THEMA théorie économique, modélisation et applications

THEMA Working Paper n°2023-18
CY Cergy Paris Université, France

Spatial multiproduct competition

Moez Kilani, André de Palma

September 2023

CY Cergy Paris Université

cnrs

Spatial multiproduct competition

Moez Kilani[a,b], André de Palma[c]

[a] *Univ. Littoral, Opal Cost, 220, av. de l'université, Dunkerque, F-59640, France*
[b] *CNRS, UMR 6060, BSE – Bordeaux School of Economics, Pessac, F-33600 Bordeaux, France*
[c] *CY Cergy Paris Université, CNRS UMR 8184, THEMA, 33, boulevard du Port, F-95000 Cergy, France*

Abstract

We analyze spatial competition on a circle between firms that have multiple outlets and face quadratic transport costs. The equilibrium is a two-stage Nash game: first, firms decide on their locations and then set their prices. We are able to solve analytically simple multi-outlet cases, but for the general case, we require an algorithm to enumerate all non-isomorphic configurations. While price equilibria are explicit and unique, solving the full two-stage game requires numerical methods. In the location game, we consider two scenarios: either firms cannot jump one outlet over a competitors' outlet, or firms have the flexibility to locate outlets anywhere on the circle. The solution involves a balance between cannibalization, market protection, and spatial monopoly power. We compare prices, profits, and transport costs for all possible configurations. With flexible locations, the firms' market areas are contiguous. In this case, surprisingly, each firm acts as a spatial monopoly. If regulations enforce that each firm must set the same price for its outlets, head-to-head competition prevails, leading to decreased profits for the firms but to a better-off situation for consumers.

Keywords: Spatial competition, circle, multi-product oligopoly, price-location equilibria, coin change problem.
JEL: L13, R32, R53

1. Introduction

Hotelling introduced explicitly space in economics in the seminal article "Stability in competition" (cf. Hotelling, 1929). He considers two firms located on a line (two ice-cream sellers on the beach), competing in locations and prices. Products sold at different geographical locations are spatially differentiated. Space matters, because consumers incur transport costs. Space also matters for firms, since a firm selling a product at a given location has some spatial monopoly power, which increases with higher transport costs. Firms face a trade-off: if a firm locates near consumers, it has better access to the market but it faces higher price competition; if it locates far away from the median consumer, it is far away from competitors and price competition is softer. Hotelling argues in this set-up that market attraction is stronger than the avoidance of price competition.

In the seminal Hotelling's article of spatial competition, the space is a segment (street market), transport costs are linear and incurred by the customers, and marginal production costs are constant and the same for both firms. He considers a two-stage location-then-price Nash equilibrium. This equilibrium illustrates the principle of minimum differentiation: both firms tend to converge towards the center of the segment. Exactly 50 years later, d'Aspremont et al. (1979) pointed out an error in Hotelling's mathematical reasoning. With linear transport costs, no price equilibrium exists, when the firms are too close to each other. They restored the two-stage equilibrium with quadratic transport costs, but in their setting, firms locate at the boundary of the segment: maximum spatial differentiation prevails. The principle of minimum differentiation was restored, with linear transport cost, by de Palma et al. (1985) and de Palma et al. (1986), when product differentiation (other than spatial differentiation) is large enough.

Many papers have extended the original Hotelling's model. Hotelling's framework can accommodate Cournot or Bertrand competition, different qualities, non-linear transport cost, sequential entry, entry deterrence, elastic demand, incomplete information, advertising and many other generalizations. Comprehensive surveys on these extensions can be found in Labbé et al. (1995), Eiselt and Marianov (2011) and Eiselt et al. (2019).

The space (the segment, the line or the circle, or multi-dimensional spaces) can represent almost anything. As a matter of fact, it is difficult to think of anything with no spatial representation. Space can denote a characteristic space, as well as time or as the left-right political spectrum, while customers are voters. Moreover, "firm" can be any organization, like a club or a church as suggested by Hotelling. Some aspatial models, such as the standard discrete product differentiation models, can also be given a representation in the characteristic space (see Anderson et al., 1989).

The space has not to be bounded and linear, and asymmetrical (like the segment). Salop (1979) introduced the circular space model, with linear transport costs, which was used far beyond location theory and in particular in industrial organization, political science and marketing. The solution around a circle of circumference L for n single product firms competing in location and then in price was derived by Economides (1989). He assumed quadratic transport costs $c\ x^2$, where x is the distance travelled by a consumer to patronize a firm and c is the unit transport cost. He showed that at equilibrium the outlets are symmetrically located on the circle and that each outlet charges the same price: $c\,(L/n)^2$.

However, in the real world firms rarely have a single point of sale. In this paper, we consider an arbitrarily number of firms managing, each, an arbitrarily number of outlets. The outlets are located around the circle, to avoid border effects. To ensure the existence of equilibria, we adopt the assumption of quadratic transport cost. We also assume that each consumer buys a single unit from the least cost (price plus transport cost) outlet. The objective is to compute and compare equilibria for any number of firms and for any number of outlets for each firm. Note that when locations are fixed, the location of an outlet is not necessarily in its market area, even when locations are endogenous (cf. Section 6.2). Clearly, the Economides' solution is obtained when each outlet is managed by a distinct firm.

Very little has been written in economics on multi-facility locations in the spatial context, despite the fact that most firms in the real world belong to a chain. Two-stage location-then-price equilibrium with multi-facility is not a simple generalization of a single outlet firms. The number of configurations on a circle with n firms and M outlets in total is large (and equal to $(M-1)!$), and for any configuration there are M equilibrium prices. Configurations are compared along with consumer's surplus, producer's surplus and total welfare. The analysis of these equilibria and their description involves the full enumeration of all possible arrangements of the outlets over space. This problem is unsolved, which may explain why it has not been taken into account so far in the literature of location theory.

Only specific and simple multi-outlet configurations were studied so far. Some contributions are briefly described below and shed some light on multi-outlet firms. In this extended multi-outlet setting the firms face new trade-offs. On one hand, the firms may compete head-to-head and locate their outlets near the adjacent rivals. On the other hand, competition may be interlaced, and the outlets of one firm alternate with the outlets of its competitors. The general case is an equilibrium, combining head-to-head configurations and interlaced configurations. For simple cases, Klemperer (1992) analyzed price competition with multiproduct firms assuming that the products are symmetrically located at specific points on the circumference of a circle. He assumes that each consumer values variety and buys several products each, and compares head-to-head and interlaced competition.

Martinez-Giralt and Neven (1988) considered two firms with two potential outlets each. If a firm adds a second outlet, (i) it increases its market share for fixed prices (but not necessarily if prices are flexible); (ii) it decreases prices; and finally (iii) the new outlet can cannibalize the other outlets of the same firm. This simple setting highlights the three forces. At equilibrium, the first one is positive and the last two are negative. In this 2×2 case, even if the fixed cost to add a new outlet is null, each firm may choose to offer only one outlet. As we will show, multi-outlet firms can be sustainable for large number of outlets per firm. Bensaid and de Palma (1994) explore simple multi-outlets configurations, as discussed in the next section.

The paper is organized as follows. In Section 2, we study analytically simple multi-outlet configurations. In Section 3, we provide an algorithm to enumerate all topologically different configurations with n firms and M outlets. In Section 4, we prove the existence of a price equilibrium for an arbitrary number of outlets with given and fixed locations. In Section 5, we characterize location-then-price equilibria for multi-outlet configurations. We consider no-jumping and jumping equilibria (in the last case, each firm can locate its outlets anywhere on the circle). In Section 6, we analyze clusters of outlets (a set of adjacent outlets managed by the same firm). In Section 7, a regulation is applied where each firm must sell its products at the same price across all its outlets. Section 8 concludes. Some technical materials and proofs are relegated to the appendices.

2. Simple configurations (analytical analysis)

We start by examining simple configurations that can be characterized analytically. Best-response strategies are discussed in Section 2.1 and equilibria in Section 2.2. In this preliminary analysis, we also introduce some concepts and definitions. The formal analysis is postponed to Section 4.

2.1. Best-responses

We first consider the case of two firms where each one can hold one or two outlets. The four possible configurations are provided in Figure 1. We use the concept of two-stage, location-then-price, equilibrium.

If two firms, with one outlet each, compete on location and price, there exists a unique equilibrium where the outlets are diametrically opposed (Configuration 1 in Figure 1). Each firm charges the same price equal to $p^* = c\,L^2/4$, in line with d'Aspremont et al. (1979) who consider two competing firms on a line segment. They show that the equilibrium of the two-stage game involves one firm at each side of the market, charging $c\,L^2/4$. In the reminder of the paper, and unless specified otherwise, we assume w.l.o.g that $c = 1$ and $L = 1$.

Consider then the simplest situation involving a one-outlet firm competing with a two-outlet firm (Configuration 2 in Figure 1). Assume that outlet 1 of the one-outlet firm is located at 12 o'clock and that outlets 2 and 3 of the other firm are symmetrically located w.r.t. 6 o'clock and far apart by a distance Δ with $0 \le \Delta \le 1$. Then, the equilibrium prices are $p_1^* = (1-\Delta)(3-\Delta)/12$ and $p_2^* = p_3^* = (1-\Delta)(3-\Delta)/12 \ge p_1^*$. It can be proved that the two-outlet firm wishes to relocate its outlets towards 6 o'clock to decrease price competition. When the three outlets are equispaced over the circle, $\Delta = 1/3$, and equilibrium prices are $p_1^* = 4/27$ and $p_2^* = p_3^* = 5/27$. As expected, the price charged by the firm with two outlets is higher since it faces cannibalisation, which reduces competition.

When locations are endogenous, the two-outlet firm chooses to increase its profit by reducing the distance between its two outlets and we end up with Configuration 1: two outlets with a single outlet for each. For any positive Δ, we have $\partial(\pi_2+\pi_3)/\partial\Delta = -(1+3\,\Delta)(3+\Delta)/72 < 0$. That is, when $\Delta > 0$, Firm 2 can always increase its profit by reducing the distance between its two outlets. So, more outlets do not necessarily yield competitive advantage since they involve higher price competition, leading the two-outlet firm to withdraw one of its outlets.

Consider now the case of four outlets, two managed by each firm. We distinguish two types of configurations. In the first one, referred to as *non-interlaced*, the two outlets of each firm are direct neighbours. In the second one, referred to as *interlaced*, the ownership of the outlet differs for adjacent outlets.

First, consider the non-interlaced configuration (Configuration 3 in Figure 1). Assume that the outlets of each firm are separated by distance Δ and that they are symmetrically located. The price equilibrium is given by $p^* = 3(2-\Delta)(1-2\Delta)(2-3\Delta)/4(1-\Delta)(4-11\Delta+15\Delta^2)$. Routine computation shows that each firm has an incentive to move one of its outlets toward

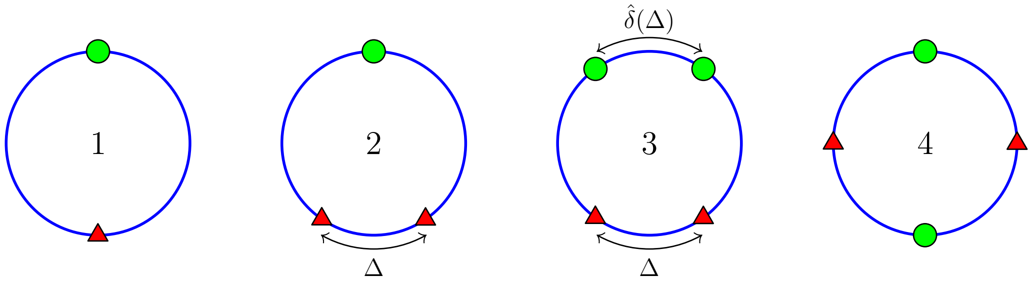


Figure 1: All four configurations with two firms and one or two outlets for each. Firm 1 is managing circle outlets and Firm 2 is managing triangle outlets.

its other one. As a result, the two firms end up with one outlet each; the two outlets are symmetrically located (as in Configuration 1).

Second, consider the interlaced configuration (Configuration 4 in Figure 1). Assume that outlets of Firm 1 located at 12 o'clock and 6 o'clock, while outlets of Firm 2 are located at 3 o'clock and 9 o'clock. The equilibrium price is equal to 1/16 (consistently with Economides, 1989),

Let's reconsider the two-firm, two-outlet case. It can be shown that the symmetrical configuration is a *no-jumping equilibrium* (i.e. for a formal definition of jumping and no-jumping equilibria see Section 5.3 below). The only jumping equilibrium involves two single-outlet firms where the outlets are symmetrically located (cf. Configuration 1).

So far, we have been able to find equilibria of single-outlet firms up to four outlets. This result is general for two firms and any number of outlets per firm (including different number of outlets per firms). Indeed, there exist equilibria where all the outlets have distinct locations, including when two or more outlets of the same firm are adjacent (we'll refer to these as *non-degenerate* equilibria), as shown in Section 5.

Let $\hat{\delta}(\Delta)$ be the best response, in location choice, of Firm 1 to the choice of Δ by Firm 2. This function is formally defined in Appendix A.[1] Observe that $\lim_{\Delta\to 0} \hat{\delta}'(\Delta) = 1/3$, which means that for Δ too small, $\hat{\delta}(\Delta)$ is increasing: when the rivals' outlets are too close and get slightly away from each other, the best-response of Firm 1 is to increase the distance between its two outlets, i.e. a form of head-to-head competition. There are two forces in play when Δ increases. The first reaction of Firm 1 is to reduce δ so that it moves away its outlets from the rivals' outlets. This allows Firm 1 to keep its market power. But, at the same time, when Δ increases, the prices of the rival outlets decrease (higher price competition) providing more incentives for Firm 1 to increase its market share by increasing δ. In general, the former force dominates the latter one, but this analysis shows that for small Δ the first impact dominates. Function $\hat{\delta}(\Delta)$ is increasing for $\Delta < 0.01513$ and decreasing for $\Delta > 0.01513$. If Firm 1 locates one outlet at 12 o'clock and a second one at 6 o'clock, its profit is 1/32, the rivals' prices are

[1] See also Bensaid and de Palma (1994).

both equal to $\Delta(1-\Delta)/4$, the price of the outlet at 12 o'clock is $\Delta/8$ and the one at 6 o'clock is $(1-\Delta)/8$. In the perfectly symmetric configuration, $\Delta = 1/2$, and all prices are equal to $1/16$.

2.2. Location-then-price equilibrium

The following result is derived from the above expression and based on Bensaid and de Palma (1994), for $c = 1$ and $L = 1$.

Proposition 1. *Consider two firms, each owning two outlets. There exists two two-stage location-then-price equilibria. Either each firm locates its outlets at the same point, directly opposite to its rivals' two outlets; equilibrium prices and profits are 1/4 and 1/8, respectively (jumping equilibrium), or Firm 1 locates its outlets at 12 and 6 o'clock, and Firm 2 locates its outlets at 3 and 9 o'clock; equilibrium prices and profits are 1/16 and 1/32, respectively (no-jumping equilibrium).*

The second equilibrium with four equispaced outlets yields the minimum transport costs (see Configurations 1 and 4 in Figure 1).

In the general case, the number of configurations explodes when the number of firms and outlets increases and a systematic enumeration procedure (described below) is needed.

3. Enumeration of potential configurations

We clarify which configurations are distinct in their structure (non-isomorphic) and which have the same structure (isomorphic). We also need a procedure to generate all the relevant ownership structures and configurations. We start by describing and enumerating possible ownerships of the outlets by the firms (Section 3.1), and then define and enumerate non-isomorphic configurations (Section 3.2).[2]

3.1. Ownership

We define the market type by a specific allocation of a number of outlets to each firm. Recall that the products (one outlet is a product) sold by each firm are identical except for space. For example, it is the same to have Firm 1 managing m_1 outlets, and Firm 2 managing m_2 outlets or Firm 1 managing m_2 outlets, and Firm 2 managing m_1 outlets (i.e. firms are anonymous). If all the other firms have the same number of outlets in both cases, the set of spatial configurations will be the same in both cases, i.e. equilibrium profit, prices and consumer surplus are the same. Therefore, we assume w.l.o.g. that $m_1 \geq \ldots \geq m_n$, given M and n.

Consider a total number of M outlets. We wish to compute the number of different market ownerships satisfying $m_1 \geq \ldots \geq m_n$. More formally, we wish to find all possible $m_1, \ldots, m_n$ such that

$$\begin{cases} \sum_{i=1}^{n} m_i & = M \\ \text{s.t.} & m_1 \geq \ldots \geq m_n \\ & m_1, \ldots, m_n \in \mathbb{N}^+, \end{cases}$$

[2]This section is interesting in itself, but it can be glanced by the reader who is not interested in how the configurations are generated.

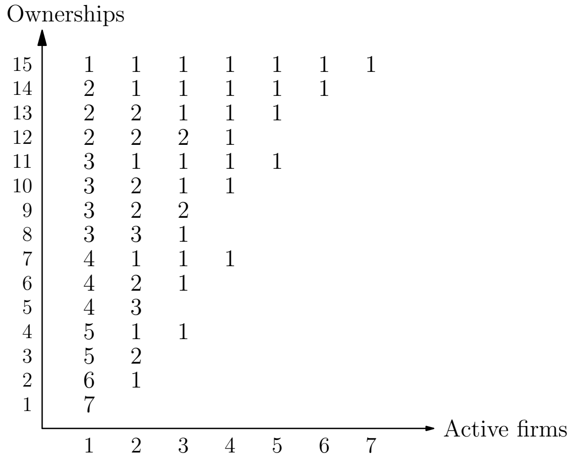


Figure 2: All possible ownerships with a total number of seven outlets ($M = 7$).

where $\mathbb{N}^+$ is the set of strictly positive integers.

To solve this problem we can use the algorithms for the *coin change problem* (Pisinger and Toth, 1998). This algorithm consists in the partition of a given integer as the sum of some other positive integers, using an iterative (and recursive) procedure. Figure 2 provides all possible ownerships for seven outlets. In the first ownership (1 on the $y-$axis), only one firm is managing the seven outlets. There are no other ownerships with a single firm. To obtain the next ownership, we remove one outlet from that firm and assign it to a second firm (2 on the $y-$axis). There are no other ownership for the lonely outlet (i.e. it is not possible to introduce a third firm).

Then, we remove a second outlet from Firm 1 and assign it to Firm 2. Now, the question is: what are all the possible ownerships for the two outlets? There are two: either Firm 2 owns both outlets (3 on the $y-$axis) or we introduce a third firm and assign one outlet to Firm 2 and one to Firm 3 (4 on the $y-$axis). Next, remove a third outlet from Firm 1 and find all possible ownerships to allow for the three outlets. There are three cases: Firm 2 owns the three outlets (5 on the $y-$axis), Firm 2 owns two outlets and Firm 3 one outlet (6 on the $y-$axis), or the three firms (2, 3 and 4) own one outlet each (7 on the $y-$axis). The number of outlets per firm is constrained to be decreasing, as a function of the firms' indices, to avoid repetitions. The algorithm continues with this recursive procedure. For the case of seven outlets we find fifteen possible ownerships types, as reported in Figure 2. Notice that the output can be easily filtered to select only the ownerships with a given number of firms. For example, only Configurations 4, 8 and 9 correspond to ownerships with exactly three firms. Algorithms for generating integer partitions are discussed in Kelleher and O'Sullivan (2009) and Stojmenović and Zoghbi (1998). The algorithm we have used is described in Appendix C.1.

3.2. Enumeration of non-isomorphic spatial configurations

Consider n firms, where Firm i has m_i outlets and $M = \sum_{i=1}^{n} m_i$ are strictly spatially separated. Our objective is to generate the different spatial configurations consistent with specific ownership of outlets satisfying $m_1 \geq \ldots \geq m_n$. Two configurations, which entail the same sequence of ownership are *isomorphic* if only the sequence of the outlets matters, not their exact position. We consider the following formal definition.

Definition 1 (Isomorphic and non-isomorphic configurations)**.** *Two configurations C_1 and C_2 are isomorphic if, and only if, there exists a clockwise (or counterclockwise) rotation of the location of the outlets or a permutation of the identity of the firms holding the same number of outlets applied on C_1, or on the inverted C_1 (outlets of C_1 placed in reverse order), and yielding to C_2. Otherwise, they are non-isomorphic.*

We need to characterize spatial equilibria for *non-isomorphic* configurations only. There are two cases: either the number of outlets per firm are strictly different or at least two firms have the same number of outlets. In the former case, the identification of all non-isomorphic configurations is straightforward. With n firms it is equal to $n!$ corresponding to the number of possible permutations of the firms' labels. When two or more firms manage the same number of outlets, the enumeration is more difficult. A formal discussion is relegated to Appendix C.2.

For illustration, we consider a simple example with three firms, $i = 1, 2, 3$, where $m_3 = 3 > m_2 = 2 > m_1 = 1$ ($M = 6$). We denote a corresponding configuration as $C_1 = \{1, 2, 2, 3, 3, 3\}$, which means that, turning clockwise and starting by outlet of Firm 1, we first encounter the two outlets of Firm 2 and then the three outlets of Firm 3. This configuration is isomorphic to two configurations: one clockwise rotation $C_2 = \{3, 1, 2, 2, 3, 3\}$, or one counter-clockwise rotation $C_3 = \{2, 2, 3, 3, 3, 1\}$ (there are five configurations obtained by these rotations). Configurations C_1, C_2 and C_3 are thus isomorphic.

The economic intuition is as follows. Consider two isomorphic configurations. If the inter-distance (in order) between the outlets is the same in the two configurations, the equilibrium price (which is unique as we prove in Section 4) will be the same. In the above example, this means that the inter-distance (in order) between the first and the second outlets in C_1 is the same as the inter-distance between the second and the third outlet in C_2, which is the same inter-distance between the sixth and first outlet in C_3. A similar comparison can be made for the inter-distance between outlets two and three in C_1 the corresponding inter-distances in C_2 and C_3, and so on.

The second case where there exists one or several groups of firms with the same number of outlets involves permutations generating extra isomorphic configurations. For example, consider again three firms but with $m_3 = 3 > m_2 = m_1 = 2$. Clearly, the configuration $\{3, 1, 1, 3, 2, 2, 3\}$ is isomorphic to $\{1, 3, 3, 1, 2, 2, 1\}$, which is obtained by the permutation: $1 \to 3$, $2 \to 2$ and $3 \to 1$. When such permutation exists, the corresponding configurations are isomorphic.

Consider now the four configurations illustrated in Figure 3. We can see that the arrangement in Configuration 2 can be obtained through a simple rotation of the outlets in Configuration 1, i.e. Configurations 1 and 2 are isomorphic. It is less obvious that Configuration 3 is isomorphic to the first two. Indeed, it can be obtained through a permutation, where the triangles are replaced by circles, circles by squares and squares by triangles, followed by the application of a convenient rotation. Configuration 4 in Figure 3, however, is non-isomorphic to

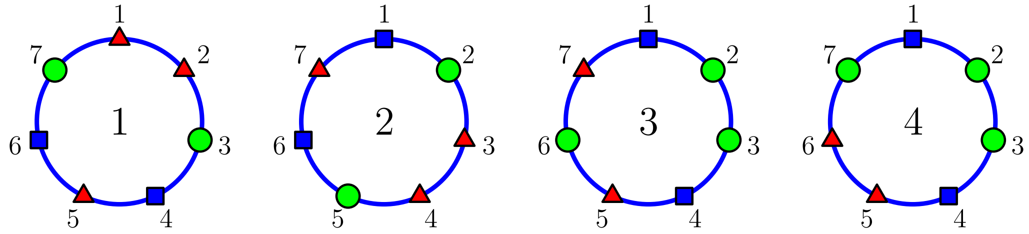


Figure 3: Four examples of configurations with a total number of seven outlets and three firms. Configurations 1, 2 and 3 are isomorphic, but they are all non-isomorphic with Configuration 4.

the other three: there are no combination of rotations and permutations that can transform it to match the arrangement in the other three. We can check that with seven outlets and three firms there is a total of 31 distinct ownerships and non-isomorphic configurations. In Appendix C.2 we provide a general procedure to generate all non-isomorphic configurations. The idea is the following. We start with $m_1, \ldots, m_n$ outlets managed by firms $i = 1, \ldots, n$, respectively. We start with an empty list. We generate random configurations consistent with firm ownership of outlets, and when a new configuration is not isomorphic to any configuration in the list, we add it to the list. This iterative procedure yields, with a probability arbitrarily close to one, to a list of all non-isomorphic configurations consistent with $m_1, \ldots, m_n$.

The enumeration of all non-isomorphic configurations uses this heuristic (we did not find in the literature a systematic procedure to implement it). While this procedure is fast for configurations with less than ten outlets and less than four firms, the computation time significantly increases for larger values. For example, for four firms managing each three outlets, it takes more than one minute to find all the 707 non-isomorphic configurations while, on the same machine, it takes less than one second to find the 25 non-isomorphic configuration for three firms managing each three outlets. With more than four firms and more than four outlets computation time is measured in hours. This is because the number of non-isomorphic configurations increases dramatically. In Figure 4, we report the total number of configurations for the first values of n and M. For example, for $n = 4$ and $M = 7$ there is a total of 33 non-isomorphic configurations: 4 for the ownership $\{4, 1, 1, 1\}$, 18 for the ownership $\{3, 2, 1, 1\}$ and 11 for the ownership $\{2, 2, 2, 1\}$. The corresponding ownerships are represented in Figure 2 where, along the $y-$axis, they correspond to ownerships 7, 10 and 12, respectively.

Some specific configurations can be enumerated easily. For example, when $M = n+1$, there is only one possible ownership where one firm manages two outlets and the other $n-1$ firms manage one outlet each. It is straightforward to see that their number is $n/2$ if n is even and $(n+1)/2$ if n is odd, as shown in Figure 4. For the general case with arbitrarily values of n and M, however, the systematic enumeration is more complex.

4. Price equilibrium

We first introduce the model (and the notations) and then compute the equilibrium price for any given configuration.

M \ n	1	2	3	4	5	6	7	8	9	10
10	1	43	538	1851	2262	1248	322	54	5	1
9	1	22	202	478	206	171	37	4	1	
8	1	17	82	137	85	27	4	1		
7	1	8	31	33	16	3	1			
6	1	8	14	11	3	1				
5	1	3	5	2	1					
4	1	3	2	1						
3	1	1	1							
2	1	1								
1	1									

Figure 4: Enumeration of all non-isomorphic configurations for small values of n and M.

4.1. The setting

There are M outlets on a circle of circumference L and selling a homogeneous product. The notations are summarized in Table 3 in Appendix D and partly illustrated in Figure 5. Let x_j be the location of outlet j, with $0 \leq x_1 \leq \ldots \leq x_M \leq L$. There are n firms, Firm i is managing m_i outlets, $i = 1, \ldots, n$. The outlets are ranked clockwise by increasing order along the circle. Index of the *successor* of outlet j is denoted $j+1$, with by convention $M+1 = 1$. Likewise, the *predecessor* of outlet j is denoted $j-1$, with by convention $1-1 = M$. The distance between outlets j and $j+1$ is denoted by x_j^+, i.e. $x_j^+ = x_{j+1} - x_j$. Likewise, the distance between outlets j and $j-1$ is denoted by x_j^-, i.e. $x_j^- = x_j - x_{j-1}$. Therefore, the distance between x_{j-1} and x_{j+1} is equal to $x_j^- + x_j^+$ and is denoted by x_j^{++}.

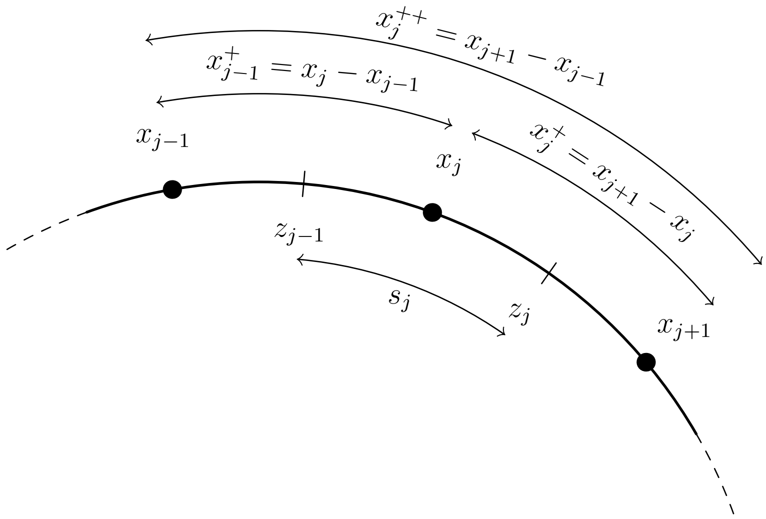


Figure 5: Notations on the circle

For a given outlet j, $F(j)$ corresponds to the index of the managing firm. Let $\mathfrak{F}_i = \{j = 1, \ldots, n$ such that $F(j) = i\}$ be the set of outlets managed by Firm i.

Consumers are uniformly distributed on the circle. Each one buys one unit of the good from one outlet. The transport cost of the consumer located at x and purchasing from the outlet located at x_j is quadratic in distance and is given by $c\,(x-x_j)^2$, where c is transport rate. The corresponding delivered price is

$$\widehat{p}_j(x) = p_j + c\,(x-x_j)^2, \quad j = 1,\ldots,M,$$

where p_j is the mill price of outlet j, with $p_j \in \mathfrak{S}_j = [0,\infty)$, $j \in \mathfrak{F}_i$. The marginal production cost is constant and set to zero (w.l.o.g.), since total demand is inelastic. Firm i has to locate each of its m_i outlets and sets, for each one, the corresponding mill price, $i = 1,\ldots,n$.

4.2. Computation and characterization of equilibrium prices

We consider a given configuration and wish to compute the Nash equilibrium in price. There are M outlets and the set of outlets belonging to Firm i is denoted by $\mathfrak{F}_i$. The ownership of outlet j is denoted by $F(j) = i$. Let the binary variable $\mathbb{I}_{j,j'}$ be defined as follows: $\mathbb{I}_{j,j'} = 1$ if $F(j) = F(j')$ and $\mathbb{I}_{j,j'} = 0$ if not, for $j, j' = 1,\ldots,M$. Then, define the binary variable $I_j = 2$ when $F(j) = F(j+1)$ and $I_j = 1$, otherwise; i.e. $I_j = 1 + \mathbb{I}_{j,j+1}$ for $j = 1,\ldots,M$.

Consider that the locations of the M outlets are given. The consumer who is indifferent between outlet j and outlet $j+1$ is located at z_j with

$$z_j = \frac{x_j + x_{j+1}}{2} + \frac{p_{j+1} - p_j}{2c(x_{j+1} - x_j)}, \qquad \text{for } j = 1,\ldots,M. \tag{1}$$

Therefore, the market share of outlet j is $s_j = z_j - z_{j-1}$ or

$$s_j = \frac{1}{2}\left(x_j^{++} + \frac{p_{j+1} - p_j}{c\,x_j^{+}} - \frac{p_j - p_{j-1}}{c\,x_j^{-}} \right), \qquad \text{for } j = 1,\ldots,M. \tag{2}$$

Notice that for Equations (1) and (2) to be defined, outlets j and $j+1$ should not be at the same location, i.e. $x_{j+1} > x_j$ or $x_j^{+} > 0$, and may be managed by different firms. The profit of outlet j is $\pi_j = p_j\,s_j$, for $j = 1\ldots,M$, and the profit of Firm i is

$$\Pi_i = \sum_{j=1}^{M} \mathbb{I}_{ij}\,\pi_j, \qquad \text{for } i = 1,\ldots,n. \tag{3}$$

We have the following property for the profit functions (the proof is in Appendix B.1):

Proposition 2. *Consider a circle with outlets located at strictly distinct given locations. Transport costs are quadratic. The prices of the outlets managed by the rival firms are given. The profit function of each Firm is strictly concave in its prices.*

The first-order conditions for all firms is given by the following system of linear equations

$$\frac{\partial \Pi_i}{\partial p_j} = \frac{\mathbb{I}_{ij}}{2}\left[\frac{\mathbb{I}_{ij-1}\,p_{j-1}}{c\,x_{j-1}^{+}} + x_j^{++} + \frac{p_{j+1} - 2\,p_j}{c\,x_j^{+}} - \frac{2\,p_j - p_{j'-1}}{c\,x_j^{-}} + \frac{\mathbb{I}_{ij+1}\,p_{j+1}}{c\,x_{j+1}^{+}} \right],$$

for $j = 1, \ldots, M$. These first-order conditions can be written in a matrix form $\mathbf{A} \cdot \mathbf{p} = \mathbf{b}$:

$$\begin{pmatrix} \frac{2x_1^{++}}{x_M^+ x_1^+} & -\frac{I_1^+}{x_1^+} & 0 & \cdots & 0 & -\frac{I_M^+}{x_M^+} \\ -\frac{I_1^+}{x_1^+} & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -\frac{I_{M-1}^+}{x_{M-1}^+} \\ -\frac{I_M^+}{x_M^+} & 0 & \cdots & 0 & -\frac{I_{M-1}^+}{x_{M-1}^+} & \frac{2x_M^{++}}{x_{M-1}^+ x_M^+} \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ \vdots \\ p_M \end{pmatrix} = c \begin{pmatrix} x_1^{++} \\ x_2^{++} \\ x_3^{++} \\ \vdots \\ \vdots \\ x_M^{++} \end{pmatrix}, \tag{4}$$

where we can observe that equilibrium prices are proportional to parameter c.[3] Since the profit functions are strictly concave (cf. Proposition 2), it remains to show that the linear system in (4) has a unique solution with positive prices. We have:

Proposition 3. *Consider n firms with m_i outlets managed by Firm i, for $i = 1, \ldots, n$. Locations are fixed. The Nash equilibrium in prices is the unique solution to the linear system (4). At equilibrium, the price and the market share of each outlet are strictly positive.*

The proof of Proposition 3, given in Appendix B.2, builds on the properties of matrix $\mathbf{A}$ in (4). Since, it is a diagonal dominant matrix, the Gershgorin circle theorem is used to prove that the inverse of the matrix exists. The Farkas Lemma (Boyd and Vandenberghe, 2004) is then used to guarantee the existence of a solution with positive prices by proving that the dual problem does not have a solution.

From the system of equations in (4) we can characterise equilibrium prices when two outlets gets too close to each other. Consider first the case of two outlets j and $j+1$ managed by the same firm. When $x_j^+ \to 0$, Equation j in (4) writes $-(I_{j-1}^+/x_{j-1}^+)\, p_{j-1} + 2x_j^{++}/(x_j^- x_j^+)\, p_j - (I_j^+/x_j^+)\, p_{j+1} = c\, x_j^{++}$. But, when $x_j^+ \to 0$ we have $x_j^{++} \to x_j^-$, so that the last expression simplifies to

$$\frac{2(p_j - p_{j-1})}{x_j^+} = c\, x_j^{++} + \frac{I_{j-1}^+}{x_{j-1}^+}\, p_{j-1},$$

because $I_j^+ = 2$. The right hand side of this equation is positive and finite, but the left hand side has a finite limit only when $(p_j - p_{j-1}) \to 0$. So, at the limit, the prices set by two outlets managed by the same firm are identical, i.e. this is equivalent to two outlets merged in a single outlet.

Consider next the case where outlets j and $j+1$ are managed by two distinct firms, i.e. $I_j^+ = 1$. Using the same procedure, Equations j and $j+1$ in system (4) reduce to

$$\begin{cases} \frac{2p_j - p_{j-1}}{x_j^+} & = c\, x_j^{++} + \frac{I_{j-1}^+}{x_{j-1}^+}\, p_{j-1}, \\ \frac{-p_j + 2p_{j-1}}{x_j^+} & = c\, x_{j+1}^{++} + \frac{I_{j+1}^+}{x_{j+1}^+}\, p_{j+2}. \end{cases}$$

[3] From the same condition, if all x_j^+, for $j = 1, \ldots, M$, are multiplied by the same constant, equilibrium prices are multiplied by the square of that constant. Thus, equilibrium prices are proportional to L^2, and we can assume, without loss of generality, that $c = 1$ and $L = 1$.

The right hand sides in these two equations are both positive and finite. The left hand sides are finite only when $(2\,p_j - p_{j-1}) \to 0$ and $(-p_j + 2\,p_{j-1}) \to 0$, respectively. This is possible only when $p_j \to 0$ and $p_{j-1} \to 0$. So, when two outlets managed by two distinct firms get too close, their prices simultaneously converge to zero, as expected.

If the outlets are equidistant ($x_j^+ = L/M$ for all $j = 1, \ldots, M$) and no two adjacent outlets belong to the same firm ($I_j = 1$ for all $j = 1, \ldots, M$), then

$$\begin{pmatrix} 4M/L & -M/L & 0 & \cdots & 0 & -M/L \\ -M/L & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -M/L \\ -M/L & 0 & \cdots & 0 & -M/L & 4M/L \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ \vdots \\ p_M \end{pmatrix} = c \begin{pmatrix} 2L/M \\ 2L/M \\ 2L/M \\ \vdots \\ \vdots \\ 2L/M \end{pmatrix},$$

which yields the standard solution derived by Economides (1989):

$$p_i^e = c\,(L/M)^2\,, \text{ for } i = 1, \ldots, M. \tag{5}$$

When the locations of the outlets are given, the equilibrium in prices is easily computed from Equation (4). The market area of each outlet is then obtained from (1). In general, the boundaries of the market area extend around the location of the corresponding outlet, so that each outlet is located inside its market area. This ensures higher market power for the managing firm. But, this pattern does not holds in all cases. With some given (and fixed) locations of the outlets, the market area of one (or more) outlet can cover an area away from the location of the corresponding outlet. An illustration is provided in Figure 6. In the left panel, Configuration 1, each outlet is located inside its market area. Notice that the size of the market share of each outlet increases when distance with neighbouring outlets gets larger. An outlet which is located between two outlets managed by the same firm has, in general, a relatively small market share, as it is the case for outlets 2 and 5 in this example. Indeed, such local monopolies exercise strong market power because they are not directly competing with rival firms. In the example shown in the right panel (Figure 6), we keep the same sorting of the outlets, but locate outlets 6 and 7 further to the right and keep the other outlets at the same locations. In the new equilibrium, outlets 1, 2 and 6 are now located outside their market areas.

In the next section we compute the location equilibrium (first stage of the game) for the jumping and the no-jumping equilibria.

5. Two-stage location-then-price equilibrium

We consider now the first stage of the game, where each firm decides where to locate its outlets. We provide the equilibrium conditions and then illustrate some specific cases.

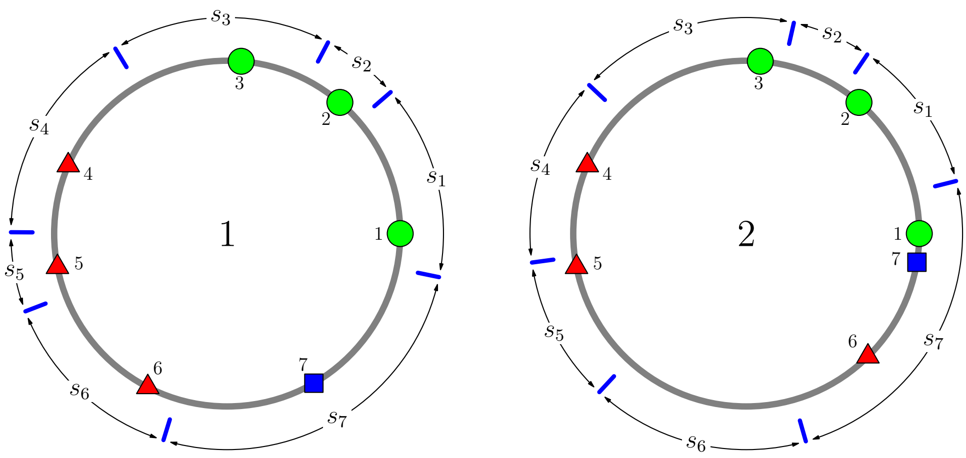


Figure 6: Two examples of equilibria in prices. Configuration 1 on the left corresponds to a standard equilibrium where each outlet is within is in its market areas. In Configuration 2 on the right, outlets 1, 2 and 6 are located outside their market areas.

5.1. Equilibrium conditions (bunching and no bunching)

We define two sets $\mathfrak{F}_i^+ = \{j = 1, \ldots, M \text{ such that } F(j) \neq i \text{ and } F(j+1) = i\}$ and $\mathfrak{F}_i^- = \{j = 1, \ldots, M \text{ such that } F(j) \neq i \text{ and } F(j-1) = i\}$. The first-order condition with respect to the location of outlet j is

$$\frac{\mathrm{d}\Pi_i}{\mathrm{d}x_j^+} = \frac{\partial \Pi_i}{\partial x_j^+} + \sum_{k \in \mathfrak{F}_i^+} \frac{p_{k+1}}{2\, c\, x_k^+} \frac{\mathrm{d}p_k}{\mathrm{d}x_j^+} + \sum_{k \in \mathfrak{F}_i^-} \frac{p_{k-1}}{2\, c\, x_k^+} \frac{\mathrm{d}p_k}{\mathrm{d}x_j^+}, \text{ for } i = 1 \ldots, n \text{ and } j = 1, \ldots, M. \tag{6}$$

In this expression, the impact of a change of the location of an outlet j has two main impacts on the profit of Firm i (managing outlet j). The first one is a direct impact on profit. In this case the prices of the outlet managed by Firm i adjust optimally and the envelope theorem applies. The second impact is induced by the impact on the prices of the neighbour outlets that are managed by competitors (the second and third terms). Prices of these outlets change and impact the market share and profits of the corresponding outlets managed by Firm i.

Consider Configuration 1 in Figure 3. What is the impact of small change in the location of outlet 3 on the profit of the firm managing outlet 3 and outlet 7? The first impact is equal to the partial derivative of the total profit of the firm with respect to x_3^+ (the first term in (6)). But there is a second impact, since the change in x_3^+ will also change all the prices $p_1, \ldots, p_7$, with p_3 and p_7 adjusted optimally, while the others not. From the previous section, we know that the equilibrium price depends on the prices and the locations of only the neighbouring outlets. Here, the locations of outlets 1 and 6 do not change, but the corresponding prices p_1 and p_6, respectively, adjust given the best-response of the corresponding rival firms. The second and third terms in the right hand side of (6) take into account these changes on the profit of the firm managing outlets 3 and 7.

The first term in Eq. (6) can be expanded as follows

$$\frac{\partial \Pi_k}{\partial x_j^+} = \begin{cases} p_j \left(-\frac{p_{j+1}-p_j}{2\,c\,(x_j^+)^2} + \frac{p_{j-1}-p_j}{2\,c\,(x_j^-)^2} \right) & \text{if } k = j, \\ p_{j+1} \left(\frac{1}{2} + \frac{p_j - p_{j+1}}{2\,c\,(x_j^+)^2} \right) & \text{if } k = j+1, \\ p_{j-1} \left(\frac{1}{2} - \frac{p_j - p_{j-1}}{2\,c\,(x_j^-)^2} \right) & \text{if } k = j-1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if Firm i is managing outlet j, the direct impact of a change in x_j^+ on the profit of Firm i is given by

$$\frac{\partial \Pi_i}{\partial x_j^+} = \mathbb{I}_{j,j-1} \cdot \frac{\partial \pi_{j-1}}{\partial x_j^+} + \frac{\partial \pi_j}{\partial x_j^+} + \mathbb{I}_{j,j+1} \cdot \frac{\partial \pi_{j+1}}{\partial x_j^+}. \tag{7}$$

To complete expression (7), we need to compute the gradient of the prices with respect to the locations of the outlets. Let $\boldsymbol{\nabla}_j\, \mathbf{p}$ denote the gradient of $\mathbf{p}$ with respect to x_j^+, i.e. the vector with component of $\mathbf{p}$ differentiated by x_j. We define $\boldsymbol{\nabla}_j\, \mathbf{b}$ and $\boldsymbol{\nabla}_j\, \mathbf{A}$ similarly. By differentiating (4), we have $\mathbf{A} \cdot \boldsymbol{\nabla}_j\, \mathbf{p} + \boldsymbol{\nabla}_j \mathbf{A} \cdot \mathbf{p} = \boldsymbol{\nabla}_j \mathbf{b}$, or

$$\boldsymbol{\nabla}_j\, \mathbf{p} = \mathbf{A}^{-1} \cdot \Big(\boldsymbol{\nabla}_j \mathbf{b} - \boldsymbol{\nabla}_j \mathbf{A} \cdot \mathbf{p} \Big). \tag{8}$$

Note that all the entries of matrix $\boldsymbol{\nabla}_j\, \mathbf{A}$ are zero, except the two entries (j, j) and $(j+1, j+1)$, which are both equal to $-2/(x_j^+)^2$, and the two entries $(j+1, j)$ and $(j, j+1)$, which are both equal to $I_j/(x_j^+)^2$. Similarly, notice that all the entries of vector $\boldsymbol{\nabla}_j\, \mathbf{b}$ are zero, except the two entries j and $j+1$, which are both equal to c. It follows then that all the entries of the column vector $\boldsymbol{\nabla}_j\, \mathbf{b} - \boldsymbol{\nabla}_j\, \mathbf{A} \cdot \mathbf{p}$ are zero, except entry j which is $c + 2/(x_j^+)^2 \cdot p_j - I_j/(x_j^+)^2 \cdot p_{j+1}$ and entry $j+1$, which is $c - I_j/(x_j^+)^2 \cdot p_j + 2/(x_j^+)^2 \cdot p_{j+1}$. This completes the computation of price gradient in (8). The inverse matrix $\mathbf{A}^{-1}$ is already computed in the solution of the linear system (4).

To solve numerically for a location-then-price equilibrium we proceed as follows. For a given configuration, we start with equispaced locations of the outlets, i.e. $x_i^+ = L/M$, for $i = 1, \ldots, M$. We then consider all outlets j such that $j \in \mathfrak{F}_1$ (the outlets that belong to Firm 1), write the corresponding profit maximizing problem and the related gradient (as explained above). Then, pass this information to a nonlinear solver that updates the locations for Firm 1, until the profit of Firm 1 is optimized, given that the locations of the outlets that do not belong to Firm 1 remain fixed. We then proceed similarly for Firm 2, and so on until Firm n. When this inner loop is completed we compare the updated locations of the outlets with the initial locations. If the differences are higher than a fixed threshold (we consider a quadratic error threshold equal to 10^{-9} in our computations), a new iteration is performed. When the difference is below the target threshold, the computation stops. The locations and prices obtained by this algorithm correspond to the numerically computed equilibrium.

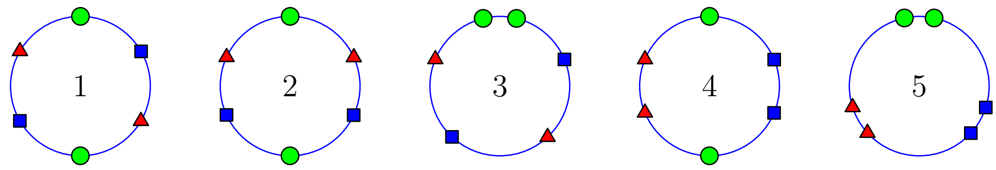


Figure 7: Location-then-price equilibrium for the five non-isomorphic configurations with three firms, two outlets each.

5.2. Some specific configurations

The first set of configurations we consider corresponds to competition between three firms managing two outlets each.[4] Location-then-price equilibria for all the five possible configurations are given in Figure 7, with some corresponding statistics reported in Table 1.

From left to right, we move from markets with high competition to markets with high concentration. In Configuration 1, the location-then-price equilibrium yields a perfectly symmetric pattern where each pair of outlets from the same firm are diametrically opposed. Prices (and profits) and transport costs are low. Contrarily, in Configuration 5 each firm has a significant market power. Prices (and profits) and transport costs are high.

	Prices			Transport costs		Full prices		Profits		
Id	min	max	Avg	max	Avg	max	Avg	min	max	Avg
1	2.78	2.78	**2.78**	0.69	**0.23**	3.47	3.01	9.26	9.26	9.26
2	2.62	2.97	2.74	1.01	0.25	3.97	**2.98**	8.84	9.68	**9.12**
3	3.27	4.14	3.62	1.44	0.31	5.58	3.93	7.39	14.41	12.07
4	4.30	5.08	4.72	1.31	0.28	6.39	5.00	13.75	19.69	15.73
5	8.84	8.84	**8.84**	1.76	**0.47**	10.59	**9.31**	29.46	29.46	**29.46**

Table 1: Location-then-price equilibrium and summary statistics for the five non-isomorphic configurations with three firms, two outlets each. For the "average" columns (Avg), the minimum and maximum values are highlighted in bold for the four aggregates. The numerical computations use $L = 10$ (and $c = 1$) to avoid small values of the aggregates.

Transport costs are minimized in Configuration 1 because the six outlets are equispaced. Equilibrium prices are also small, but slightly higher than in Configuration 2. In the latter case, all the outlets are in direct competition with outlets from the rival firms. Price competition is high in this case. In Configuration 3, one firm has two adjacent outlets, which locate near each other, while the other outlets are more spaced on the remaining part of the circle. Prices, as well as transport costs, are higher here and Firm 1 has the lowest profit. In Configuration 4, two firms cluster their outlets together and one firm locates its outlets symmetrically on the circle. The latter firm has the highest profit and the transport cost is smaller than in the previous configuration. Efficiency is not monotonic in the number of firms that cluster their outlets

[4]The computer code to find the location-then-price equilibrium for any configuration, including the examples in this paper, can be found in the git link https://gogs.univ-littoral.fr/mkilani/outlets.

together. Configuration 5, contrasts with Configuration 1 since the three firms cluster their outlets together and keep away from their rival firms. Maximum differentiation (Configuration 1) is the most socially efficient configuration, while minimum differentiation (Configuration 5) is the worse socially, but the most profitable for the firms. Interestingly, the fully symmetric Configuration 1 is not the most competitive configuration. Configuration 2 is more competitive since the triangle firm and square firm could coordinate their prices, and so increase price competition and therefore decrease market profitability. As a result, average transport cost increases under Configuration 2, since both consumers and firms are worse off, compared with Configuration 1.

For the case of three firms managing three outlets each, there are twenty-five non-isomorphic configurations, illustrated in Figure 8; the statistics on the equilibria are reported in Table 4 in Appendix E. There are comparable observations for the case of three firms managing two outlets each. Indeed, from Configuration 1 to Configuration 25 we move from a fully symmetric configuration to progressively clustered configurations, and finally to a configuration where each firm clusters its three outlets.

Among the twenty-five configurations in Figure 8, there are five configurations, where the firms' labels can be interchanged without any impact on their profits and relative locations vis-à-vis to competitors: 1, 4, 15, 17 and 25 (firms are interchangeable). The location-then-price equilibrium corresponding to these configurations can be computed more efficiently since, by symmetry, the number of unknowns can be reduced significantly. For example, for Configuration 4 there are two unknown prices and two locations to check for.

The vertical lines separate distinct patterns of clusters. For example, in the first four configurations there no adjacent outlets from the same outlet. From Configurations 5 to 8, there are only two outlets from one firm that are adjacent, and so on. For several configurations here, one or two firms find it optimal to merge two of its outlets, and even its three outlets as in Configuration 19. Interestingly, this occurs only when the two outlets (that merge) are located between two outlets from distinct firms. Configuration 2 here is the least profitable, as it is the case for Configuration 2 in Figure 7. The intuition of this result goes along the same line as above.

The configurations are sorted with respect to increasing market-power. In the last configuration, each firm concentrates its outlets away from its rivals' outlets. Market prices and profits are high in this case. The reported configurations are all in location-then-price equilibria without jumps. In this case, we find that fifteen out from the twenty-five configurations correspond to such an equilibria. Degenerate configurations produce the same equilibrium locations and prices, and thus profits and transport costs, as configurations with less outlets.

5.3. Analysis of no-jumping and jumping equilibria

To characterize the location and price equilibrium, we define formally the concept of jumping and no-jumping equilibria. For, a *no-jumping equilibrium*, at the first stage, no firm is allowed to move one or more of its outlets beyond the location of an opponent's outlet. For a *jumping*

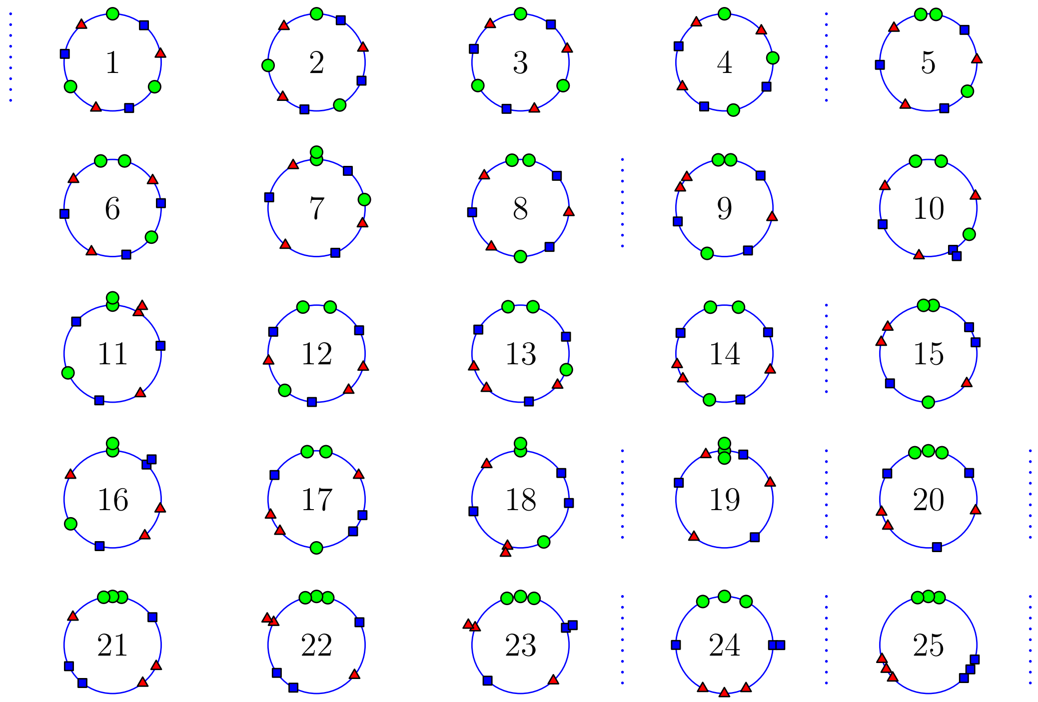


Figure 8: Location-then-price equilibrium for three firms, three outlets each. All the twenty five configurations are non-isomorphic. The vertical lines separate distinct patterns of clusters (for 1 to 4, there are no adjacent outlets from the same firm; for 5 to 8 just one firm has two adjacent outlets, and so on).

equilibrium, the firms are allowed to relocate their outlets everywhere along the circle. We will see that only some very simple patterns correspond to jumping equilibria.

In the previous subsection, we have discussed equilibria under the assumption that the order of the outlets along the circle does not change (these are no-jumping equilibria). In a more general setting, a firm may alter the order of the outlets by operating a jump over a rival's firm outlet. Under equilibria meeting this criteria, i.e. jumping equilibria, no firm wishes to change the order of the outlets in the existing configuration. Identifying jumping equilibria is straightforward when all the no-jumping equilibria (for all possible permutations) are computed. In each case, we check if one firm at least can increase its benefit by sorting differently its outlets. For the case reported in Figure 8, only Configuration 25 corresponds to a jumping equilibrium; it involves minimum spatial differentiation, larger profitability and transport costs.

To identify the jumping equilibria for a limited number of outlets and firms, we consider all the configurations with less than ten outlets and less than three firms, where each one is managing three outlets at most. In addition, we consider the case of a total number of ten outlets, where one firm is managing four outlets and the other two are managing three outlets each.

The corresponding equilibria are reported in Figure 9. The most simple one is with two firms managing each a single outlet. It is located on the left. The numbers under parenthesis indicate the number of outlets managed by each firm (here one for each). The subscript indicates the number of non-isomorphic configurations and the exponent corresponds to the number

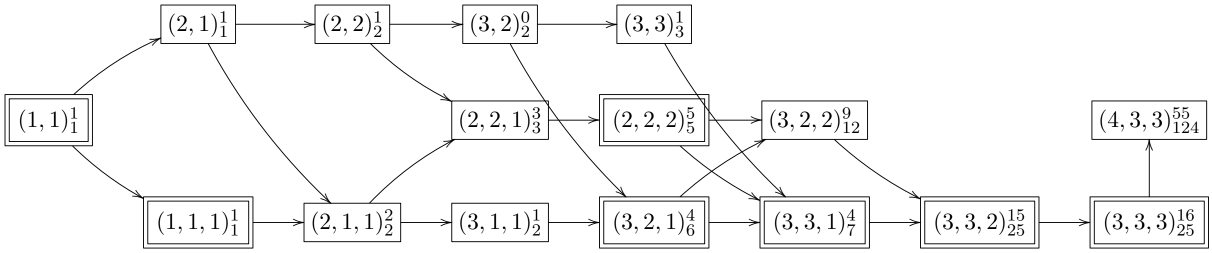


Figure 9: Equilibria for configurations with less than ten outlets and less than three firms. The numbers under parenthesis indicate the number of outlets managed by each firm. The subscript number indicates the number of non-isomorphic configurations and the superscript number indicates the number of non-isomorphic and non-degenerate equilibria. The double frames indicate the existence of a jumping equilibrium and the arrows indicate the addition of an outlet.

of non-isomorphic and non-degenerate equilibria. As discussed in Section 2, there is one equilibrium where the outlets are located in the opposite of each other. Clearly, this configuration corresponds to an equilibrium even when jumping is allowed. Such equilibria are highlighted in Figure 9 by the double frames. The configurations are organized on three rows and eight columns. As we move by one column to the right, one outlet is added (except the last one to keep the figure size manageable). So, from left to right, we start with two outlets and end up with ten outlets. Jump-equilibria, when they exist, are unique and always correspond to configurations where all firms cluster their outlets.

The only jumping equilibria are those where all the firms locate their outlets together. For the case of three firms managing each two outlets, only Configuration 5 in Figure 7 satisfies this criteria. For the case of three firms managing each three outlets, only Configuration 25 in Figure 8 satisfies this criteria. So, in these jumping equilibria only the two peripheral outlets face direct competition with a competitor's outlet. Systematic numerical computations show that configurations with outlets from the same firm clustered together are the only sorting of outlets that allows each firm to secure its market area. We conjecture that this is a general results (although, we were unable to prove it analytically). For all the other configurations (with no compact market areas), one firm at least has an incentive to leapfrog one of its outlets. With three muti-outlet firms, with a different number of outlets per firm, there are four non-isomorphic configurations. Notice that the minimum differentiation equilibria are the less socially desirable ones (this is in line with the Hotelling's linear city).

6. Hotelling model revisited

In Configurations 20 to 25 in Figure 8 each outlet between two outlets from the same firm is located exactly in the middle of these two. Similar symmetric configurations with more outlets and more firms confirms this pattern. In this section, we derive a general property describing how a firm managing adjacent outlets adjusts to outside competition. This allows to characterize equilibria involving local monopolies with equal number of clustered outlets. We start with a simple case in Section 6.1 and then discuss more general results in Section 6.2.

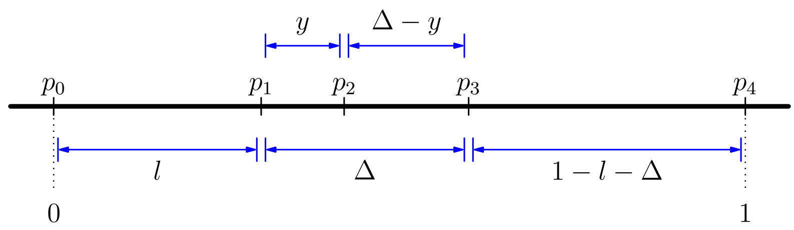


Figure 10: A simple cluster with outlets 1, 2 and 3 managed by the same firm and outlets 0 and 4 are managed by one or two distinct competing firms. Decision variables are y, p_1, p_2 and p_3.

6.1. Three outlets monopoly

From Figure 8, and as noticed above for all equilibria configurations with a cluster of three outlets, the central outlet is always located just in the middle of the other two outlets. This raises two questions: (i) Whether this obtains independently of the distance from the rivals' outlets? (ii) Whether this pattern generalises to clusters with more than three outlets?

The second part of the question is discussed in the next subsection. For the first part, we consider the problem illustrated in Figure 10. The firm managing outlets 1, 2 and 3 chooses prices p_1, p_2 and p_3 as well as the location of outlet 2, i.e. the value of y. The values of all the other variables (l, Δ, p_0 and p_4) are exogenously given for the managing firm.

For the location-price equilibrium, using Eqs. (2) and (3) and considering the first-order conditions, we obtain $p_1 = [(1-l)\,p_0 + l\,p_4 + (1-l)\,l]/2$, $p_2 = [(1-y-l)\,p_0 + (y+l)\,p_4 + (1-y-l)\,(y+l)]/2$ and $p_3 = [(1-l-\Delta)\,p_0 + (l+\Delta)\,p_4 + (1-l-\Delta)\,(l+\Delta)]/2$. Substituting for p_1, p_2 and p_3 in the total profit of outlets 1, 2 and 3 and taking the first-order condition with respect to y yields the optimum location for outlet 2 as

$$y = \Delta/2. \tag{9}$$

Therefore, the firm managing outlets 1, 2 and 3 maximizes its profit by locating outlet 2 exactly halfway between outlets 1 and 3.

The equilibrium location given in Equation (9) is obtained for the simple configuration considered. This result is generalized below.

6.2. General cluster

Consider a situation where Firm i has $r > 2$ outlets in the submarket $[0, L]$. There is a competitor located at 0, charging p_0 and one (the same or a different one) located at L, charging p_L. Those prices are fixed. This amounts to considering a Hotelling model for one firm, which has to locate r outlets and which faces exogenously given prices of two firms located at the left and right boundaries. As such, it is a separate problem. As r increases the optimal location of the first outlet 1 decreases, while the optimal spread, measured by $(x_r - x_1)$ increases with r, but at a decreasing speed. The outlets $2, \ldots, r-1$, are referred to as the *inframarginal* outlets. Interestingly, the inframarginal outlets are regularly located on $[x_1, x_r]$. This property is true even if the outlets x_1 and x_r are not optimally located. We have:

Lemma 1. *Consider the Hotelling model, with quadratic transport cost, where one firm located at* 0 *and one firm located at* L*, charging prices* p_0 *and* p_{r+1}*, respectively. A third rival firm is managing* r *outlets located between* 0 *and* L *to maximize its profit. Locations are fixed. The optimal prices of the* r *outlets are given by:*

$$p_j = \frac{1}{2}\left[\frac{(L - x_j)\,p_0 + x_j\,p_{r+1}}{L} + (L - x_j)\,x_j\right], \qquad \textit{for} \quad j = 1, \ldots, r. \tag{10}$$

Proof. By solving the first-order conditions for the profit of the firm managing the cluster with r outlets, we obtain the best-response (10). □

The best-response in (10) shows that the price of each outlet in the cluster depends only on the prices of the outlets of the rival firm (located at 0 and L, respectively) and the distance separating the outlet itself from the same two outlets of the rival firm. Stated differently, the price of each outlet in the cluster does not depend on the prices of the other outlets in the cluster, neither on the locations of the infra marginal outlets. As a consequence of this price-setting rule, we derive a specific location pattern for the inframarginal outlets in the cluster (the proof is relegated to Appendix B.3).

Proposition 4. *Consider the Hotelling model, with quadratic transport costs, where one firm located at* 0 *and one firm located at* L*, charging prices* p_0 *and* p_{r+1}*, respectively. Consider a rival firm that wishes to locate* r *outlets between* 0 *and* L *to maximize its profit. Then, (i) the inframarginal outlets are equally spaced on* $[x_1, x_r]$*, and (ii) the market share of the inframarginal outlets are equal to* $(x_r - x_1)/[2(r-1)]$*.*

If the firm with the r outlets acts as a public monopoly, and if $p_0 = p_l$, it will set regularly those outlets on $[0, 1]$ and charge p_M, with $p_M = p_0 = p_l$ to minimize total transport cost, so each outlet will have the same market share.

The intriguing feature is that those properties are maintained when prices p_0 and p_L are different. The firm which has to locate r outlets acts as a private, profit maximizing, firm. The prices of the inframarginal outlet have a parabolic shape (increasing, decreasing or bell-shaped). The inframarginal outlet are still regularly spaced and the market shares are still the same. However, the total transport cost of the consumer is not minimized, because the price difference induces market share where some consumers have to travel longer distances to get the minimum full cost. In this case, the rival firm mitigates the objective to minimize total transport cost with the objective to face optimally the prices charged by its competitor, p_0 and p_L. As a consequence, Firm i no more charges equal prices for its $r - 2$ inframarginal outlets.

When the number of outlets increases, the profits decrease because price competition is higher (market shares do not change under symmetric equilibria). The consumers, however, benefit from this large number of outlets since prices are lower and transport costs are lower.

Based on the expression of price equilibrium given in (10), we can characterize the symmetric equilibrium. Consider that M firms are managing r adjacent outlets each along the circle. Outlets in each cluster are indexed $j = 1, \ldots, r$ from left to right, so that outlets 1 and r are at the edges and outlets $2, \ldots, r - 1$ are inframarginals. At a symmetric equilibrium, the inframarginal distance between two adjacent outlets, denoted δ, is the same in all the M clusters. The equilibrium prices of the outlets at the edges are all the same. The distance between two adjacent clusters is $1/M - (r-1)\delta$. We start by computing p_1 and p_r, the equilibrium prices of the outlets at the edge. Notice that the fraction in (10) is simply a weighted average of p_0 and p_{r+1}, the prices of the nearest outlets at the edges of the two neighbouring clusters. Since, in a symmetric equilibrium, prices of all the outlets at the edges of the clusters are the same, we have $p_1 = (1/2)\,[p_1 + \Delta((r-1)\delta + \Delta)]$, or

$$p_1 = \frac{1-(r-1)\delta M}{M^2}.$$

The conditions for equilibrium prices of the inframarginal outlets are obtained from a similar condition, which involves only the update of the distances to the nearest outlets from the neighbouring clusters. We can check that for outlet $j = 2, \ldots, r-1$, $p_j = (1/2)\big[p_1 + \big(\Delta + (j-1)\,\delta\big)\big((r-j)\,\delta + \Delta\big)\big]$, yielding the general solution

$$p_j = \frac{1}{M^2} - \delta\,\frac{(r-1) - (j-1)(r-j)\,\delta\,M}{M}, \qquad \text{for} \quad j = 1, \ldots, r. \tag{11}$$

Equilibrium prices in (11) are the lowest for the outlets at the edges, $j = 1$ and $j = r$, and increase as we move to the center of cluster. From (11), we can derive the recurrence relation $p_{j+1} = p_j + (r - 2\,j)\,\delta^2/2$, for $j = 1, \ldots, r-1$. This shows that prices increase as long as j is smaller than $r/2$ (the center of the cluster), and decrease after. Notice that with $r = 1$, all prices are equal to $1/M^2$, so that (11) is a generalization of the Economides (1989) solution, given in (5), to the case of clusters with several outlets for each firm.

To close this discussion we briefly show how the above result can be used improve the computation of the location-then-price equilibrium for the symmetric case. Assume that M firms locate symmetrically their clusters along the circle. The distance between two inframarginal outlets is δ. Now, consider the case where $M-1$ firms hold the locations of their outlets fixed, and Firm i deviates by choosing δ' as the distance between its inframarginal outlets. Firm i chooses δ' to maximize its profit. Following the deviation δ', the locations of the outlets of the $M-1$ other firms do not change but their prices are adjusted optimally, i.e. rule (10) holds. The distance between the nearest outlets from two adjacent clusters of the $M-1$ clusters is $\Delta = 1/M - (r-1)\delta$. The distance between outlets r and outlet $r+1$ (or outlets 1 and outlet $M\,r$) is $\Delta' = 1/M - (r-1)(\delta + \delta')$. The solution of the equilibrium in prices is performed in two steps. The first step involves the solution of a linear system of $2\,n$ equations, that relate the prices of the outlets at the edges in each cluster, instead of $n\,M$ linear equations if (4) is used. Notice that, from (10) the equilibrium prices of the outlets $j = 1, \ldots, r$ (cluster managed by Firm i) depends only on prices p_{r+1} and p_{Mr}. In the second step, we use this solution to compute equilibrium prices for outlets $j = 2, \ldots, r-1$. We can thus write the profit of Firm i (stage 1) as a function of δ' and δ. Then, the first-order condition with respect to δ' yields the best-response function of Firm i to the other competitors setting δ. By replacing δ' by δ and solving for δ (a polynomial equation), we obtain the symmetric solution.

7. Constrained prices

In this section, we assume that each firm is constrained to set the same price for all its outlets. This may be either a marketing strategy (shared by all the firms) or imposed by the regulation authorities.

When the prices of the outlets belonging to the same firm are equal, the equilibrium prices are the solution to a linear system comparable to (4) but of dimension n (the number of firms) instead of M (the number of outlets). In matrix form, it is $\mathbf{A_n} \cdot \mathbf{p_n} = \mathbf{b_n}$ (the subscript n is used to distinguish the notation from that used in Equation (4)), where the entries of matrix $\mathbf{A_n}$ are given by

$$\mathbf{A_n}(i,i) = \sum_{\substack{k \in \mathfrak{F}_i \text{ and } k+1 \notin \mathfrak{F}_i \\ \text{or } k \notin \mathfrak{F}_i \text{ and } k+1 \in \mathfrak{F}_i}} \frac{2}{x_k^+} \qquad \text{and} \qquad \mathbf{A_n}(i,j) = \sum_{\substack{k+1 \in \mathfrak{F}_j \text{ and } k \in \mathfrak{F}_i \\ \text{or } k+1 \in \mathfrak{F}_i \text{ and } k \in \mathfrak{F}_j}} \frac{1}{x_k^+}, \tag{12}$$

for $i = 1, \dots, n$, $j = 1, \dots, n$ and $j \neq i$. The entries of the column vector $\mathbf{b_n}$ are given by $\mathbf{b_n}(i) = c \sum_{k \in \mathfrak{F}_i \text{ or } k+1 \in \mathfrak{F}_i} x_k^+$, and $\mathbf{p_n}$ the column vector of prices $p_1, \dots, p_n$ set by each Firm $i = 1, \dots, n$. We have the following result.

Proposition 5. *Consider n firms with m_i outlets for Firm i with given and fixed locations x_j on the circle, $j = 1, \dots, M$. The Nash equilibrium in price, where each Firm i charges the same price $p_i > 0$, $i = 1, \dots, n$, for all its outlets j, i.e. $j = 1, \dots, M$ and $j \in \mathfrak{F}_i$, is the unique solution to the linear system $\mathbf{A_n} \cdot \mathbf{p_n} = \mathbf{b_n}$.*

The proof is relegated to Appendix B.4. The main difference with Proposition 3, is that the firm does not necessarily have a positive market share for all its outlets. Indeed, decreasing the price to attract consumers to a given outlet may reduce demand to other outlets and can lead to an overall negative impact on the profit of the firm. In some configurations, a firm may keep its price sufficiently high so that one (or more) of its outlets has zero sale.

We now compare the new equilibria with those obtained with the flexible price setting. Location-then-price equilibria for the case of three two-outlet firms are reported in Table 2. For a simple comparison, we report in the first row the equilibria from Figure 7 next to the corresponding new equilibria in the second row (denoted by a “prime”).

For the perfectly symmetric case, Configurations 1 (flexible prices) and 1’ (equal prices for all outlets managed by the same firm), are identical. In the initial model, when the outlets are equispaced, and when each outlet faces two different competitors, from (4) the equilibrium price is the same for all the outlets. It is then clear why Configurations 1 and 1’ are identical and transport costs are minimized.

More interestingly, Configuration 2’ where all the outlets have two neighbours managed by a rival firm yields an equispaced (optimal) distribution of outlets. Recall that when prices at each outlet are not necessarily the same, we do not obtain symmetric locations. In Configuration 3’, one firm has two adjacent outlets, and the corresponding equilibria is quite different, from that corresponding to Configuration 3, since the distance between these two outlets is significantly larger. The profit of the firm managing these two outlets is higher than those of its two rivals.

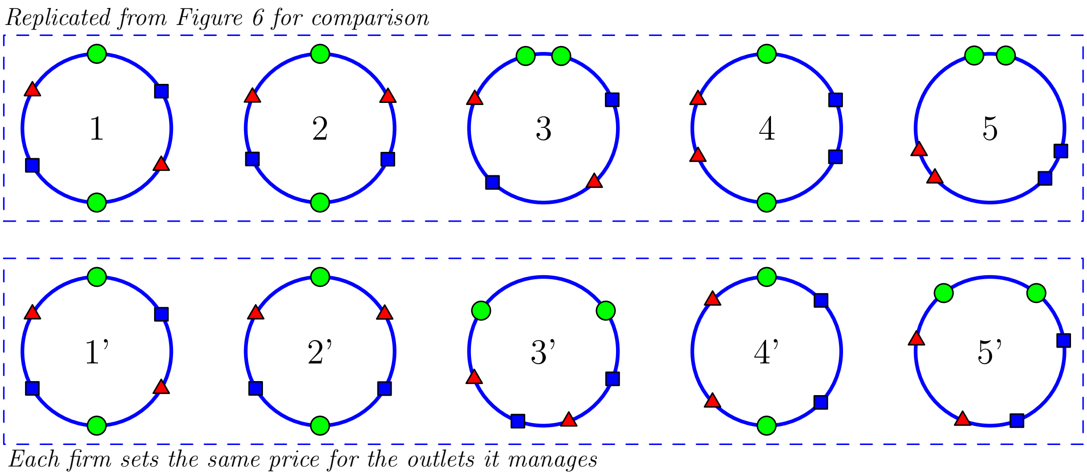


Figure 11: Location-then-price equilibrium for the five non-isomorphic configurations with three firms, two outlets each: a comparison between equilibria reported in Figure 7 (first row, flexible prices) and equilibria in Table 2 (second row, equal prices for all outlets managed by the same firm).

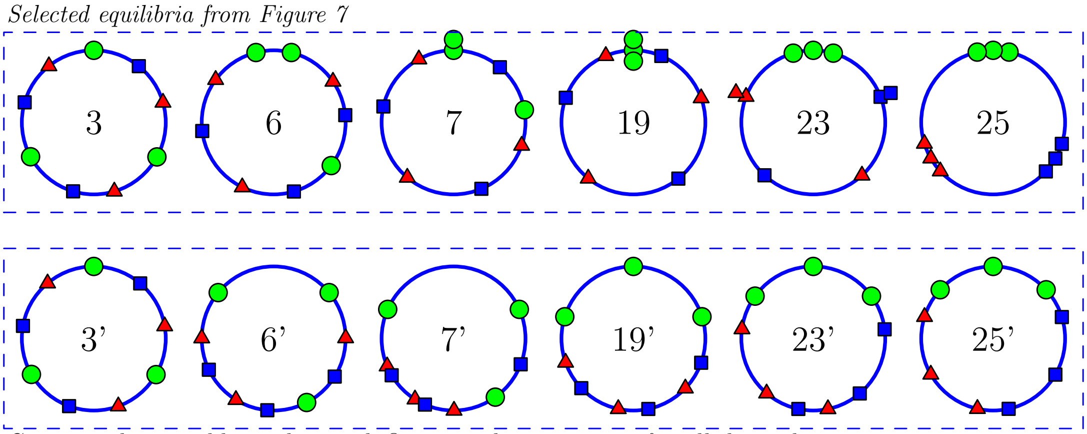
	Prices			Trans. costs		Full prices		Profits		
Id	min	max	Avg	max	Avg	max	Avg	min	max	Avg
1’	2.78	2.78	**2.78**	0.69	**0.23**	3.47	**3.01**	9.26	9.26	**9.26**
2’	2.78	2.78	**2.78**	0.69	**0.23**	3.47	**3.01**	9.26	9.26	**9.26**
3’	1.78	1.88	1.85	1.45	0.31	3.33	2.16	4.90	6.82	6.18
4’	2.08	2.74	2.36	2.50	0.39	5.24	2.75	5.99	11.65	7.88
5’	1.56	1.56	**1.56**	1.14	**0.29**	2.70	**1.84**	5.19	5.19	**5.19**

Table 2: Location-then-price equilibrium and summary statistics for the five non-isomorphic configurations with three firms, two outlets each (see Figure 11). Each firm set a single price at all the outlets that it manages. For average values, the minimum and maximum are highlighted in bold. The numerical computations use $L = 10$ (and $c = 1$) to avoid small values of the aggregates.

A similar pattern obtains when two firms have two adjacent outlets as shown in Configuration 4’. The square and triangle firms move their outlets near the outlets of the circle firm (head-to-head competition), while in Configuration 4, the same outlets are kept clustered far away from those of the circle firm. The comparison between 5 and 5’ confirms further these differences. In Configuration 5, each firm behaves as a local monopoly and cluster its two outlets away from the competitors. But, the equilibrium in Configuration 5’ is different since each firm locates its outlets just next to the rivals outlets. Overall, the two cases where there are no adjacent outlets managed by the same firm (Configurations 1’ and 2’) yield an equispaced location of the outlets. When outlets from the same firm are adjacent they locate away from each other. In the case of flexible prices, the users who face high transport costs are those who are located between clusters of outlets manager by distinct firms, while in the new setting those who face high transport costs are also those located between two outlets managed by the same firm.

To confirm these observations, we consider in Figure 12 some selected configurations for the case of three firms managing three outlets each. The first row in Figure 11 are selected

Selected equilibria from Figure 7



Corresponding equilibria when each firm sets the same price for all the outlets it manages

Figure 12: Examples of location-price equilibrium and comparison between the standard setting (Figure 8, first row) and the case where each firm is imposed to set the set same price at all the outlets it manages (second row).

equilibria from Figure 8 and the second row reports corresponding equilibria when each firm sets the same price at all the outlets it manages. In Configuration 3 there are no adjacent outlets managed by the same firm, and the locations are not equispaced. But, when the price constraint is active the equilibrium locations, as shown for Configuration 3', are equispaced. The same outcome is observed for Configurations 2 and 4 (in Figure 8) for which locations are not equispaced, but when the equal prices constraint is active the outlets are equispaced in the corresponding equilibria. In Configuration 6, one firm has two adjacent outlets and all the other have neighbours managed by rival firms. As for the similar case above, the distance between the two outlets increases when the firms set the same price at the outlets they manage, as shown by Configuration 6', and these two outlets occupy a large part of the market. In Configuration 7, two outlets are merged in a single one, but with equal prices the same firm sets the two outlets far apart as shown in Configuration 7'. This is confirmed by comparing 19 and 19', where in the former configuration three outlets merge in a single one. For Configuration 23, one firm has three adjacent outlets and each of the other two has a two adjacent outlets. Again, the equilibrium in 23' shows that the distance between these outlets is large while the other isolated outlets are clustered in a small space. Finally, Configurations 25 and 25' where, instead of clustering its outlets away for the competitors, with equal price, each outlet locates near a rival's outlet.

To sum-up: with equal prices for the outlets of the same firm, price competition is softer, but spatial competition increases. With constrained price setting, the firms adopt a head-to-head competition with their neighbours. Equilibrium prices and profits are significantly lower in this case and the outlets are, on average, more evenly distributed along the circle. In this case, the consumers are clearly better off and the firms worse off. Notice that Configuration 5 in Table 1 has the maximum average price, contrasting with Configuration 5' in Table 2, that has the minimum average price. This yields smaller profits for Configuration 5'.

In the jumping equilibrium with fixed prices, the location of the inframarginal outlets of a firm does not affect its profit (see, for example, Configuration 25' in Figure 12). To solve this indeterminacy, assume that the firm locates the inframarginal outlets uniformly over space, as for the flexible price case. Such solution decreases the average transport cost incurred by the consumers. By doing so, we can verify that Configuration 25' is a jumping equilibrium. Note that such strategy leads also more robust configuration with respect to the entry of new outlets (not modeled here).

8. Conclusion

In this paper, we consider explicitly spatial equilibria on the circle. Firms compete in a two-stage location-then-price. The spatial setting represents the geographical space or any characteristic space. Transport costs are quadratic. The enumeration of all configurations is an open problem. It is based on coin change problem, and involves Monte Carlo simulations to obtain the isomorphic (topological different) situations for n firms and m_i outlet for Firm i. The number of configurations explode with n and m_i, and we provide a heuristic approach to address this problem.

Simple situations have been studied analytically and show that the firms may wish to shut-down some outlets to reduce price competition. The social optimal solutions (equal space configurations) occur if the firms locate their outlets symmetrically, each outlet being surrounded by two competitors (belonging to the same firm or not). By far, it is not the norm at equilibrium, so resources are wasted. Non-symmetric equilibria are prevalent, and characterized. The configurations obtained extend the concept of interlaced and non-interlaced competition introduced by Klemperer (1992). Transport costs, mill prices and profits are compared. We have shown that given outlet's locations there exists a unique price equilibrium, but the full game can only be studied analytically with specific examples. Otherwise, they can be studied numerically. The symmetric equilibria are preferred by the consumers since they yield maximum spatial differentiation, while the clustering (minimal spatial differentiation) is preferred by the firms.

We can reduce the number of location-then-price equilibria, using the concept of "jumping equilibrium", i.e. allowing a firm to leapfrog a competitor's outlet, and be free to locate it anywhere. In this case, the set of location-then-price equilibria is surprisingly small and simple: at equilibrium, each firm locates all its outlets on the same geographical area and only its two external outlets face competitors. The organization of inframarginal (strictly interiors) outlets is unexpectedly simple, since they are equally spaced, yet prices are not identical (they are bell shaped). Finally, we have considered the case where the firms charge equal prices to all consumers. Symmetric equilibria remain unchanged, but the equilibrium price change since deviation of the location of an outlet imply different price responses. For non-symmetric local equilibrium, imposing the same price for all outlets can increase or decrease firms' profitability. Some firms can loose and others can win, but often firms are worse off with this regulation, since they compete more spatially having their hand locked for setting prices of their outlet.

We have in a sense close the topic for the simple version of the multiproduct firms on the circle, for inelastic demand and quadratic transport costs. But the space is not always a circle. Our approach can be envisaged for a linear segment involving boundary effects. Except of the enumeration part, all the derivations should be adapted (for fixed locations, the existence and uniqueness of a price equilibrium could be derived along the same lines). More realistic spatial settings are likely to be harder. The 2-D dimension, for a road and public transport network is needed to better understand the location strategies of chains of stores. For fixed locations, the market area are polygons based on the true (shortest path) generalized cost distances. Yet this topic is likely to require operation research techniques to deal with realistic situations (see e.g. Drezner et al., 2002).

Yet much remains to be done beyond this theoretical approach. The truth is neither purely non-localized competition (described by discrete choice models, for example), neither pure localized competition (described by pure spatial model, à la Hotelling). Combining localized model (pure spatial models) and non-localized competition (with, e.g. the nested logit model, for fixed individual demand or the nested CES, for variable consumptions) is feasible, but involves more numerical computations, at least for non-symmetric situations (no closed form solutions for the equilibrium prices exist). Anderson and de Palma (2000) explored this problem for single product firms using the Logit demand model. Multi-outlet firms with differentiated product, combined with Economides (1989) approach, are likely to be challenging, but needed for empirical applications. Bensaid and de Palma (1994) explored for some simple multi-outlets examples the sequential entry. In this case, the firms can use strategic locations to preempt further entry. Preemption is possible when there is an outlet entry cost. Finally, our assumption of fixed market demand is only made for convenience.

Competition is predominantly multiproduct and does not necessarily lead to optimal market solutions (Laffont and Tirole, 1990). Equilibrium within differentiated multiproduct firms (in a non-spatial setting), has been thoroughly examined by various researchers, notably Anderson and de Palma (2006), extended by Nocke and Schutz (2018) who specifically delve into the existence of a price equilibrium — a complex endeavor. A substantial body of empirical literature relies on multiproduct non-spatial models and product differentiation, as evidenced in works such as De Loecker (2011) and Mayer et al. (2021). Geographic space matters (but not so much localized competition) in the new economic geography tradition (see e.g. Fujita and Krugman, 2004). While transport costs hold a crucial role in this trade literature, it does not account for localized competition as in Hotelling's models in a multi-outlet setting. In spatial models (contrarily to the product differentiation setting), considered here, competition is localized and the outlets of a firm compete with its two immediate neighbors (or cannibalize its own products). The findings of this paper can illuminate new perspectives within the field of trade literature.

The current model can be extended to the study of home delivery versus fixed relay points, also known as the last-mile problem in the supply chain literature. In this case, either the firm only, or the firm and the consumer pay for the transport cost. The number and the location of relay points are chosen by the firm, and in this case product differentiation is needed to guarantee a price equilibrium. Firms may charge different prices when some consumers order at home or pick the parcel at a relay point. This extension is not straightforward but is doable and builds on the current analysis.

Acknowledgements

We are thankful for Simon Anderson, Bernard Bensaid, Horst Eiselt and Robin Lindsey for useful discussions. André de Palma is also grateful to MAAT (project no. 18356856) and AFFINITE (Projet-ANR-20-CE22-0014) projects for their financial supports. An early version of the paper has been presented in the "CY Transport and Urban Seminar" (March 2021). We are thankful to the participants for comments and suggestions.

References

Anderson, S. P. and de Palma, A. (2000), 'From local to global competition', *European Economic Review* **44**(3), 423–448.

Anderson, S. P. and de Palma, A. (2006), 'Market performance with multiproduct firms', *The Journal of Industrial Economics* **54**(1), 95–124.

Anderson, S. P., de Palma, A. and Thisse, J.-F. (1989), 'Demand for differentiated products, discrete choice models, and the characteristics approach', *The Review of Economic Studies* **56**(1), 21–35.

Bensaid, B. and de Palma, A. (1994), Spatial multiproduct oligopoly, Notes d'études et de recherche, Banque de France (unpublished draft).

Boyd, S. P. and Vandenberghe, L. (2004), *Convex optimization*, Cambridge university press.

d'Aspremont, C., Gabszewicz, J. J. and Thisse, J.-F. (1979), 'On hotelling's" stability in competition"', *Econometrica* **47**(5), 1145–1150.

De Loecker, J. (2011), 'Product differentiation, multiproduct firms, and estimating the impact of trade liberalization on productivity', *Econometrica* **79**(5), 1407–1451.

de Palma, A., Ginsburgh, V., Papageorgiou, Y. Y. and Thisse, J.-F. (1985), 'The principle of minimum differentiation holds under sufficient heterogeneity', *Econometrica* **53**(4), 767–781.

de Palma, A., Labbé, M., Thisse, J. and Norman, G. (1986), On the existence of price equilibria under mill and uniform delivered price policies, *in* G. Norman, ed., 'Spatial pricing and differentiated markets', Pion Limited, pp. 30–42.

Drezner, T., Drezner, Z. and Salhi, S. (2002), 'Solving the multiple competitive facilities location problem', *European Journal of Operational Research* **142**(1), 138–151.

Economides, N. (1989), 'Symmetric equilibrium existence and optimality in differentiated product markets', *Journal of Economic Theory* **47**(1), 178–194.

Eiselt, H. A. and Marianov, V. (2011), Pioneering developments in location analysis, *in* H. Eiselt and V. Marianov, eds, 'Foundations of location analysis', Springer, pp. 3–22.

Eiselt, H. A., Marianov, V. and Drezner, T. (2019), Competitive location models, *in* G. Laporte, S. Nickel and F. S. da Gama, eds, 'Location science', Springer, pp. 391–429.

El-Mikkawy, M. E. (2004), 'On the inverse of a general tridiagonal matrix', *Applied Mathematics and Computation* **150**(3), 669–679.

Feingold, D. G. and Varga, R. S. (1962), 'Block diagonally dominant matrices and generalizations of the gerschgorin circle theorem.', *Pacific Journal of Mathematics* **12**(4), 1241–1250.

Fujita, M. and Krugman, P. (2004), 'The new economic geography: Past, present and the future', *Papers in Regional Science* **83**, 139–164.

Hotelling, H. (1929), 'Stability in competition', *The Economic Journal* **39**(153), 41–57.

Kelleher, J. and O'Sullivan, B. (2009), 'Generating all partitions: a comparison of two encodings', *arXiv preprint arXiv:0909.2331* .

Klemperer, P. (1992), 'Equilibrium product lines: Competing head-to-head may be less competitive', *The American Economic Review* **82**(4), 740–755.

Labbé, M., Peeters, D. and Thisse, J.-F. (1995), Location on networks, *in* M. Ball, T. Magnanti, C. Monma and G. Nemhausera, eds, 'Handbooks in operations research and management science', Vol. 8, Elsevier, pp. 551–624.

Laffont, J.-J. and Tirole, J. (1990), 'The regulation of multiproduct firms: Part I: Theory', *Journal of Public Economics* **43**(1), 1–36.

Martinez-Giralt, X. and Neven, D. J. (1988), 'Can price competition dominate market segmentation?', *The Journal of Industrial Economics* **36**(4), 431–442.

Mayer, T., Melitz, M. J. and Ottaviano, G. I. (2021), 'Product mix and firm productivity responses to trade competition', *Review of Economics and Statistics* **103**(5), 874–891.

Nocke, V. and Schutz, N. (2018), 'Multiproduct-firm oligopoly: An aggregative games approach', *Econometrica* **86**(2), 523–557.

Pisinger, D. and Toth, P. (1998), Knapsack problems, *in* D.-Z. Du and P. M. Pardalos, eds, 'Handbook of Combinatorial Optimization', Vol. 1, Springer, pp. 299–428.

Salop, S. C. (1979), 'Monopolistic competition with outside goods', *The Bell Journal of Economics* **10**(1), 141–156.

Stojmenović, I. and Zoghbi, A. (1998), 'Fast algorithms for genegrating integer partitions', *International Journal of Computer Mathematics* **70**(2), 319–332.

Appendix A Best-reply, two-firms, three outlets

The following is a formal characterization of the best-reply correponding to the simple case of two firms, three outlets discussed in Section 2.

Lemma 2 ($c = 1$ and $L = 1$)**.** *The best reply of Firm 1 facing two outlets managed by Firm 2 is to locate the single outlet in the middle of the large side of the market. The best reply of Firm 1 facing two outlets managed by two competing firms and symmetrically located with respect to 12 o'clock and spaced by distance $\Delta > 0$ is (i) to locate two outlets symmetrically to 6 o'clock if $\Delta < 0.4298$, (ii) to locate a single outlet at 6 o'clock if $0.4298 \leq \Delta < 0.46739$, and (iii) to locate two outlets, one at 12 o'clock and one at 6 o'clock if $\Delta \geq 0.46739$.*

The proof of the first part follows from the analysis above. Let us consider the second case, where Firm 1 is in competition with Firm 2 managing one outlet, located at $\Delta/2$, and Firm 3 managing one outlet, located at $1-\Delta/2$. The two outlets of Firm 1 are located symmetrically at the opposite side at distance δ from each other. After solving for the equilibrium prices (stage 2) and substituting, the profit of firm in stage 1 is $(1-\delta-\Delta)\big(1+4\,\Delta-(\delta-\Delta)^2\big)^2/32\,(1-\delta+2\,\Delta)$. The first-order condition with respect to δ is a polynomial of degree three, $3\,\delta^3-(10\,\Delta+7)\,\delta^2+(14\,\Delta+5)\,\delta+4\,\Delta^3+13\,\Delta^2-4\,\Delta-1$, and has a unique root between 0 and $1-\Delta$, that we denote $\hat{\delta}(\Delta)$. Let Δ^* be the unique root, between 0 and 1/2, of equation $4\,\Delta^3+13\,\Delta^2-4\,\Delta-1=0$ (the previous polynomial with $\delta = 0$). Numerically, we have $\Delta^* \simeq 0.4298$. Then, we check that, for $\Delta < \Delta^*$, $\hat{\delta}(\Delta) > 0$. For $\Delta > \Delta^*$, Firm 1 maximizes the profit given above by setting $\delta = 0$, i.e. the two outlets at the same location, and thus merge into a single outlet. In this case, the profit is $(1-\Delta)\big(1+4\,\Delta-\Delta^2\big)^2/32\,(1+2\,\Delta)$. At the same time, the profit of Firm 1 when it locates one outlet at 12 o'clock and the second one at 6 o'clock is equal to 1/32. By a direct computation, we can check that this is higher than the above profit for $\Delta > \Delta^{**}$, where Δ^{**} is the unique real root of polynomial $3+2\,\Delta-22\,\Delta^2+9\,\Delta^3-\Delta^4=0$. Numerically, we have $\Delta^{**} \simeq 0.46739$.

Appendix B Proofs

B.1 Proof of Proposition 2

From the expression of the market shares the profit π_j of outlet j depends on the price of the outlet itself, p_j, and the prices of the neighbouring outlets p_{j-1} and p_{j+1}. Thus, when the prices of the outlets managed the the rival firms are fixed, the profit Π_i of Firm i is additively separable in the profits of clusters (sets of adjacent outlets belonging to the same firm) of its outlets. It suffices then to prove that the sum of profits of any cluster, from outlet j to j', is strictly concave in the prices of the outlets in that cluster. The Hessian matrix for the cluster is obtained by differentiating the corresponding submatrix of $\mathbf{A}$ in Equation (13). We have

$$\begin{pmatrix} -X_0 - X_1 & X_1 & 0 & \cdots & 0 & 0 \\ X_1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & X_{l-1} \\ 0 & 0 & \cdots & 0 & X_{l-1} & -X_{l-1} - X_l \end{pmatrix}, \tag{13}$$

where $X_0 = 1/x_{j-1}, \ldots, X_l = 1/x_{j'}$ and $l = j' - j + 1$. The principal minors of the tridiagonal symmetric matrix given in (13) satisfy the recursive relation (cf. El-Mikkawy, 2004) $f_r = -(X_r + X_{r-1})f_{r-1} - X_{r-1}f_{r-2}$ for $r = 2, \ldots, l$. We have

$$\begin{aligned} -(X_r + X_{r-1})f_{r-1} = &- X_1\, X_2 \ldots X_{r-1}\,(X_r + X_{r-1}) \\ &- X_0\, X_2\, \ldots X_{r-1}\,(X_r + X_{r-1}) \\ &- \ldots \\ &- X_0\, X_1\, \ldots X_{r-3}\, X_{r-1}\,(X_r + X_{r-1}) \\ &- X_0\, X_1\, \ldots X_{r-3}\, X_{r-2}\,(X_r + X_{r-1}) \end{aligned} \tag{14}$$

and

$$\begin{aligned} -X_{r-1}^2 f_{r-2} = &\; X_1\, X_2\, \ldots X_{r-2}\, X_{r-1}^2 \\ &+ X_0\, X_2\, \ldots X_{r-2}\, X_{r-1}^2 \\ &+ \ldots \\ &+ X_0\, X_1\, \ldots X_{r-4}\, X_{r-2}\, X_{r-1}^2 \\ &+ X_0\, X_1\, \ldots X_{r-4}\, X_{r-3}\, X_{r-1}^2. \end{aligned} \tag{15}$$

Summing (14) and (15), we obtain the determinant of the Hessian matrix

$$\begin{aligned} f_r &= -(X_1\, \ldots\, X_r) - (X_0\, X_2\, \ldots\, X_r) - \ldots - (X_0\, \ldots\, X_{r-1}) \\ &= -\sum_{j=0}^{r} \left(\prod_{k=0;k\neq j}^{r} X_k \right). \end{aligned}$$

So, if $f_{r-2} = -\sum_{j=0}^{r-2}\left(\prod_{k=0;k\neq j}^{r-2} X_k\right)$ and $f_{r-1} = \sum_{j=0}^{r-1}\left(\prod_{k=0;k\neq j}^{r-1} X_k\right)$ then, we have $f_r = -\sum_{j=0}^{r}\left(\prod_{k=0;k\neq j}^{r} X_k\right)$. It is straightforward to show that if f_{r-2} and f_{r-1} are both multiplied by (-1) then f_r has the same value multiplied by (-1). Since these relations are satisfied for $r = 2$ and $r = 3$, by recursion, they are satisfied for all positive integer r. The minors of the Hessian matrix alternate signs, the first one being negative, we conclude that the Hessian matrix is negative definite and the profit function strictly concave in its price.

B.2 Proof of Proposition 3

First, notice that, by construction, $\mathbf{A}$ is a weakly chained diagonally dominant matrix (WCDD). This holds by comparing each diagonal element to the sum of the other non zero elements from the same row (or the same column). Since there are at least two firms, one of the I_j is necessarily equal to 1 and the inequality is strict for one row, at least. So, matrix $\mathbf{A}$ is nonsingular. This is a consequence of the Gershgorin circle theorem (see Feingold and Varga, 1962, for example). Then, we need to show that the solution prices are positive. To ease notation, let $X_j = 2/x_j^+$ and $X'_j = I_j/x_j^+$ for $j = 1, \ldots, n$. Notice that $X'_j \leq X_j$ for $j = 1, \ldots, n$. We need to show that $\mathbf{A}\,\mathbf{p} = \mathbf{b}$, with $\mathbf{p} \geq 0$ has solution. By *Farkas Lemma*, this is true if, and only if, there are no row vector $\mathbf{q}$ such that $\mathbf{q}\,\mathbf{A} \geq 0$ and $\mathbf{q}\,\mathbf{b} < 0$. Notice first that $\mathbf{q}\,\mathbf{A} \geq 0$ is a set of inequalities

$$q_1\left(X_M + X_1\right) \geq q_M X'_M + q_2 X'_1, \tag{16a}$$

$$q_2\left(X_1 + X_2\right) \geq q_1 X'_1 + q_3 X'_2 \tag{16b}$$

and so on, until

$$q_M(X_{M-1} + X_M) \geq q_{M-1} X'_{M-1} + q_1 X'_M, \tag{16c}$$

after arranging terms. Then, the second inequality $\mathbf{q}\,\mathbf{b} < 0$ can be written as $q_1\, x_1^{++} + q_2\, x_2^{++} + \ldots + q_M\, x_M^{++} < 0$, which cannot hold if all $q_1, \ldots, q_M$ are positive. Assume that there exists only one negative q_i; without loss of generality assume it is q_1. From (16a), we can see that this is not possible, because, if q_M and q_2 are both positive, the inequality does not hold. So, at either q_2 or q_M are negative, if not both. Let us assume that only q_2 is negative. Thus, from (16a), $|q_2| > |q_1|$. But in this case, for (16b) to hold, q_3 should be negative. Now, let us assume that both q_M and q_2 are negative. Thus, from (16c), q_{M-1} should be negative. When all q_i are negative, $|q_1| < |q_2|$, $|q_2| < |q_3|$, ..., $|q_{M-1}| < |q_M|$ and $|q_M| < |q_1|$: a contradiction. Thus, the system $\mathbf{qA} \geq 0$ and $\mathbf{qA} < 0$ does not have a solution. By Farkas Lemma, $\mathbf{Ap} = \mathbf{b}$, with $\mathbf{p} \geq 0$, i.e. Eq. (4), has a solution.

Finally, we prove that the market share of each outlet is strictly positive. Assume not. That is, assume that there exists one outlet j with a market share that is zero. Notice that if the two neighbours (outlets $j-1$ and $j+1$ belong to the same firm), then there exists $\epsilon > 0$, small enough, such that the firm managing this outlet can set a price to attract consumers in $(x_i - \epsilon, x_i + \epsilon)$ and increases its profit. If the two neighbours belong to a rival firm, then clearly the managing firm can undercut price, gain a positive market share and increase its profit. If the two outlets belong to the same firm (managing outlet j), it is clear that it can increase its market power by charging a higher delivered price (still lower than the full price charged by outlets $j-1$ and $j+1$) at least for users just around x_j. Consider now the case where one of the outlets, say $j+1$ belongs to the same firm and displays a high price, but outlet $j-1$ belongs to a rival firm and displays a small price. In this case, undercutting price can attract consumers from the market share of outlet $j+1$ and, so, it remains to show which price deviation increase its profit. In this case, we use the expression of the market share given (2) and solve $s_j = 0$ for p_j. We denote the solution by p_j^*. We then, consider the impact of a price deviation on the profit $\partial(p_j\, s_j + p_{j+1}\, s_{j+1})/\partial p_j$ and replace p_j by $p_j^* - \epsilon$. We find that the profit of the firm managing outlet j increases by $\epsilon\,(1/x_{j-1}^+ + 1/x_j^+)$. Thus, independently of the firms managing its neighbouring outlets, there exists a profitable price deviation for the firm managing outlet j, contradicting that we are in a price equilibrium.[5]

B.3 Proof of Proposition 4

When prices satisfy (10), then

$$\frac{p_{j+1} - p_j}{x_{j+1} - x_j} = \frac{1}{2}\left[\frac{p_{r+1} - p_0}{L} + \frac{(L - x_{j+1})\, x_{j+1} - (L - x_j)\, x_j}{x_{j+1} - x_j}\right],$$

and

$$\frac{p_j - p_{j-1}}{x_j - x_{j-1}} = \frac{1}{2}\left[\frac{p_{r+1} - p_0}{L} + \frac{(L - x_j)\, x_j - (L - x_{j-1})\, x_{j-1}}{x_j - x_{j-1}}\right],$$

so that the market share of outlet j is

$$s_j = \frac{1}{2}\left[x_{j+1} - x_{j-1} + \frac{(L - x_{j+1})\, x_{j+1} - (L - x_j)\, x_j}{2\,(x_{j+1} - x_j)} - \frac{(L - x_j)\, x_j - (L - x_{j-1})\, x_{j-1}}{2\,(x_j - x_{j-1})}\right].$$

substituting in the profit $\sum_{k=1}^{r} p_k\, s_k$ and evaluating the first-order condition with respect to x_j, we find the optimal location

$$x_j = \frac{x_{j-1} + x_{j+1}}{2},$$

i.e. each inframarginal outlet in the cluster is in the middle of its two neighbours. Thus, the r outlets are equispaced between x_1 and x_r. The market share follows from the computation of s_j in (2).

B.4 Proof of Proposition 5

By definition in (12), matrix $\mathbf{A}_n$ is strictly diagonal dominant matrix with positive elements on the diagonal. It is then nonsingular. To prove that solution prices are positive, we use the Farkas Lemma. Consider the system $\mathbf{q}\,\mathbf{A}_n \geq 0$ and $\mathbf{q}\,\mathbf{b}_n < 0$. Notice first that $\mathbf{q}\,\mathbf{A}_n \geq 0$ is a

[5] As we notice below, it is possible that for a price equilibrium a firm (some firms) is not located within its market area. But, the argument used in this proof only states that there is a profitable deviation, not an equilibrium, where each outlet can attracts the consumers that are too close to its location.

set of inequalities

$$q_i \sum_{\substack{k \in \mathfrak{F}_i \text{ and } k+1 \notin \mathfrak{F}_i \\ \text{or } k \notin \mathfrak{F}_i \text{ and } k+1 \in \mathfrak{F}_i}} \frac{2}{x_k^+} \quad \geq \quad \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j \left(\sum_{\substack{k+1 \in \mathfrak{F}_j \text{ and } k \in \mathfrak{F}_i \\ \text{or } k+1 \in \mathfrak{F}_i \text{ and } k \in \mathfrak{F}_j}} \frac{1}{x_k^+} \right), \tag{17}$$

for $i = 1, \dots, n$. Then, the condition that $\mathbf{q}\,\mathbf{b}_n < 0$ is $\sum_{i=1}^{n} q_i \left(\sum_{k \in \mathfrak{F}_i \text{ or } k+1 \in \mathfrak{F}_i} x_k^+ \right) < 0$. As in the proof of Proposition 3, we can notice that all the q_i cannot be all negative, otherwise the last inequality cannot hold. Any configuration where some q_i are positive yields a contradiction. When all q_i are negative, consider the sum of all the terms on the left hand side in (17). Notice that they correspond to elements on the diagonal of matrix $\mathbf{A}_n$. Since the matrix $\mathbf{A}_n$ is strictly diagonal dominant with all elements on the diagonal being positive, this sum is necessarily smaller than the sum of the members of right hand side of (17), which contradicts the inequality itself. We conclude that there are no solution vector $\mathbf{q}$ and, by Farkas Lemma, the linear system $\mathbf{A_n} \cdot \mathbf{p_n} = \mathbf{b_n}$, with $\mathbf{p_n} > 0$, has a solution.

Appendix C Algorithms for the enumeration of the relevant configurations

This appendix provides a description of the algorithms we have used in the enumeration of non-isomorphic configurations. The computer codes implementing these algorithms are available in the git link https://gogs.univ-littoral.fr/mkilani/outlets.

C.1 The coin change problem

The basic step in the enumeration of non-isomorphic configuration is the identification of all possible groups of outlets a given firm can generate from a total number of outlets its manages. For example a firm managing one outlet, has only one possibility. If it has two outlets it can either form one group of two outlets or two groups one outlet each. With three outlets, there are three possibilities: one group of three outlets; two groups, one of two outlets and the other of one outlet; and, three groups of one outlet each. By increasing the number of outlets, the number of possibilities increases. Figure 2 reports all 15 possibilities for seven outlets. The enumeration of these cases is known as the coin change problem, which is well documented in the operational research literature.

The version we deal with in this problem is to find all possible ways to write a positive integer as the sum of other positive integers. Algorithm 1 implements a function that return a table containing the solution once provided the number of outlets.

Algorithm 1: The coin change problem

```
Data: m: a positive integer number of outlets.
Result: Two dimensional table 𝒫 containing all possible arrangements.
begin
    N ⟵ 0; K ⟵ 1; P(K) ⟵ m
    while True do
        R ⟵ 0; N ⟵ N + 1
        for I ∈ 1 ..., K do
            𝒫(N, I) ⟵ P(I)
        while K ≥ 1  and  P(K) == 1 do
            R ⟵ R + P(K); K ⟵ K − 1;
        if K < 1 then
            Stop
        P(K) ⟵ P(K) − 1; R ⟵ R + 1;
        while R > P(K) do
            P(K + 1) ⟵ P(K); R ⟵ R − P(K); K ⟵ K + 1;
        P(K + 1) ⟵ R
        K ⟵ K + 1;
```

C.2 Enumerating non-isomorphic configurations

The enumeration of some simple configurations can be stated formally. For example, consider the case of M outlets and $n = M - 1$ firms. In this case, necessarily on firm has two outlets and the each one of all the others has one outlet. Without loss of generality let us consider that

Firm 1 is the one that has two outlets. Non-isomorphic configurations are thus those where the number of outlets between the two outlets of Firm 1. Their enumeration is straightforward and their number we can check that their number is equal to the integer part of $M/2$, i.e. $M/2$ when M is even and $(M-1)/2$ if M is odd (cf. corresponding cases in Figure 4).

Other cases are more complex. To illustrate, consider the case of two firms denoted F_1 and F_2, with F_1 having $m_1 = 7$ outlets and F_2 having $m_2 \geq 7$. We start by placing the m_2 outlets of F_2 along the circle. This leaves more then 7 locations where F_1 can locate its seven outlets. From Figure 2, F_1 has 15 possibilities to group its outlets. First, it can form a single group of seven outlets and place it in one available location. There is one possibility here. Then, it can form two groups and there are now three possibilities (6,1), (5,2) or (4,3) that correspond to Configurations 2, 3 and 5 in Figure 2, respectively. For each of these possibilities, we have the same problem discussed above and we need only to care about the distance (the number of outlets from F_2) between the two groups of the outlets to identify the number of non-isomorphic configurations. Thus, for each case it is the integer part of $m/2$. Next, we have the cases where F_1 forms three clusters and there are now four possibilities provided by Configurations 4, 6, 8 and 9 in Figure 2. The number of possibilities is now more complex because we have to find all pairs of distances that are distinct. The procedure should be continued for higher values. Some other cases are simple to enumerate. For example, when F_1 forms seven groups isolated outlets (Configuration 15 in Figure 2) we easily see that there only one corresponding non-isomorphic configuration. Deriving a formal expression for the general case is not straightforward, as well as the enumeration of all non-isomorphic configurations.

For our practical purpose, we adopt a Monte Carlo simulation approach. We generate a large number of configurations and iteratively test the new configurations that are not isomorphic to those that have been identified. The used algorithm performs the following operations:

1. The input data is the number of outlets per firm: m_i, for $i = 1, \ldots, n$. We have $M = \sum_{i=1}^{n} m_i$.
2. Create the list $\mathcal{L}$ of m_1 successive 1's, m_2 successive 2's, ..., m_n successive n's.
3. Initiate the list $\mathfrak{L}$ of non-isomorphic configurations to an empty list.
4. Generate a list of M quasi-random numbers in the interval $(0, 1)$.
5. Sort these numbers and collect the permutation inducing this permutation.
6. Apply this permutation to list $\mathcal{L}$ and call it Configuration C.
7. For each element of $\mathfrak{L}$ check whether it is isomorphic to C .
8. If the test is true go to step 4 (if max number of iterations is reached exit).
9. If the test is false add C to list $\mathfrak{L}$ (if max number of iterations is reached exit).

Upon exit this procedure returns $\mathfrak{L}$, the set of non-isomorphic configurations. To insure that all of these configurations have been drawn, this procedure should be applied with large number of iterations. In our case of nine and ten outlets. The process stops when step 4 is repeated 5,000 without drawing any new element in $\mathfrak{L}$.

To check that C is non-isomorphic to a configuration in $\mathfrak{L}$, we perform the following operations:

1. Check if there is a permutation that makes the two configurations identical;
2. Check if a clockwise rotation makes the two Configurations identical;
3. Check if a counter-clockwise rotation makes the two configurations identical.

Operation 1 is conducted for all possible rotations described in steps 2 and 3.

Appendix D The table of notations

Variables	Definitions	Values
L	Circumference of the circle	1
M	Total number of outlets	≥ 2
n	Number of firms	≥ 2
i	Index of the firms	$1, \ldots, n$
j	Index of the outlets	$1, \ldots, M$
$j+1$	Index of the successor of outlet j, with $M+1=1$	
$j-1$	Index of the predecessor of outlet j, with $1-1=M$	
m_i	Number of outlets managed by Firm i	$\lvert\mathfrak{F}_i\rvert$
$F(j)$	Index of the firm managing outlet j	$1, \ldots, n$
$\mathfrak{F}_i$	Set of outlets managed by Firm i	
$\mathbb{I}_{i,j}$	Binary variable equal to one if outlets i and j are managed by the same firm, zero otherwise.	
I_j	Binary variable for adjacency between j and $j+1$	$I_j = 1 + \mathbb{I}_{j,j+1}$
x	Locations over the circle	$[0, l]$
x_j	Location of outlet j	$[0, l]$
x_j^+	Distance between outlets j and $j+1$	$[0, l]$
x_j^-	Distance between outlets j and $j-1$	$[0, l]$
x_j^{++}	Distance between outlets $j-1$ and $j+1$	$x_j^- + x_j^+$
z_j	Location of the consumer indifferent between outlets j and $j+1$	
s_j	Market share of outlet j	$z_j - z_{j-1}$
p_j	Mill price of outlet j	
$\mathfrak{S}_j$	Strategy space for p_j	$[0, \infty)$
c	Transportation rate	1
$c\,(x-x_j)^2$	Transportation cost (consumer at x, outlet at x_j)	
$\widehat{p}_j(x)$	Delivered price	$p_j + c\,(x-x_j)^2$
π_j	Profit of outlet j	$p_j\, s_j$
Π_i	Profit Firm i	$\sum_{j\in\mathfrak{F}_i} \pi_j$

Table 3: Glossary. Notice that the notations in Sections 6 and 7 slightly departs from the one described in this table.

Appendix E Equilibria for configurations in Figure 8

	Prices			Trans. costs		Full prices		Profits		
Id	min	max	Avg	max	Avg	max	Avg	min	max	Avg
1	1.23	1.23	1.23	0.31	**0.10**	1.54	1.34	4.12	4.12	4.12
2	1.07	1.47	1.22	0.55	0.12	2.03	1.33	3.68	4.25	4.06
3	1.12	1.38	1.23	0.47	0.11	1.86	1.34	3.99	4.28	4.09
4	1.11	1.26	**1.21**	0.47	0.11	1.73	**1.32**	4.03	4.03	**4.03**
5	1.17	1.92	1.49	0.73	0.13	2.59	1.62	3.49	5.74	4.96
6	1.05	1.75	1.45	0.69	0.13	2.35	1.58	4.01	5.48	4.82
*7	1.05	2.56	1.62	1.02	0.20	3.58	1.81	2.36	6.92	5.39
8	1.21	1.60	1.47	0.56	0.12	2.14	1.59	3.85	5.41	4.89
9	1.77	1.95	1.86	0.64	0.14	2.59	1.99	5.49	7.58	6.19
*10	1.14	2.85	1.91	1.15	0.19	3.21	2.11	4.20	8.18	6.38
*11	1.46	2.34	2.02	0.91	0.21	3.26	2.23	5.47	9.23	6.72
12	1.23	2.25	1.75	0.88	0.14	3.13	1.88	5.36	6.74	5.82
13	0.90	2.47	1.80	1.08	0.18	3.10	1.98	5.37	7.26	6.00
14	1.37	2.37	1.77	0.85	0.14	2.70	1.91	4.75	6.98	5.91
15	2.16	2.60	2.40	0.70	0.15	3.30	2.56	7.75	8.14	8.01
*16	2.22	3.23	2.49	1.18	0.21	4.40	2.71	6.96	10.32	8.32
17	2.12	2.40	2.27	0.61	0.14	3.01	2.41	7.57	7.57	7.57
*18	2.22	3.23	2.49	1.18	0.21	4.40	2.71	6.96	10.32	8.32
*19	0.71	3.60	2.27	1.71	0.37	5.31	2.64	0.81	10.97	7.58
20	1.82	3.03	2.37	1.24	0.20	3.94	2.57	3.97	10.93	7.92
21	2.42	4.32	3.26	1.62	0.27	5.94	3.53	4.75	13.94	10.88
*22	2.86	4.47	3.39	1.57	0.28	6.02	3.67	6.08	14.94	11.31
*23	3.22	4.04	3.57	1.40	0.30	5.44	3.86	7.66	14.00	11.89
*24	3.98	5.11	4.47	1.27	0.25	6.38	4.72	13.39	17.94	14.91
^{s}25	8.74	8.81	**8.75**	1.72	**0.45**	10.52	**9.20**	29.15	29.15	**29.15**

Table 4: Location-then-price equilibrium and summary statistics for the five non-isomorphic configurations with three firms, two outlets each. The stars in the first column indicate that, at least, two outlets of the same firm merge at location equilibrium. The "s" on the left of Configuration 25 refers to jumping equilibria. For average values, the minimum and maximum are highlighted in bold. The numerical computations use $L = 10$ (and $c = 1$) to avoid small values of the aggregates.