-3,05
Kurtosis	44,72	39,94	51,00	39,88	44,72	37,07	51,00	38,08	44,72	36,17	51,00	38,55
Omega ratio	1,00	0,99	1,03	1,00	1,00	0,99	1,03	0,99	1,00	0,98	1,03	0,98
Margin	4,98%	3,38%	6,89%	2,80%	4,98%	2,27%	6,89%	1,76%	4,98%	1,72%	6,89%	1,48%

(c) Sensitivity to the trigger parameter.

Quantile level	0,25%				1,00%				5,00%			
Trigger	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**
Risk control	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**
Total return	4,67%	4,20%	6,16%	4,46%	4,67%	4,20%	6,16%	4,46%	6,44%	4,89%	4,14%	3,77%
Avg. return (an.)	0,77%	0,72%	0,96%	0,77%	0,77%	0,72%	0,96%	0,77%	1,00%	0,81%	0,70%	0,69%
Volatility (an.)	7,87%	8,72%	7,62%	9,25%	7,87%	8,72%	7,62%	9,25%	7,61%	8,69%	8,06%	9,39%
Skewness	-3,45	-2,95	-3,77	-2,99	-3,45	-2,95	-3,77	-2,99	-3,82	-2,91	-3,44	-3,05
Kurtosis	44,72	37,07	51,00	38,08	44,72	37,07	51,00	38,08	51,92	37,01	44,34	38,35
Omega ratio	1,00	0,99	1,03	0,99	1,00	0,99	1,03	0,99	1,03	1,00	1,00	0,98
Margin	4,98%	2,27%	6,89%	1,76%	4,98%	2,27%	6,89%	1,76%	7,24%	2,58%	4,33%	1,44%

(d) Sensitivity to the quantile parameter.

Table 6: Bootstrap statistics based on the SX5E index for the period 2004-2012.

Threshold	1,00%				2,50%				5,00%			
Trigger	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**
Risk control	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**
Total return	39,08%	44,96%	39,08%	43,01%	39,08%	44,96%	39,08%	47,38%	39,08%	44,96%	39,08%	51,58%
Avg. return (an.)	8,09%	9,62%	8,09%	9,14%	8,09%	9,62%	8,09%	10,23%	8,09%	9,62%	8,09%	11,18%
Volatility (an.)	13,27%	14,55%	13,27%	14,11%	13,27%	14,55%	13,27%	15,10%	13,27%	14,55%	13,27%	15,95%
Skewness	-2,82	-1,79	-2,82	-1,98	-2,82	-1,79	-2,82	-1,59	-2,82	-1,79	-2,82	-1,42
Kurtosis	27,01	13,80	27,01	15,76	27,01	13,80	27,01	11,87	27,01	13,80	27,01	10,57
Omega ratio	1,20	1,30	1,20	1,28	1,20	1,30	1,20	1,32	1,20	1,30	1,20	1,34
Margin	9,38%	8,02%	9,38%	8,35%	9,38%	8,02%	9,38%	7,40%	9,38%	8,02%	9,38%	6,39%

(a) Sensitivity to the threshold parameter.

Multiple	3				6				8			
Trigger	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**
Risk control	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**
Total return	31,30%	31,46%	31,30%	31,63%	39,08%	44,96%	39,08%	47,38%	57,17%	55,98%	50,30%	49,77%
Avg. return (an.)	7,06%	7,10%	7,06%	7,15%	8,09%	9,62%	8,09%	10,23%	12,26%	12,00%	10,82%	10,66%
Volatility (an.)	10,18%	10,21%	10,18%	10,25%	13,27%	14,55%	13,27%	15,10%	17,10%	16,99%	15,90%	15,98%
Skewness	-1,72	-1,69	-1,72	-1,66	-2,82	-1,79	-2,82	-1,59	-1,24	-1,29	-1,60	-1,58
Kurtosis	11,87	11,60	11,87	11,41	27,01	13,80	27,01	11,87	11,09	10,76	12,29	12,03
Omega ratio	1,30	1,31	1,30	1,31	1,20	1,30	1,20	1,32	1,35	1,34	1,32	1,32
Margin	9,38%	9,25%	9,38%	9,11%	9,38%	8,02%	9,38%	7,40%	5,61%	5,49%	7,47%	6,71%

(b) Sensitivity to the multiple parameter.

Trigger level	1,00%				5,00%				10,0%			
Trigger	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**
Risk control	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**
Total return	39,08%	43,69%	39,08%	45,62%	39,08%	44,96%	39,08%	47,38%	39,08%	45,22%	39,08%	47,24%
Avg. return (an.)	8,09%	9,31%	8,09%	9,78%	8,09%	9,62%	8,09%	10,23%	8,09%	9,69%	8,09%	10,18%
Volatility (an.)	13,27%	14,29%	13,27%	14,72%	13,27%	14,55%	13,27%	15,10%	13,27%	14,68%	13,27%	15,16%
Skewness	-2,82	-1,94	-2,82	-1,76	-2,82	-1,79	-2,82	-1,59	-2,82	-1,72	-2,82	-1,59
Kurtosis	27,01	15,49	27,01	13,63	27,01	13,80	27,01	11,87	27,01	13,02	27,01	11,73
Omega ratio	1,20	1,29	1,20	1,30	1,20	1,30	1,20	1,32	1,20	1,30	1,20	1,31
Margin	9,38%	8,44%	9,38%	7,90%	9,38%	8,02%	9,38%	7,40%	9,38%	7,53%	9,38%	7,01%

(c) Sensitivity to the trigger parameter.

Quantile level	0,25%				1,00%				5,00%			
Trigger	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**
Risk control	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**
Total return	39,08%	44,96%	39,08%	47,38%	39,08%	44,96%	39,08%	47,38%	39,08%	44,96%	39,08%	47,94%
Avg. return (an.)	8,09%	9,62%	8,09%	10,23%	8,09%	9,62%	8,09%	10,23%	8,09%	9,62%	8,09%	10,36%
Volatility (an.)	13,27%	14,55%	13,27%	15,10%	13,27%	14,55%	13,27%	15,10%	13,27%	14,55%	13,27%	15,22%
Skewness	-2,82	-1,79	-2,82	-1,59	-2,82	-1,79	-2,82	-1,59	-2,82	-1,79	-2,82	-1,56
Kurtosis	27,01	13,80	27,01	11,87	27,01	13,80	27,01	11,87	27,01	13,80	27,01	11,60
Omega ratio	1,20	1,30	1,20	1,32	1,20	1,30	1,20	1,32	1,20	1,30	1,20	1,32
Margin	9,38%	8,02%	9,38%	7,40%	9,38%	8,02%	9,38%	7,40%	9,38%	8,02%	9,38%	7,27%

(d) Sensitivity to the quantile parameter.

Table 7: Bootstrap statistics based on the S&P 500 index for the period 2018-2022.

Threshold	1,00%				2,50%				5,00%			
Trigger	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**
Risk control	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**
Total return	8,56%	8,37%	7,10%	7,32%	9,97%	9,88%	7,93%	7,66%	10,12%	10,01%	8,97%	8,39%
Avg. return (an.)	2,15%	2,09%	1,70%	1,77%	2,56%	2,54%	1,96%	1,87%	2,61%	2,57%	2,28%	2,09%
Volatility (an.)	13,55%	13,56%	12,63%	12,97%	14,66%	14,65%	13,09%	13,27%	14,76%	14,73%	13,84%	13,84%
Skewness	-2,19	-2,18	-2,97	-2,60	-1,90	-1,90	-2,48	-2,37	-1,88	-1,88	-2,06	-2,15
Kurtosis	20,45	20,31	30,17	25,19	17,88	17,76	23,78	22,27	17,81	17,69	19,10	19,71
Omega ratio	1,02	1,02	0,97	0,99	1,04	1,04	1,01	1,01	1,04	1,04	1,03	1,02
Margin	5,89%	5,63%	7,55%	6,25%	4,58%	4,48%	6,59%	5,71%	4,42%	4,35%	5,52%	5,10%

(a) Sensitivity to the threshold parameter.

Multiple	3				6				8			
Trigger	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**
Risk control	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**
Total return	4,02%	4,21%	4,02%	4,31%	9,97%	9,88%	7,93%	7,66%	5,86%	5,94%	7,00%	7,75%
Avg. return (an.)	1,00%	1,06%	1,00%	1,09%	2,56%	2,54%	1,96%	1,87%	1,21%	1,23%	1,59%	1,82%
Volatility (an.)	8,55%	8,62%	8,55%	8,71%	14,66%	14,65%	13,09%	13,27%	14,33%	14,38%	13,86%	14,26%
Skewness	-2,46	-2,31	-2,46	-2,22	-1,90	-1,90	-2,48	-2,37	-3,04	-3,01	-2,78	-2,58
Kurtosis	20,66	19,15	20,66	18,25	17,88	17,76	23,78	22,27	29,42	29,02	27,68	25,32
Omega ratio	0,98	0,99	0,98	1,00	1,04	1,04	1,01	1,01	0,92	0,92	0,96	0,98
Margin	9,37%	8,77%	9,37%	8,34%	4,58%	4,48%	6,59%	5,71%	5,30%	5,16%	6,09%	5,45%

(b) Sensitivity to the multiple parameter.

Trigger level	1,00%				5,00%				10,0%			
Trigger	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**
Risk control	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**
Total return	9,97%	9,96%	7,93%	7,88%	9,97%	9,88%	7,93%	7,66%	9,97%	9,76%	7,93%	7,96%
Avg. return (an.)	2,56%	2,56%	1,96%	1,94%	2,56%	2,54%	1,96%	1,87%	2,56%	2,50%	1,96%	1,96%
Volatility (an.)	14,66%	14,66%	13,09%	13,10%	14,66%	14,65%	13,09%	13,27%	14,66%	14,64%	13,09%	13,65%
Skewness	-1,90	-1,90	-2,48	-2,47	-1,90	-1,90	-2,48	-2,37	-1,90	-1,89	-2,48	-2,23
Kurtosis	17,88	17,88	23,78	23,58	17,88	17,76	23,78	22,27	17,88	17,48	23,78	20,61
Omega ratio	1,04	1,04	1,01	1,01	1,04	1,04	1,01	1,01	1,04	1,04	1,01	1,01
Margin	4,58%	4,57%	6,59%	6,50%	4,58%	4,48%	6,59%	5,71%	4,58%	4,33%	6,59%	4,95%

(c) Sensitivity to the trigger parameter.

Quantile level	0,25%				1,00%				5,00%			
Trigger	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**
Risk control	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**
Total return	9,97%	9,88%	7,93%	7,66%	9,97%	9,88%	7,93%	7,66%	10,04%	9,91%	8,12%	7,72%
Avg. return (an.)	2,56%	2,54%	1,96%	1,87%	2,56%	2,54%	1,96%	1,87%	2,58%	2,54%	2,02%	1,88%
Volatility (an.)	14,66%	14,65%	13,09%	13,27%	14,66%	14,65%	13,09%	13,27%	14,69%	14,66%	13,22%	13,36%
Skewness	-1,90	-1,90	-2,48	-2,37	-1,90	-1,90	-2,48	-2,37	-1,95	-1,93	-2,38	-2,32
Kurtosis	17,88	17,76	23,78	22,27	17,88	17,76	23,78	22,27	18,71	18,32	22,54	21,63
Omega ratio	1,04	1,04	1,01	1,01	1,04	1,04	1,01	1,01	1,04	1,04	1,01	1,01
Margin	4,58%	4,48%	6,59%	5,71%	4,58%	4,48%	6,59%	5,71%	4,47%	4,39%	6,39%	5,58%

(d) Sensitivity to the quantile parameter.

Table 8: Bootstrap statistics based on the SX5E index for the period 2018-2022.

Finally, the analysis of the strategies payoff profile (for the parameters in the green rows in the table 3 and 4) provides additional interesting results. The payoff profile in the figure 3 represents the relationship between the percentile of the total returns of the simulated price path and the percentile of the total returns of the strategies. From this figure, we note that the use of soft triggering mechanisms provide better participation to upward price scenarios compare to the hard triggering strategies. For instance, the payoff of hard triggering strategies converge toward soft triggering strategies payoff only for very high underlyer's total returns. However, this better upward participation comes at an higher cost. Indeed, when the underlyer shows negative total returns the strategies using soft triggering events provide greater losses. Applying the same analysis over the risk control allows to show that when combined to soft triggering events the strategies provides better upward participation but also at an higher cost. On the other hand when associated to the hard triggering events there are almost no differences in the payoff profiles for the different risk controls.

The analysis of the cumulative distribution function of the strategies total returns reinforce the previous results. The introduction of soft triggering events increase the probability of reaching negative total returns but provides greater probability to reach higher positive returns in all configuration compare to hard triggering events. The same results are obtain for the use of the variation risk control regarding the level risk control. The intensity of these results depends mainly on the price scenario used for the simulations. For instance, for scenarios generated from the S&P 500 index over the period 2018-2022 the use of soft triggering events greatly increase the probability of reaching higher positive returns while the probability of ending in negative territory is slightly raised. Conversely, when considering scenario with less defined upward trend as for the Eurostoxx 50 index over the period 2018-2022 differences between strategies are less distinct.

[INSERT CONCLUSION]

5.2 Empirical application

In this part we apply the previous strategy configurations (green rows in the tables 3 and 4) over the two indexes. The empirical application allows to assess the strategies under real market conditions. In our application, we assume that we can directly trade the indexes, the quantities bought or sold are integers and fees of 0.5 basis points of the traded amount are applied. Then for simplicity purpose the return quantile is estimated using a rolling empirical quantile based on a 2 years moving window.

5.2.1 Margin effect

The figure 5 illustrates the strategy values with respect to their floor process. As we can see hard triggering events compare to soft triggering mechanism (see figures 5a and 5c) are more likely to be locked to their floor without breaching them. This behavior reduces drastically the ability of floor based strategies to benefit from market recovery. Additionally, the introduction of the variation risk control provides greater floor reduction which allows to participate more quickly to upward markets. However, this greater reduction introduces more risk. Indeed, the market recovery might not materialized and also a greater part of the margin is consumed limiting the ability to support other downside periods. For instance, the figure 5b shows that the use of the risk variation control combined with soft margin provides a relatively good upward participation from the 2009-2011 recovery. However, the gains generated during this period are immediately offset for the period 2011-2012 and the strategy value only slightly differs from the other configuration and the margin is significantly consumed.

The performance analysis highlights these behavior, for every time period with a drawdown recovery the use of soft triggering events allows to participate to the recovery (see figure 6a, 6b and 6d). However, this participation does not always yields to better performances. It only indicates that these strategies are more reactive to price variation. For instance, for the S&P 500 index over the period 2004-2012 (table 9a) the performance gains of the strategies using soft triggering events with a level risk control does not generates better performances than the strategies using the hard triggering events.

The choice of the risk control impacts significantly the performance for periods with a more pronounced recovery (see the period 2018-2022 for tables 9a and 9b). Since the variation risk control implies greater floor reductions it also provides a greater gain in the underlyer exposure and thus a better upward participation. Reciprocally, the gain in exposure for period without recovery (see the period 2004-2012 for the table 9b) conducts to lower the performances.

Period	31/12/2003 - 31/01/2012				01/01/2018 - 01/01/2022			
Trigger	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**
Risk control	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**
Total return	2,72%	2,38%	2,34%	4,12%	-3,72%	39,11%	-3,72%	55,18%
Avg, return (an.)	0,76%	0,76%	0,73%	1,16%	-0,46%	9,14%	-0,46%	12,06%
Volatility (an.)	9,19%	9,70%	9,43%	11,47%	9,63%	13,46%	9,63%	14,86%
Skewness	-0,99	-0,88	-0,93	-0,77	-3,62	-1,40	-3,62	-1,10
Kurtosis	7,23	6,07	6,65	5,75	25,71	8,88	25,71	7,50
Omega ratio	1,04	1,03	1,03	1,04	0,97	1,31	0,97	1,38
Margin	7,50%	6,00%	6,27%	4,20%	9,38%	7,50%	9,38%	6,20%

(a) S&P 500 index

Period	31/12/2003 - 31/01/2012				01/01/2018 - 01/01/2022			
Trigger	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**	**Hard**	**Soft**
Risk control	**Level**	**Level**	**Var.**	**Var.**	**Level**	**Level**	**Var.**	**Var.**
Total return	-5,54%	-5,60%	-4,70%	-5,19%	21,66%	21,66%	-5,43%	-5,43%
Avg, return (an.)	0,30%	0,32%	0,38%	0,32%	5,82%	5,82%	-1,05%	-1,05%
Volatility (an.)	14,01%	14,25%	13,82%	13,89%	13,69%	13,69%	8,20%	8,20%
Skewness	-1,26	-1,20	-1,30	-1,28	-0,40	-0,40	-2,32	-2,32
Kurtosis	11,05	10,34	11,63	11,39	7,24	7,24	13,36	13,36
Omega ratio	1,01	1,01	1,01	1,01	1,20	1,20	0,95	0,95
Margin	0,00%	0,00%	0,48%	0,00%	0,00%	0,00%	6,63%	6,63%

(b) Eurostoxx 50 index

Table 9: Performance table of the strategies applied to the S&P 500 and Eurostoxx 50 index. The backtest includes trading fees of 0.5 basis points.

References

[1] Rama Cont. Empirical properties of asset returns: Stylized facts and statistical issues. *Quantitative Finance*, 1:223–236, 2001.

[2] HE Leland and M Rubinstein. The evolution of portfolio insurance. 1976.

[3] Hans Follmer and Peter Leukert. Quantile hedging. *Finance and Stochastics*, 3:251–273, 5 1999.

[4] Hans Föllmer and Peter Leukert. Efficient hedging: Cost versus shortfall risk. *Finance and Stochastics*, 4:117–146, 2 2000.

[5] M Strassberger. Capital requirement, portfolio risk insurance, and dynamic risk budgeting. *papers.ssrn.com*, 3:78, 2006.

[6] Peter Carr, Hongzhong Zhang, and Olympia Hadjiliadis. Maximum drawdown insurance. *International Journal of Theoretical and Applied Finance*, 14:1195–1230, 12 2011.

[7] Maciej J. Capiński. Hedging conditional value at risk with options. *European Journal of Operational Research*, 242:688–691, 4 2015.

[8] Fischer Black and AndréF F. Perold. Theory of constant proportion portfolio insurance. *Journal of Economic Dynamics and Control*, 16:403–426, 1992.

[9] Rama Cont and Peter Tankov. Constant proportion portfolio insurance in the presence of jumps in asset prices. *Mathematical Finance*, 19:379–401, 7 2009.

[10] Alexei Chekhlov, Stanislav P. Uryasev, and Michael Zabarankin. Portfolio optimization with drawdown constraints. *SSRN Electronic Journal*, 4 2000.

[11] Alexei Chekhlov, Stanislav Uryasev, and Michael Zabarankin. Drawdown measure in portfolio optimization. *International Journal of Theoretical and Applied Finance*, 8:13–58, 2005.

[12] H. Ben Ameur and J. L. Prigent. Portfolio insurance: Gap risk under conditional multiples. *European Journal of Operational Research*, 236:238–253, 7 2014.

[13] Sanford J. Grossman and Zhongquan Zhou. Optimal investment strategies for controlling drawdowns. *Mathematical Finance*, 3:241–276, 1993.

[14] Jean-François Boulier and Arw Kanniganti. Expected performance and risks of various portfolio insurance strategies. *actuaries.org*, page 1093, 2005.

[15] H. Ben Ameur and J. L. Prigent. Risk management of time varying floors for dynamic portfolio insurance. *European Journal of Operational Research*, 269:363–381, 8 2018.

A Circular block bootstrap illustration

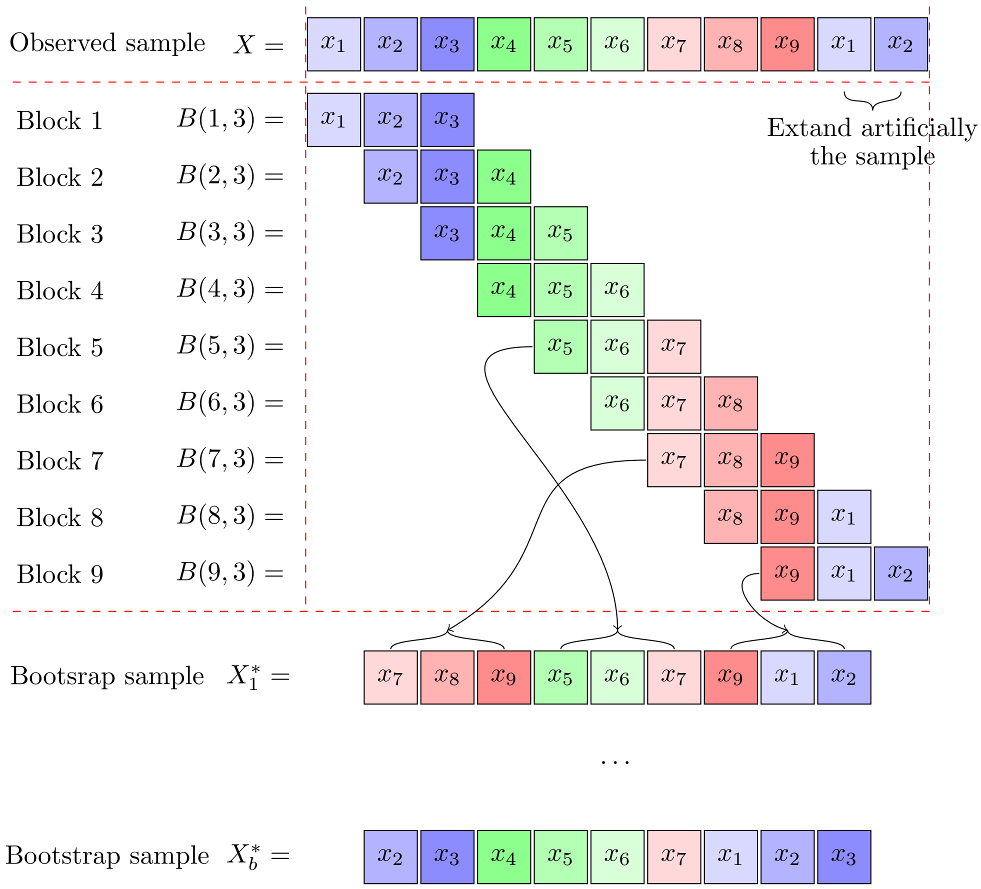


Figure 7: Representation of the circular block Bootstrap for a sample of 9 elements and a block size of 3 elements.

B Sensitivity analysis

B.1 CPPI with margin effects

Figures/Data introduction/spx_2003_2012.pdf

(a) SPX Index from 31/12/2003 to 31/01/2012

Figures/Data introduction/spx_2018_2022.pdf

(b) SPX Index from 01/01/2018 to 01/01/2022

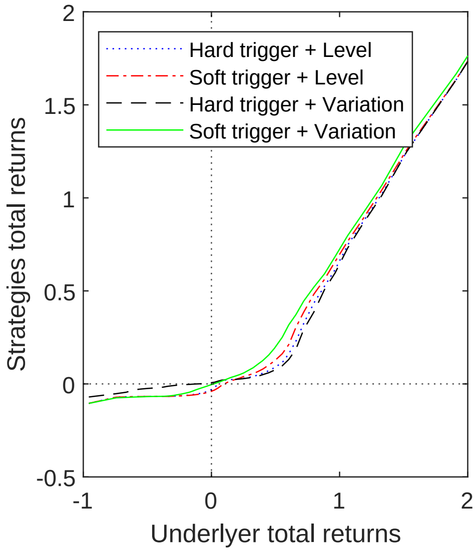
Figures/Data introduction/sx5e_2003_2012.pdf

(c) SX5E Index from 31/12/2003 to 31/01/2012

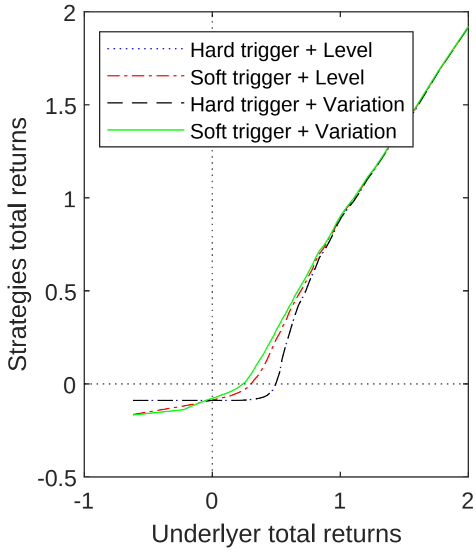
Figures/Data introduction/sx5e_2018_2022.pdf

(d) SX5E Index from 31/12/2018 to 31/01/2022

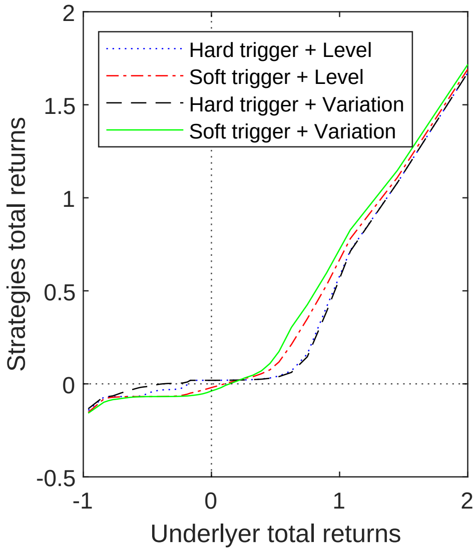
Figure 1: Price index of the S&P 500 Index and the Eurostoxx 50 Index over the two considered time period on a weekly basis.



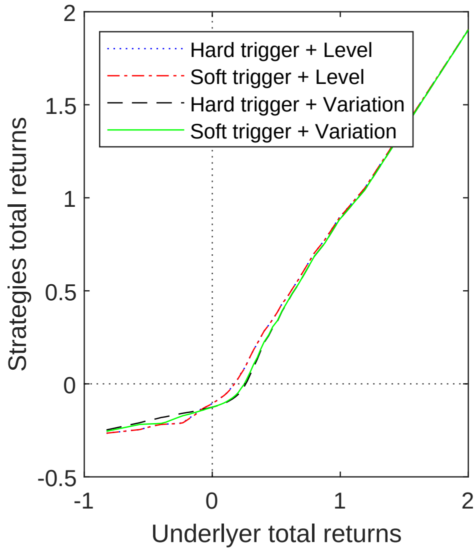
(a) SPX Index from 31/12/2003 to 31/01/2012



(b) SPX Index from 01/01/2018 to 01/01/2022

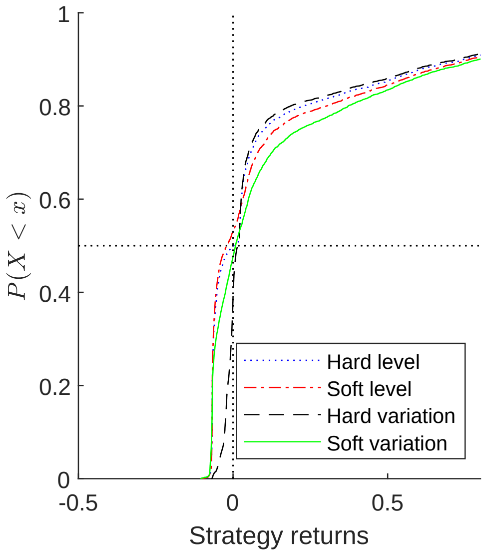


(c) SX5E Index from 31/12/2003 to 31/01/2012

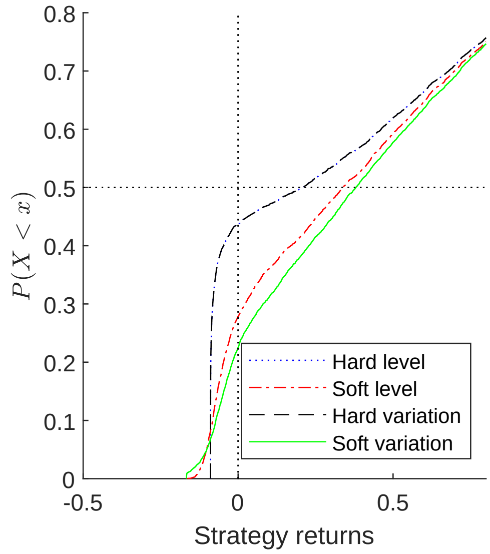


(d) SX5E Index from 31/12/2018 to 31/01/2022

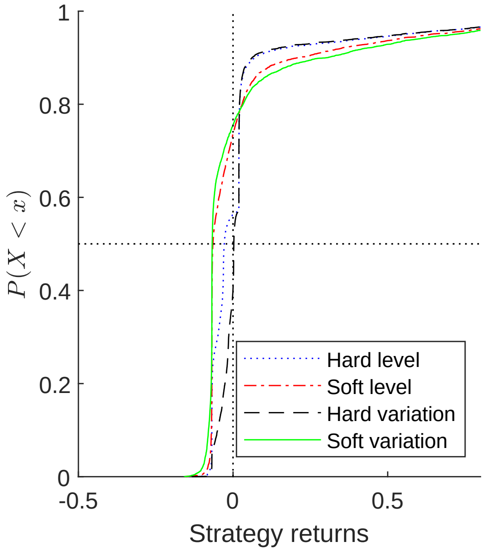
Figure 3: Representation of the estimated payoff profile of the different strategies over the different dataset. The payoff profile represents the relationship between the percentile of simulated price path total returns and the percentile of the strategies total returns. In this figure there 50 percentiles evenly ranging from the 0-th to the 100-th percentile.



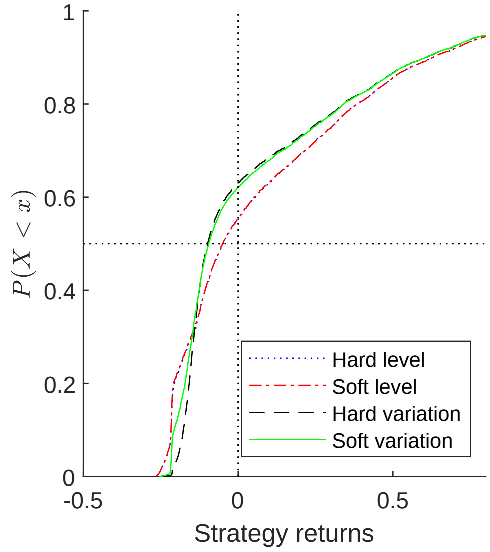
(a) SPX Index from 31/12/2003 to 31/01/2012



(b) SPX Index from 01/01/2018 to 01/01/2022

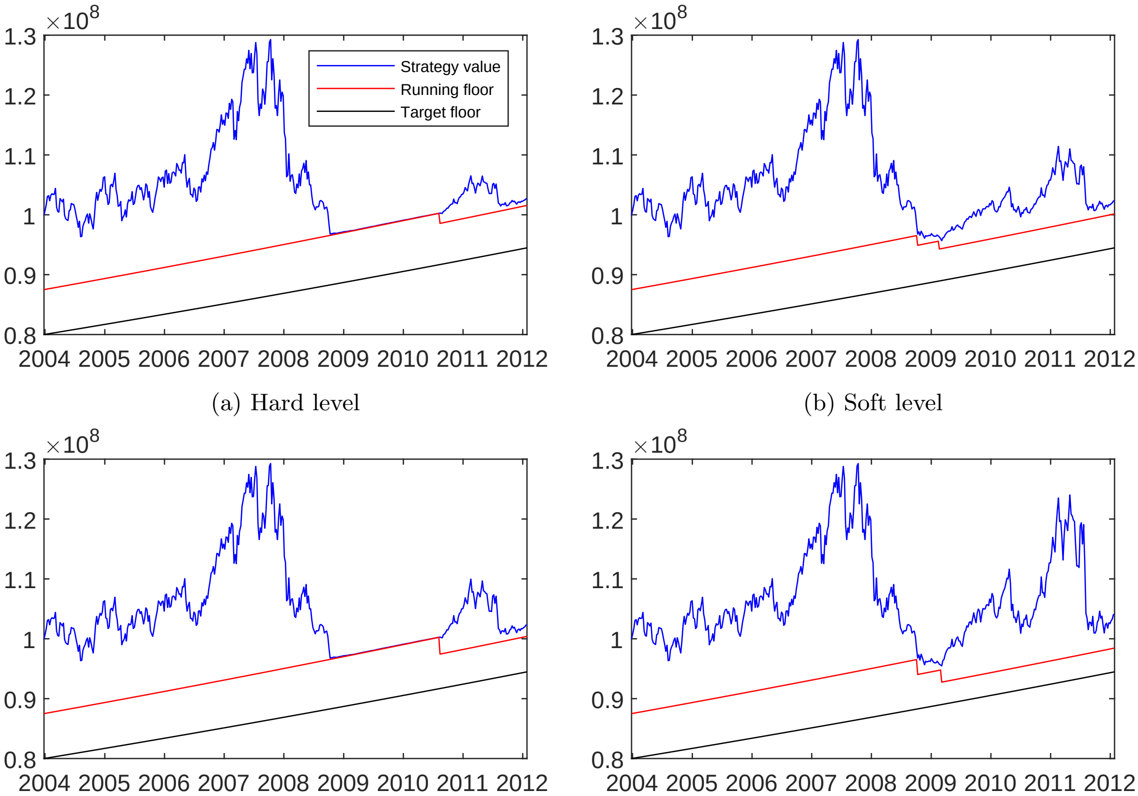


(c) SX5E Index from 31/12/2003 to 31/01/2012



(d) SX5E Index from 31/12/2018 to 31/01/2022

Figure 4: Representation of the estimated cumulative distribution function of the different strategies over the different dataset. The CDF is based on the total returns of the simulated strategies.



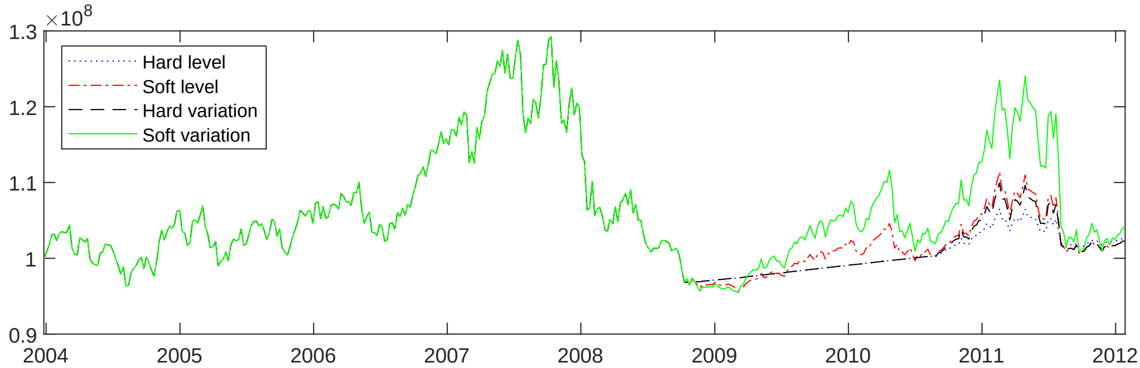
(a) Hard level

(b) Soft level

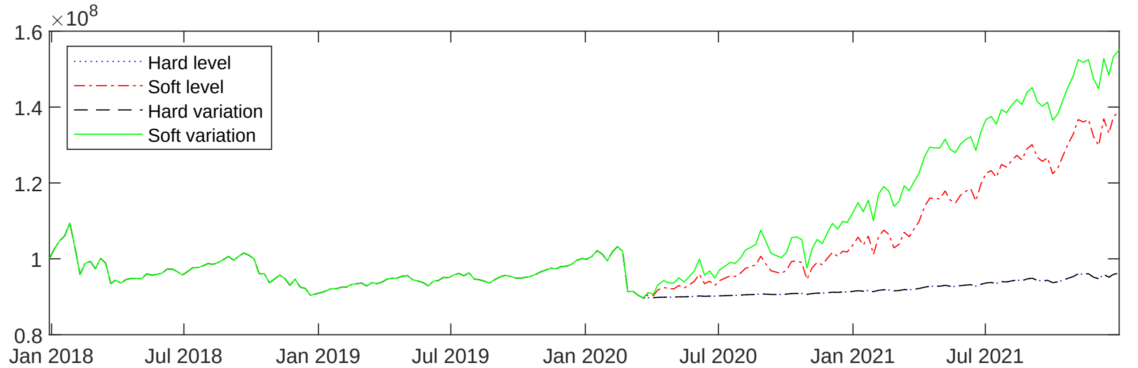
(c) Hard variation

(d) Soft variation

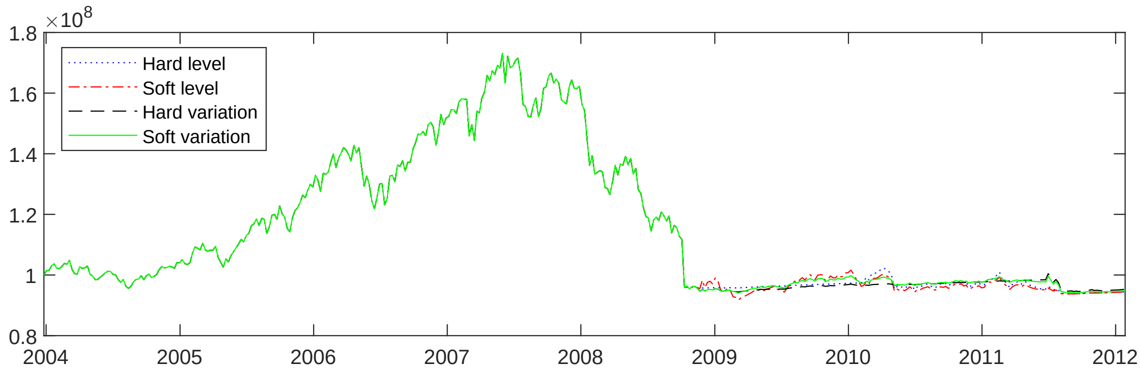
Figure 5: Representation of the CPPI with margin effects over the S&P 500 index from 31/12/2003 to 31/01/2012.



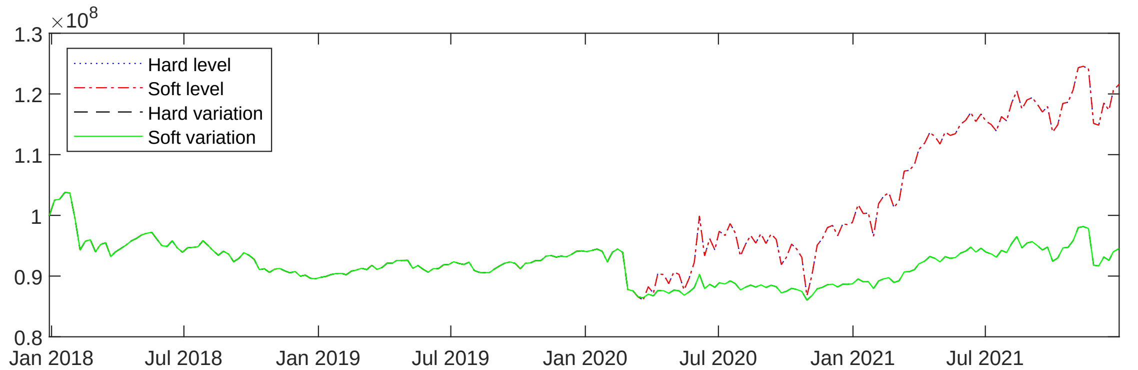
(a) S&P 500 index from 31/12/2003 to 31/01/2012



(b) S&P 500 index from 01/01/2018 to 01/01/2022



(c) Eurostoxx 50 index from 31/12/2003 to 31/01/2012



(d) Eurostoxx 50 index from 01/01/2018 to 01/01/2022

Figure 6: Representation of the different strategy configurations

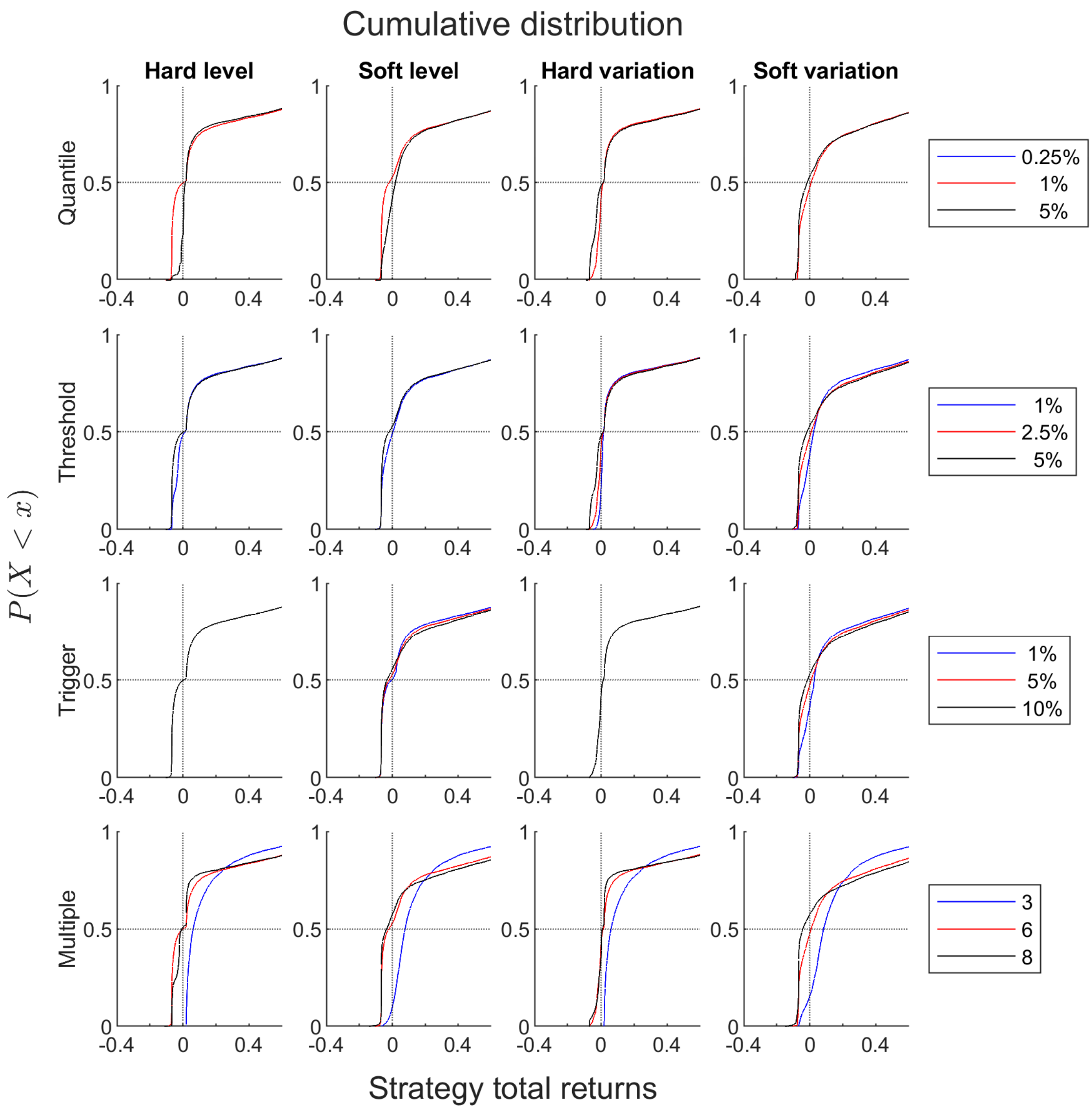


Figure 8: Representation of the estimated cumulative distribution function of the different strategies over the SPX Index from 31/12/2003 to 31/01/2012. The CDF is based on the total returns of the simulated strategies.

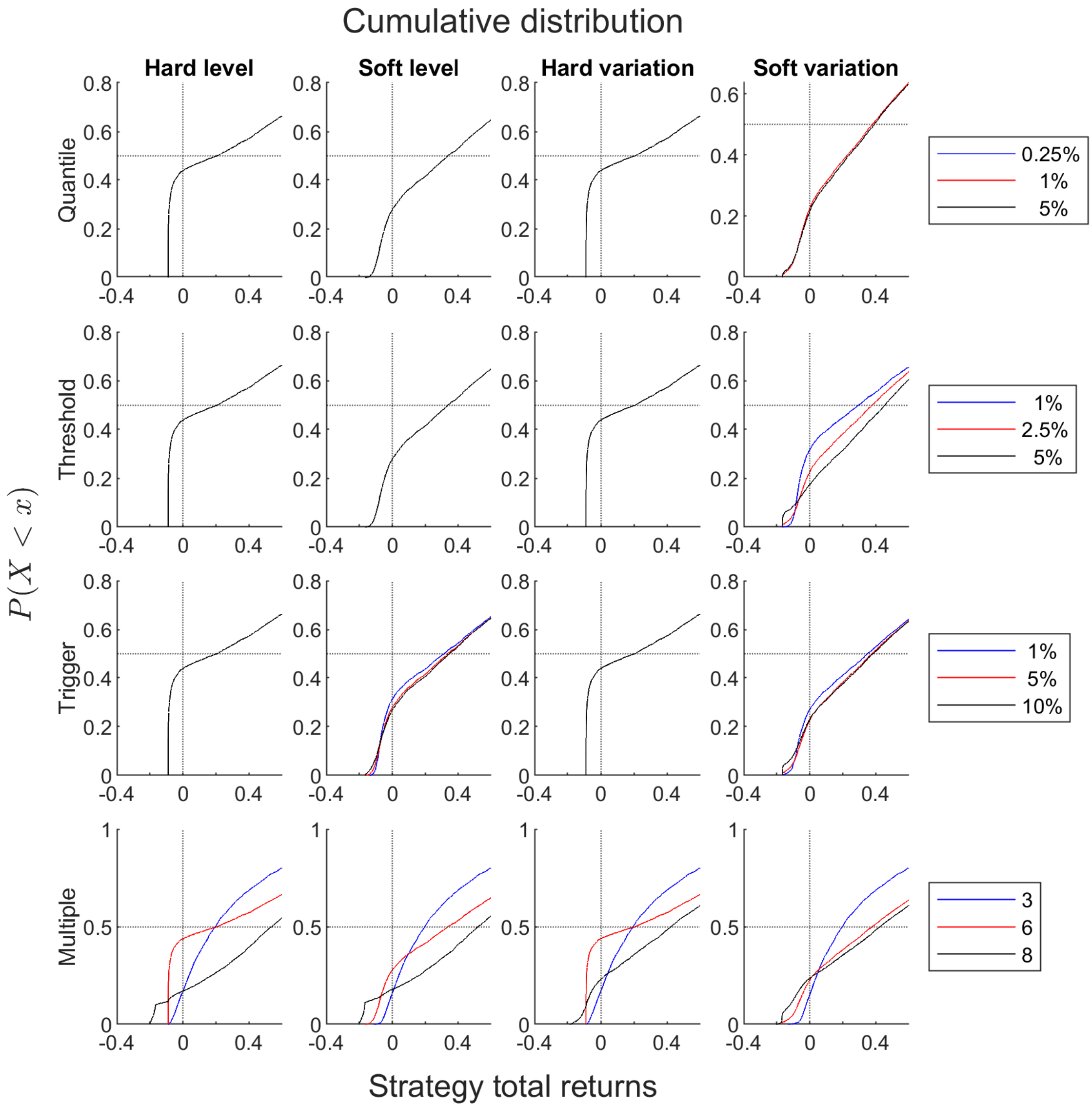


Figure 9: Representation of the estimated cumulative distribution function of the different strategies over the SPX Index from 01/01/2018 to 01/01/2022. The CDF is based on the total returns of the simulated strategies.

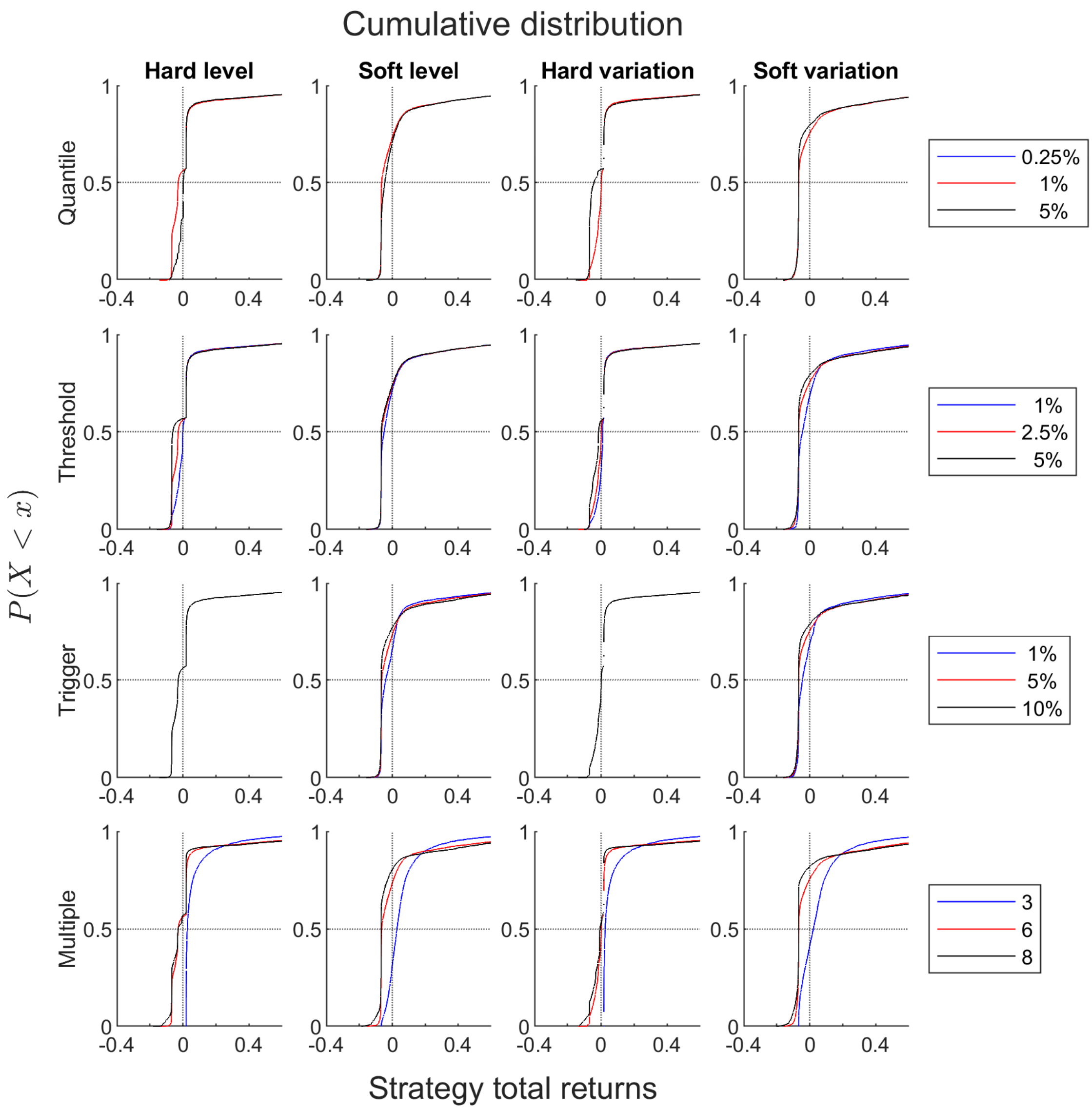


Figure 10: Representation of the estimated cumulative distribution function of the different strategies over the SX5E Index from 31/12/2004 to 31/01/2012. The CDF is based on the total returns of the simulated strategies.

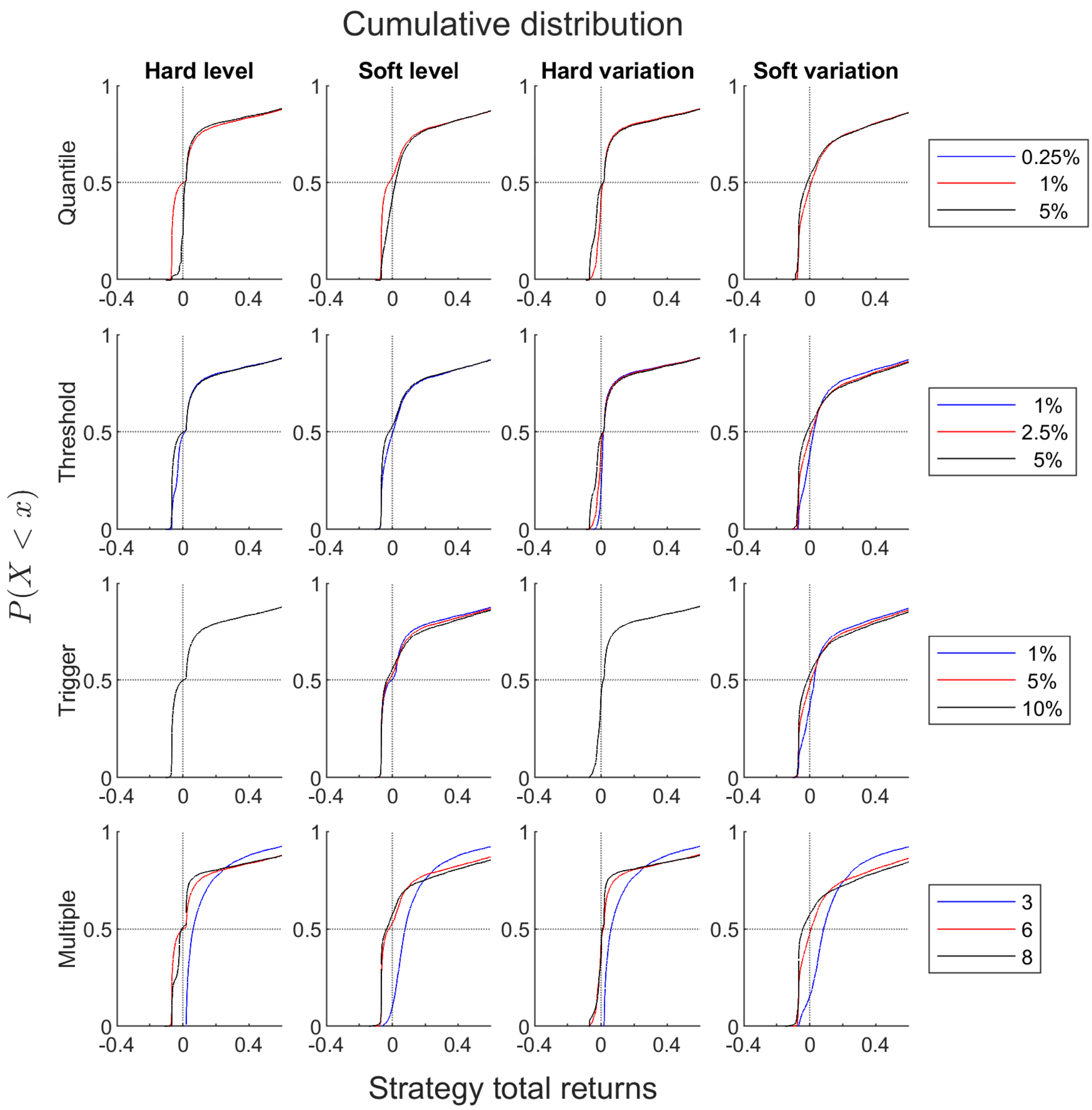


Figure 11: Representation of the estimated cumulative distribution function of the different strategies over the SX5E Index from 01/01/2018 to 01/01/2022. The CDF is based on the total returns of the simulated strategies.