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Zhi-Chun Li [a], De-Ping Yu [a], André de Palma [b]

[a] *School of Management, Huazhong University of Science and Technology, Wuhan 430074, China*

[b] *CY Cergy-Paris University, Thema, France*

Abstract

We propose an analytically solvable model for the residential location choices of heterogeneous households in a linear monocentric city corridor with bottleneck congestion. Residents have heterogeneous income, and make a joint choice of residential location and departure time. There is a bottleneck with a fixed or with a stochastic location between central downtown and adjacent suburb of the city. The urban system equilibrium and the effects of bottleneck capacity expansion on the city system are analytically investigated, together with the design of the bottleneck capacity. We show that the residents spatially sort themselves along the city corridor from CBD outward in a descending order of their values of time. Expanding bottleneck capacity leads to an increase in the commuting costs of the downtown residents but a decrease in the commuting costs of the suburban residents. All residents of the city benefit from the bottleneck capacity expansion, with the highest benefit for the relatively mid-income residents, and the lowest benefit for the lowest-income or the highest-income residents, depending on the status quo of the bottleneck capacity. Expanding the bottleneck capacity leads to urban sprawl, and a decrease in total net land rent. Ignoring the effects of the bottleneck capacity expansion on the urban spatial structure overestimates the social surplus. The bottleneck location's stochasticity smoothes the residential distribution, increases the system's transportation cost, and decreases household utility and social surplus.

Keywords: Residential location choice; linear monocentric city; heterogeneous residents; stochastic bottleneck; bottleneck capacity expansion.

JEL classification: R13, R14, R41, R42

1. Introduction

It has been widely recognized that commuting cost significantly affects households' residential location choices, whereas traffic congestion dynamics due to diversities of commuters' departure time choices directly affect commuting cost. Dynamic traffic congestion during peak periods may thus be a vital factor influencing households' residential location choices. Therefore, as pointed out by Ross and Yinger (2000), there is a need to incorporate the interactions between traffic congestion dynamics (time dimension) and households' residential location choices (spatial dimension) in the urban models.

In the literature, there are several studies involving both dynamic bottleneck congestion and urban spatial equilibrium models. These studies can be classified into two major types of modeling methods: discrete spatial approaches (e.g., Arnott, 1998; Xu et al., 2018; Fosgerau and Kim, 2019) and continuum spatial approaches (e.g., Fosgerau and de Palma, 2012; Gubins and Verhoef, 2014; Takayama and Kuwahara, 2017; Fosgerau et al., 2018; Takayama, 2020). The discrete approaches usually concern two discrete residential zones (a downtown and a suburb) connected by a bottleneck-constrained highway. Urban residents make residential location choices and trip schedules between the two zones. The spatial aspects (e.g., housing/land space and price at each location) within each zone in the discrete approaches are generally neglected in order to focus on the inter-zone transportation. By contrast, the continuous approaches consider a city as a continuum, in which the spatial structure of the city is explicitly treated. The discrete approaches can thus be seen as a special case of the continuous ones.

We have summarized in Table 1 some major contributions to the combined model of dynamic bottleneck congestion and residential location choice in terms of the modeling approach, form of solution, resident's heterogeneity, stochasticity of bottleneck locations, and bottleneck capacity expansion and design. Table 1 shows that most of the existing studies considered residents to be homogeneous in terms of income or value of time (VOT). More realistically, residents' income/VOT levels are generally different, depending on various factors, such as occupations and skills. This leads to differential tastes in the residential location choice and the response to transport policies (Li and Peng, 2016; Li et al., 2020). Two recent studies by Takayama and Kuwahara (2017) and Takayama (2020) considered the resident's VOT heterogeneity in a discrete form, in which the city's residents are classified into limited discrete groups, and the residents in the same group are assumed to be homogeneous. However, their studies cannot derive a closed-form solution, and thus the relations among variables or parameters cannot explicitly be identified.

Table 1 Contributions to a combination of dynamic bottleneck and residential location choice.

Reference	**Modeling approach**	**Analytical solution**	**Income heterogeneity**	**Stochastic bottleneck location**	**Bottleneck capacity expansion and design**
Arnott (1998)	Discrete	×	×	×	×
Fosgerau and de Palma (2012)	Continuous	×	×	×	×
Gubins and Verhoef (2014)	Continuous	×	×	×	×
Takayama and Kuwahara (2017)	Continuous	√	Discrete	×	×
Fosgerau et al. (2018)	Continuous	×	×	×	×
Xu et al. (2018)	Discrete	√	×	×	×
Fosgerau and Kim (2019)	Discrete	×	×	×	×
Takayama (2020)	Continuous	√	Discrete	×	×
This paper	Continuous	√	Continuous	√	√

The previous studies about the combined issue of the dynamic bottleneck and the residential location choice often considered a deterministic case. Although one can find some bottleneck model studies involving the randomness of bottleneck capacity, travel demand, or travel time (e.g., see Arnott et al., 1999; Fosgerau and Lindsey, 2013; Xiao et al., 2014, 2015; Tian and Huang, 2015), the bottleneck's location was usually assumed to be fixed, like for bridges and tunnels. However, in practice, the bottleneck on a road may be caused by various random factors, such as adverse weather, road works, traffic accidents, and vehicle lane change. The location of the bottleneck may thus stochastically change on a roadway (e.g., a stretch of narrow highway with a larger capacity in its upstream and downstream links) due to random incidents by time of day, day of week, and season. In essence, the deterministic case of bottleneck congestion location falls into a class of recurrent congestion, whereas the stochastic case belongs to non-recurrent congestion. Both types of congestion are widespread in the real life. Skabardonis et al. (2003) conducted an empirical study of three highways in Los Angeles and San Francisco, and found that 13%-30% of the rush-hour delays belong to non-recurrent congestion. A report released by the World Bank in 2010 showed that for 11 corridors in Cairo, Egypt, 60% of the commuting delay costs were caused by nonrecurrent congestion.[1] A recent program implemented by the U.S. Department of Transportation further disclosed that about half of the congestion experienced by Americans is nonrecurrent.[2] Therefore, besides the fixed bottleneck, there is also a need to investigate the case of stochastic bottleneck locations, which is never involved in the previous related studies.

In addition, the effects of the bottleneck capacity expansion on the corridor system were not analytically explored, and the analytical expression for the optimal bottleneck capacity that maximizes the social surplus was not determined in the previous studies. Although a simulation method can be used to numerically serve these purposes, the conclusions obtained about the relations among the bottleneck capacity expansion, household relocation behavior (household utility), and the system performance (e.g., housing/land price, city size, social surplus) may not be robust because they depend very much on the values of model's input parameters. Consequently, it is meaningful to analytically investigate the effects of the bottleneck capacity expansion, and to derive the analytical solution for the optimal bottleneck capacity.

In view of the above discussion, this paper addresses the combined issue of the residential location choices of heterogeneous households and the dynamic bottleneck congestion in a linear monocentric city corridor. The main contributions of this paper are threefold. First, an analytically continuous solvable model that simultaneously incorporates the bottleneck congestion and residential location choices of heterogeneous households is proposed. The heterogeneity of residents' VOTs is considered as a continuous form. The urban system equilibrium is formulated as a solution of a system of differential equations. Second, in our setting, bottleneck congestion may occur at a fixed location (recurrent congestion) or a stochastic location on a road segment (non-recurrent congestion). The stochastic bottleneck location case has never been investigated in the literature. This paper attempts to fill this research gap. Third, the effects of bottleneck capacity expansion on the residents and the urban system are investigated, and the optimal bottleneck capacity that maximizes the social surplus of the system is determined. Analytical solutions for the bottleneck capacity

[1] https://documents.worldbank.org/en/publication/documents-reports/documentdetail/650141468248419267/egypt-cairo-traffic-congestion-study-phase-1.

[2] https://ops.fhwa.dot.gov/program_areas/reduce-non-cong.htm.

expansion and the optimal bottleneck capacity design are derived for the fixed bottleneck location case. The proposed methodology provides a new framework for investigating the urban system equilibrium with heterogeneous residents and bottleneck congestion and for identifying the effects of various urban policies on the urban spatial structure.

The remainder of this paper is organized as follows. In the next section, the urban system equilibrium with heterogeneous residents and a fixed bottleneck is formulated. Section 3 carries out the comparative statics analysis of bottleneck capacity expansion. Section 4 designs the optimal bottleneck capacity to maximize the social surplus of the system. In Section 5, the urban system equilibrium is analyzed for the stochastic bottleneck case. In Section 6, a numerical study is provided to illustrate the properties and applications of the proposed model. Section 7 concludes this paper and provides suggestions for further studies. Several proofs and mathematical derivations are given in the appendices.

2. Urban system equilibrium with heterogeneous residents and a fixed bottleneck

2.1. Basic setup

Consider a transportation corridor located in a closed, linear, and monocentric city, with a population size of *N*. The city's residents continuously distribute along the corridor. Residents have different income levels and thus different values of time (VOT). We represent α as the resident's VOT, with $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ in which $\underline{\alpha}$ and $\overline{\alpha}$ are the lower and upper bounds of resident's VOT, respectively. Suppose that there is a bottleneck with a capacity of q at location a of the corridor from the CBD, as shown in Fig. 1. This bottleneck divides the corridor into two areas: downtown area (i.e., $[0, a)$), and suburb area (i.e., $[a, B]$). Traffic congestion during the commuting peak period occurs at the bottleneck due to its limited capacity. The length of the corridor (or city boundary) is *B*, endogenously determined.

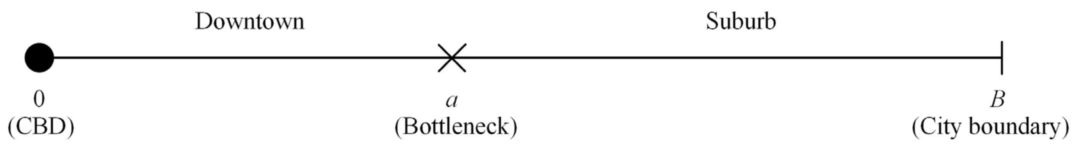

Fig. 1. A transportation corridor with a fixed bottleneck.

All job opportunities are located in the CBD area. Every morning, commuters travel from their home locations to the CBD along the bottleneck-constrained corridor. The commuters' travel costs depend on their home locations in the corridor. Commuters originating in the downtown do not need to pass through the bottleneck, and thus do not face bottleneck congestion during their commutes. However, the commuters originating in the suburb suffer from bottleneck congestion during their commutes. In the following, we define the commuting costs of the commuters originating in the downtown and suburban areas. For ease of presentation, the words "commuter", "resident" and "household" are interchangeable in this paper.

Since the commuters residing in the downtown area do not encounter the bottleneck during their commutes, all of them prefer punctual arrivals at the workplace without causing any schedule delay so as to minimize their commuting costs. Their commuting costs thus include only the free-flow travel time cost along the corridor. Let $c_D(x,\alpha)$ be the (one-way) commuting cost of the commuters with VOT α residing at location x of the corridor ($0 \le x \le a$) from the CBD in the downtown area. The downtown commuting cost $c_D(x,\alpha)$ is given as

$$c_D(x,\alpha) = \tau x\alpha, \tag{1}$$

where the subscript "*D*" represents the downtown area, and τ denotes the free-flow travel time per unit of distance along the corridor, and thus τx represents the free-flow travel time from location x to the CBD. The VOT parameter α is used to convert time units into equivalent monetary cost units.

By contrast, the commuters residing in the suburban area need to traverse the bottleneck on their way to work, thus incurring a queuing delay. Similar to most of bottleneck congestion studies, the queue at the bottleneck is assumed to be vertical and has no physical length.[3] For simplicity, we assume that no late arrivals are permitted, and thus commuters may arrive early or punctually. Therefore, their departure time choices will be based on a trade-off between the bottleneck queuing delay and the schedule delay of arriving early. Let $c_S(x,\alpha)$ be the commuting cost of the commuters with VOT α residing at location x of the corridor

[3] For the model with considering physical vehicle queue length, readers can refer to Mun (1999).

($a \le x \le B$). According to the bottleneck model theory with continuous VOT distribution (see e.g., Xiao and Zhang, 2014; Xiao et al., 2011; Van den Berg and Verhoef, 2011), $c_S(x,\alpha)$ can be expressed as

$$c_S(x,\alpha) = \frac{\hat{N}_S}{q}\beta + \tau x\alpha = \frac{\hat{N}_S}{q}\eta\alpha + \tau x\alpha, \tag{2}$$

where the subscript "*S*" represents the suburban area. $\hat{N}_S$ is the total number of commuters in the suburban area (i.e., those who need to traverse the bottleneck), which is endogenously determined. β is the value of early arrival time. For presentation purpose, we denote by η the ratio of the value of early arrival time β to the VOT α, i.e., $\eta = \beta/\alpha$. In this paper, we assume η is a constant across residents, as in some previous bottleneck models (e.g., Vickrey, 1973; Arnott et al., 1994; Xiao and Zhang, 2014). The first term on the right-hand side of Eq. (2) is the equilibrium bottleneck congestion cost (i.e., the sum of the bottleneck congestion delay cost and the schedule delay cost of early arrival). The second term is the free-flow travel time cost. The detailed derivation of Eq. (2) is provided in Appendix A.

To sum up, the commuting cost, $c(x,\alpha)$, of the commuters with VOT α at any location *x* of the corridor can be represented as

$$c(x,\alpha) = \begin{cases} c_D(x,\alpha), & \text{for } 0 \le x < a, \\ c_S(x,\alpha), & \text{for } a \le x \le B. \end{cases} \tag{3}$$

2.2. Equilibrium household residential distribution

The traditional urban models usually assume that all households in the city are homogeneous in terms of their income or VOTs. Income across households varies and depends on their occupations and skills. This translates into different values of time, explicitly modeled in this paper, and into different budget constraints. In the following, we explore the difference of households' residential location choice behavior due to their heterogeneities in VOT.

Household utility depends on land/housing consumption and non-land goods consumption. The price of the non-land good is normalized to one, and that of the land is endogenously determined by the model. The utility of a household with VOT α residing at location *x*, denoted as $u(x,\alpha)$, follows a hyperbolic utility function, expressed as

$$u(x,\alpha) = z(x,\alpha) - \frac{k}{2h(x,\alpha)}, \tag{4}$$

where $z(x,\alpha)$ is the consumption of the non-land goods (numéraire), measured in monetary units. $h(x,\alpha)$ is the land consumption, measured in land areas consumed. A positive constant, *k*, indicates the preference of households for land. A larger value of *k* represents a stronger preference for land consumption, and vice versa. The second term on the right-hand side of Eq. (4) represents the households' utility derived from land consumption, measured in monetary units. Such a hyperbolic utility function has been adopted in some previous household residential location choice models (see e.g., Mossay and Picard, 2011; Picard and Tabuchi, 2013; Blanchet et al., 2016; Akamatsu et al., 2017; Picard and Tran, 2021).[4] Eq. (4) measures the utility in monetary units, i.e., cardinal utility, which facilitates the comparison of utilities of households with different VOTs, and the calculation of social surplus.

Since households' income is spent on non-land goods consumption, land consumption, and commuting, the household income budget constraint can be expressed as

$$w(\alpha) = z(x,\alpha) + p(x)h(x,\alpha) + c(x,\alpha), \tag{5}$$

where $w(\alpha)$ is the income of the households with VOT α. Following Becker (1965) and Small (2012), households' income is assumed to be proportional to their VOTs, i.e., $w(\alpha) = \varphi\alpha$ in which φ is a positive constant. $p(x)$ is the rental price per unit of land, and thus $p(x)h(x,\alpha)$ is the land consumption of the household with VOT α at location *x*, measured in monetary units. It is assumed that the condition $w(\alpha) - c(x,\alpha) > 0$ always holds, meaning that household income can at least cover the commuting cost regardless of residential location and VOT α.

Each household chooses a residential location, land area, and amount of non-land goods consumption to maximize its utility subject to the income budget constraint. From Eqs. (4) and (5), the utility maximization problem for the household with VOT α residing at location *x* can be expressed as

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[4] The hyperbolic and logarithmic preferences (see e.g., Beckmann, 1976; Fujita and Thisse, 2002) for the land are two frequent instances of the same class of preferences $(h^{1-\rho} - 1)/(1-\rho)$ where ρ = 2 and $\rho \to 1$ respectively, which yield iso-elastic demands for residential space with price elasticity equal to 1/2 and 1, respectively. Therefore, the present hyperbolic preference represents an intermediate case between Beckmann's demand and the inelastic demand for residential space that is standard in urban economics.

$$\max_h \; u(x,\alpha) = w(\alpha) - p(x)h(x,\alpha) - c(x,\alpha) - \frac{k}{2h(x,\alpha)} . \tag{6}$$

From the first-order optimality condition of maximization problem (6), i.e., $du/dh = 0$, one can easily obtain the expression of the consumed land area as

$$h(x) = \sqrt{\frac{k}{2p(x)}}. \tag{7}$$

Eq. (7) shows that for a given location x, the demand for land area $h(x)$ is only related to the land rental price $p(x)$ and independent of household's VOT α, i.e., the households with any VOTs at a given location consume the same amount of land. For this reason, we skip the argument α in the function $h(\cdot)$. Substituting Eq. (7) into Eq. (6) yields the household indirect utility as

$$u(x,\alpha) = w(\alpha) - c(x,\alpha) - \sqrt{2kp(x)} . \tag{8}$$

At equilibrium, no household has an incentive to unilaterally change its residential location, i.e., $\partial u(x,\alpha)/\partial x = 0$ holds. From Eq. (8), we have

$$\frac{dp(x)}{dx} = -\tau\alpha\sqrt{\frac{2p(x)}{k}} < 0 . \tag{9}$$

Eq. (9) shows that as the distance from the CBD increases, the land rental price decreases. Note that for a given residential location, the commuting cost of a household is determined by Eqs. (1) and (2). According to Eqs. (1), (2) and (9), given the land/housing area consumed, one can find that for any commuter with VOT α, the ratio, $c(x,\alpha)/p(x)h(x)$, of the travel cost $c(x,\alpha)$ to the housing consumption cost $p(x)h(x)$ increases with the increase in the distance x from the CBD. That is, there is a trade-off between commuting cost and land rental price: a residential location closer to the CBD can reduce the commuting cost but suffers a higher housing cost, whereas a residential location farther from the CBD may benefit from a lower housing cost but bears a higher commuting cost.

Using the indirect utility function and the classical bid-rent theory in urban economics (Fujita, 1989), we immediately derive the following important property about households' residential location choices.

Proposition 1. At equilibrium, households spatially sort themselves in a descending order of VOTs from the CBD outward, i.e., a household with a higher VOT would reside closer to the CBD, while a household with a lower VOT would reside closer to the suburb.

The proof of Proposition 1 is relegated to Appendix B. Proposition 1 indicates that the household's residential location is determined by its VOT. Specifically, the higher a household's VOT is, the stronger its willingness to live closer to the CBD, and vice versa. This is because households with a higher VOT prefer to avoid a higher commuting time cost compared to a higher land rental price. A similar residential sorting has also been shown in Takayama and Kuwahara (2017), but with a discrete households' VOT distribution. This sorting is also optimal, and specific to some Chinese and European cities, like Beijing, Shanghai, Paris or London where high-income residents mainly live near or in the CBD.

2.3. Equilibrium household residential density

We have derived the law for household residential sorting along the corridor in the previous section. In this section, we further derive the household residential density along the corridor. According to Eqs. (1) and (2), the commuting cost $c(x,\alpha)$ for the downtown and suburban areas is discontinuous at the bottleneck (an upward jump). Thereby, the household residential densities (i.e., the number of households per unit of land area) for these two areas are different, which are in turn derived as follows.

Let $n_D(x)$ be the household residential density at location x in the downtown area ($0 \le x < a$), and $N_D(x)$ be the cumulative number of households from the CBD to location x, with $dN_D(x)/dx = n_D(x)$. The land supply at any location of the city is assumed to be a constant. Without loss of generality, it is normalized to 1. The relationship between $h(x)$ and $n_D(x)$ can thus be expressed as:

$$h(x) = \frac{1}{n_D(x)}. \tag{10}$$

Substituting Eq. (10) into Eq. (7), the land rental price $p(x)$ can be expressed as a function of the residential density $n_D(x)$:

$$p(x) = \frac{k}{2}\left(n_D(x)\right)^2. \tag{11}$$

Substituting Eq. (11) into the equilibrium condition Eq. (9), one can obtain the following first-order ordinary differential equation of the residential density:

$$\tau\alpha + k\frac{dn_D(x)}{dx} = 0. \tag{12}$$

It is assumed that households' VOTs are uniformly distributed, i.e., $\alpha \sim U[\underline{\alpha}, \overline{\alpha}]$. Since households are spatially distributed in a descending order of VOTs outward (see Proposition 1), there is a one-to-one correspondence between location x and VOT α. The VOTs of the households living at the CBD and at the city boundary are exactly $\overline{\alpha}$ and $\underline{\alpha}$, respectively. Let α^* be the VOT of the households at the bottleneck (also referred to as critical VOT). It is endogenously determined. For ease of presentation, we introduce a concept of VOT's density as the number of households per unit of VOT. Given that VOT is uniformly distributed, the VOT's density is thus a constant, denoted as b, i.e., $b = N/\left(\overline{\alpha} - \underline{\alpha}\right)$. We can then represent the total number of households in the downtown and suburban areas as

$$\hat{N}_D = b(\overline{\alpha} - \alpha^*), \text{ and } \hat{N}_S = b(\alpha^* - \underline{\alpha}), \tag{13}$$

where $\hat{N}_D$ and $\hat{N}_S$ are the number of households residing in the downtown and suburb, respectively.

For any downtown location x with VOT α, the cumulative number of households, $N_D(x)$, from CBD to x is $N_D(x) = b(\overline{\alpha} - \alpha)$. Substituting it into Eq. (12) to remove variable α, and using the relationship $dN_D/dx = n_D$, one can obtain the following important second-order ordinary differential equation for the cumulative number of households $N_D(x)$:

$$\frac{d^2N_D(x)}{dx^2} - \frac{\tau}{kb}N_D(x) = -\frac{\tau\overline{\alpha}}{k}, x \in (0, a). \tag{14}$$

Note that the cumulative number of households over an infinitesimal slice at the CBD is 0, and the cumulative number of households from the CBD to the bottleneck location a is $\hat{N}_D$, as shown in Eq. (13). Hence, the boundary conditions for Eq. (14) is $N_D(0) = 0$ and $N_D(a) = \hat{N}_D = b(\overline{\alpha} - \alpha^*)$. The analytical solution for $N_D(x)$ in Eq. (14) can thus be further derived as

$$N_D(x) = c_1 e^{rx} + c_2 e^{-rx} + b\overline{\alpha},\ x \in (0, a), \tag{15}$$

where r, c_1, and c_2 are constants, with

$$r = \sqrt{\frac{\tau}{kb}},\ \ c_1 = \frac{b\overline{\alpha} - e^{ra} b\alpha^*}{e^{2ra} - 1}, \text{ and } \ c_2 = \frac{e^{ra}(-e^{ra} b\overline{\alpha} + b\alpha^*)}{e^{2ra} - 1}. \tag{16}$$

From Eq. (16), $r > 0$ and $c_2 < 0$ hold, while the sign of c_1 is ambiguous. Taking the first-order derivative of $N_D(x)$ in Eq. (15) with respect to *x*, one can obtain the residential density function of any location x in the downtown area as

$$n_D(x) = c_1 r e^{rx} - c_2 r e^{-rx},\ x \in (0, a). \tag{17}$$

Similarly, for the suburban area, a one-to-one correspondence exists between location *x* and VOT α. Let $n_S(x)$ be the household residential density at location x ($a < x < B$) with VOT α in the suburban area, and $N_S(x)$ be the cumulative number of households from the CBD to location x with VOT α. $N_S(x)$ can be expressed as

$$N_S(x) = b(\overline{\alpha} - \alpha). \tag{18}$$

This leads to a second-order differential equation for $N_S(x)$:

$$\frac{d^2 N_S(x)}{dx^2} - \frac{\tau}{kb} N_S(x) = -\frac{\tau\overline{\alpha}}{k},\ x \in (a, B), \tag{19}$$

with the boundary conditions: $N_S(a) = \hat{N}_D = b(\overline{\alpha} - \alpha^*)$, $N_S(B) = N = b(\overline{\alpha} - \underline{\alpha})$, and $p(B) = r_A$. Here r_A is the exogenous agricultural land rent, and $p(B)$ is the land rental price at the city boundary *B*, equal to $kn_S^2(B)/2$ by Eq. (11). These boundary conditions mean that the cumulative number of households from the CBD to the bottleneck is the total number of households $\hat{N}_D$ in the downtown area, and that from the CBD to the city boundary is the total number of households N in the city corridor, and the land rental price at the city boundary is equal to the exogenous agricultural land rent r_A.

Based on Eq. (19) and the associated boundary conditions, the cumulative number of households $N_S(x)$ can be solved as

$$N_S(x) = c_3 e^{rx} + c_4 e^{-rx} + b\overline{\alpha},\ \ x \in (a, B), \tag{20}$$

where c_3 and c_4 are constants, given by

$$c_3 = e^{-ra}\frac{-b\alpha^* + \sqrt{(b\alpha^*)^2 - (b\underline{\alpha})^2 + 2r_A/(kr^2)}}{2} \text{ and } c_4 = e^{ra}\frac{-b\alpha^* - \sqrt{(b\alpha^*)^2 - (b\underline{\alpha})^2 + 2r_A/(kr^2)}}{2}. \quad (21)$$

From Eq. (21), $c_4 < 0$ holds, while the sign of c_3 is undetermined.

The resultant city boundary B can be solved as

$$B = \frac{1}{r}\ln\left(\frac{-b\underline{\alpha} + \sqrt{2r_A/(kr^2)}}{2c_3}\right). \quad (22)$$

Since $dN_S/dx = n_S$, one can obtain the residential density at any suburban location x as

$$n_S(x) = c_3 re^{rx} - c_4 re^{-rx}, x \in (a, B). \quad (23)$$

Thus far, we have solved the equilibrium household residential density in the downtown and suburban areas for the given critical VOT α^* that is the VOT of the households at the bottleneck. In the following, we determine the critical VOT α^*. Substituting the expression for $p(x)$ into Eq. (8) yields the indirect utilities for the downtown and suburban residents as

$$u_i(x,\alpha) = w(\alpha) - c_i(x,\alpha) - kn_i(x),\ i = D, S. \quad (24)$$

Note that at the equilibrium, the utilities of households with VOT α^* at the bottleneck are equal regardless of the downtown and suburb areas, i.e., $u_D(a,\alpha^*) = u_S(a,\alpha^*)$. According to Eqs. (1), (2), (17), (23), and (24), the equilibrium condition for VOT α^* can be expressed as

$$kr\frac{e^{2ra}+1}{e^{2ra}-1}b\alpha^* + kr\sqrt{(b\alpha^*)^2 - (b\underline{\alpha})^2 + 2r_A/(kr^2)} + \frac{\eta}{q}b(\alpha^* - \underline{\alpha})\alpha^* = kr\frac{2e^{ra}b\overline{\alpha}}{e^{2ra}-1}. \quad (25)$$

The critical VOT α^* can be uniquely determined by Eq. (25) because the left-hand side of Eq. (25) is monotonously increasing with regard to α^*. Once α^* is solved by Eq. (25), one can then uniquely determine the equilibrium cumulative number of households, and the equilibrium residential density at any location according to Eqs. (15), (17), (20), and (23). In light of the above, we have the following property. Its proof is relegated to Appendix C.

Proposition 2. For the fixed bottleneck case, the equilibrium household residential density and land rental price along the city's corridor, determined by Eqs. (17), (23), and (11), respectively, monotonically decrease with the distance from the CBD, and are discontinuous at the bottleneck (a downward jump).

Proposition 2 indicates that the bottleneck congestion cost does have an important impact on the land/housing price, as expected. The bottleneck congestion reduces the accessibility to the CBD for the suburban households, thus lowering their willingness to pay for the suburban land. Such discontinuity of the accessibility at the bottleneck eventually causes the discontinuity (a downward jump) of the land/housing price and residential density. Similar discontinuity has also been observed when implementing cordon toll. For example, Mun et al. (2003) and De Lara et al. (2013) showed that the cordon toll causes a discontinuity in the household density and the land rent. Tang (2016) empirically found that after imposing a cordon toll in the Western Extension Zone London, households moving into the cordoned zone pay more for houses than those outside the cordoned zone, which means a downward discontinuity of housing price and residential density from the cordon outward.

2.4. Equilibrium residential location and utility

In the previous section, we have determined the equilibrium household residential density. However, for an arbitrary resident with VOT α, where he/she resides and how about the utility he/she gains at equilibrium are still unresolved. In what follows, we derive the equilibrium residential location and utility for each resident.

Due to a one-on-one correspondence between VOT α and location x, we denote $x_D(\alpha)$ and $x_S(\alpha)$ as the residential location of the households with any VOT α in the downtown and suburban areas, respectively. Combining Eqs. (13) and (15), we can solve $x_D(\alpha)$ as

$$x_D(\alpha) = \frac{1}{r}\ln\left(\frac{-b\alpha + \sqrt{(b\alpha)^2 - 4c_1c_2}}{2c_1}\right), \alpha \in (\alpha^*, \overline{\alpha}), \tag{26}$$

where α^* is determined by Eq. (25), and c_1 and c_2 are given by Eq. (16).

Combining Eqs. (13) and (20) can yield the solution of $x_S(\alpha)$:

$$x_S(\alpha) = \frac{1}{r}\ln\left(\frac{-b\alpha + \sqrt{(b\alpha)^2 - (b\underline{\alpha})^2 + 2r_A/(kr^2)}}{2c_3}\right), \alpha \in (\underline{\alpha}, \alpha^*), \tag{27}$$

where c_3 and c_4 are given by Eq. (21).

From Eqs. (26) and (27), one can easily obtain $dx_i(\alpha)/d\alpha < 0$, $i = D, S$. This means that the residential location is negatively related to VOT α, regardless of the downtown or suburban area. This is consistent with Proposition 1.

We denote $u_D(\alpha)$ as the utility of the households with any VOT α in the downtown area, and $u_S(\alpha)$ as the utility of the households with any VOT α in the suburban area. From Eq. (24), the equilibrium utilities $u_D(\alpha)$ and $u_S(\alpha)$ can be given as

$$\begin{cases} u_D(\alpha) = w(\alpha) - c_D\left(x_D(\alpha), \alpha\right) - kn_D\left(x_D(\alpha)\right), \text{ for } \alpha \in (\alpha^*, \overline{\alpha}), \\ u_S(\alpha) = w(\alpha) - c_S\left(x_S(\alpha), \alpha\right) - kn_S\left(x_S(\alpha)\right), \text{ for } \alpha \in (\underline{\alpha}, \alpha^*), \end{cases} \tag{28}$$

where n_D and n_S are given by Eqs. (17) and (23), and x_D and x_S are given by Eqs. (26) and (27), respectively.

Based on the above discussion, the households with different VOTs have different tastes for residential location and land consumption. In particular, low-VOT households bear a relatively low land/housing price since they reside farther from the CBD. However, they suffer a long commuting distance. This raises one interesting question: how do the fractions of a household's income devoted to commuting (i.e., $c(x,\alpha)/w(\alpha)$) and to housing consumption (i.e., $p(x)h(x)/w(\alpha)$) change with the household's VOT/income? The following proposition answers this question, and its proof is relegated to Appendix D.

Proposition 3. Income allocations of heterogeneous households between commuting costs and housing consumption have the following properties:

(i) For the city system, the residents with a higher VOT will devote a smaller fraction of their income to commuting, compared to those with a lower VOT.

(ii) For the downtown area, as $c_1 = 0$, all the downtown residents would have the same fraction of income for housing; as $c_1 > 0$ (or $c_1 < 0$), the downtown residents with a higher VOT will devote a smaller (or a bigger) fraction of their income to housing. For the suburban area, as $c_3 = 0$, all the suburban residents would have the same fraction of income for housing; as $c_3 > 0$ (or $c_3 < 0$), the suburban residents with a higher VOT will devote a smaller (or a bigger) fraction of their income to housing.

As stated in Proposition 1, the low-income households reside farther from the CBD. Hence, the low-income residents would definitely use a bigger fraction of their income for commuting than the high-income residents due to their longer travel distances, as shown in Proposition 3. However, the relationship between the ratio of housing/land consumption to income and the VOT is ambiguous, depending on the signs of the coefficients c_1 and c_3. In fact, from Eqs. (16), (21) and (25) we can judge that both c_1 and c_3 increase with the exogenous agricultural land rent r_A, and vice versa. Accordingly, given the downtown or suburban area, for an enough big r_A that leads to positive values of c_1 and c_3, the households with a lower VOT (those who reside farther from the CBD) will use a greater fraction of income for housing compared to the households with a higher VOT; and for a sufficiently small r_A causing negative values of c_1 and c_3, the households with a higher VOT (those who reside closer to the CBD) will use a greater fraction of income for housing compared to the households with a lower VOT.

3. Comparative statics analysis of bottleneck capacity expansion

The bottleneck congestion is a deadweight loss for the society, and the bottleneck capacity expansion is considered to be an efficient measure to alleviate the bottleneck congestion. In a long run, the bottleneck capacity expansion may change household's commuting schedule and residential location choice, and thus the urban spatial structure. This section aims to look at the effects of bottleneck capacity expansion.

3.1. Effect on residential migration, bottleneck congestion, and commuting cost

It is plausible that the bottleneck capacity expansion may lead some residents in the downtown area to have a motivation to migrate into the suburban area due to improved bottleneck capacity. To confirm this, we check the sign of the marginal effect of capacity expansion on the critical VOT using Eq. (25), expressed as

$$\frac{d\alpha^*}{dq}=\left(kr\frac{e^{2ra}+1}{e^{2ra}-1}b+kr\frac{b^2\alpha^*}{\sqrt{(b\alpha^*)^2-(b\underline{\alpha})^2+2r_A/(kr^2)}}+\frac{\eta b(2\alpha^*-\underline{\alpha})}{q}\right)^{-1}\frac{\eta b(\alpha^*-\underline{\alpha})\alpha^*}{q^2}>0\,. \quad (29)$$

This means that as the bottleneck capacity expands, the critical VOT α^* of households at the bottleneck increases. From Proposition 1, households spatially sort themselves in a descending order of VOTs from the CBD outward. This shows that some high-VOT households would like to migrate into the suburban area when the bottleneck capacity increases. As a result, the number of commuters passing through the bottleneck during the commuting peak period increases due to the increased number of households in the suburb. This raises an interesting question of whether the bottleneck congestion during the peak period is alleviated after the bottleneck capacity expansion.

Note that the peak-period duration $\hat{N}_S/q$, which is equal to $b(\alpha^* - \underline{\alpha})/q$, can be used as a proxy of the bottleneck congestion degree. In order to examine the effect of the bottleneck capacity expansion on the bottleneck congestion degree, one needs to check the sign of $d\left(b(\alpha^* - \underline{\alpha})/q\right)/dq$. From Eq. (25), one can obtain:

$$\frac{d\left(b(\alpha^* - \underline{\alpha})/q\right)}{dq} = -\frac{1}{\eta\alpha^*}\left(kr\frac{e^{2ra}+1}{e^{2ra}-1}b + kr\frac{b^2\alpha^*}{\sqrt{(b\alpha^*)^2 - (b\underline{\alpha})^2 + 2r_A/(kr^2)}} + \frac{\eta}{q}b(\alpha^* - \underline{\alpha})\right)\frac{d\alpha^*}{dq} < 0. \tag{30}$$

This implies that the bottleneck congestion duration is reduced after the bottleneck capacity expansion.

We now examine the effects of the bottleneck capacity expansion on the residential locations and commuting costs of residents at any locations along the corridor. From Eqs. (26) and (27), we can derive the change of residential locations per unit of capacity increase for the downtown households as

$$\frac{dx_D(\alpha)}{dq} = \frac{e^{ra}b}{e^{2ra}-1}\frac{-2c_1c_2 - 2c_1^2 + (b\alpha)^2 - b\alpha\sqrt{(b\alpha)^2 - 4c_1c_2}}{rc_1\left(-b\alpha + \sqrt{(b\alpha)^2 - 4c_1c_2}\right)\sqrt{(b\alpha)^2 - 4c_1c_2}}\frac{d\alpha^*}{dq} > 0, \tag{31}$$

and that for the suburban households as

$$\frac{dx_S(\alpha)}{dq} = \frac{1}{r}\frac{b}{\sqrt{(b\alpha^*)^2 - (b\underline{\alpha})^2 + 2r_A/(kr^2)}}\frac{d\alpha^*}{dq} > 0. \tag{32}$$

Based on Eqs. (1), (2), and (29)-(32), we further obtain the change of commuting costs per unit of capacity increase for the downtown households as

$$\frac{dc_D(x,\alpha)}{dq} = \tau\alpha\frac{dx_D(\alpha)}{dq} > 0, \tag{33}$$

and that for the suburban households as

$$\frac{dc_S(x,\alpha)}{dq} = -\frac{\alpha}{\alpha^*}\left(krb\frac{e^{2ra}+1}{e^{2ra}-1}+\frac{\eta}{q}b(\alpha^*-\underline{\alpha})\right)\frac{d\alpha^*}{dq}<0, \tag{34}$$

where $dx_D(\alpha)/dq$ is given by Eq. (31). The derivations of Eqs. (31)-(34) are provided in Appendix E. Summarizing Eqs. (29)-(34), we have the following proposition.

Proposition 4. After the bottleneck capacity is expanded,
(i) All households reside farther from the CBD. Particularly, the downtown residents living nearby the bottleneck will migrate to the suburb.
(ii) The bottleneck congestion duration will be decreased.
(iii) Commuting cost is increased for the downtown residents, but is decreased for the suburban residents.

Proposition 4 shows that every household will migrate outward in response to the improved bottleneck capacity and thus the number of bottleneck users increases. However, the decreased congestion effect due to the expanded bottleneck capacity dominates the increased congestion effect due to the increased number of bottleneck users. As a result, the bottleneck congestion duration decreases. By item (iii), the bottleneck capacity expansion has an entirely opposite impact on the downtown and suburban households in terms of the commuting cost. The commuting cost of the downtown residents always increases due to an increased travel distance under the outward migration after the bottleneck capacity expansion. However, for the suburban residents, the decreased bottleneck congestion would dominate the increased travel distance, and thus their commuting costs would decrease.

3.2. Effect on city's spatial structure

We now look at the effect of bottleneck capacity expansion on city's spatial structure, including city boundary, residential density, and land rental price.

Actually, according to the outward migration of residents (see item (i) in Proposition 4), we can obtain that bottleneck capacity expansion would make the city boundary move outward, i.e., urban sprawl. To further confirm this, from Eq. (22) we derive:

$$\frac{dB}{dq} = \frac{b}{r}\frac{1}{\sqrt{(b\alpha^*)^2-(b\underline{\alpha})^2+2r_A/(kr^2)}}\frac{d\alpha^*}{dq}>0\,, \tag{35}$$

where $d\alpha^*/dq$ is given by Eq. (29) and is larger than 0. Eq. (35) quantifies the movement of the city boundary with increased bottleneck capacity. A similar phenomenon regarding urban sprawl has also been observed in Gubins and Verhoef (2014), in which the bottleneck is assumed to be at the entrance to the CBD and the bottleneck capacity expansion leads the city boundary to move outward. This also theoretically confirms the result of empirical studies, such as Tennøy et al. (2019), which verified that road capacity expansion causes urban sprawl.

We next examine the effect of bottleneck capacity expansion on the residential density. From Eqs. (16) and (17), one can derive the change of the downtown residential density with the bottleneck capacity expansion as

$$\frac{dn_D(x)}{dq} = \frac{dn_D(x)}{d\alpha^*}\frac{d\alpha^*}{dq} = \left(-\frac{e^{ra}b}{e^{2ra}-1}re^{rx} - \frac{e^{ra}b}{e^{2ra}-1}re^{-rx}\right)\frac{d\alpha^*}{dq} < 0. \tag{36}$$

This shows that the residential density at any downtown location $x\ (x < a)$ decreases with the larger bottleneck capacity.

Similarly, from Eqs. (21) and (23) one can derive the change of the suburban residential density with the improved bottleneck capacity as

$$\frac{dn_S(x)}{dq} = \frac{dn_S(x)}{d\alpha^*}\frac{d\alpha^*}{dq} = \frac{-br}{\sqrt{(b\alpha^*)^2 - (b\underline{\alpha})^2 + 2r_A/(kr^2)}}\left(c_3e^{rx} + c_4e^{-rx}\right)\frac{d\alpha^*}{dq} > 0. \tag{37}$$

The sign of Eq. (37) can be judged from Eq. (29) and $c_3e^{rx} + c_4e^{-rx} < 0$ in terms of $N_S(x) < N$ (referring to Eq. (20)). Eq. (37) implies that the residential density at any suburban location $x\ (x \geq a)$ increases with larger bottleneck capacity.

Note that the land rental price and the residential density are a one-to-one relationship with the same changing direction in terms of Eq. (11). We thus have the following proposition.

Proposition 5. After bottleneck capacity expansion, the residential density and the land rental price at *any* downtown location decrease, whereas those at *any* suburban location increase.

As stated in Proposition 2, the residential density and the land rental price in the downtown are larger than in the suburb. From Proposition 5, we can deduce that the differences in residential density and in land rental price between the downtown and suburban areas would decrease after the bottleneck capacity expansion. In other words, increasing bottleneck capacity would flatten the residential density and land rental price.

3.3. Effect on equilibrium household utility

According to Eqs. (1) and (17), the utility function for the downtown households in Eq. (28) can be written as:

$$u_D(\alpha) = w(\alpha) - \tau\alpha x_D(\alpha) - kr\left(c_1 e^{rx_D(\alpha)} - c_2 e^{-rx_D(\alpha)}\right), \tag{38}$$

where $x_D(\alpha)$ is given by Eq. (26).

Based on Eq. (38), we can identify the sign of the derivative of utility function for the downtown households with regard to the bottleneck capacity expansion

$$\frac{du_D(\alpha)}{dq} = \left(\frac{-b\alpha + \sqrt{(b\alpha)^2 - 4c_1c_2}}{2c_1} + \frac{2c_1}{-b\alpha + \sqrt{(b\alpha)^2 - 4c_1c_2}}\right)\frac{e^{ra}krb}{e^{2ra}-1}\frac{d\alpha^*}{dq} > 0,\ \alpha \in [\alpha^*, \bar{\alpha}]. \tag{39}$$

Similarly, according to Eqs. (2) and (23), the utility function for the suburban households in Eq. (28) can be rewritten as:

$$u_S(\alpha) = w(\alpha) - \tau\alpha x_S(\alpha) - \frac{b(\alpha^* - \underline{\alpha})}{q}\eta\alpha - kr\left(c_3 e^{rx_S(\alpha)} - c_4 e^{-rx_S(\alpha)}\right). \tag{40}$$

After some operations, one can identify the sign of the derivative of utility function for the suburban households with regard to the bottleneck capacity expansion

$$\frac{du_S(\alpha)}{dq} = \left(\frac{e^{2ra}+1}{e^{2ra}-1}krb + \frac{\eta b(\alpha^* - \underline{\alpha})}{q}\right)\frac{\alpha}{\alpha^*}\frac{d\alpha^*}{dq} > 0,\ \alpha \in [\underline{\alpha}, \alpha^*]. \tag{41}$$

The detailed derivations of Eqs. (39) and (41) are provided in Appendix F.

To judge who benefit more after the bottleneck capacity expansion, from Eqs. (39) and (41), one can further obtain the derivatives of the marginal utilities with respect to VOT α as

$$\frac{d^2u_D(\alpha)}{dqd\alpha} = d\left(\frac{-b\alpha + \sqrt{(b\alpha)^2 - 4c_1c_2}}{2c_1} + \frac{2c_1}{-b\alpha + \sqrt{(b\alpha)^2 - 4c_1c_2}}\right)\frac{1}{d\alpha}\frac{e^{ra}krb}{e^{2ra}-1}\frac{d\alpha^*}{dq} < 0, \tag{42}$$

and

$$\frac{d^2u_S(\alpha)}{dqd\alpha} = \left(\frac{e^{2ra}+1}{e^{2ra}-1}krb + \frac{\eta b(\alpha^* - \underline{\alpha})}{q}\right)\frac{1}{\alpha^*}\frac{d\alpha^*}{dq} > 0. \tag{43}$$

The above results can be summarized as follows.

Proposition 6. The utility of each household in the city increases after the bottleneck capacity expansion. However, the marginal utility increments (i.e., the utility increments caused by one unit of bottleneck capacity expansion) are different for the downtown and suburban households. Specifically, for the downtown households, the lower the VOT is, the larger the marginal utility increment is. However, for the suburban households, the higher the VOT is, the larger the marginal utility increment is.

Note that the residents sort themselves in a descending order of VOT along the corridor, as stated in Proposition 1. The residents with VOTs $\overline{\alpha}$ and $\underline{\alpha}$ thus reside at the CBD and the city boundary, respectively. Let $\alpha=\overline{\alpha}$ in Eq. (39) and $\alpha=\underline{\alpha}$ in Eq. (41), and compute the marginal utility increments of the residents with the highest VOT and with the lowest VOT, respectively. Based on Proposition 6, we have the following property.

Corollary 1. After the bottleneck capacity expansion,

(i) The marginal utility increment for the relatively mid-VOT residents residing nearby the bottleneck is the highest, whereas that for other residents is smaller.

(ii) If the condition $\left(\frac{e^{2ar}+1}{e^{2ar}-1}kr+\frac{\eta(\alpha^{*}-\underline{\alpha})}{q}\right)\frac{\underline{\alpha}}{\alpha^{*}}>\frac{2e^{ar}kr}{e^{2ar}-1}$ holds, then the marginal utility increment for the CBD residents with the highest VOT is the smallest; otherwise, that for the residents at the city boundary with the lowest VOT is the smallest.

Corollary 1 (ii) provides a sufficient and necessary condition to compare the marginal benefits from bottleneck capacity expansion between the highest-VOT and lowest-VOT residents. Note that the term on its right-hand side is independent of bottleneck capacity q, while the term on its left-hand side decreases with the increased capacity q according to Eqs. (29) and (30). Therefore, there must exist a unique critical capacity $\overline{q}$ which satisfies the equation: $\left(\frac{e^{2ar}+1}{e^{2ar}-1}kr+\frac{\eta(\alpha^{*}-\underline{\alpha})}{\overline{q}}\right)\frac{\underline{\alpha}}{\alpha^{*}}=\frac{2e^{ar}kr}{e^{2ar}-1}$. We further have the following property.

Corollary 2. If the bottleneck capacity $q>\overline{q}$, then the marginal utility increment for the lowest-VOT resident is the smallest; otherwise, that for the highest-VOT resident is the smallest.

Corollary 2 shows among the richest and the poorest residents, who benefit more from the bottleneck capacity expansion depends on the status quo of capacity q. If the status-quo capacity q is enough large (i.e., $q > \overline{q}$), the richest benefit more; otherwise, the poorest benefit more. These properties will also be illustrated in the later numerical study.

4. Optimal design of bottleneck capacity

In this section, we discuss the optimal design issue of bottleneck capacity, with an objective to maximize the social surplus of the system from the society's perspective. The social surplus of the urban system is defined as the sum of the total utility of all households and the aggregate net land rent received by absentee landlords, minus the bottleneck capacity construction cost.

The total utility of all households, TU, is defined as

$$TU = \hat{N}_D \int_{\alpha^*}^{\overline{\alpha}} \frac{1}{\overline{\alpha} - \alpha^*} u_D(\alpha) d\alpha + \hat{N}_S \int_{\underline{\alpha}}^{\alpha^*} \frac{1}{\alpha^* - \underline{\alpha}} u_S(\alpha) d\alpha , \tag{44}$$

where $\hat{N}_D$, $\hat{N}_S$, $u_D(\alpha)$ and $u_S(\alpha)$ are given by Eqs. (13) and (24), respectively, and α^* is determined by Eq. (25). $1/(\overline{\alpha} - \alpha^*)$ and $1/(\alpha^* - \underline{\alpha})$ are the conditional density functions of VOT α of the downtown and suburban residents, respectively. The first term is the total utility of all the households in the downtown area, and the second term is the total utility of all the households in the suburban area.

The aggregate net land rent, LR, is the total land rent within the city minus the total agricultural land rent (or land opportunity cost), expressed as

$$LR = \int_0^B (p(x) - r_A) dx , \tag{45}$$

where the land rent $p(x)$ is given by Eq. (11), and the city boundary B is given by Eq. (22).

The bottleneck capacity construction cost, CC, is assumed to be positively related to the bottleneck capacity. Following Arnott et al. (1990), a linear bottleneck capacity construction cost function is adopted in this paper, expressed as

$$CC = \delta q , \quad (46)$$

where δ is the marginal cost per unit of bottleneck capacity improvement.

As shown in Proposition 6, the bottleneck capacity expansion can increase the total utility of all households. The effect of the bottleneck capacity expansion on the aggregate net land rent is below:

$$\frac{dLR}{dq} = \left(\frac{krb}{2} \frac{\left(e^{2ra}+1\right)b\alpha^* - 2e^{ra}b\overline{\alpha}}{e^{2ra}-1} + \frac{krb\left((b\alpha^*)^2 - (b\underline{\alpha})^2\right)}{2\sqrt{(b\alpha^*)^2 - (b\underline{\alpha})^2 + 2r_A/(kr^2)}} + kr^2 a \frac{-e^{ra}\left(e^{2ra}+1\right)b^2\overline{\alpha} + 2e^{2ra}b^2\alpha^*}{\left(e^{2ra}-1\right)^2} \right) \frac{d\alpha^*}{dq} < 0, \quad (47)$$

where α^* is determined by Eq. (25), and $d\alpha^*/dq$ is given by Eq. (29).

The total effect of the bottleneck capacity expansion on the sum of the aggregate net land rent and households' utility is

$$\frac{d\left(LR+TU\right)}{dq} = \left(\frac{krb}{2} \frac{e^{2ra}+1}{e^{2ra}-1}\left(b\alpha^* - \frac{b\underline{\alpha}^2}{\alpha^*}\right) + \frac{krb\left((b\alpha^*)^2 - (b\underline{\alpha})^2\right)}{2\sqrt{(b\alpha^*)^2 - (b\underline{\alpha})^2 + 2r_A/(kr^2)}} + \frac{\eta\left((b\alpha^*)^2 - (b\underline{\alpha})^2\right)\left(\alpha^* - \underline{\alpha}\right)}{2\alpha^* q} \right) \frac{d\alpha^*}{dq} > 0. \quad (48)$$

The detailed derivations of Eqs. (47) and (48) are provided in Appendix G.

The above results can be summarized as follows.

Proposition 7. Bottleneck capacity expansion would increase the total utility of households, but reduce the aggregate net land rent revenue. As a result of the trade-off, the sum of total household utility and aggregate net land rent revenue still increases.

Proposition 7 shows that the bottleneck capacity expansion would benefit all the households in the city, but has an adverse effect on the absentee landlords due to the decreased total net land rent revenue. This is because after expanding the bottleneck capacity, the residents migrate outward according to Proposition 5, thus leading the land rent in the downtown to decrease, but the land rent in the suburb to increase. The decreased downtown land rent dominates the increased suburban land rent. As a result, the total net land rent revenue decreases.

The optimal bottleneck capacity design issue aims to determine the optimal bottleneck capacity so as to maximize the social surplus of the system, formulated as

$$\max_q SS = TU + LR - CC , \quad (49)$$

where SS represents the social surplus of the urban system.

The first-order optimality condition of the maximization problem (49) requires $dSS/dq = 0$, i.e.,

$$\left(\frac{krb}{2}\frac{e^{2ra}+1}{e^{2ra}-1}\left(b\alpha^* - \frac{b\underline{\alpha}^2}{\alpha^*}\right) + \frac{krb\left((b\alpha^*)^2 - (b\underline{\alpha})^2\right)}{2\sqrt{(b\alpha^*)^2 - (b\underline{\alpha})^2 + 2r_A/(kr^2)}} + \frac{\eta\left((b\alpha^*)^2 - (b\underline{\alpha})^2\right)\left(\alpha^* - \underline{\alpha}\right)}{2\alpha^* q}\right)\frac{d\alpha^*}{dq} - \delta = 0, \quad (50)$$

where $d\alpha^*/dq$ is given by Eq. (29). The first term on the left-hand side of Eq. (50) represents the marginal contribution per unit of bottleneck capacity expansion on the sum of households' utility and aggregate net land rent revenue, while the second term is the marginal cost of bottleneck capacity expansion. Eq. (50) implies that the bottleneck capacity should be optimally designed such that the marginal contribution to the society equals the marginal cost of bottleneck capacity expansion.

From the optimality condition Eq. (50), one can solve the optimal capacity q^* as

$$q^* = \frac{2\delta\eta b\left(2\alpha^* - \underline{\alpha}\right) - \eta^2 b^3\left((\alpha^*)^2 - \underline{\alpha}^2\right)\left(\alpha^* - \underline{\alpha}\right)^2}{2\eta b\left(\alpha^* - \underline{\alpha}\right)\alpha^*\xi_1 - 2\delta\xi_2}, \quad (51)$$

where α^* is determined by Eq. (25), and ξ_1 and ξ_2 are the parameters related to α^*, defined as

$$\begin{cases} \xi_1 = \dfrac{krb}{2}\dfrac{e^{2ra}+1}{e^{2ra}-1}\left(b\alpha^* - \dfrac{b\underline{\alpha}^2}{\alpha^*}\right) + \dfrac{krb\left((b\alpha^*)^2 - (b\underline{\alpha})^2\right)}{2\sqrt{(b\alpha^*)^2 - (b\underline{\alpha})^2 + 2r_A/(kr^2)}}, \\ \xi_2 = krb\dfrac{e^{2ra}+1}{e^{2ra}-1} + \dfrac{krb^2\alpha^*}{\sqrt{(b\alpha^*)^2 - (b\underline{\alpha})^2 + 2r_A/(kr^2)}}. \end{cases} \quad (52)$$

Combining Eqs. (51) and (25), one can immediately obtain the optimal bottleneck capacity q^* together with the critical VOT α^* at the bottleneck.

Traditional bottleneck studies without urban spatial structure consideration, e.g., Arnott et al. (1990, 1993), have shown that the optimal bottleneck capacity should be such that the marginal construction cost of the bottleneck capacity expansion is equal to the marginal benefit to commuters/households. However, when involving urban spatial structure, such an optimality condition would not hold since the bottleneck capacity expansion also affects the aggregate net land revenue (a negative effect, see Eq. (47)), besides the effects on the household utility and the capacity construction cost. In other words, traditional studies may overestimate the benefits of bottleneck capacity expansion since they ignored such a negative effect on land rent, thus causing an over-investment in the bottleneck capacity construction. Therefore, there is indeed a need to incorporate the effects on the urban spatial structure in the design of the bottleneck capacity.

So far, for the fixed bottleneck location case, it has been shown that residents sort themselves along the corridor in a descending order of VOT/income, and the equilibrium residential densities have been derived. The effects of bottleneck capacity expansion have analytically been examined, and the uneven benefits for different-income residents have been revealed and explained. The bottleneck capacity design problem has also been addressed. In the next section, we turn to the stochastic bottleneck location case.

5. Stochastic bottleneck

In the previous sections, we have analyzed the equilibrium problem of an urban continuum system with a fixed bottleneck and the issues of the bottleneck capacity expansion. However, the bottleneck location may randomly change on a road segment due to various non-recurrent random incidents, such as traffic accidents, vehicle breakdowns, road works, signal failures, adverse weather, and earthquakes. In this section, we thus investigate the transportation corridor equilibrium with stochastic bottleneck location.

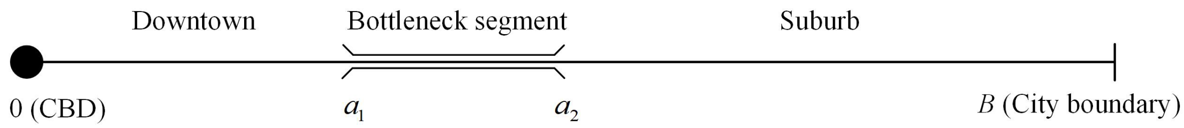


Fig. 2. A transportation corridor with stochastic bottleneck.

Referring to Fig. 2, suppose that the bottleneck congestion stochastically occurs at a segment of the corridor, denoted as $[a_1, a_2]$.[5] For example, it may be a stretch of road which has a smaller capacity than its upstream and downstream roads. This is often the case at the entrance of the downtown area where road capacity is typically smaller. The bottleneck

[5] "$a_1 = a_2$" means that the stochastic bottleneck segment length is 0, and thus the stochastic bottleneck case is reduced to the fixed bottleneck case. This means that the fixed bottleneck case is a special case of the stochastic bottleneck case.

segment $[a_1, a_2]$ divides the corridor into three segments: downtown area $[0, a_1]$, bottleneck segment $[a_1, a_2]$, and suburban area $[a_2, B]$. We use the subscripts "*D*", "*M*", and "*S*" to represent these three areas, respectively. Here, "*M*" means the middle part of the corridor, i.e., the bottleneck segment. It is assumed that different locations on the bottleneck segment $[a_1, a_2]$ have a uniform capacity of *q*. The congestion stochastically occurs within the bottleneck segment $[a_1, a_2]$, and the commuters only know the distribution of the bottleneck congestion locations, but do not know the specific location of the bottleneck congestion occurring at any time. Define $g(x)$ as the probability density that the bottleneck congestion occurs at location $x \in [a_1, a_2]$, satisfying $\int_{a_1}^{a_2} g(x)dx = 1$.

The commuting costs for the three segments are defined as follows. The households in the downtown area never face the bottleneck congestion, and their commuting costs only include the free-flow travel time cost, which is given by Eq. (1). The households on the bottleneck segment face the bottleneck congestion with a certain probability. The cumulative probability of the bottleneck congestion occurring for the households residing at location $x \in [a_1, a_2]$ is $\int_{a_1}^{x} g(y)dy$. Let $N_M(x)$ be the cumulative number of commuters from the CBD to location *x* at the bottleneck segment. If the bottleneck congestion occurs at $x \in [a_1, a_2]$, then the total number of commuters passing through the bottleneck is $N - N_M(x)$, and thus the bottleneck congestion cost for a commuter with VOT α is $(N - N_M(x))\eta\alpha/q$ in terms of Eq. (2). Therefore, the expected commuting cost $c_M(x, \alpha)$ for a commuter with VOT α residing at location *x* can be expressed as

$$c_M(x, \alpha) = \int_{a_1}^{x} \frac{N - N_M(y)}{q} \eta\alpha g(y)dy + \tau x\alpha, \; x \in [a_1, a_2], \tag{53}$$

where the first term on the right-hand side represents the expected bottleneck congestion cost, while the second term is the free-flow travel time cost.

The households located in the suburban area definitely suffer the bottleneck congestion with a variable level of congestion, since the location of the bottleneck is uncertain. For a suburban household with VOT α at location $x \in [a_2, B]$, the expected commuting cost can be represented as

$$c_S(x,\alpha)=\int_{a_1}^{a_2}\frac{N-N_M(y)}{q}\eta\alpha g(y)dy+\tau x\alpha,\ x\in[a_2,B]. \tag{54}$$

The first term on the right-hand side is the expected bottleneck congestion cost which is a constant independent of location x, and the second term is the free-flow travel time cost.

From Eqs. (1), (53) and (54), one can easily show that the commuting cost function of the corridor is continuous and monotonically increasing with respect to location x, which is different from the fixed bottleneck location case where the commuting cost function is discontinuous at the bottleneck. Following a similar procedure presented in Section 2.2, one can obtain the indirect utility function $u(x,\alpha)$ (c.f. Eq. (8)), and the residents in the city corridor still sort in a decreasing order of VOTs from the CBD to the city boundary. Similar to Section 2.3, one can derive the cumulative number of households at any location x for the downtown area, bottleneck segment, and suburban area, as follows.

Let α_1^* and α_2^* be the equilibrium critical VOTs at locations a_1 and a_2, respectively. Similar to the fixed bottleneck case, we derive the equilibrium differential equation for the cumulative number of households at any downtown location, using the condition $\partial u(x,\alpha)/\partial x=0$, as

$$\frac{d^2N_D(x)}{dx^2}-\frac{\tau}{kb}N_D(x)=-\frac{\tau\bar{\alpha}}{k},\ \text{for}\ x\in(0,a_1), \tag{55}$$

with boundary conditions $N_D(0)=0$ and $N_D(a_1)=b(\bar{\alpha}-\alpha_1^*)$.

The cumulative number of households at any location of the suburban area satisfies the following second-order differential equation:

$$\frac{d^2N_S(x)}{dx^2}-\frac{\tau}{kb}N_S(x)=-\frac{\tau\bar{\alpha}}{k},\ \text{for}\ x\in(a_2,B), \tag{56}$$

with three boundary conditions $N_S(a_2)=b(\bar{\alpha}-\alpha_2^*)$, $N_S(B)=b(\bar{\alpha}-\underline{\alpha})$, and $p(B)=r_A$, where $p(B)$ is the land rental price at the city boundary, equal to $kn_S^2(B)/2$ according to Eq. (11).

However, the cumulative number of households at the bottleneck segment $N_M(x)$ is closely related to the probability density function $g(x)$ of the bottleneck location. In this paper, two types of probability density functions are considered below: uniform distribution and exponential distribution.

(a) Uniformly distributed bottleneck location

Under the uniform distribution, the probability density function $g(x)$ of the bottleneck congestion occurring at any location x is

$$g(x)=\frac{1}{a_2-a_1}, x\in[a_1,a_2]. \tag{57}$$

As an example, Fig. 3 gives a uniformly distributed probability density function $g(x)$ with $a_1=8$ and $a_2=12$, yielding an expected bottleneck location of 10 km.

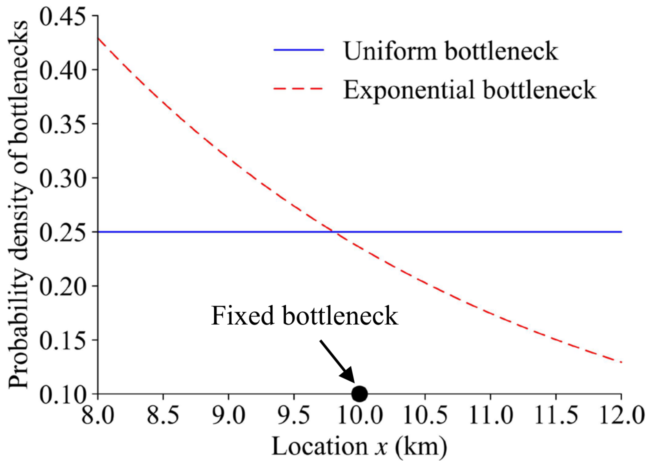


Fig. 3. Probability density function of stochastic bottlenecks: $a_1=8$ and $a_2=12$ km for uniform bottleneck; and $\lambda=0.3$, $a_1=8$, and $a_2=12$ km for exponential bottleneck.

Substituting Eqs. (53) and (57) into (24), the indirect utility function of the households on the bottleneck segment $[a_1,a_2]$ can be rewritten as:

$$u_M(x,\alpha)=w(\alpha)-\int_{a_1}^{x}\frac{1}{a_2-a_1}\frac{N-N_M(y)}{q}\eta\alpha dy-\tau x\alpha-kn_M(x), \tag{58}$$

where $n_M(x)$ is the residential density of location x on the bottleneck segment $[a_1,a_2]$. Using the equilibrium conditions $\partial u_M(x,\alpha)/\partial x=0$ and $N_M(x)=b(\bar{\alpha}-\alpha)$, we derive the following second-order differential equation with regard to $N_M(x)$:

$$\frac{d^2N_M(x)}{dx^2}+\frac{\eta}{(a_2-a_1)qbk}N_M^2(x)-\left(\frac{\tau}{kb}+\frac{\eta N}{(a_2-a_1)qbk}+\frac{\eta\bar{\alpha}}{(a_2-a_1)qk}\right)N_M(x)=-\left(\frac{\tau}{k}+\frac{\eta N}{(a_2-a_1)qk}\right)\bar{\alpha}, \tag{59}$$

with the boundary conditions $N_M(a_1) = b(\bar{\alpha} - \alpha_1^*)$ and $N_M(a_2) = b(\bar{\alpha} - \alpha_2^*)$.

Since the commuting cost function is continuous over the corridor according to Eqs. (1), (53), and (54), the equilibrium conditions of the whole corridor system are given by

$$\frac{dN_D(x)}{dx}\bigg|_{x=a_1} = \frac{dN_M(x)}{dx}\bigg|_{x=a_1}, \text{ and } \frac{dN_M(x)}{dx}\bigg|_{x=a_2} = \frac{dN_S(x)}{dx}\bigg|_{x=a_2}, \tag{60}$$

which means that the residential densities at locations a_1 and a_2 are continuous. From Eq. (60), the critical VOTs α_1^* and α_2^* can be solved.

(b) Exponentially distributed bottleneck location

In reality, the traffic density at the CBD area is generally higher than that at the suburban area, and thus the probability that bottleneck congestion occurs at the CBD area is generally larger than that at the suburban area. In order to model the realism, we introduce a truncated exponential bottleneck congestion probability density function $g(x)$:

$$g(x) = \frac{\lambda e^{-\lambda x}}{e^{-\lambda a_1} - e^{-\lambda a_2}}, x \in [a_1, a_2], \tag{61}$$

where λ is a positive parameter.

Obviously, $dg(x)/dx < 0$ holds, implying that the probability of the bottleneck congestion occurring is decreasing with location *x*. For illustration purpose, Fig. 3 shows an example of an exponentially distributed bottleneck congestion probability density function with $\lambda = 0.3$, $a_1 = 8$ and $a_2 = 12$ km.

Substituting Eqs. (53) and (61) into (24), the indirect utility function for the households on the bottleneck segment can be rewritten as:

$$u_M(x, \alpha) = w(\alpha) - \int_{a_1}^{x} \frac{\lambda e^{-\lambda y}}{e^{-\lambda a_1} - e^{-\lambda a_2}} \frac{N - N_M(y)}{q} \eta\alpha dy - \tau x\alpha - kn_M(x). \tag{62}$$

Again, using the equilibrium conditions $\partial u_M(x,\alpha)/\partial x = 0$ and $N_M(x) = b(\bar{\alpha} - \alpha)$, one can obtain the equilibrium differential equation about $N_M(x)$ as:

$$\frac{d^2 N_M(x)}{dx^2} + \frac{\eta\lambda e^{-\lambda x}}{kqb\left(e^{-\lambda a_1} - e^{-\lambda a_2}\right)} N_M^2(x) - \left(\frac{\tau}{kb} + \frac{\eta\lambda\left(2\bar{\alpha} - \underline{\alpha}\right)e^{-\lambda x}}{kq\left(e^{-\lambda a_1} - e^{-\lambda a_2}\right)}\right) N_M(x) = -\left(\frac{\tau\bar{\alpha}}{k} + \frac{\eta\lambda N\bar{\alpha} e^{-\lambda x}}{kq\left(e^{-\lambda a_1} - e^{-\lambda a_2}\right)}\right), \tag{63}$$

subject to boundary conditions $N_M(a_1) = b(\bar{\alpha} - \alpha_1^*)$ and $N_M(a_2) = b(\bar{\alpha} - \alpha_2^*)$. Besides, the equilibrium conditions for the whole urban corridor system are given by Eq. (60).

With stochastic bottleneck locations, given α_1^* and α_2^*, then N_D and N_S can be analytically solved by Eqs. (55) and (56). However, the closed-form solution for N_M is unavailable for the uniformly and exponentially distributed stochastic bottleneck cases, because the differential equations about N_M in Eqs. (59) and (63) involve a non-linear term $N_M^2(x)$. Therefore, a numerical approach, such as finite difference method (see e.g., Atkinson et al., 2011), is used for solving N_M. The step-by-step procedure of the finite difference method for solving the cumulative number of households (N_D, N_M, and N_S) and the critical VOTs (α_1^* and α_2^*) for the stochastic bottleneck case is not provided here in order to save paper space, but is available from authors upon request.

Once the equilibrium cumulative number of households is solved, the residential location $x(\alpha)$ and the household utility $u(\alpha)$ can then be obtained. With stochastically distributed bottleneck locations, it is still true that residents sort themselves in a descending order of VOT from the CBD outward. However, the commuting cost function is continuous with regard to location x, which is different from the fixed bottleneck case where the commuting cost function is discontinuous at the bottleneck. As a result, the household residential density function is also continuous, which can be determined by a second-order ordinary differential equation system. Nevertheless, the optimal bottleneck capacity design issue cannot be analytically derived, and a numerical method (e.g., grid search) is thus needed. Numerical illustrations are provided in the next section.

6. Numerical study

6.1. Parameter specifications

We now apply a numerical example to illustrate the properties of the proposed model. Consider one transportation corridor connecting the suburban area with the CBD. The total number of households along the corridor is 20,000 households, i.e., $N = 20,000$. For the fixed bottleneck case, the bottleneck is assumed to be located at 10 km from the CBD, i.e., $a = 10\,\text{km}$, as shown in Fig. 3. For the stochastic cases (uniform and exponential), the bottleneck congestion stochastically occurs at the corridor's segment of 8 to 12 km from the CBD. The parameter λ in the exponential density function takes 0.3. Bottleneck capacity q and the agricultural rent r_A are, respectively, assumed to be $q = 4000$ veh/h (two lanes) and $r_A = \$200/\text{day}$. The average auto free-flow travel speed is 40 km per hour, i.e., $\tau = 1/40$ h/km. The ratio, η, of the value of early arrival time to the VOT is 0.6, as in Arnott et al. (1990). The parameter φ in the income function $w(\alpha)$ is assumed to be 8, i.e., $w(\alpha) = 8\alpha$. The lower and upper bounds of the daily wage are assumed to be \$240 and \$800 per day, thus leading the lower and upper bounds of VOT α to be $\underline{\alpha} = \$30$ and $\overline{\alpha} = \$100$ per hour, respectively. The parameter k in the hyperbolic utility function (see Eq. (4)) is assumed to be 0.05. Besides, the discounted daily construction cost per unit of bottleneck capacity expansion is \$30 per day, i.e., $\delta = \$30/\text{day}$. These input data are used as the baseline values.

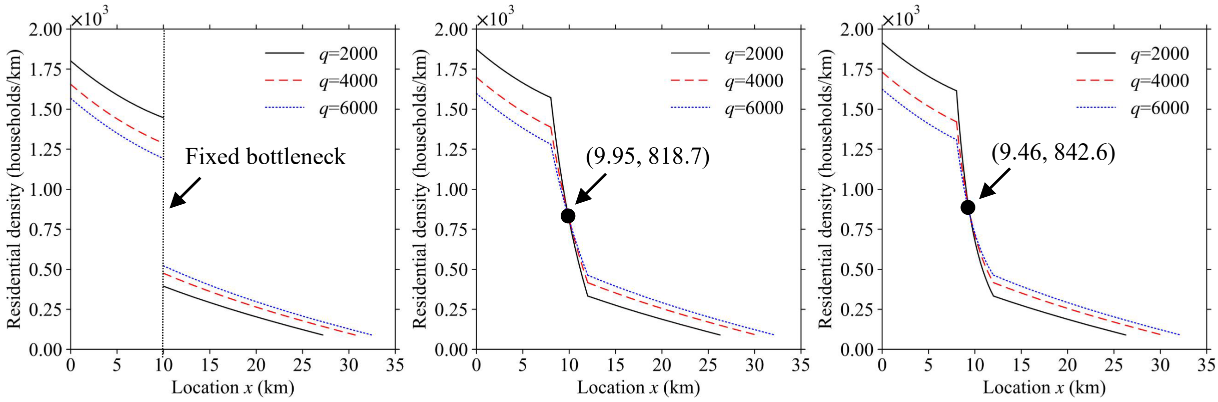
6.2. Discussion of results

Fig. 4 shows the effects of the bottleneck capacity expansion on the residential density under the fixed, uniformly distributed, and exponentially distributed bottleneck locations. The bottleneck capacities considered here include three levels: 4000 veh/h (base case), 2000 veh/h (scaling the base capacity 0.5 time down) and 6000 veh/h (scaling the base capacity 0.5 time up). The main findings are presented as follows.

(i) The residential density monotonically decreases as the distance from CBD, regardless of the fixed or stochastic bottleneck. The residential density curves are continuous for the stochastic bottleneck case (see Figs. 4b and c), but not continuous for the fixed bottleneck case with the bottleneck location of $x = 10\,\text{km}$ as the discontinuity point (see Fig. 4a). These illustrate the results of Proposition 2. It shows the stochasticity of the bottleneck location can smooth the residential density curve.

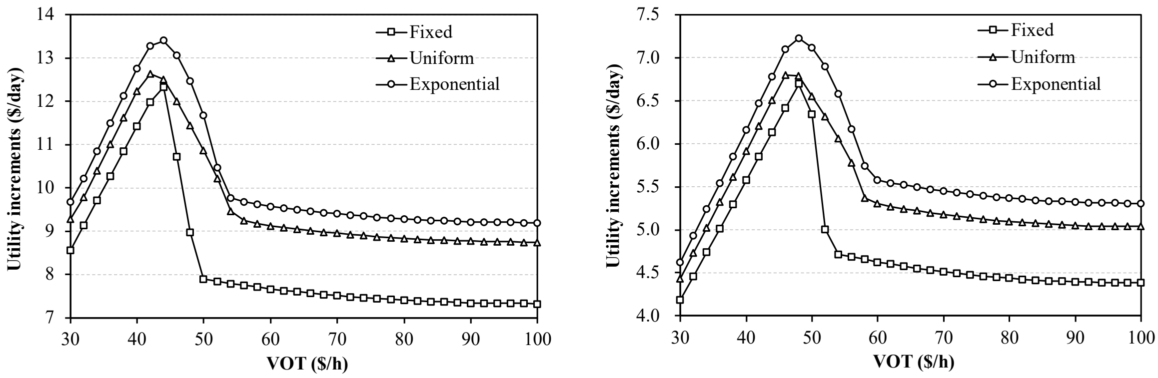
(ii) The bottleneck capacity expansion increases the length of the corridor. It can be seen in Fig. 4 that regardless of the fixed or stochastic bottleneck case, as the bottleneck capacity expands from 2000 to 6000 vehicles per hour, the city boundary moves from about 27 km to about 33 km from the CBD outward.

(iii) The bottleneck capacity expansion leads to a decrease in the residential densities in the downtown area, but an increase in the suburban area. The distance of the critical location from the CBD for such a residential density switch is different for different bottleneck distributions. Specifically, the critical location is exactly 10 km for the fixed bottleneck case, 9.95 km for the uniform distribution case, and 9.46 km for the exponential distribution case.



(a) (b) (c)

Fig. 4. Residential densities under different bottleneck capacities: (a) fixed bottleneck, (b) uniformly distributed bottleneck, and (c) exponentially distributed bottleneck.



(a) (b)

Fig. 5. Change of household utility with bottleneck capacity expansion: (a) from 2000 to 4000 veh/h, and (b) from 4000 to 6000 veh/h.

For illustrating the effects of bottleneck capacity expansion, Fig. 5 shows the changes of different-VOT households' utilities when the bottleneck capacity is scaled 0.5 time down and up (i.e., q changes from 2000 to 4000 veh/h, and from 4000 to 6000 veh/h) for the fixed and stochastic bottleneck cases, respectively. Some insightful findings can be summarized as follows.

(i) Regardless of fixed or stochastic bottlenecks, the utility change for all households is positive. This implies that all households can benefit from the bottleneck capacity expansion, as stated in Proposition 6. Particularly, the relatively mid-income residents can always obtain the largest benefits. Taking the fixed bottleneck case as an example, as q expands from 2000 to 4000 veh/h, the households with a VOT of \$44/h gain the largest utility increment by \$12.3 (see Fig. 5a). As q continues to expand to 6000 veh/h, the largest utility increment is \$6.7 for the households with a VOT of \$48/h (see Fig. 5b). The increasing VOT from \$44 to \$48 per hour means that as the bottleneck capacity increases, the rich households gradually benefit more than the poor. This is because as the bottleneck capacity expands, more residents migrate to the suburb from the downtown, and thus the suburb becomes increasingly crowded. As a result, the effects of the bottleneck capacity expansion on the bottleneck congestion alleviation marginally decrease, and thus the rich living the downtown area gradually benefit more than the poor because they do not need to pass through the bottleneck.

(ii) Those who obtain the smallest utility increments are the lowest-VOT or highest-VOT households, depending on the value of the status-quo bottleneck capacity. As the status-quo capacity is relatively small (e.g., q is 2000 veh/h), the largest-VOT households benefit the least (see Fig. 5a). However, when the status-quo capacity is relatively large (e.g., q is 4000 veh/h), the smallest-VOT households benefit the least (see Fig. 5b).

(iii) Among the three utility increment curves, the curve for the exponential bottleneck case would be at the top, that for the fixed bottleneck case would be at the bottom, and that for the uniform bottleneck case is in between. This means that the marginal benefit from the bottleneck capacity expansion ranks in an order of the exponential, uniform, and fixed bottleneck cases. This is attributed to the large spatial heterogeneity in congestion occurring probability along the corridor.

Fig. 6 indicates the effects of bottleneck capacity expansion on the total net land rent revenue, the sum of the total household utility and the total net land rent revenue, and the social surplus. Some main findings are as follows.

(i) The bottleneck capacity expansion would lead to a decrease in the total net land rent revenue (see Fig. 6a), but an increase in the sum of the total household utility and the total net land rent revenue (see Fig. 6b), regardless of the fixed or stochastic bottleneck cases.

(ii) The social surplus curves are concave regardless of the fixed or stochastic bottleneck cases, meaning that there exists an optimal bottleneck capacity to maximize the social surplus, as shown in Fig. 6c. The optimal bottleneck capacities that maximize the social surplus are 3400, 4300, and 4600 veh/h for fixed, uniform, and exponential bottleneck cases, respectively.

(iii) The stochasticity of the bottleneck congestion location brings a loss to the urban corridor system. As a matter of fact, given a bottleneck capacity, the aggregate land rent revenue for the fixed bottleneck case is lower than that for the stochastic bottleneck cases (see Fig. 6a), whereas the sum of total household utility and land rent revenue for the fixed bottleneck case is higher than that for the stochastic bottleneck cases (see Fig. 6b). This means that the total household utility for the fixed case is larger than that for the stochastic cases. As a result, the social surplus under the fixed case for a given bottleneck capacity is highest among the three cases (see Fig. 6c). This is attributed to the adoption of a risk-averse utility function with regard to housing/land consumption (i.e., $d^2u/dh^2 < 0$, see Eq. (4)), and thus the households prefer to reside in the downtown area under the stochastic bottleneck cases, causing a higher housing/land price and thus a higher total net land rent revenue. By contrast, under the fixed bottleneck case, some households would like to live in the suburb area such that they can enjoy large housing spaces, which bring a high household utility and social surplus as well.

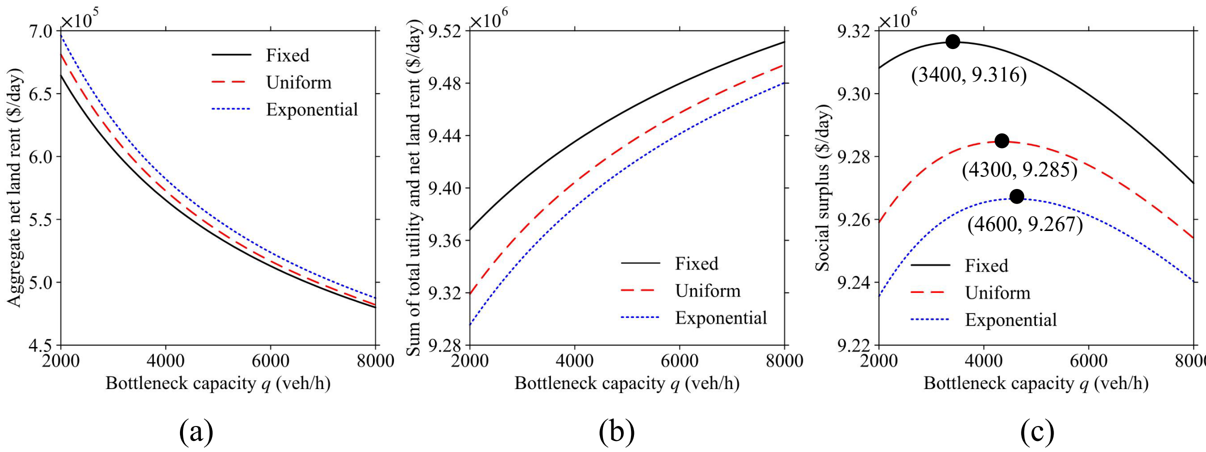
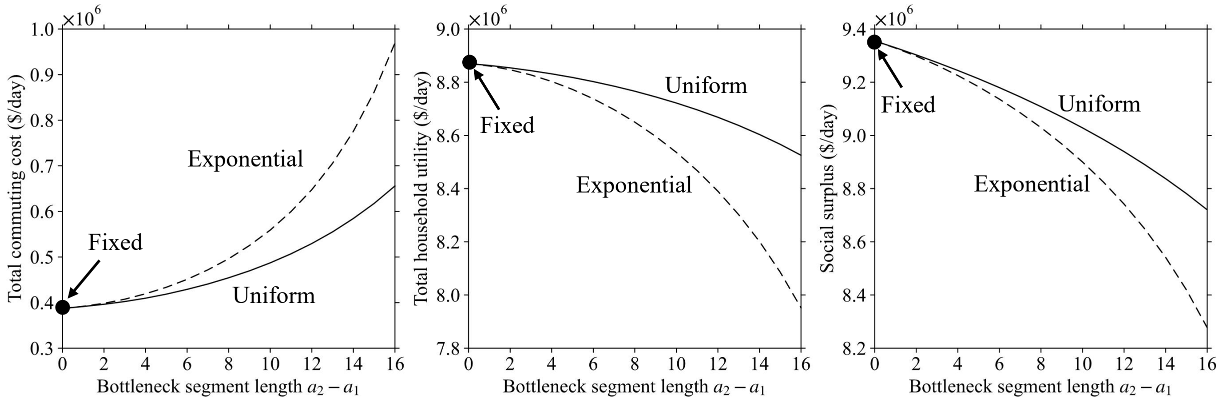


Fig. 6. Effects of bottleneck capacity expansion on: (a) aggregate net land rent revenue, (b) sum of total household utility and aggregate net land rent revenue, and (c) social surplus.

In order to examine the effects of the stochastic bottleneck's coverage, we conduct a sensitivity analysis of the bottleneck segment length $a_2 - a_1$ by symmetrically extending its length with regard to the location of 10 km. Fig. 7 shows the effects of $a_2 - a_1$ on the total commuting cost, total household utility, and social surplus. It can be seen that the total commuting cost of the corridor system increases with the bottleneck segment length (see Fig. 7a), which leads to a decrease in the household utility and thus in the social surplus (see Figs. 7b and c). The fixed bottleneck case (which is associated with a zero bottleneck segment length) induces the highest social surplus. For a given bottleneck segment length, the uniform bottleneck case always leads to a lower commuting cost, a higher household utility, and thus a higher social surplus compared to the exponential bottleneck case. These observations further illustrate the properties of the model and the results previously presented.



(a) (b) (c)

Fig. 7. Effects of bottleneck segment length on: (a) total commuting cost, (b) total household utility, and (c) social surplus.

To sum up, the above numerical studies have verified some important properties of the proposed model in this paper, and have also revealed some new insights. It shows that the bottleneck capacity expansion can benefit all households, but the benefit level is different across households. Particularly, when the status-quo bottleneck capacity is large, the poorest residents gain the least benefit from the bottleneck capacity expansion. Therefore, the authority should carefully make the decision of the bottleneck capacity expansion and especially care for the inequity issue due to the expansion. The distribution of stochastic bottleneck congestion locations significantly affects the urban system performance. In particular, in terms of the social surplus, the fixed bottleneck case is the most efficient, the exponential case is the least efficient, and the uniform case is in between. For all the cases, the social surplus curve is concave about the bottleneck capacity. In addition, the bottleneck congestion segment length also plays an important role in the household residential distribution and urban system performance. Therefore, the authority should strengthen the real-time dynamic monitoring of bottleneck congestion locations so as to timely take actions to control the bottleneck congestion.

7. Conclusion and further studies

We present a novel modeling framework that combines the household residential location choice and bottleneck congestion. The residents were assumed to be continuously distributed along the linear city corridor. There is a bottleneck with a fixed or with a stochastic location in the city corridor, causing traffic congestion during the morning commute. Residents have different values of time. For the fixed bottleneck case, we derived analytical solutions for the urban system equilibrium and the bottleneck capacity design, together with the effects of the bottleneck capacity expansion. For the stochastic bottleneck case, we considered uniform and exponential bottleneck location distributions. We established differential equations for the urban system equilibrium, and examined numerically the effects of the bottleneck capacity expansion and the optimal bottleneck capacity design.

The following findings were obtained. First, residents spatially sort themselves outwards in a descending order of value of time along the corridor, and low-income residents would use a larger fraction of their income for commuting than high-income residents. The residential density and land rental price decrease with the distance from the CBD, and they are not continuous for the fixed bottleneck case. Second, all residents benefit from the bottleneck capacity expansion, but not equally. The mid-income residents residing nearby the bottleneck benefit the most, whereas those who benefit the least are the richest or the poorest, depending on the status-quo bottleneck capacity. Specifically, if the status-quo bottleneck capacity is enough small, then the richest residents benefit the least, and otherwise the poorest residents benefit the least. Third, the bottleneck capacity expansion leads to an increase in the city size (i.e., urban sprawl), total household utility and the bottleneck capacity investment cost, but to a decrease in the aggregate net land rent revenue. As a result, ignoring the effects of the bottleneck capacity expansion on the urban spatial structure (e.g., aggregate net land rent) leads to an overestimate of the social surplus. With the bottleneck capacity expanded, the bottleneck congestion duration decreases; the commuting costs of the downtown residents increase, while those of the suburban residents decrease. Fourth, the stochasticity of bottleneck location harms the system due to a decreased social surplus. Such stochasticity smooths the residential distribution, and increases the transportation cost of the system. As a result, the household utility and the social surplus are decreased compared to the fixed bottleneck case. The proposed model can help understand the interrelationship between bottleneck congestion and urban spatial structure, and can serve as a useful tool for efficient evaluation and design of anti-congestion policies in practice, such as congestion toll and auto ownership rationing.

Although theoretical frameworks proposed in this paper can be used to model the relationship between bottleneck congestion and households' residential location choice, and to design the optimal bottleneck capacity from the social perspective, some extensions should be made as follows. First, the household utility function was assumed to be a quasi-linear utility function of land consumption. Such an assumption needs to be further justified using real survey data. Moreover, empirical calibrations of the utility function are also beneficial for the applications of the proposed model in realistic cases. Second, a uniform household income distribution was assumed in this paper. It may be more realistic to consider the effects of other income distributions, such as lognormal distribution. Of course, it may be difficult to derive a closed-form solution for a general distribution, and thus a simulation method may be needed in this case. Third, this paper considered a linear form of urban structure. However, in reality many cities have a radial and/or a circular structure (Li et al., 2013; Li and Wang, 2018). There is thus a need to extend the proposed model to consider other urban forms. Fourth, this paper focused on a monocentric city in which all business and commercial activities occur at the CBD area. However, modern cities usually have multiple business and commercial centers and thus a polycentric urban model should be developed (see e.g., Anas and Kim, 1996; Anas and Xu, 1999). Finally, this paper did not explicitly consider the commuter parking price and parking space availability, which can significantly affect the commuter parking behavior and commuting schedule (Arnott et al., 1991). Therefore, it is meaningful to extend the proposed model to take into account the parking issue in a further study.

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Appendix A: Derivation of Eq. (2)

For the resident residing at location $x(x > a)$ with VOT α, his/her commuting cost $c_S(x,\alpha)$ is composed of three parts: queuing delay cost at the bottleneck, the early-arrival penalty, and the free-flow travel time cost, expressed as

$$c_S(x,\alpha) = \alpha \cdot m(t) + \beta(\alpha) \cdot (t_1 - t) + \tau x \alpha, \tag{A.1}$$

where $\beta(\alpha)$ is the value of early arrival time for the resident with VOT α, t is the arrival time at the CBD, t_1 is the desired work time, and $m(t)$ is the queuing delay time at the bottleneck. As assumed in Section 2.1, the residents with different VOTs have the same ratio, η, of value of early arrival time to VOT, i.e.,

$$\eta = \frac{\beta(\alpha)}{\alpha}, \forall \alpha \in [\underline{\alpha}, \overline{\alpha}]. \tag{A.2}$$

At equilibrium, any resident in the suburb cannot unilaterally change his/her schedule so as to reduce commuting cost, i.e., $\frac{dc_S(x,\alpha)}{dt} = 0$ holds. Combining it with Eq. (A.1) yields

$$\alpha \frac{dm(t)}{dt} - \beta(\alpha) = 0. \tag{A.3}$$

Let t_0 be the arrival time of the earliest arriver at the CBD. Obviously, there is no queue for the first arriver, i.e., $m(t_0) = 0$. Using this condition, one can derive $m(t)$ from Eqs. (A.2) and (A.3) as

$$m(t) = \eta \cdot (t - t_0). \tag{A.4}$$

Since late arrival is not allowed, the work start time t_1 is also the latest arrival time at the CBD, and thus the peak period lasts for $t_1 - t_0$ units of time. The bottleneck runs at the full capacity during the peak period. We thus have

$$\frac{\hat{N}_S}{q} = t_1 - t_0. \tag{A.5}$$

Substituting Eqs. (A.2), (A.4), and (A.5) into (A.1), the commuting cost for the suburban residents residing at x with VOT α can be obtained as

$$c_S(x,\alpha) = \frac{\hat{N}_S}{q} \eta \alpha + \tau x \alpha. \tag{A.6}$$

This completes the derivation of Eq. (2).

Appendix B: Proof of Proposition 1

According to the bid-rent theory, all residents in the city compete for the limited land subject to a budget constraint, and the land is offered to the residents with the highest bid. Let $\bar{u}(\alpha)$ be the equilibrium utility of the residents with VOT α at the urban system equilibrium. By the budget constrain in Eq. (5), the bid-rent function $\bar{p}(x,\alpha)$ for the residents with VOT α at location *x*, i.e., the highest bid for the land they are willing to pay at location *x*, can be written as

$$\bar{p}(x,\alpha,\bar{u}(\alpha)) = \max_{h,z} \frac{w(\alpha) - z(x,\alpha) - c(x,\alpha)}{h(x,\alpha)}\Big|_{u(x,\alpha)=\bar{u}(\alpha)} . \tag{B.1}$$

Using $u(x,\alpha) = \bar{u}(\alpha)$ to replace variable $z(x,\alpha)$, Eq. (B.1) can be rewritten as

$$\bar{p}(x,\alpha,\bar{u}(\alpha)) = \max_{h} \frac{w(\alpha) - k/(2h) - \bar{u}(\alpha) - c(x,\alpha)}{h} . \tag{B.2}$$

From the first-order optimality condition, one can obtain

$$k/h = w(\alpha) - \bar{u}(\alpha) - c(x,\alpha) . \tag{B.3}$$

Substituting Eq. (B.3) into (B.2), one can obtain the bid-rent function as

$$\bar{p}(x,\alpha,\bar{u}(\alpha)) = \frac{1}{2k}\left(w(\alpha) - \bar{u}(\alpha) - c(x,\alpha)\right)^2 . \tag{B.4}$$

In the following, we derive the residential order of households. We first consider any two residents simultaneously from the downtown area or the suburban area, called "Resident 1" and "Resident 2". They are associated with VOTs α_1 and α_2, and equilibrium utility levels $\bar{u}(\alpha_1)$ and $\bar{u}(\alpha_2)$, respectively. Without loss of generality, we assume $\alpha_1 > \alpha_2$. Suppose that at location *x*, the bid-rent functions of these two residents are equal, i.e.,

$$\frac{1}{2k}\left(w(\alpha_1) - \bar{u}(\alpha_1) - c(x,\alpha_1)\right)^2 = \frac{1}{2k}\left(w(\alpha_2) - \bar{u}(\alpha_2) - c(x,\alpha_2)\right)^2 . \tag{B.5}$$

We thus have

$$w(\alpha_1) - \bar{u}(\alpha_1) - c(x,\alpha_1) = w(\alpha_2) - \bar{u}(\alpha_2) - c(x,\alpha_2) . \tag{B.6}$$

By Eq. (B.6), we have

$$\frac{1}{k}\left(w(\alpha_1) - \bar{u}(\alpha_1) - c(x,\alpha_1)\right)\cdot(\tau\alpha_1) > \frac{1}{k}\left(w(\alpha_2) - \bar{u}(\alpha_2) - c(x,\alpha_2)\right)\cdot(\tau\alpha_2) . \tag{B.7}$$

From Eqs. (B.4) and (B.7), one obtains

$$-\frac{\partial \overline{p}(x,\alpha_1,\overline{u}(\alpha_1))}{\partial x} > -\frac{\partial \overline{p}(x,\alpha_2,\overline{u}(\alpha_2))}{\partial x}. \tag{B.8}$$

According to Fujita (1989), Eq. (B.8) means that the bid-rent function of Resident 1 is steeper than that of Resident 2, which shows that Resident 1 with a higher VOT resides closer to the CBD than Resident 2 with a lower VOT.

We now consider the case in which any two residents are from different areas. Suppose that Resident 1 residing at location x_1 with VOT α_1 is from the downtown area, i.e., $x_1 < a$, and Resident 2 residing at location x_2 with VOT α_2 is from the suburban area, i.e., $x_2 > a$. Their equilibrium utilities are $u(x_1,\alpha_1)$ and $u(x_2,\alpha_2)$ respectively. Since any resident's utility under equilibrium is maximized, interchanging their locations inevitably reduces their total utility, i.e.,

$$u(x_1,\alpha_1)+u(x_2,\alpha_2) > u(x_2,\alpha_1)+u(x_1,\alpha_2). \tag{B.9}$$

Substituting Eq. (8) into (B.9) yields

$$\left(\frac{\hat{N}_S}{q}\eta+\tau(x_2-x_1)\right)(\alpha_1-\alpha_2) > 0. \tag{B.10}$$

Owing to $x_2 > x_1$, $\alpha_1 > \alpha_2$ holds, which means that the downtown residents has a higher VOT than the suburban residents. This completes the proof of Proposition 1.

Appendix C: Proof of Proposition 2

Taking the derivative of $n_D(x)$ with respect to x in terms of Eq. (17) yields

$$\frac{dn_D(x)}{dx} = r^2(c_1 e^{rx} + c_2 e^{-rx}). \tag{C.1}$$

Since $N_D(x) < \hat{N}_D$ always holds, comparing Eqs. (13) and (15) yields

$$c_1 e^{rx} + c_2 e^{-rx} < 0. \tag{C.2}$$

From Eq. (C.1), one immediately obtains

$$\frac{dn_D(x)}{dx} < 0. \tag{C.3}$$

Similarly, one obtains

$$\frac{dn_S(x)}{dx} < 0. \tag{C.4}$$

Since $u_D(a, \alpha^*) = u_S(a, \alpha^*)$ holds, from Eqs. (1), (2), and (24), one obtains

$$n_D(a) - n_S(a) = \frac{1}{k}\frac{\hat{N}_S}{q}\eta\alpha^* > 0. \tag{C.5}$$

Eq. (C.5) shows that the residential density is not continuous (a downward jump) at the bottleneck. From Eqs. (C.3)-(C.5), it can be seen that the residential density is monotonically decreasing for the downtown and suburban areas, respectively.

According to Eq. (11), for each of the downtown and suburban areas, the land rent and residential density have a one-to-one relation. They have the same continuity and monotonicity for each area. This completes the proof of Proposition 2.

Appendix D: Proof of Proposition 3

For the downtown residents, from Eq. (1) the fraction of commuting cost to income is

$$\frac{c_D(x,\alpha)}{w(\alpha)}=\frac{\tau x_D(\alpha)\alpha}{\varphi\alpha}=\frac{\tau}{\varphi}x_D(\alpha). \tag{D.1}$$

From Eq. (26), one can further derive

$$\frac{d\,\tau x_D(\alpha)/\varphi}{d\alpha}<0. \tag{D.2}$$

For the suburban residents, from Eq. (2) the fraction of commuting cost to income is

$$\frac{c_S(x,\alpha)}{w(\alpha)}=\frac{\tau}{\varphi}x_D(\alpha)+\frac{\hat{N}_S}{\varphi q}\eta. \tag{D.3}$$

From Eq. (27), one can further derive

$$\frac{d\left(\tau x_D(\alpha)/\varphi+\hat{N}_S\eta/(\varphi q)\right)}{d\alpha}<0. \tag{D.4}$$

Eqs. (D.2) and (D.4) show that the fraction of commuting cost to income decreases with VOT α. This completes the proof of Proposition 3(i).

We next prove Proposition 3(ii). Considering that VOT α and location x have a one-to-one correspondence, i.e., $N_i(x)=b(\overline{\alpha}-\alpha),\ i=D,S$, from Eqs. (15) and (20), we can obtain the VOT of a household at any location x as

$$\alpha(x)=\begin{cases}-\dfrac{c_1e^{rx}+c_2e^{-rx}}{b}, & \text{for } x\in(0,a),\\[2ex] -\dfrac{c_3e^{rx}+c_4e^{-rx}}{b}, & \text{for } x\in(a,B).\end{cases} \tag{D.5}$$

For the downtown residents at location x, the housing consumption is $p(x)h(x)$ and the income is $\varphi\alpha(x)$. From Eqs. (10), (11), (17) and (D.5), the ratio of housing consumption to income is

$$\frac{p(x)h(x)}{w(\alpha(x))}=\frac{kn_D(x)/2}{\varphi\alpha(x)}=\frac{krb}{2\varphi}\left(1+\frac{2c_1}{-c_1-c_2e^{-2rx}}\right),\ x\in(0,a). \tag{D.6}$$

From Eq. (D.6), if $c_1=0$, all downtown residents have the same ratio of housing consumption to income, equal to $\dfrac{krb}{2\varphi}$. If $c_1\neq 0$, we further derive

$$\begin{cases} \dfrac{d\left(p(x)h(x)/w(\alpha(x))\right)}{d\alpha} = \dfrac{d\left(p(x)h(x)/w(\alpha(x))\right)}{dx}\dfrac{dx_D(\alpha)}{d\alpha} < 0, \text{ if } c_1 > 0, \\ \dfrac{d\left(p(x)h(x)/w(\alpha(x))\right)}{d\alpha} = \dfrac{d\left(p(x)h(x)/w(\alpha(x))\right)}{dx}\dfrac{dx_D(\alpha)}{d\alpha} > 0, \text{ if } c_1 < 0. \end{cases} \tag{D.7}$$

Similarly, for the suburban residents, the ratio of housing consumption to income can be written as

$$\frac{p(x)h(x)}{w(\alpha(x))} = \frac{kn_S(x)/2}{\varphi\alpha(x)} = \frac{krb}{2\varphi}\left(1 + \frac{2c_3}{-c_3 - c_4 e^{-2rx}}\right), \; x \in (a, B). \tag{D.8}$$

Again, from Eq. (D.8), if $c_3 = 0$, all suburban residents have the same ratio of housing consumption to income, equal to $\frac{krb}{2\varphi}$. If $c_3 \neq 0$, we further derive

$$\begin{cases} \dfrac{d\left(p(x)h(x)/w(\alpha(x))\right)}{d\alpha} = \dfrac{d\left(p(x)h(x)/w(\alpha(x))\right)}{dx}\dfrac{dx_S(\alpha)}{d\alpha} < 0, \text{ if } c_3 > 0, \\ \dfrac{d\left(p(x)h(x)/w(\alpha(x))\right)}{d\alpha} = \dfrac{d\left(p(x)h(x)/w(\alpha(x))\right)}{dx}\dfrac{dx_S(\alpha)}{d\alpha} > 0, \text{ if } c_3 < 0. \end{cases} \tag{D.9}$$

This completes the proof of Proposition 3(ii).

Appendix E: Derivations of Eqs. (31)-(34)

From Eqs. (16) and (26), we can derive

$$\begin{aligned}\frac{dx_D(\alpha)}{dq} &= \frac{dx_D(\alpha)}{d\alpha^*}\frac{d\alpha^*}{dq} \\ &= \frac{1}{r}\frac{c_1}{-b\alpha+\sqrt{(b\alpha)^2-4c_1c_2}}\frac{1}{c_1^2}\left(\frac{\left(-4\frac{dc_1}{d\alpha^*}c_2-4c_1\frac{dc_2}{d\alpha^*}\right)c_1}{2\sqrt{(b\alpha)^2-4c_1c_2}}-\left(-b\alpha+\sqrt{(b\alpha)^2-4c_1c_2}\right)\frac{dc_1}{d\alpha^*}\right)\frac{d\alpha^*}{dq} \\ &= \frac{e^{ra}b}{e^{2ra}-1}\frac{-2c_1c_2-2c_1^2+\left(b\alpha\right)^2-b\alpha\sqrt{(b\alpha)^2-4c_1c_2}}{rc_1\left(-b\alpha+\sqrt{(b\alpha)^2-4c_1c_2}\right)\sqrt{(b\alpha)^2-4c_1c_2}}\frac{d\alpha^*}{dq}.\end{aligned} \tag{E.1}$$

In terms of Eq. (26), we can judge $c_1\left(-b\alpha+\sqrt{(b\alpha)^2-4c_1c_2}\right)>0$, and thus the sign of $\frac{dx_D(\alpha)}{dq}$ in Eq. (E.1) relies on the sign of $-2c_1c_2-2c_1^2+\left(b\alpha\right)^2-b\alpha\sqrt{(b\alpha)^2-4c_1c_2}$.

From Eq. (16), we have $c_2<0$, but the sign of c_1 is ambiguous. If $c_1>0$, we can derive $\frac{d\left[\left(b\alpha\right)^2-b\alpha\sqrt{(b\alpha)^2-4c_1c_2}\right]}{d\alpha}>0$.

Since $\alpha\in[\alpha^*,\overline{\alpha}]$, we have

$$-2c_1c_2-2c_1^2+\left(b\alpha\right)^2-b\alpha\sqrt{(b\alpha)^2-4c_1c_2}\geq -2c_1c_2-2c_1^2+\left(b\alpha^*\right)^2-b\alpha^*\sqrt{(b\alpha^*)^2-4c_1c_2}. \tag{E.2}$$

Since $x_D(\alpha^*)=a$, using $c_1>0$ and Eq. (26), we can further derive Eq. (E.2) as

$$\begin{aligned}-2c_1c_2-2c_1^2+\left(b\alpha\right)^2-b\alpha\sqrt{(b\alpha)^2-4c_1c_2} &\geq -2c_1c_2-2c_1^2-2b\alpha^*e^{ra}c_1 \\ &= 2c_1\left(b\overline{\alpha}-b\alpha^*e^{ra}\right)>0.\end{aligned} \tag{E.3}$$

If $c_1<0$, we can derive $\frac{d\left[\left(b\alpha\right)^2-b\alpha\sqrt{(b\alpha)^2-4c_1c_2}\right]}{d\alpha}<0$. Similar to Eq. (E.2), we have

$$-2c_1c_2-2c_1^2+\left(b\alpha\right)^2-b\alpha\sqrt{(b\alpha)^2-4c_1c_2}\geq -2c_1c_2-2c_1^2+\left(b\overline{\alpha}\right)^2-b\overline{\alpha}\sqrt{(b\overline{\alpha})^2-4c_1c_2}. \tag{E.4}$$

Since $x_D(\overline{\alpha})=0$, using Eq. (26), we can further derive Eq. (E.4) as

$$-2c_1c_2-2c_1^2+\left(b\alpha\right)^2-b\alpha\sqrt{(b\alpha)^2-4c_1c_2}\geq -2c_1c_2-2c_1^2-2c_1b\overline{\alpha}=0. \tag{E.5}$$

Combining Eqs. (E.3) and (E.5), we can judge $\frac{dx_D(\alpha)}{dq}$ in Eq. (E.1) is larger than 0. This completes the derivation of Eq. (31).

From Eqs. (21) and (27), we can derive

$$\frac{dx_S(\alpha)}{dq}=\frac{dx_S(\alpha)}{d\alpha^*}\frac{d\alpha^*}{dq}=-\frac{1}{r}\frac{1}{c_3}\frac{dc_3}{d\alpha^*}\frac{d\alpha^*}{dq}=\frac{1}{r}\frac{b}{\sqrt{(b\alpha^*)^2-(b\underline{\alpha})^2+2r_A/(kr^2)}}\frac{d\alpha^*}{dq}>0. \quad \text{(E.6)}$$

This completes the derivation of Eq. (32)

In Eq. (1), taking the first-order derivative of $c_D(x,\alpha)$ with respect to q can yield Eq. (33).

According to Eq. (2), the effect of capacity expansion on the commuting cost of suburban households is

$$\frac{dc_S(x,\alpha)}{dq}=\tau\alpha\frac{dx_S(\alpha)}{dq}+\eta\alpha\frac{d\left(b(\alpha^*-\underline{\alpha})/q\right)}{dq}. \quad \text{(E.7)}$$

Substituting Eqs. (E.6) and (30) into (E.7), we have

$$\frac{dc_S(x,\alpha)}{dq}=\tau\alpha\left(\frac{b}{r}\frac{1}{\sqrt{(b\alpha^*)^2-(b\underline{\alpha})^2+2r_A/(kr^2)}}\right)\frac{d\alpha^*}{dq}-\frac{\alpha}{\alpha^*}\left(kr\frac{e^{2ra}+1}{e^{2ra}-1}b+kr\frac{b^2\alpha^*}{\sqrt{(b\alpha^*)^2-(b\underline{\alpha})^2+2r_A/(kr^2)}}+\frac{\eta}{q}b(\alpha^*-\underline{\alpha})\right)\frac{d\alpha^*}{dq} \quad \text{(E.8)}$$
$$=-\frac{\alpha}{\alpha^*}\left(kr\frac{e^{2ra}+1}{e^{2ra}-1}b+\frac{\eta}{q}b(\alpha^*-\underline{\alpha})\right)\frac{d\alpha^*}{dq}<0.$$

This completes the derivation of Eq. (34).

Appendix F: Derivations of Eqs. (39) and (41)

We first derive Eq. (39). From Eq. (38), we have

$$\frac{du_D(\alpha)}{dq}=\frac{\partial u_D(\alpha)}{\partial x_D(\alpha)}\frac{dx_D(\alpha)}{dq}+\frac{\partial u_D(\alpha)}{\partial c_1}\frac{dc_1}{d\alpha^*}\frac{d\alpha^*}{dq}+\frac{\partial u_D(\alpha)}{\partial c_2}\frac{dc_2}{d\alpha^*}\frac{d\alpha^*}{dq}. \tag{F.1}$$

At equilibrium, $\frac{\partial u_D(x,\alpha)}{\partial x}=0$ holds. One can further derive

$$\frac{du_D(\alpha)}{dq}=\left(-e^{rx_D(\alpha)}\frac{dc_1}{d\alpha^*}+e^{-rx_D(\alpha)}\frac{dc_2}{d\alpha^*}\right)kr\frac{d\alpha^*}{dq}. \tag{F.2}$$

According to Eqs. (16) and (26), after some operations, Eq. (F.2) can be further calculated as

$$\frac{du_D(\alpha)}{dq}=\left(\frac{-b\alpha+\sqrt{(b\alpha)^2-4c_1c_2}}{2c_1}+\frac{2c_1}{-b\alpha+\sqrt{(b\alpha)^2-4c_1c_2}}\right)\frac{e^{ra}krb}{e^{2ra}-1}\frac{d\alpha^*}{dq}>0. \tag{F.3}$$

We next derive Eq. (41). According to Eq. (21), we can derive

$$\begin{cases}\dfrac{dc_3}{d\alpha^*}=\dfrac{-b}{\sqrt{(b\alpha^*)^2-(b\underline{\alpha})^2+2r_A/(kr^2)}}c_3,\\[2ex] \dfrac{dc_4}{d\alpha^*}=\dfrac{b}{\sqrt{(b\alpha^*)^2-(b\underline{\alpha})^2+2r_A/(kr^2)}}c_4.\end{cases} \tag{F.4}$$

At equilibrium, $\frac{\partial u_S(x,\alpha)}{\partial x}=0$. Taking the first-order derivative of $u_S(\alpha)$ in Eq. (40) yields

$$\frac{du_S(\alpha)}{dq}=\frac{\partial u_S(\alpha)}{\partial x_S(\alpha)}\frac{dx_S(\alpha)}{dq}-\eta\alpha\frac{d\left(b(\alpha^*-\underline{\alpha})/q\right)}{dq}-kr\left(\frac{dc_3}{d\alpha^*}e^{rx_S(\alpha)}-\frac{dc_4}{d\alpha^*}e^{-rx_S(\alpha)}\right)\frac{d\alpha^*}{dq}. \tag{F.5}$$

Substituting Eqs. (27), (30), and (F.4) into (F.5) yields

$$\frac{du_S(\alpha)}{dq}=\frac{krb}{2}\left(\begin{array}{l}\dfrac{e^{2ra}+1}{e^{2ra}-1}\dfrac{2\alpha}{\alpha^*}+\dfrac{2\eta}{kr}\dfrac{\alpha^*-\underline{\alpha}}{q}\dfrac{\alpha}{\alpha^*}+\dfrac{2b\alpha}{\sqrt{(b\alpha^*)^2-(b\underline{\alpha})^2+2r_A/(kr^2)}}\\[2ex] +\dfrac{-b\alpha+\sqrt{(b\alpha)^2-(b\underline{\alpha})^2+2r_A/(kr^2)}}{\sqrt{(b\alpha^*)^2-(b\underline{\alpha})^2+2r_A/(kr^2)}}\\[2ex] -\dfrac{-(b\underline{\alpha})^2+2r_A/(kr^2)}{\sqrt{(b\alpha^*)^2-(b\underline{\alpha})^2+2r_A/(kr^2)}}\cdot\dfrac{1}{-b\alpha+\sqrt{(b\alpha)^2-(b\underline{\alpha})^2+2r_A/(kr^2)}}\end{array}\right)\frac{d\alpha^*}{dq}. \tag{F.6}$$

After some operations, one can derive that the sum of the final three terms in the bracket of Eq. (F.6) is equal to 0, and thus Eq. (F.6) can further be simplified as

$$\frac{du_S(\alpha)}{dq} = \left(\frac{e^{2ra}+1}{e^{2ra}-1} krb + \eta \frac{b(\alpha^* - \underline{\alpha})}{q} \right) \frac{\alpha}{\alpha^*} \frac{d\alpha^*}{dq} > 0. \tag{F.7}$$

This completes the derivation of Eq. (41).

Appendix G: Derivations of Eqs. (47) and (48)

We first derive Eq. (47). From Eq. (11), the total net land rent LR in Eq. (45) can be rewritten as

$$LR=\int_0^a \frac{k}{2}n_D^2(x)dx+\int_a^B \frac{k}{2}n_S^2(x)dx-r_A B. \tag{G.1}$$

From Eqs. (16) and (17), we have

$$\begin{aligned}&\int_0^a \frac{k}{2}n_D^2(x)dx=\int_0^a \frac{k}{2}r^2\left(c_1^2e^{2rx}+c_2^2e^{-2rx}-2c_1c_2\right)dx\\&=\frac{k}{4}r^2c_1^2\left(e^{2ra}-1\right)-\frac{k}{4}r^2c_2^2\left(e^{-2ra}-1\right)-kr^2c_1c_2a\\&=\frac{kr}{4}\frac{\left(e^{2ra}+1\right)\left((b\overline{\alpha})^2+(b\alpha^*)^2\right)-4e^{ra}b^2\overline{\alpha}\alpha^*}{e^{2ra}-1}+kr^2a\frac{e^{2ra}(b\overline{\alpha})^2-\left(e^{3ra}+e^{ra}\right)b^2\overline{\alpha}\alpha^*+e^{2ra}(b\alpha^*)^2}{\left(e^{2ra}-1\right)^2}.\end{aligned} \tag{G.2}$$

Similarly, from Eqs. (21) and (23), we have

$$\begin{aligned}&\int_a^B \frac{k}{2}n_S^2(x)dx=\int_a^B \frac{k}{2}r^2\left(c_3^2e^{2rx}+c_4^2e^{-2rx}-2c_3c_4\right)dx\\&=\frac{k}{4}rc_3^2\left(e^{2rB}-e^{2ra}\right)-\frac{k}{4}rc_4^2\left(e^{-2rB}-e^{-2ra}\right)-kr^2c_3c_4\left(B-a\right)\\&=\frac{kr}{4}\left(-b\underline{\alpha}\sqrt{2r_A/(kr^2)}\right)+\frac{kr}{4}b\alpha^*\sqrt{(b\alpha^*)^2-(b\underline{\alpha})^2+2r_A/(kr^2)}-kr^2\frac{(b\underline{\alpha})^2-2r_A/(kr^2)}{4}\left(B-a\right).\end{aligned} \tag{G.3}$$

Substituting Eqs. (G.2) and (G.3) into (G.1), and using equilibrium condition Eqs. (25) and (35), one can derive

$$\begin{aligned}&\frac{dLR}{dq}=\frac{dLR}{d\alpha^*}\frac{d\alpha^*}{dq}\\&=\left(\frac{krb}{2}\frac{\left(e^{2ra}+1\right)b\alpha^*-2e^{ra}b\overline{\alpha}}{e^{2ra}-1}+\frac{krb\left((b\alpha^*)^2-(b\underline{\alpha})^2\right)}{2\sqrt{(b\alpha^*)^2-(b\underline{\alpha})^2+2r_A/(kr^2)}}+kr^2a\frac{-e^{ra}\left(e^{2ra}+1\right)b^2\overline{\alpha}+2e^{2ra}b^2\alpha^*}{\left(e^{2ra}-1\right)^2}\right)\frac{d\alpha^*}{dq}\\&<\left(\frac{krb}{2}\frac{\left(e^{2ra}+1\right)b\alpha^*-2e^{ra}b\overline{\alpha}}{e^{2ra}-1}+\frac{krb}{2}\sqrt{(b\alpha^*)^2-(b\underline{\alpha})^2+2r_A/(kr^2)}+kr^2a\frac{-e^{ra}\left(e^{2ra}+1\right)b^2\overline{\alpha}+2e^{2ra}b^2\alpha^*}{\left(e^{2ra}-1\right)^2}\right)\frac{d\alpha^*}{dq},\end{aligned} \tag{G.4}$$

where $\dfrac{d\alpha^*}{dq}$ is given by Eq. (29).

According to equilibrium condition Eq. (25), one can obtain

$$\frac{krb}{2}\frac{\left(e^{2ra}+1\right)b\alpha^*-2e^{ra}b\overline{\alpha}}{e^{2ra}-1}+\frac{krb}{2}\sqrt{(b\alpha^*)^2-(b\underline{\alpha})^2+2r_A/(kr^2)}<0. \tag{G.5}$$

Owing to $\overline{\alpha} > \alpha^*$, we thus have

$$kr^2 a \frac{-e^{ra}\left(e^{2ra}+1\right)b^2\overline{\alpha}+2e^{2ra}b^2\alpha^*}{\left(e^{2ra}-1\right)^2}<0. \tag{G.6}$$

In terms of Eqs. (G.4), (G.5) and (G.6), one can immediately obtain $\frac{dLR}{dq}<0$.

We now look at Eq. (48). According to Eqs. (28) and (D.5), the utility of the household at location x can be written as

$$u(x)=w(\alpha(x))-c(x,\alpha(x))-kn(x), \tag{G.7}$$

where $n(x)=n_D(x)$ for $x\in(0,a)$, and $n(x)=n_S(x)$ for $x\in(a,B)$. $c(x,\alpha)$ is determined by Eq. (3).

From Eq. (G.7), the total utility of all households in the city can thus be given as

$$TU=\int_0^B u(x)n(x)dx=\int_0^B w(\alpha(x))n(x)dx-\int_0^B c(x,\alpha(x))n(x)dx-\int_0^B kn^2(x)dx. \tag{G.8}$$

Combining it with Eq. (G.1), we have

$$TU+LR=\int_0^B w(\alpha(x))n(x)dx-\int_0^B c(x,\alpha(x))n(x)dx-LR-2r_A B. \tag{G.9}$$

From Eqs. (1), (2), (13), (17), and (23), we have

$$\begin{aligned}
&\int_0^B c(x,\alpha(x))n(x)dx\\
&=\int_0^a \frac{\tau r}{b}x\left(c_1e^{rx}+c_2e^{-rx}\right)\left(c_2e^{-rx}-c_1e^{rx}\right)dx+\int_a^B \frac{r}{b}\left(\tau x+\frac{b(\alpha^*-\underline{\alpha})\eta}{q}\right)\left(c_3e^{rx}+c_4e^{-rx}\right)\left(c_4e^{-rx}-c_3e^{rx}\right)dx\\
&=-\left(\begin{aligned}
&\frac{\tau a}{2b}\frac{2e^{2ra}(b\overline{\alpha})^2+(e^{4ra}+1)(b\alpha^*)^2-2e^{ra}(e^{2ra}+1)b\overline{\alpha}b\alpha^*}{\left(e^{2ra}-1\right)^2}+\frac{\tau}{4br}\frac{4e^{ra}b\overline{\alpha}b\alpha^*-(e^{2ra}+1)\left((b\overline{\alpha})^2+(b\alpha^*)^2\right)}{e^{2ra}-1}\\
&+\frac{\tau}{4b}\left((b\underline{\alpha})^2+2r_A/(kr^2)\right)B-\frac{\tau a}{4b}\left(2(b\alpha^*)^2-(b\underline{\alpha})^2+2r_A/(kr^2)\right)+\frac{(\alpha^*-\underline{\alpha})\eta}{2q}\left((b\underline{\alpha})^2-(b\alpha^*)^2\right)\\
&+\frac{\tau}{4br}\left(b\underline{\alpha}\sqrt{2r_A/(kr^2)}-b\alpha^*\sqrt{(b\alpha^*)^2-(b\underline{\alpha})^2+2r_A/(kr^2)}\right)
\end{aligned}\right).
\end{aligned} \tag{G.10}$$

According to Eqs. (30), (35) and (G.10), one can derive

$$\frac{d\left(\int_0^B c(x,\alpha(x))n(x)dx\right)}{dq}=-\left(\begin{aligned}
&\frac{krb}{2}\frac{e^{2ra}+1}{e^{2ra}-1}\left(b\alpha^*-\frac{b\underline{\alpha}^2}{\alpha^*}\right)+\frac{\eta\left((b\alpha^*)^2-(b\underline{\alpha})^2\right)\left(\alpha^*-\underline{\alpha}\right)}{2\alpha^*q}-\frac{b\left(\alpha^*-\underline{\alpha}\right)\eta}{q}b\alpha^*\\
&+\tau a\frac{2e^{2ra}b\alpha^*-e^{ra}(e^{2ra}+1)b\overline{\alpha}}{\left(e^{2ra}-1\right)^2}+\frac{krb}{2}\frac{-\left(e^{2ra}+1\right)b\alpha^*+2e^{ra}b\overline{\alpha}}{e^{2ra}-1}
\end{aligned}\right)\frac{d\alpha^*}{dq}. \tag{G.11}$$

In addition, from Eq. (35), we have

$$2r_A \frac{dB}{dq} = \frac{krb}{2} \frac{4 r_A/(kr^2)}{\sqrt{(b\alpha^*)^2 - (b\underline{\alpha})^2 + 2r_A/(kr^2)}} \frac{d\alpha^*}{dq}. \tag{G.12}$$

Note that the first integral term on the right-hand side of Eq. (G.9) represents the total income of all households, which is independent of the bottleneck capacity q. We thus have

$$\frac{d(TU+LR)}{dq} = -\frac{d\left(\int_0^B c(x,\alpha(x))n(x)dx\right)}{dq} - \frac{dLR}{dq} - 2r_A \frac{dB}{dq}. \tag{G.13}$$

Substituting Eqs. (G.4), (G.11) and (G.12) into Eq. (G.13) and using the equilibrium condition Eq. (25), one can obtain

$$\frac{d(TU+LR)}{dq} = \left(\frac{krb}{2} \frac{e^{2ra}+1}{e^{2ra}-1} \left(b\alpha^* - \frac{b\underline{\alpha}^2}{\alpha^*} \right) + \frac{krb\left((b\alpha^*)^2 - (b\underline{\alpha})^2\right)}{2\sqrt{(b\alpha^*)^2 - (b\underline{\alpha})^2 + 2r_A/(kr^2)}} + \frac{\eta\left((b\alpha^*)^2 - (b\underline{\alpha})^2\right)\left(\alpha^* - \underline{\alpha}\right)}{2\alpha^* q} \right) \frac{d\alpha^*}{dq} > 0, \tag{G.14}$$

where $\frac{d\alpha^*}{dq}$ is given by Eq. (29).

This completes the derivations of Eqs. (47) and (48).