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Abstract

In this paper, we propose a method to disaggregate very small time series by fitting them with higher frequency related series using a cointegration regression with multiple partial endogenous structural breaks. We allow any coefficient to change at up to two dates of structural break and three related series and provide critical values for the test of cointegration corrected for the very small sample size. We find that increasing the number of related series drastically improves the power of the test by allowing for increased flexibility in the cointegration model. The simulated power of the test is shown to be very high even in very small sample sizes such as fifteen observations. This flexibility also mildly improves the accuracy of the disaggregation method when the sample size is as small as thirty-five observations. An application to the Chinese national accounts data is provided and allows the study of the Chinese business cycles stylized facts. We find that household consumption, public spending, and trade surpluses are the main driver of the business cycle.

JEL classification codes: C32, E17, E37

Keywords: Time series, macroeconomic forecasting, disaggregation, structural change, business cycles, emerging economies

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Introduction

Tests of cointegration are critical in finding the best linear model to link an aggregated macroeconomic time series of interest to its indicators. Taking into account structural change in the parameters helps tackle the issue of instability in emerging countries' data, whether a source can be exogenous shocks affecting the relationships between the series or changes in the methodology used to measure economic data. Many alternatives for modeling parameter instability can be found in the literature, from the simplest piece-wise linear relationships (Chow (1960)) to state-space representations with time-varying parameters (Primiceri (2005)). However, the very small sample sizes of emerging countries' data (less than fifty observations for the common dates between low-frequency time series of interest and related high-frequency indicators) prevent the use of the later and more sophisticated representations. In order to find a stationary relationship with structural change, a residual-based test *à la* Gregory and Hansen (1996) is simple and computationally feasible enough to be a satisfactory tool for practitioners willing to analyze emerging economies data. This test allows all coefficients to break and have low power in very small samples of, say, less than thirty observations, notably when considering several indicators because the number of parameters increases significantly while the sample size stays relatively small. Allowing for partial structural breaks, as suggested by Kejriwal and Perron (2010) in a cointegrated setting, can potentially save degrees of freedom for the estimation and help in isolating variables responsible for the rejection of the null hypothesis. Facilitating the estimation of a cointegration relationship eventually can help find a stationary relationship between the macroeconomic series and, in the end, have more candidate models to disaggregate low-frequency series of interest using higher frequency indicators by the method of Chow and Lin (1971). In this chapter, we extend Gregory and Hansen (1996)'s test of cointegration with endogenous structural breaks by allowing for partial structural breaks. The goal is to increase the power of the test in very small sample (Bai et al. (1998)) and make the use of structural break models more consistently reliable as a model selection process for disaggregating very small sample time series. In the first section, we present the upgraded disaggregation procedure. The second section provides the corresponding limiting distributions for the test of cointegration with endogenous partial structural breaks and the approximate size-corrected critical values of the test statistics. The third section evaluates the power of the test and the predictive performance of the disaggregation procedure. The last section applies the procedure to the Chinese national accounts data and we undertake a study of the business cycles stylized facts *à la* Backus et al. (1992).

1 The Setup

We extend the baseline framework of Chow and Lin (1971) with partial structural breaks in the linear coefficients. We define the high-frequency series of interest $\mathbf{y}$ ($fT \times 1$), their m related indicators $\mathbf{x}$ ($fT \times m$), and their respective low-frequency aggregates $\mathbf{Y} = \mathbf{A}\mathbf{y}$ and $\mathbf{X} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}= \mathbf{I}_T \otimes 1'_f$ is a $(T \times fT)$ aggregating matrix, with $\mathbf{I}_p$ the p-dimension identity matrix and $\mathbf{1}_p$ a p-dimensional column vector of ones. They are linked by the linear relationship with b structural changes:

$$\mathbf{y} = \begin{pmatrix}\mathbf{1}_{fT} & \mathbf{x}\end{pmatrix}\begin{pmatrix}\mu \\ \beta\end{pmatrix} + \sum_{i=1}^{b} \mathbf{B}_i \begin{pmatrix}\mathbf{1}_{fT} & \mathbf{x}\end{pmatrix} \mathbf{J}_i \begin{pmatrix}\mu_i \\ \beta_i\end{pmatrix} + \mathbf{u} \tag{1}$$

where $\mathbf{B}_i$ is a $(fT \times fT)$ diagonal matrix[1] of dummy variables applying the date t_i of structural change to $\begin{pmatrix}\mathbf{1}_{fT}, & \mathbf{x}\end{pmatrix}$ and $\mathbf{J}_i$ $((m+1)\times(m+1))$ a diagonal matrix[2] of dummy variables selecting which components $\begin{pmatrix}\mathbf{1}_{fT}, & \mathbf{x}\end{pmatrix}$ to be subject to the structural change at t_i. $\mathbf{y}$ and $\mathbf{x}$ are I(1), $\mathbf{u}$ are I(0). In a compact form:

$$\mathbf{y} = \mathbf{z}\Gamma + \mathbf{u} \tag{2}$$

where $\mathbf{z}$ is the vector of regressors and Γ the vector of coefficients. As we only observe $\mathbf{Y}$ and $\mathbf{x}$, we can predict $\mathbf{y}$ by applying the Fernandez (1981) general disaggregation formula:

$$\begin{gathered}\hat{\mathbf{y}} = \mathbf{z}\hat{\Gamma} + (\mathbf{D}'\mathbf{D})^{-1}\mathbf{A}'(\mathbf{A}(\mathbf{D}'\mathbf{D})^{-1}\mathbf{A}')^{-1}(\mathbf{Y} - \mathbf{A}\mathbf{z}\hat{\Gamma}) \\ \text{where} \quad \hat{\Gamma} = (\mathbf{z}'\mathbf{A}'\mathbf{A}\mathbf{z})^{-1}\mathbf{z}'\mathbf{A}'\mathbf{Y}\end{gathered} \tag{3}$$

where $\mathbf{D}$ is a $((fT-1) \times fT)$ matrix that converts the high frequency time series to its first differences, such that

$$\mathbf{D} = \begin{pmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 1 \end{pmatrix}.$$

The baseline model without structural change is such that $\mathbf{z} = \begin{pmatrix}1_{fT}, & \mathbf{x}\end{pmatrix}$ and $\Gamma = \begin{pmatrix}\mu, & \beta\end{pmatrix}'$. We allow any coefficients μ or components of β to break at the dates of structural change. In practice, the underlying true model is unknown. A rule of thumb for the practitioners is to find a satisfying model for which the estimated residuals $\hat{\mathbf{U}} = \mathbf{Y} - \mathbf{Z}\hat{\Gamma}$, where $\mathbf{Z}$=$\mathbf{Az}$, are stationary, so that predictions can be done for both interpolation and distribution (when $\mathbf{Y}$ and $\mathbf{x}$ are observed on the same period), but also extrapolation (when $\mathbf{x}$ are observed outside the time period of $\mathbf{Y}$). Testing for a unit root in the residuals is therefore critical for model selection. The Augmented Dickey-Fuller (ADF) test statistic is preferred for our sample sizes of interest ($T \leq 50$). When there is no structural change, conventional cointegration between $\mathbf{Y}$ and $\mathbf{X}$ can be tested. When the intercept μ or every coefficient μ and β break at the same but unknown dates, cointegration with endogenous structural breaks *à la* Gregory and Hansen (1996) can be tested [3] by selecting the dates of structural change $\{t_i\}_{1\leq i\leq b}$ for which minimizes

[1] The diagonal components are $\{\mathbb{1}_{t\geq t_i}\}$, $t = 1, \cdots, fT$.

[2] For example, if $m = 3$ and the intercept and first slope coefficient break at the first date of structural change t_1 and the intercept and the second and third slope coefficients break at the second date of structural change t_2, we have $\mathbf{J}_1 = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$ and $\mathbf{J}_2 = \begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$. Effectively, the associated coefficients β_{12}, β_{13} and β_{21}, where $\beta_i = \begin{pmatrix}\beta_{i1}, & \cdots, & \beta_{im}\end{pmatrix}'$, are not estimated.

[3] This corresponds to the unrestricted cases of our framework, i.e. the $\mathbf{J}_i$ are then identity matrices.

the ADF-test statistic for testing the unit root for $\hat{\mathbf{U}}$. In order to gain power, we generalize it to allow restricted cases where only a subset of the coefficients break, which drastically increases the number of candidate models. In the next section, we provide the limiting distributions and the approximate critical values for our extended framework.

2 Test of cointegration with partial endogenous structural breaks

Let us consider the following linear relationship analogously to the previous section:

$$\mathbf{Y} = \mathbf{Z}\Gamma + \mathbf{U} \tag{4}$$

where $\mathbf{Y}$ and the column-vectors of $\mathbf{Z}$, except for the intercept, are I(1) processes. The columns of $\mathbf{Z}$ depend on the type of model of structural change. To test the null of no cointegration, i.e. $\mathbf{Z} = \begin{pmatrix} 1_T, & \mathbf{X} \end{pmatrix}$ and $\mathbf{U}$ is a I(1) process, against the alternative of cointegration with partial endogenous structural breaks, i.e. $\mathbf{U}$ is a I(0) process, the following test statistic is calculated for a number b of structural changes :

$$ADF^* = \inf_{\{t_i\}_{1 \le i \le b}} ADF(\hat{\mathbf{U}}) \tag{5}$$

where $ADF(\hat{\mathbf{U}})$ is the ADF-test statistic for a unit root in $\hat{\mathbf{U}}$, given the dates of structural change $\{t_i\}_{1 \le i \le b}$ and type of model. The ADF models for the tests are selected such that the number of lags is the minimal number necessary to remove the serial correlation in the residuals of the ADF models.

2.1 Asymptotic distributions

Gregory and Hansen (1996) derive the asymptotic distributions of the Z_t^* test statistics based on the Phillips (1987) portmanteau test, and the ADF^* test is expected to have the same asymptotic properties. In an analogous way[4], we extend the expression of the limiting distributions under the null to the case of up to two partial structural breaks as follows:

$$Z_t^* \to_d \inf_{\tau} \frac{\int_0^1 W_\tau dW_\tau}{\left[\int_0^1 W_\tau^2\right]^{\frac{1}{2}} \left[1 + \kappa_\tau' \mathbf{D}_\tau \kappa_\tau\right]^{\frac{1}{2}}}$$

where

$$W_\tau(r) = W_1(r) - \int_0^1 W_1 W_{2\tau}' \left[\int_0^1 W_{2\tau} W_{2\tau}'\right]^{-1} W_{2\tau}(r)$$
$$\kappa_\tau = \left[\int_0^1 W_{2\tau} W_{2\tau}'\right]^{-1} \int_0^1 W_{2\tau} W_1$$

[4] See Gregory and Hansen (1996) for the precise expressions of the functions W_1 and W_2 of Brownian motions. Their proof can be adapted by mapping the vectors of regressors adequately such that we consider each of the structural break i to affect q_i slope coefficients and get the limiting distributions to depend on functions $W_{2i} \equiv BM(\mathbf{I}_{q_i})$ which are independent to W_1.

and $W_{2\tau}(r)$ and $\mathbf{D}_\tau$ depend on the model.
If $b = 1$, then $\tau = \tau_1$, $q_1 \leq m$ and

$$W_{2\tau} = 1, \phi_{\tau_1}(r), W_2'(r), W_{21}'(r)\phi_{\tau_1}(r)]'$$

where $\phi_{\tau_i}(r) = \mathbb{1}\{r \geq \tau_i\}$ and

$$\mathbf{D}_\tau = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m & (1-\tau_1)\mathbf{E}_1 \\ \mathbf{0} & (1-\tau_1)\mathbf{E}_1' & (1-\tau_1)\mathbf{I}_{q_1} \end{pmatrix}$$

with $\mathbf{E}_i$ a $(m \times q_i)$ matrix depending on the model[5].
If $b = 2$, then $\tau = \{\{\tau_1, \tau_2\}, \tau_1 < \tau_2\}$, $q_1 \leq m$, $q_2 \leq m$ and

$$W_{2\tau} = [1, \phi_{\tau_1}(r), \phi_{\tau_2}(r), W_2'(r), W_{21}'(r)\phi_{\tau_1}(r), W_{22}'(r)\phi_{\tau_2}(r)]'$$

where $\phi_{\tau_i}(r) = \mathbb{1}\{r \geq \tau_i\}$ and

$$\mathbf{D}_\tau = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m & (1-\tau_1)\mathbf{E}_1 & (1-\tau_2)\mathbf{E}_2 \\ \mathbf{0} & (1-\tau_1)\mathbf{E}_1' & (1-\tau_1)\mathbf{I}_{q_1} & (1-\tau_2)\mathbf{E}_1'\mathbf{E}_2 \\ \mathbf{0} & (1-\tau_2)\mathbf{E}_2' & (1-\tau_2)\mathbf{E}_2'\mathbf{E}_1 & (1-\tau_2)\mathbf{I}_{q_2} \end{pmatrix}.$$

Similarly to Gregory and Hansen (1996) and Hatemi-j (2008), the limiting distributions are dependent to the type of model, i.e. the number of variables $1+m$ and the number of structural breaks b. Allowing for $q_i < m$ slope coefficients to change at structural break i naturally makes D_τ dependent on the q_i. Additionally in the case $b = 2$, for a given number $q_1 + q_2$ of slope coefficients affected by structural changes, having more slope coefficients affected by the first break increases the magnitude of $\mathbf{D}_\tau$ because $\tau_1 < \tau_2$. Therefore, the absolute value of the test statistics is on average lower if the slope coefficients change at the first break instead of the second one.

2.2 Approximate critical values

The test of cointegration with endogenous structural break of Gregory and Hansen (1996) tests the null of no cointegration between $\mathbf{Y}$ and $\mathbf{X}$ with a unit root of the residuals against the alternative of cointegration between $\mathbf{Y}$ and $\mathbf{X}$ with a structural break in the intercept or the intercept and all slope coefficients at an unknown date[6], i.e. $\{q_i\}_{i \leq b}$ and $b = 1$. We extend it to two unknown dates as in Hatemi-j (2008) (more would be less relevant for the considered small samples) and allow subsets of the regressors to be subject to the structural changes. This amounts to considering a variety of restrictions on the model of Gregory and Hansen

[5] $\mathbf{E}_i$ is a selection of the column-vectors of $\mathbf{I}_m$, and $\mathbf{E}_i'\mathbf{E}_i = \mathbf{I}_{q_i}$. For example, if $m = 3$ and the first and third component of $\mathbf{x}$ are subject to the first structural change, we have $\mathbf{E}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$. The case $W_{21}(r) = W_2(r)$ and $\mathbf{E}_1 = \mathbf{I}_m$ is exactly model 4 of Gregory and Hansen (1996).

[6] Respectively models 2 and 4 of Gregory and Hansen (1996).

(1996), which we classify by numbers $\{q_i\}_{i\leq b}$ of breaking slope coefficients per date of structural change, accordingly to the limiting distributions expressed above. Since they have no closed form expressions, the distributions of the test statistics are simulated under the null to compute approximate critical values. Our setting considers smaller sample sizes ($T < 50$ whereas the literature computes simulations of sample sizes $T \geq 50$) of both unrestricted and restricted cases. 5 000 to 10 000 replications for $T = 15, 20, 25, 30, 35, 40, 45, 50, 100, 200, 500$ are simulated under the null, using the following data generating model[7]:

$$\begin{cases} Y_t &= \mu + \mathbf{X}_t'\beta + U_t, \quad U_t = U_{t-1} + \varepsilon_t \\ \mathbf{X}_t &= \theta + \mathbf{V}_t, \qquad\quad \mathbf{V}_t = \mathbf{V}_{t-1} + \eta_t \end{cases} \tag{6}$$

where ε and η are stationary. The ADF^*-test statistics are calculated the residuals of every replication for every type of structural break model and sample size, after which cumulative distribution are computed in each case. Quantiles can then be retrieved. Considering the high rate of convergence of residual-based tests and the very small sample considered here, the values can be significantly different depending on the sample size. Therefore, instead of directly using the simulated quantiles as critical values, critical values for the test are estimated by using MacKinnon (1991)'s surface response methodology. Functions of polynomials of $1/T$'s for every quantile, number of regressor m, number of structural breaks b and numbers q_i of slope coefficients subject to the i-th structural break are estimated by linear regression:

$$Crt(\text{quantile}, T, m, b, \{q_i\}_{i\leq b}) = \psi_\infty + \sum_{k=1}^{K} \psi_k T^{-k} + \text{error}$$

where the order of polynomial K is selected by minimizing the Akaike Information Criterion ($K \leq 6$). The intercept ψ_∞ represents the asymptotical critical value for the test (as $T \to \infty$) and all size-corrected critical values are obtained by simply applying the polynomial function of $1/T$'s to the desired sample size T.

Table 1 reports the approximate critical values at the 5% nominal level for the test. The particular cases of a structural break in either only the constant only ($q_1 = q_2 = 0$) or the constant and every slope coefficient ($q_1 = q_2 = m$) are the ones considered in Gregory and Hansen (1996), for which we find the same asymptotical critical values (with an 0.1 absolute value difference at worst). As expected, the critical values are more negative the more parameters are to be estimated, that is to say when the number of regressors m, the number of structural breaks b or the number $q_1 + q_2$ of affected slope coefficients increases. However, separating the number of breaking slope coefficients by date of structural break as suggested by the limiting distributions doesn't yield significantly different critical values except for the very small sample sizes. Generally, allowing for a smaller number of slope parameters to be affected by the structural changes makes the rejection of the null hypothesis easier, in addition to adding flexibility to the regression model. Consequently, the rejection of the unit root in the residuals when there is actually none (i.e. the power of the test) is expected to be higher when the number of candidate regression models considered is larger.

[7]The details for calibration are reported in Appendix A.

regression				$T=$				
m	b	q_1	q_2	15	20	30	50	∞
1	0	-	-	-3.87	-3.76	-3.64	-3.54	-3.4
	1	0	-	-5.53	-5.54	-5.26	-4.98	-4.67
		1	-	-5.99	-5.92	-5.64	-5.3	-4.87
	2	0	0	-6.82	-6.91	-6.36	-6.03	-5.61
		1	0	-7.22	-7.26	-6.65	-6.34	-5.89
		0	1	-7.44	-7.3	-6.75	-6.37	-5.82
		1	1	-7.78	-7.64	-6.99	-6.63	-6.23
2	0	-	-	-4.36	-4.33	-4.13	-3.98	-3.86
	1	0	-	-6.09	-6	-5.7	-5.41	-5.08
		1	-	-6.52	-6.38	-6.03	-5.72	-5.36
		2	-	-6.91	-6.68	-6.32	-5.96	-5.57
	2	0	0	-7.45	-7.35	-6.74	-6.38	-5.95
		1	0	-7.76	-7.7	-7.04	-6.65	-6.17
		0	1	-8.03	-7.79	-7.12	-6.7	-6.19
		2	0	-8.19	-8.07	-7.31	-6.89	-6.33
		1	1	-8.45	-8.16	-7.4	-6.97	-6.43
		0	2	-8.71	-8.23	-7.44	-6.96	-6.41
		2	1	-8.95	-8.57	-7.68	-7.21	-6.66
		1	2	-9.23	-8.63	-7.75	-7.24	-6.7
		2	2	-9.84	-9.07	-8	-7.45	-6.93
3	0	-	-	-4.82	-4.76	-4.59	-4.34	-4.12
	1	0	-	-6.59	-6.44	-6.07	-5.78	-5.42
		1	-	-7.12	-6.83	-6.39	-6.08	-5.65
		2	-	-7.59	-7.21	-6.67	-6.32	-5.89
		3	-	-8.1	-7.58	-6.92	-6.54	-6.07
	2	0	0	-7.98	-7.78	-7.09	-6.71	-6.2
		1	0	-8.33	-8.18	-7.4	-6.98	-6.47
		0	1	-8.72	-8.28	-7.47	-7.01	-6.48
		2	0	-8.87	-8.54	-7.67	-7.22	-6.66
		1	1	-9.17	-8.68	-7.77	-7.27	-6.71
		0	2	-9.42	-8.8	-7.8	-7.27	-6.69
		3	0	-9.42	-8.94	-7.95	-7.42	-6.81
		2	1	-9.75	-9.13	-8.05	-7.51	-6.94
		1	2	-10	-9.23	-8.12	-7.53	-6.92
		0	3	-10.11	-9.26	-8.09	-7.49	-6.89
		3	1	-10.41	-9.6	-8.35	-7.71	-6.99
		2	2	-10.77	-9.72	-8.41	-7.76	-7.1
		1	3	-10.93	-9.76	-8.45	-7.76	-7.1
		3	2	-11.99	-10.24	-8.7	-7.97	-7.24
		2	3	-12.43	-10.27	-8.75	-8.01	-7.28
		3	3	-16.01	-10.75	-9.05	-8.2	-7.37

Table 1: 5% size-corrected approximate critical values for the ADF^* test of cointegration with endogenous partial structural breaks

2.3 Power of the test

6 000 replications are simulated under the alternative of stationary residuals for every cointegration model with structural change for sample sizes $T = 15, 20, \ldots, 45, 50$, following the general parametrization[8]:

$$\begin{cases} Y_t &= \mu + \mathbf{X}_t'\beta + \sum_{i=1}^{b} \begin{pmatrix} 1 & \mathbf{X}_t' \end{pmatrix} \mathbf{J}_i \begin{pmatrix} \mu_i \\ \beta_i \end{pmatrix} \mathbb{1}_{t \geq t_i} + U_t, \quad U_t = \rho U_{t-1} + \varepsilon_t \\ \mathbf{X}_t &= \theta + \mathbf{V}_t, \qquad \mathbf{V}_t = \mathbf{V}_{t-1} + \eta_t \end{cases} \tag{7}$$

where $1 \leq t_i \leq T$, $i = 1, \ldots, b$, are the dates of structural change, $|\rho| < 1$ and the residuals ε and η are stationary. For each case, we estimate every cointegration model with the same number of regressors m. The null of no cointegration is rejected if at least one of the cointegration models rejects it.

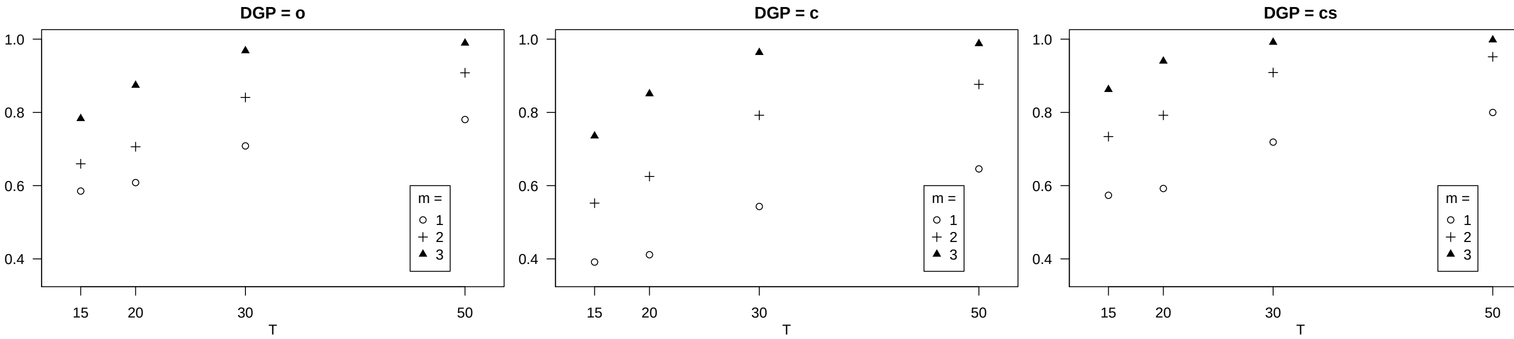

Figure 1: Empirical rejection rates (ERFs) for ADF^* of the null of no cointegration at the 5% nominal level when the alternative is true (o: no structural change, c: change in the constant, cs: change in the constant and the slope coefficients)

Figure 1 represents the probabilities of rejecting the null when the alternative is true using the 5% critical values (Appendix B details the power of the test by type of structural change model). We observe the power of the test to significantly increase with the number m of related series. This comes from the higher number of candidate cointegration models when $\mathbf{X}$ is of a higher dimension. For example, when $\mathbf{X}$ is $(T \times 1)$, there are only 7 possible cointegration models, against 21 when $\mathbf{X}$ is $(T \times 2)$ or 73 when $\mathbf{X}$ is $(T \times 3)$. As a consequence when m increases it is more likely to fit at least one cointegration model with structural change for which the residuals are stationary, even if the selected regression models are not the same as the data generating one (Appendix C details the power of testing each type of cointegration model). In the end, we obtain a very reasonable power for a very small sample size as low as 15 observations, from around 50% when $m = 1$ to around 80% when $m = 3$, increasing significantly with sample size to almost the certainty of rejecting the null when the alternative is true for $n = 30$ and $m = 3$.

[8] See Appendix A for details.

3 Predictive performance of the methodology

We show in the previous section that even for a very small sample as $15 \leq T \leq 50$, we can find a cointegration model which fits the observed data regardless of the true data generating process, and this is all the more the case if we consider several indicators, thus several candidate models. In order to assess the performance of the full prediction procedure, we simulate $\mathbf{y}$ using a wide array of data-generating processes as functions of the simulated indicators $\mathbf{x}$, with $(\mathbf{y}, \mathbf{x})$ being series of higher frequency and sample size fT, following a general parametrization similar to section 2.3:

$$\begin{cases} y_t = \mu + \mathbf{x}_t'\beta + \sum_{i=1}^{b} \begin{pmatrix} 1 & \mathbf{x}_t' \end{pmatrix} \mathbf{J}_i \begin{pmatrix} \mu_i \\ \beta_i \end{pmatrix} \mathbb{1}_{t \geq t_i} + u_t, & u_t = \rho u_{t-1} + \varepsilon_t \\ \mathbf{x}_t = \theta + \mathbf{v}_t, & \mathbf{v}_t = \mathbf{v}_{t-1} + \eta_t \end{cases} \tag{8}$$

The simulated $(\mathbf{y}, \mathbf{x})$ are aggregated into lower frequency data $(\mathbf{Y}, \mathbf{X})$ with sample size T. Then, we can see how well we can predict $\mathbf{y}$ when only the lower frequency $\mathbf{Y}$ and the high-frequency indicators $\mathbf{x}$ are observed. In order to do so, we fit every possible cointegration model with up to two structural breaks $(0 \leq b \leq 2)$ and retain the ones for which the unit root in the residuals is rejected when using the approximate critical values computed in Section 2. The estimated cointegration models can be expressed in a general form as a linear relationship with structural breaks between the aggregated series

$$\mathbf{Y} = \mathbf{Z}\Gamma + \mathbf{U}, \tag{9}$$

and their residuals $\hat{\mathbf{U}}$ can be retrieved. Among the valid models, we select the one which minimizes the Akaike Information Criterion (AIC)[9], i.e. the one which yields the best in-sample prediction while penalizing for the number of regressors. We predict the higher frequency series of interest $\hat{\mathbf{y}}$ by applying the disaggregation formula (3). Finally, we can compare the prediction $\hat{\mathbf{y}}$ to the real values $\mathbf{y}$ by computing the root mean square error of prediction :

$$\text{RMSE}(\hat{\mathbf{y}}) = \sqrt{(fT)^{-1}(\hat{\mathbf{y}} - \mathbf{y})'(\hat{\mathbf{y}} - \mathbf{y})}.$$

For practitioners, a common way to correct for a unit root in the residuals is to differentiate the data if stationary residuals can't be obtained. This corresponds to the more general case[10] of the one with I(1) residuals considered by Fernandez (1981), in which we also allow for an intercept in the model in first difference. We use it as a benchmark to assess the relative predictive performance of our methodology. The higher frequency prediction $\hat{\mathbf{y}}$ for the benchmark is obtained by applying the following disaggregation equation (see Appendix D):

$$\begin{gathered} \hat{\mathbf{y}}_{\mathbf{diff}} = \hat{y}_1 \mathbf{1}_{fT} + \mathbf{z}_\Delta \Gamma + (\mathbf{D}'\mathbf{D})^{-1}\mathbf{A}'(\mathbf{A}(\mathbf{D}'\mathbf{D})^{-1}\mathbf{A}')^{-1}(\delta\mathbf{Y} - \mathbf{Z}_\Delta\hat{\Gamma}) \\ \text{where} \quad \hat{\Gamma} = (\mathbf{Z}_\Delta'\mathbf{Z}_\Delta)^{-1}\mathbf{Z}_\Delta'\delta\mathbf{Y} \end{gathered} \tag{10}$$

[9]We use the AIC corrected for small sample size (AICc), such that $AICc = AIC + \frac{2K^2+2K}{T-K-1}$, where K is the number of parameters in the regression.

[10]We are less restrictive than Fernandez (1981) by allowing for an intercept in the differentiated model. If the intercept is proven not to be significant, its estimate should be small enough so that the resulting model isn't quantitatively very different. This implies a linear trend component in the non-differentiated model. Therefore we cannot use their disaggregation formula as the assumption of constancy of the first-year values does not hold and would considerably impact the estimates considering our small sample sizes.

where δ is a $(T-1) \times T$ matrix converting annual data to their first difference, $\mathbf{Z}_\Delta = \left(\mathbf{1}_{T-1}, \quad \delta \mathbf{A} \mathbf{x}\right)$, $\hat{y}_1 = \frac{Y_1}{f} + \left(\frac{f-1}{2f}, \quad (\mathbf{x}_1 - \mathbf{X}_1)'\right) \hat{\Gamma}$ and $\mathbf{z}_\Delta = \{\mathbf{z}'_{\Delta t}\}_t$ with $\mathbf{z}_{\Delta t} = \left(\frac{t-1}{f^2}, \quad (\mathbf{x}_t - \mathbf{x}_1)'\right)'$. We also compute the RMSE($\hat{\mathbf{y}}_{\text{diff}}$) as a predictive performance measure. Figure 2 represents the

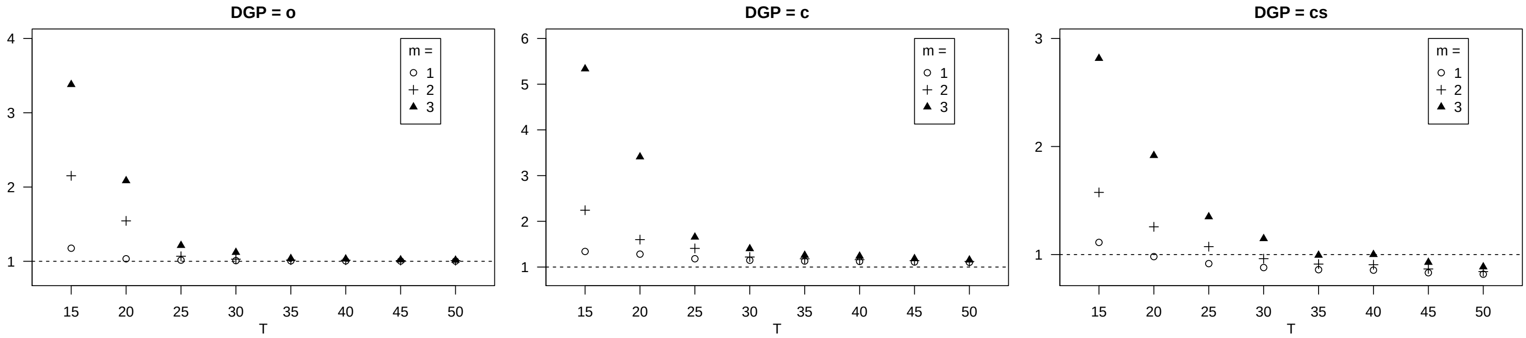

Figure 2: Predictive performance: RMSE($\hat{\mathbf{y}}$) / RMSE($\hat{\mathbf{y}}_{\text{diff}}$)

relative performance RMSE($\hat{\mathbf{y}}$) / RMSE($\hat{\mathbf{y}}_{\text{diff}}$) of our methodology to the benchmark (Appendix E reports the results for all types of DGP). The gains in accuracy differ depending on the type of underlying data generating process and sample size. When the data has no structural change or only a change in the intercept, we observe no gain. Considering an endogenous structural breaks however improves the prediction accuracy when there are breaking slope coefficients and $T \geq 20$ for $m = 1$, $T \geq 30$ for $m = 2$ and $T \geq 45$ for $m = 3$. The accuracy improves with the sample size by up to twenty percentage points when $m = 1$ or $m = 2$, or fifteen percentage points when $m = 3$. These results suggest that the efficiency of residual-based tests of cointegration for selecting a prediction model is limited very small sample size lower than a number of observations which increases with the number of indicators considered. This is explained by the tendency of such tests to select a model with minimal serial correlation at the expense of minimizing the standard error. The estimated residuals are then more likely to be centered, but as the true model is not necessarily selected, this may lead to larger estimated residuals.

4 Application: the stylized facts of the Chinese business cycles

In order to illustrate our methodology, we undertake a study of the stylized facts of the quarterly Chinese national accounts. These series are however only released every year by the National Bureau of Statistics as yearly data. Therefore, we apply our procedure to disaggregate every component — namely household consumption expenditures ($cons$), gross fixed capital formation ($gfcf$), government consumption expenditures (ge), net exports of goods and services (nx), and changes in inventories (ci) — from annual to quarterly series. Table 2 details the data which are used for our exercise. The annual series we disaggregate are available from 1952 to 2020 as part of the annual national accounts published by the National Bureau of Statistics. The quarterly indicators for the annual national accounts component are only available for more recent history, starting at the earliest in 1995 and at the latest in 2005, and ending between

2018 and 2020, at the date of writing this paper. Therefore a cointegration model can be only estimated between the national account data and the annualized indicators for the intersecting time period of availability, which correspond to between 16 and 23 years.

National accounts component		Indicator(s)[1]		Deflator	
Households consumption expenditures ($cons$)	1952-2020	x_1: Disposable income x_2: Cons. expenditures	2005Q1-2020Q4 2005Q1-2020Q4	CPI	1990M1-2021M9
Gross fixed capital formation ($gfcf$)	1952-2020	x_1: Investment in fixed assets	1998M2-2021M9	Inv. in fixed assets price index	1998Q1-2018Q4
Government cons. expenditure (ge)	1952-2020	x_1: Gov. revenue x_2: Gov. expenditure	1998Q1-2020Q4 1998Q1-2020Q4	GDP deflator*	1998Q1-2018Q4
Net exports of goods and services (nx)	1952-2020	x_1: Export of goods x_2: Import of goods	1995M1-2021M7 1995M1-2021M7	GDP deflator*	1998Q1-2018Q4
Changes in inventories (ci)	1952-2020	x_1: Quarterly $cons$* x_2: Quarterly $gfcf$*	2005Q1-2020Q4 1998Q1-2020Q4	GDP deflator*	1998Q1-2018Q4

[1] Indicators are selected from a wider array of available quarterly data. The indicators we eventually retain are the ones for which the cointegration model minimizes the AIC.

*Estimated data.

Table 2: Database for disaggregating the Chinese national accounts

Tables 3 to 7 report the cointegration models which are selected as in Section 3 between the national account series and their annualized indicators. They show that considering at least one structural break in the slope coefficients leads to a better fit for the cointegration models. Two particular ranges of selected dates of structural change $\hat{t}_i$ draw our attention. The first range is between 2008 and 2011, which corresponds to the start or the aftermath of the subprime financial global crisis. The second range is between 2014 and 2016, which corresponds to the period surrounding the Shanghai stock market crash of June 2015. If only one date of structural change is selected, the cointegration models have breaking slope coefficients at either the first or the second range of dates. If two dates are selected, the coefficients break once at a date inside each of the two ranges. Figures 3 to 7 represent the observed quarterly and annual data with the quarterly predictions we obtain from the disaggregation procedure.

In the case of the household consumption expenditures (Table 3), the quarterly survey data on the Chinese households' disposable income and consumption expenditures constitute significant[11] explaining variables for the annual fluctuations until 2014. From then, only the survey data on consumption expenditures are a good predictor. In the case of the gross fixed capital formation (Table 4), the correlation with the investment in fixed assets mildly but not significantly decreases in 2008. In the case of government consumption expenditures (Table 5), the government revenue starts as a good predictor of the annual variations before its coefficient decreases in 2016, though not significantly. The quarterly government expenditures gains some correlation with the annual data mildly in 2011 and more significantly in 2016. In the case of

[11]Conventional tests of significance for the estimated coefficients, however, cannot be applied in our case, as homoskedasticity is hardly verified for the residuals. As a rule of thumb, we will interpret a ratio of the estimated coefficient on the standard error higher than two as the significance of the coefficient.

net exports in goods and services (Table 6), the slope coefficients before 2009 are close to 1 for the exports of goods and to -1 for the import of goods. This indicates that the net exports of goods by themselves are a good approximation of the net exports of goods and services. Both indicators however almost entirely lose their correlation to the annual data in 2009, before gaining it back in 2015. Finally, in the case of the changes in inventories (Table 7), every previously predicted quarterly national accounts series are candidates as indicators. The quarterly estimates of *cons* and $gfcf$ come out as the best indicators, with the former losing predicting power and the latter gaining more of it in 2011.

Explained variable: Y

$\hat{t}_i$	-	2014	-
const.	16855 (6404)	-69199 (10664)	-
X_1	1.857 (0.367)	-1.657 (0.383)	-
X_2	-1.498 (0.530)	2.664 (0.560)	-

Table 3: Regression results for *cons*

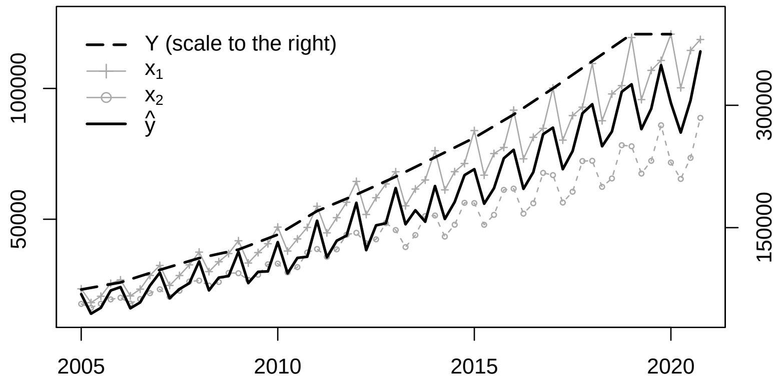

Figure 3: Quarterly predictions for *cons*

Explained variable: Y

$\hat{t}_i$	-	2008	-
const.	15694 (9465)	37711 (14865)	-
X_1	0.757 (0.156)	-0.269 (0.158)	-

Table 4: Regression results for $gfcf$

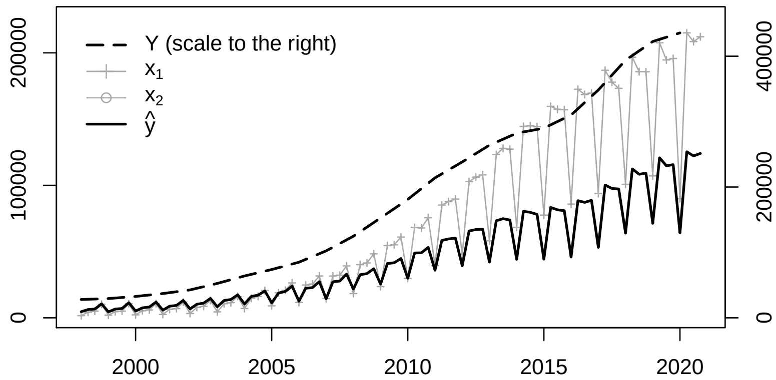

Figure 4: Quarterly predictions for $gfcf$

Explained variable: Y

$\hat{t}_i$	-	2011	2016
const.	7634 (460.3)	-6085 (3168)	-79092 (7699)
X_1	0.499 (0.081)	- -	-0.119 (0.111)
X_2	0.123 (0.080)	0.074 (0.027)	0.524 (0.074)

Table 5: Regression results for *ge*

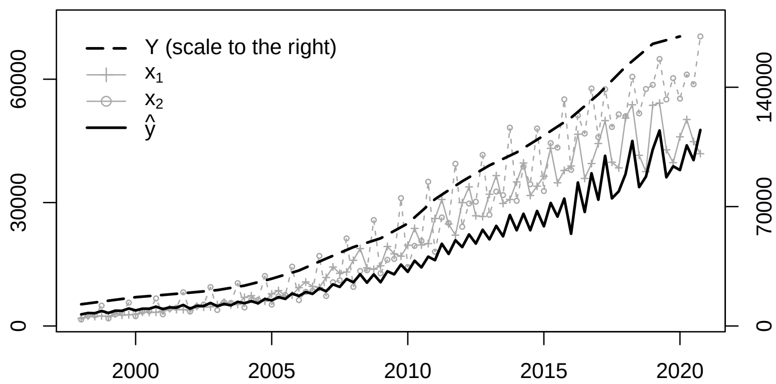

Figure 5: Quarterly predictions for *ge*

Explained variable Y

$\hat{t}_i$	-	2009	2015
const.	-521.4	17078	-54135
	(657.4)	(3156)	(8058)
X_1	1.056	-0.951	1.279
	(0.118)	(0.178)	(0.182)
X_2	-1.017	0.873	-1.159
	(0.148)	(0.210)	(0.185)

Table 6: Regression results for nx

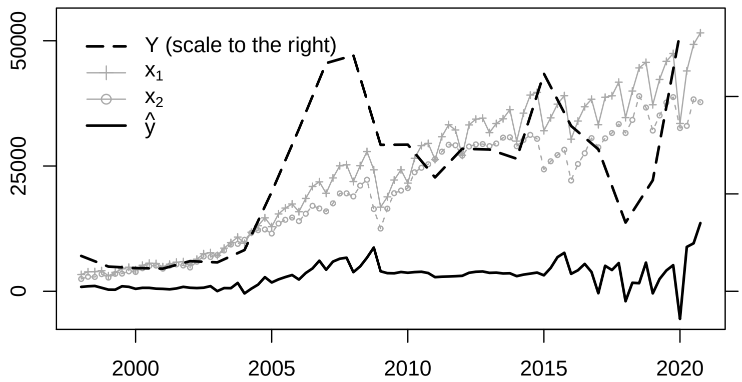

Figure 6: Quarterly predictions for nx

Explained variable: Y

$\hat{t}_i$	-	2011	-
const.	-25093	43110	-
	(7154)	(7816)	
X_1	0.713	-0.750	-
	(0.212)	(0.218)	
X_2	-0.365	-0.371	-
	(0.131)	(0.140)	

Table 7: Regression results for ci

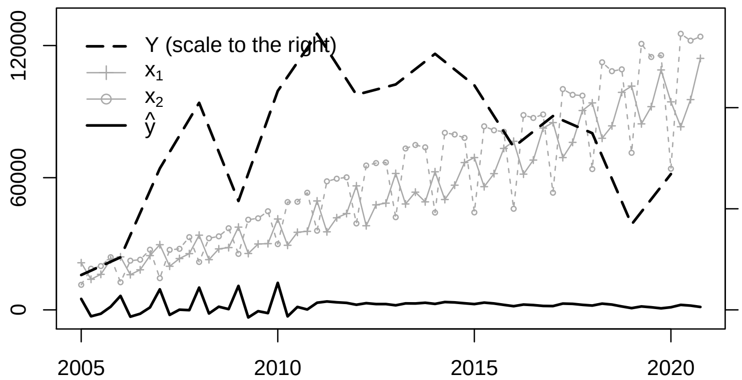

Figure 7: Quarterly predictions for ci

Now that we estimated the quarterly series for every component of the Chinese national accounts, we can compute the quarterly Chinese GDP (*gdp*) as the sum of the components. We then undertake the study of the stylized facts of the Chinese business cycles *à la* Backus et al. (1992), *i.e.* the second-order moments of the cyclical components of the national accounts in volume. We are able to deflate *cons* and $gfcf$ using respectively the Consumer Price Index and the Investment in fixed assets price index. As no deflator is available for the other components, a GDP deflator is estimated for consistency as the weighted average of the price indexes for *cons* and $gfcf$, then used to deflate *gdp*, *ge*, *nx* and *ci*. The cyclical components are finally retrieved by Hodrick-Prescott (HP) filtering[12] the deflated quarterly estimates in logarithm, except for *nx* and *ci* which are expressed as shares of *gdp*, after seasonal adjustment[13]. We end up with an intersecting period of data availability ranging between 2005 and 2018, and the years at the border are dropped in order to mitigate the effects of filtering the data. Figure 8 plots the cyclical components and Table 8 reports their second-order moments from 2006Q1 to 2017Q4.

Our results can be compared with the comparative studies between emerging and developed markets by Aguiar and Gopinath (2007) in the late twentieth century. A standard deviation for the GDP (1.59) is more in line with the most volatile developed markets than most emerging economies, for which $\sigma(gdp)$ is mostly higher than two. The relative volatility of consump-

[12]The smoothing parameter for the HP-filter is set as $\lambda = 1\ 600$.

[13]The X13-ARIMA-SEATS program by the US Census Bureau is used for seasonal adjustment.

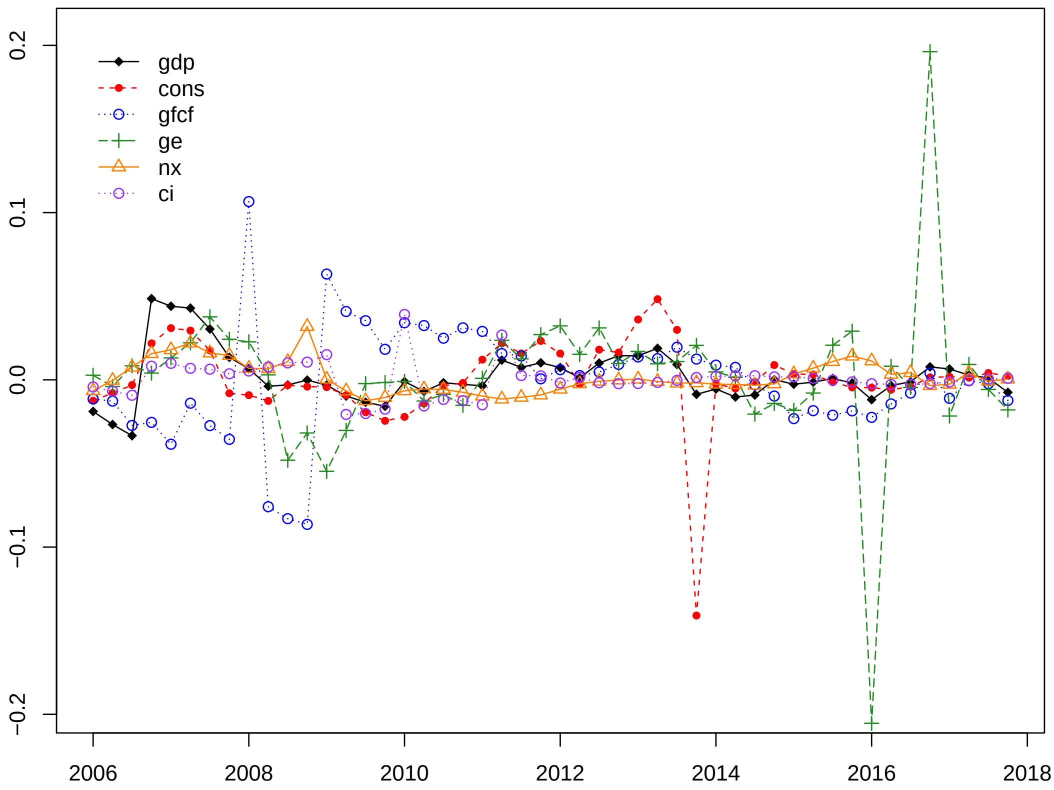

Figure 8: Cyclical components of the estimated quarterly national accounts

tion to GDP (1.64) is however more consistent with emerging markets ($\sigma(cons)/\sigma(gdp) > 1$) than most developed markets ($\sigma(cons)/\sigma(gdp) < 1$). This especially implies the absence of consumption smoothing at the national level. A striking result is the countercyclicality of investment ($\rho(gfcf_t, gdp_t) = -0.10$) in the short term. However the correlation is very small in absolute value, even though the aggregate investment constitutes almost half of the GDP. On the other hand, public spending and net exports are procyclical ($\rho(ge_t, gdp_t) = 0.28$ and $\rho(nx_t, gdp_t) = 0.47$). While the case for public spending is not surprising for emerging markets (see Kaminsky et al. (2004)), the case for net exports shows that the important trade surpluses during the last decades strongly drive the Chinese business cycles.

				$\rho(x_t, \text{gdp}_{t+k})$, where $k=$								
x	$\sigma(x)$	$\frac{\sigma(x)}{\sigma(\text{gdp})}$	$\rho_1(x)$	-4	-3	-2	-1	0	1	2	3	4
gdp	1.59	1.00	0.60	-0.29	-0.12	0.24	0.60	1.00	0.60	0.24	-0.12	-0.29
$cons$	2.60	1.64	0.17	-0.26	-0.16	-0.01	0.24	0.50	0.39	0.28	0.12	-0.05
$gfcf$	3.36	2.12	0.25	-0.07	-0.03	0.03	-0.06	-0.10	-0.21	-0.16	-0.08	0.05
ge	4.60	2.90	-0.01	-0.02	0.06	0.15	0.18	0.28	0.19	0.12	0.06	-0.02
nx/gdp	0.95	...	0.75	0.08	0.10	0.21	0.32	0.40	0.36	0.17	-0.07	-0.21
ci/gdp	1.05	...	0.01	0.14	0.07	0.07	0.15	0.37	0.14	0.02	-0.17	-0.24

Statistics are calculated on HP-filtered season-adjusted deflated data. Except for nx/gdp and ci/gdp, all series are in logarithms. Standard deviations are expressed in percentage points.

gdp : gross domestic product, $cons$: household consumption expenditure, $gfcf$: gross fixed capital formation, g: government consumption expenditure, nx: net exports of goods and services, ci: changes in inventories

Table 8: Business cycle stylized facts of the Chinese national accounts (2006Q1-2017Q4)

Concluding remarks

By allowing for endogenous partial structural breaks in the cointegration model, we significantly improve the power of the test in very small sample because of two reasons: the candidate models have on average fewer parameters to estimate, and the number of candidate models drastically increases, especially when the number of explaining variables considered is larger. Even when using size-corrected critical values, very good power is found the more we add explaining variables and for sample sizes as small as fifteen observations. In addition to the novelty of these results, the simple estimation procedure used is not computationally time-consuming in very small sample. Applying the cointegrating relationship to link low-frequency time series to higher frequency indicators allows us to find better models to disaggregate at least thirty observations, provided there are structural breaks in the linear parameters. However, for smaller sample sizes we fail to improve the prediction accuracy, especially when more explaining variables are added. This implies that finding cointegration can be at the expense of minimizing the prediction errors in very small sample. Effectively, selecting a model by minimizing the serial correlation to reject a unit root is not adequate for prediction in a situation where the serial correlation is not accurately estimated. Finding a way to correct this bias in very small sample may help find more accurate relationships among the data. Moreover, we didn't account for the serial correlation when testing the significance of the coefficients. Estimating it more accurately would allow more robust inference in the selection of individual variables.

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A Monte Carlo simulations: calibration

In order to save computation time and memory space, all simulations are set as follows, for $t = -50, \ldots, fT$, with high frequency $f = 4$ and sample size T:

$$\begin{cases} y_t &= \mu + \mathbf{x}_t'\beta + \sum_{i=1}^{b} \begin{pmatrix} 1 & \mathbf{x}_t' \end{pmatrix} \mathbf{J}_i \begin{pmatrix} \mu_i \\ \beta_i \end{pmatrix} \mathbb{1}_{t \geq t_i} + u_t, \quad u_t = \rho u_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \text{NID}(0,1) \\ \mathbf{x}_t &= \theta + \mathbf{v}_t, \quad \mathbf{v}_t = \mathbf{v}_{t-1} + \eta_t, \quad \eta_t \sim \text{NID}(0,\Sigma) \end{cases} \tag{11}$$

where the fixed parameters are drawn from the following distributions:

- Σ is a $(m \times m)$ diagonal matrix where each component of the diagonal is drawn from a uniform distribution $\mathcal{U}[0.5, 4]$
- $\theta \sim \mathcal{U}[0.5, 4]^m$
- $(\mu, \beta')' \sim \mathcal{U}[-4, 4]^{m+1}$
- $t_i = [\![\tau f T]\!]$ where $[\![\cdot]\!]$ stands for the integer part, and $\tau \sim \mathcal{U}[0.15, 0.85]$

To compute the approximate critical values (Section 2.2), 10 000 replications for sample sizes $T = 15, 20, 25, 30, 35, 40, 45, 50, 100$ and 5 000 for $T = 500$ are simulated under the null hypothesis, i.e. $\rho = 1$ and the $\mathbf{J}_i$ are matrices of zeroes. To assess the power of the test (Section 2.3) and the predictive power of the temporal disaggregation method (Section 3), 6 000 replications are simulated under the alternative hypothesis for $T = 15, 20, 25, 30, 35, 40, 45, 50$, i.e. the slope coefficients selection matrices $\mathbf{J}_i$ depend on the type of structural break model and $\rho \sim \mathcal{U}[0, 0.9]$. Then the first fifty observations are discarded. $\mathbf{Y}$=$\mathbf{Ay}$, $\mathbf{X}$=$\mathbf{Ax}$, $\mathbf{U}$=$\mathbf{Au}$ and $\mathbf{V}$=$\mathbf{Av}$ are obtained by temporal aggregation.

Note that $\mathbf{U}$ follows an ARMA(1,1) under the alternative and an ARIMA(0,1,1) under the null (as well as $\mathbf{V}$ for all cases). This doesn't qualitatively affect the cointegration tests as the ADF-test is strong against all alternatives of ARMA models.

B Power of the test : by DGP

DGP				$T =$			
m	b	q_1	q_2	15	20	30	50
1	0	-	-	0.585	0.608	0.708	0.78
	1	0	-	0.43	0.454	0.588	0.688
		1	-	0.604	0.627	0.76	0.838
	2	0	0	0.352	0.368	0.498	0.604
		1	0	0.697	0.709	0.81	0.87
		0	1	0.442	0.463	0.62	0.723
		1	1	0.552	0.57	0.685	0.769
2	0	-	-	0.66	0.706	0.841	0.908
	1	0	-	0.574	0.648	0.813	0.899
		1	-	0.711	0.782	0.915	0.959
		2	-	0.761	0.833	0.944	0.974
	2	0	0	0.531	0.602	0.771	0.854
		1	0	0.808	0.846	0.932	0.964
		0	1	0.613	0.69	0.852	0.918
		2	0	0.885	0.912	0.97	0.984
		1	1	0.711	0.77	0.891	0.943
		0	2	0.668	0.743	0.893	0.946
		2	1	0.764	0.811	0.911	0.954
		1	2	0.688	0.753	0.885	0.937
		2	2	0.73	0.782	0.896	0.939
3	0	-	-	0.783	0.874	0.968	0.989
	1	0	-	0.743	0.857	0.965	0.988
		1	-	0.831	0.927	0.989	0.997
		2	-	0.867	0.946	0.994	0.999
		3	-	0.874	0.954	0.996	1
	2	0	0	0.728	0.844	0.962	0.988
		1	0	0.902	0.952	0.992	0.998
		0	1	0.781	0.897	0.981	0.995
		2	0	0.945	0.978	0.997	0.999
		1	1	0.849	0.931	0.989	0.998
		0	2	0.81	0.917	0.99	0.997
		3	0	0.96	0.987	0.999	1
		2	1	0.881	0.948	0.992	0.998
		1	2	0.833	0.927	0.99	0.998
		0	3	0.825	0.924	0.993	0.998
		3	1	0.9	0.958	0.994	0.998
		2	2	0.856	0.936	0.991	0.998
		1	3	0.833	0.928	0.989	0.998
		3	2	0.872	0.944	0.992	0.998
		2	3	0.845	0.931	0.99	0.998
		3	3	0.862	0.936	0.991	0.998

Table 9: Empirical rejection rates of the null of no cointegration at the 5% nominal level when the alternative is true

C Power of the test : all regression models

DGP			regression			$T =$			
b	q_1	q_2	b	q_1	q_2	15	20	30	50
0	-	-	0	-	-	0.352	0.382	0.417	0.444
			1	0	-	0.292	0.288	0.361	0.457
				1	-	0.269	0.288	0.356	0.463
			2	0	0	0.228	0.211	0.333	0.423
				1	0	0.212	0.204	0.328	0.404
				0	1	0.201	0.227	0.348	0.447
				1	1	0.205	0.228	0.354	0.453
1	0	-	0	-	-	0.066	0.074	0.086	0.095
			1	0	-	0.135	0.134	0.184	0.253
				1	-	0.164	0.177	0.223	0.303
			2	0	0	0.12	0.111	0.188	0.257
				1	0	0.141	0.136	0.221	0.284
				0	1	0.141	0.157	0.247	0.329
				1	1	0.197	0.227	0.362	0.448
	1	-	0	-	-	0.073	0.081	0.095	0.105
			1	0	-	0.157	0.156	0.195	0.254
				1	-	0.217	0.228	0.286	0.375
			2	0	0	0.157	0.147	0.222	0.287
				1	0	0.219	0.21	0.317	0.385
				0	1	0.221	0.243	0.344	0.432
				1	1	0.312	0.349	0.499	0.586
2	0	0	0	-	-	0.042	0.049	0.057	0.064
			1	0	-	0.085	0.084	0.12	0.171
				1	-	0.103	0.111	0.145	0.208
			2	0	0	0.094	0.088	0.151	0.217
				1	0	0.124	0.12	0.193	0.251
				0	1	0.121	0.138	0.225	0.304
				1	1	0.158	0.175	0.282	0.355
	1	0	0	-	-	0.104	0.115	0.128	0.141
			1	0	-	0.229	0.227	0.281	0.36
				1	-	0.247	0.261	0.311	0.394
			2	0	0	0.285	0.273	0.377	0.456
				1	0	0.31	0.305	0.395	0.457
				0	1	0.353	0.375	0.484	0.567
				1	1	0.358	0.381	0.49	0.56
	0	1	0	-	-	0.058	0.066	0.075	0.083
			1	0	-	0.098	0.097	0.132	0.182
				1	-	0.122	0.131	0.173	0.253
			2	0	0	0.092	0.084	0.153	0.211
				1	0	0.128	0.121	0.216	0.28
				0	1	0.139	0.156	0.246	0.333
				1	1	0.197	0.224	0.359	0.445
	1	1	0	-	-	0.087	0.096	0.108	0.12
			1	0	-	0.138	0.136	0.177	0.232
				1	-	0.166	0.177	0.216	0.291
			2	0	0	0.15	0.139	0.216	0.281
				1	0	0.178	0.17	0.26	0.322
				0	1	0.237	0.259	0.35	0.436
				1	1	0.275	0.295	0.406	0.484

Table 10: Empirical rejection rates of the null of no cointegration at the 5% nominal level when the alternative is true, $m = 1$

DGP			regression			$T=$			
b	q_1	q_2	b	q_1	q_2	15	20	30	50
0	-	-	0	-	-	0.285	0.29	0.343	0.384
			1	0	-	0.261	0.28	0.357	0.432
				1	-	0.329	0.367	0.469	0.565
				2	-	0.237	0.278	0.359	0.454
			2	0	0	0.204	0.22	0.349	0.445
				1	0	0.273	0.286	0.441	0.551
				0	1	0.277	0.321	0.493	0.615
				2	0	0.175	0.191	0.328	0.433
				1	1	0.341	0.41	0.594	0.718
				0	2	0.17	0.242	0.406	0.531
				2	1	0.232	0.294	0.485	0.607
				1	2	0.207	0.311	0.504	0.63
				2	2	0.136	0.224	0.412	0.54
1	0	-	0	-	-	0.067	0.069	0.087	0.103
			1	0	-	0.115	0.126	0.177	0.237
				1	-	0.209	0.236	0.314	0.405
				2	-	0.155	0.187	0.249	0.326
			2	0	0	0.102	0.114	0.2	0.273
				1	0	0.176	0.183	0.307	0.404
				0	1	0.184	0.219	0.362	0.48
				2	0	0.127	0.139	0.245	0.335
				1	1	0.301	0.371	0.584	0.703
				0	2	0.129	0.181	0.313	0.423
				2	1	0.213	0.275	0.482	0.604
				1	2	0.193	0.288	0.489	0.625
				2	2	0.135	0.222	0.394	0.518
	1	-	0	-	-	0.066	0.069	0.087	0.103
			1	0	-	0.115	0.125	0.165	0.216
				1	-	0.258	0.289	0.378	0.476
				2	-	0.216	0.251	0.321	0.403
			2	0	0	0.11	0.119	0.193	0.26
				1	0	0.219	0.229	0.367	0.473
				0	1	0.238	0.283	0.445	0.569
				2	0	0.17	0.184	0.297	0.388
				1	1	0.364	0.439	0.675	0.791
				0	2	0.183	0.238	0.385	0.501
				2	1	0.279	0.345	0.588	0.716
				1	2	0.271	0.373	0.607	0.739
				2	2	0.206	0.31	0.482	0.596
	2	-	0	-	-	0.068	0.07	0.089	0.104
			1	0	-	0.118	0.127	0.166	0.22
				1	-	0.27	0.303	0.383	0.47
				2	-	0.229	0.26	0.33	0.416
			2	0	0	0.108	0.118	0.193	0.253
				1	0	0.232	0.241	0.368	0.463
				0	1	0.257	0.299	0.442	0.551
				2	0	0.182	0.196	0.31	0.399
				1	1	0.385	0.461	0.671	0.788
				0	2	0.197	0.259	0.405	0.512
				2	1	0.309	0.376	0.632	0.755
				1	2	0.294	0.395	0.637	0.769
				2	2	0.23	0.337	0.511	0.623

Table 11: Empirical rejection rates of the null of no cointegration at the 5% nominal level when the alternative is true, $m = 2$

DGP b	DGP q_1	DGP q_2	regression b	regression q_1	regression q_2	$T=$ 15	20	30	50
2	0	0	0	-	-	0.05	0.052	0.066	0.078
			1	0	-	0.07	0.078	0.115	0.161
				1	-	0.145	0.165	0.229	0.304
				2	-	0.112	0.135	0.181	0.244
			2	0	0	0.077	0.086	0.164	0.231
				1	0	0.156	0.167	0.285	0.378
				0	1	0.16	0.195	0.332	0.449
				2	0	0.116	0.128	0.225	0.306
				1	1	0.263	0.322	0.514	0.633
				0	2	0.119	0.168	0.29	0.393
				2	1	0.188	0.236	0.409	0.528
				1	2	0.183	0.272	0.44	0.576
				2	2	0.133	0.205	0.368	0.481
	1	0	0	-	-	0.087	0.09	0.111	0.128
			1	0	-	0.173	0.185	0.236	0.301
				1	-	0.295	0.324	0.409	0.501
				2	-	0.239	0.271	0.33	0.394
			2	0	0	0.2	0.214	0.32	0.402
				1	0	0.352	0.363	0.504	0.601
				0	1	0.358	0.407	0.567	0.677
				2	0	0.273	0.287	0.392	0.472
				1	1	0.475	0.54	0.721	0.818
				0	2	0.28	0.346	0.486	0.59
				2	1	0.366	0.415	0.57	0.668
				1	2	0.386	0.478	0.648	0.755
				2	2	0.297	0.364	0.502	0.601
	0	1	0	-	-	0.056	0.058	0.074	0.09
			1	0	-	0.077	0.084	0.121	0.164
				1	-	0.179	0.203	0.278	0.367
				2	-	0.145	0.174	0.229	0.302
			2	0	0	0.072	0.08	0.148	0.209
				1	0	0.146	0.156	0.28	0.379
				0	1	0.175	0.212	0.364	0.488
				2	0	0.101	0.113	0.215	0.3
				1	1	0.268	0.336	0.564	0.696
				0	2	0.137	0.188	0.33	0.436
				2	1	0.19	0.25	0.481	0.617
				1	2	0.208	0.299	0.5	0.638
				2	2	0.153	0.244	0.415	0.53
	2	0	0	-	-	0.094	0.096	0.117	0.133
			1	0	-	0.197	0.211	0.267	0.324
				1	-	0.356	0.387	0.473	0.56
				2	-	0.288	0.321	0.382	0.449
			2	0	0	0.238	0.251	0.356	0.438
				1	0	0.44	0.45	0.595	0.677
				0	1	0.42	0.469	0.627	0.727
				2	0	0.337	0.35	0.453	0.524
				1	1	0.568	0.628	0.793	0.873
				0	2	0.326	0.406	0.554	0.653
				2	1	0.432	0.476	0.631	0.724
				1	2	0.466	0.56	0.721	0.818
				2	2	0.354	0.427	0.563	0.648
	1	1	0	-	-	0.069	0.071	0.089	0.107
			1	0	-	0.103	0.112	0.156	0.207
				1	-	0.2	0.225	0.299	0.388
				2	-	0.162	0.188	0.24	0.309
			2	0	0	0.111	0.121	0.201	0.273
				1	0	0.202	0.211	0.334	0.434
2	1	1		0	1	0.242	0.284	0.435	0.557
				2	0	0.148	0.16	0.254	0.336
				1	1	0.359	0.425	0.622	0.738
				0	2	0.192	0.252	0.393	0.501
				1	2	0.308	0.402	0.58	0.694
				2	2	0.238	0.311	0.461	0.566
	0	2	0	-	-	0.059	0.06	0.08	0.096
			1	0	-	0.08	0.088	0.124	0.174
				1	-	0.198	0.226	0.31	0.4
				2	-	0.157	0.189	0.249	0.329
			2	0	0	0.071	0.08	0.144	0.2
				1	0	0.149	0.157	0.275	0.372
				0	1	0.193	0.235	0.385	0.508
				2	0	0.104	0.118	0.225	0.318
				1	1	0.283	0.352	0.591	0.717
				0	2	0.139	0.195	0.34	0.451
				2	1	0.202	0.268	0.534	0.676
				1	2	0.217	0.315	0.536	0.678
				2	2	0.162	0.264	0.447	0.559
	2	1	0	-	-	0.076	0.078	0.097	0.113
			1	0	-	0.12	0.131	0.174	0.231
				1	-	0.232	0.259	0.334	0.416
				2	-	0.181	0.207	0.259	0.328
			2	0	0	0.131	0.14	0.229	0.3
				1	0	0.241	0.253	0.384	0.478
				0	1	0.277	0.319	0.472	0.586
				2	0	0.17	0.184	0.279	0.358
				1	1	0.415	0.476	0.655	0.769
				0	2	0.231	0.292	0.432	0.534
				2	1	0.308	0.358	0.529	0.642
				1	2	0.36	0.455	0.621	0.725
				2	2	0.284	0.355	0.497	0.591
	1	2	0	-	-	0.065	0.067	0.086	0.102
			1	0	-	0.088	0.097	0.136	0.187
				1	-	0.172	0.197	0.272	0.355
				2	-	0.143	0.168	0.222	0.294
			2	0	0	0.09	0.098	0.173	0.233
				1	0	0.164	0.173	0.289	0.383
				0	1	0.205	0.245	0.389	0.51
				2	0	0.121	0.132	0.225	0.3
				1	1	0.316	0.382	0.584	0.704
				0	2	0.173	0.23	0.37	0.476
				2	1	0.234	0.286	0.483	0.608
				1	2	0.288	0.383	0.563	0.682
				2	2	0.228	0.302	0.458	0.565
	2	2	0	-	-	0.067	0.069	0.09	0.107
			1	0	-	0.097	0.108	0.15	0.203
				1	-	0.183	0.207	0.284	0.368
				2	-	0.149	0.174	0.226	0.292
				1	0	0.184	0.193	0.314	0.412
				1	0	0.184	0.193	0.314	0.412
				0	1	0.228	0.265	0.405	0.531
				2	0	0.136	0.146	0.237	0.314
				1	1	0.346	0.403	0.595	0.712
				0	2	0.201	0.26	0.396	0.498
				2	1	0.259	0.31	0.49	0.606
				1	2	0.323	0.416	0.591	0.697
				2	2	0.269	0.346	0.487	0.586

Table 12: Empirical rejection rates of the null of no cointegration at the 5% nominal level when the alternative is true, $m = 2$ (cont.)

DGP			regression			$T=$			
b	q_1	q_2	b	q_1	q_2	15	20	30	50
0	-	-	0	-	-	0.247	0.256	0.294	0.352
			1	0	-	0.239	0.27	0.357	0.436
				1	-	0.345	0.411	0.54	0.639
				2	-	0.322	0.41	0.543	0.644
				3	-	0.174	0.251	0.379	0.476
			2	0	0	0.197	0.227	0.37	0.469
				1	0	0.31	0.34	0.534	0.646
				0	1	0.3	0.385	0.599	0.729
				2	0	0.265	0.326	0.524	0.636
				1	1	0.444	0.556	0.778	0.877
				0	2	0.262	0.378	0.623	0.764
				3	0	0.13	0.182	0.337	0.454
				2	1	0.39	0.526	0.769	0.871
				1	2	0.386	0.549	0.798	0.905
				0	3	0.139	0.232	0.457	0.612
				3	1	0.204	0.321	0.581	0.725
				2	2	0.321	0.517	0.791	0.901
				1	3	0.204	0.365	0.648	0.792
				3	2	0.131	0.302	0.593	0.748
				2	3	0.124	0.342	0.619	0.772
				3	3	0.05	0.226	0.435	0.592
1	0	-	0	-	-	0.068	0.072	0.088	0.121
			1	0	-	0.111	0.131	0.187	0.245
				1	-	0.224	0.278	0.397	0.499
				2	-	0.244	0.316	0.448	0.548
				3	-	0.134	0.189	0.29	0.375
			2	0	0	0.098	0.117	0.217	0.299
				1	0	0.197	0.22	0.386	0.507
				0	1	0.199	0.278	0.474	0.612
				2	0	0.201	0.248	0.426	0.552
				1	1	0.364	0.487	0.739	0.851
				0	2	0.205	0.309	0.541	0.688
				3	0	0.106	0.144	0.278	0.389
				2	1	0.349	0.479	0.756	0.863
				1	2	0.342	0.505	0.777	0.888
				0	3	0.117	0.197	0.395	0.534
				3	1	0.201	0.31	0.569	0.72
				2	2	0.313	0.508	0.79	0.898
				1	3	0.182	0.334	0.611	0.764
				3	2	0.134	0.31	0.589	0.743
				2	3	0.122	0.328	0.602	0.752
				3	3	0.061	0.228	0.434	0.582
	1	-	0	-	-	0.063	0.067	0.081	0.109
			1	0	-	0.097	0.111	0.157	0.206
				1	-	0.261	0.321	0.439	0.537
				2	-	0.316	0.398	0.533	0.633
				3	-	0.176	0.238	0.35	0.433
			2	0	0	0.094	0.111	0.196	0.265
				1	0	0.215	0.242	0.418	0.54
				0	1	0.23	0.31	0.517	0.66
				2	0	0.23	0.278	0.47	0.593
				1	1	0.398	0.525	0.786	0.896
				0	2	0.26	0.371	0.615	0.756
				3	0	0.133	0.171	0.306	0.411
				2	1	0.406	0.545	0.82	0.916
				1	2	0.406	0.58	0.847	0.935
				0	3	0.145	0.232	0.431	0.575
				3	1	0.245	0.36	0.618	0.771
1	1	-		2	2	0.389	0.602	0.866	0.946
				1	3	0.224	0.395	0.668	0.823
				3	2	0.175	0.363	0.654	0.792
				2	3	0.154	0.383	0.668	0.803
				3	3	0.085	0.262	0.469	0.614
	2	-	0	-	-	0.062	0.066	0.081	0.107
			1	0	-	0.099	0.113	0.16	0.209
				1	-	0.263	0.321	0.433	0.527
				2	-	0.343	0.421	0.553	0.654
				3	-	0.192	0.253	0.363	0.444
			2	0	0	0.093	0.109	0.197	0.267
				1	0	0.222	0.247	0.419	0.532
				0	1	0.229	0.308	0.51	0.654
				2	0	0.243	0.292	0.484	0.605
				1	1	0.411	0.532	0.79	0.896
				0	2	0.278	0.388	0.636	0.773
				3	0	0.145	0.184	0.318	0.424
				2	1	0.425	0.565	0.834	0.929
				1	2	0.424	0.607	0.864	0.947
				0	3	0.155	0.24	0.445	0.583
				3	1	0.262	0.375	0.64	0.785
				2	2	0.411	0.632	0.887	0.959
				1	3	0.237	0.409	0.681	0.832
				3	2	0.193	0.386	0.676	0.813
				2	3	0.165	0.396	0.686	0.82
				3	3	0.097	0.277	0.484	0.625
	3	-	0	-	-	0.06	0.063	0.077	0.106
			1	0	-	0.094	0.107	0.151	0.202
				1	-	0.256	0.313	0.427	0.525
				2	-	0.35	0.422	0.547	0.646
				3	-	0.2	0.265	0.375	0.455
			2	0	0	0.092	0.106	0.196	0.255
				1	0	0.229	0.255	0.418	0.536
				0	1	0.236	0.315	0.515	0.654
				2	0	0.25	0.299	0.485	0.598
				1	1	0.417	0.546	0.787	0.893
				0	2	0.277	0.39	0.635	0.772
				3	0	0.152	0.193	0.322	0.429
				2	1	0.435	0.572	0.832	0.927
				1	2	0.426	0.609	0.861	0.947
				0	3	0.163	0.251	0.449	0.589
				3	1	0.271	0.385	0.643	0.794
				2	2	0.417	0.636	0.887	0.964
				1	3	0.242	0.414	0.683	0.835
				3	2	0.201	0.395	0.68	0.816
				2	3	0.175	0.406	0.685	0.818
				3	3	0.102	0.283	0.488	0.629
2	0	0	0	-	-	0.051	0.054	0.07	0.096
			1	0	-	0.067	0.081	0.122	0.168
				1	-	0.152	0.199	0.291	0.38
				2	-	0.175	0.235	0.351	0.442
				3	-	0.093	0.132	0.218	0.293
			2	0	0	0.067	0.084	0.174	0.25
				1	0	0.166	0.19	0.352	0.474
				0	1	0.167	0.237	0.429	0.565
				2	0	0.181	0.223	0.391	0.524
				1	1	0.328	0.438	0.696	0.818
				0	2	0.189	0.278	0.501	0.654

Table 13: Empirical rejection rates of the null of no cointegration at the 5% nominal level when the alternative is true, $m = 3$

DGP			regression			$T =$			
b	q_1	q_2	b	q_1	q_2	15	20	30	50
2	0	0	2	3	0	0.099	0.131	0.24	0.34
				2	1	0.318	0.442	0.704	0.823
				1	2	0.325	0.485	0.757	0.88
				0	3	0.106	0.178	0.354	0.492
				3	1	0.172	0.27	0.507	0.658
				2	2	0.309	0.488	0.772	0.889
				1	3	0.189	0.342	0.599	0.755
				3	2	0.145	0.298	0.565	0.728
				2	3	0.151	0.342	0.598	0.744
				3	3	0.086	0.241	0.431	0.571
2	1	0	0	-	-	0.072	0.077	0.092	0.12
			1	0	-	0.139	0.158	0.215	0.274
				1	-	0.297	0.356	0.472	0.567
				2	-	0.34	0.414	0.543	0.636
				3	-	0.188	0.242	0.33	0.4
			2	0	0	0.153	0.179	0.292	0.376
				1	0	0.347	0.376	0.568	0.685
				0	1	0.306	0.397	0.607	0.737
				2	0	0.377	0.433	0.615	0.718
				1	1	0.512	0.63	0.845	0.926
				0	2	0.318	0.436	0.676	0.799
				3	0	0.204	0.248	0.37	0.463
				2	1	0.516	0.631	0.837	0.919
				1	2	0.495	0.66	0.88	0.952
				0	3	0.182	0.278	0.485	0.618
				3	1	0.316	0.408	0.604	0.728
				2	2	0.483	0.653	0.868	0.943
				1	3	0.322	0.489	0.737	0.862
				3	2	0.278	0.424	0.64	0.764
				2	3	0.271	0.474	0.71	0.833
				3	3	0.17	0.346	0.509	0.625
	0	1	0	-	-	0.057	0.06	0.074	0.1
			1	0	-	0.07	0.083	0.125	0.171
				1	-	0.192	0.247	0.356	0.456
				2	-	0.232	0.308	0.438	0.541
				3	-	0.119	0.173	0.27	0.35
			2	0	0	0.065	0.079	0.162	0.227
				1	0	0.158	0.178	0.342	0.463
				0	1	0.175	0.251	0.458	0.609
				2	0	0.154	0.196	0.375	0.505
				1	1	0.332	0.451	0.725	0.854
				0	2	0.207	0.315	0.569	0.718
				3	0	0.083	0.116	0.237	0.341
				2	1	0.321	0.462	0.75	0.874
				1	2	0.346	0.522	0.804	0.913
				0	3	0.111	0.188	0.383	0.528
				3	1	0.174	0.283	0.545	0.71
				2	2	0.323	0.534	0.822	0.924
				1	3	0.187	0.347	0.622	0.781
				3	2	0.135	0.309	0.605	0.759
				2	3	0.135	0.342	0.621	0.77
				3	3	0.076	0.236	0.444	0.588
	2	0	0	-	-	0.078	0.082	0.098	0.125
			1	0	-	0.156	0.175	0.231	0.292
				1	-	0.345	0.41	0.526	0.617
				2	-	0.401	0.48	0.608	0.694
				3	-	0.219	0.275	0.364	0.433
			2	0	0	0.176	0.201	0.316	0.4

DGP			regression			$T =$			
b	q_1	q_2	b	q_1	q_2	15	20	30	50
2	2	0	2	1	0	0.41	0.442	0.627	0.734
				0	1	0.352	0.446	0.651	0.774
				2	0	0.45	0.507	0.679	0.773
				1	1	0.574	0.69	0.888	0.952
				0	2	0.37	0.49	0.728	0.84
				3	0	0.247	0.29	0.411	0.5
				2	1	0.583	0.694	0.877	0.944
				1	2	0.555	0.716	0.918	0.971
				0	3	0.223	0.319	0.53	0.656
				3	1	0.367	0.453	0.632	0.746
				2	2	0.542	0.707	0.897	0.959
				1	3	0.365	0.54	0.783	0.892
				3	2	0.328	0.469	0.671	0.783
				2	3	0.319	0.525	0.747	0.856
				3	3	0.208	0.389	0.545	0.657
	1	1	0	-	-	0.062	0.066	0.081	0.108
			1	0	-	0.088	0.104	0.152	0.204
				1	-	0.209	0.263	0.374	0.47
				2	-	0.238	0.308	0.433	0.531
				3	-	0.13	0.181	0.267	0.336
			2	0	0	0.091	0.109	0.2	0.274
				1	0	0.21	0.234	0.406	0.526
				0	1	0.217	0.3	0.505	0.646
				2	0	0.218	0.264	0.443	0.561
				1	1	0.399	0.518	0.769	0.878
				0	2	0.241	0.349	0.596	0.738
				3	0	0.117	0.152	0.266	0.362
				2	1	0.403	0.528	0.777	0.882
				1	2	0.417	0.59	0.84	0.932
				0	3	0.14	0.223	0.418	0.554
				3	1	0.23	0.325	0.556	0.704
				2	2	0.41	0.6	0.846	0.933
				1	3	0.271	0.434	0.693	0.828
				3	2	0.219	0.38	0.635	0.769
				2	3	0.227	0.432	0.679	0.808
				3	3	0.137	0.307	0.487	0.614
	0	2	0	-	-	0.056	0.059	0.074	0.1
			1	0	-	0.072	0.085	0.127	0.174
				1	-	0.207	0.265	0.379	0.475
				2	-	0.258	0.338	0.479	0.583
				3	-	0.131	0.191	0.296	0.378
			2	0	0	0.068	0.082	0.159	0.223
				1	0	0.158	0.178	0.338	0.458
				0	1	0.186	0.262	0.466	0.61
				2	0	0.159	0.206	0.392	0.521
				1	1	0.334	0.459	0.736	0.86
				0	2	0.231	0.335	0.591	0.741
				3	0	0.085	0.115	0.244	0.355
				2	1	0.333	0.477	0.773	0.892
				1	2	0.36	0.541	0.818	0.929
				0	3	0.12	0.201	0.396	0.539
				3	1	0.176	0.285	0.563	0.73
				2	2	0.332	0.556	0.845	0.938
				1	3	0.192	0.355	0.634	0.79
				3	2	0.138	0.323	0.63	0.778
				2	3	0.135	0.351	0.635	0.781
				3	3	0.074	0.239	0.454	0.594
	3	0	0	-	-	0.079	0.085	0.101	0.127

Table 14: Empirical rejection rates of the null of no cointegration at the 5% nominal level when the alternative is true, $m = 3$ (cont.)

DGP			regression			$T =$			
b	q_1	q_2	b	q_1	q_2	15	20	30	50
2	3	0	1	0	-	0.152	0.173	0.231	0.289
				1	-	0.364	0.432	0.546	0.635
				2	-	0.427	0.506	0.627	0.712
				3	-	0.236	0.293	0.384	0.447
			2	0	0	0.183	0.206	0.318	0.4
				1	0	0.436	0.467	0.652	0.757
				0	1	0.369	0.46	0.666	0.783
				2	0	0.489	0.543	0.71	0.798
				1	1	0.603	0.712	0.901	0.958
				0	2	0.395	0.51	0.742	0.854
				3	0	0.272	0.317	0.432	0.519
				2	1	0.607	0.722	0.894	0.951
				1	2	0.59	0.74	0.927	0.975
				0	3	0.249	0.352	0.552	0.676
				3	1	0.402	0.487	0.652	0.757
				2	2	0.577	0.737	0.912	0.963
				1	3	0.394	0.57	0.807	0.905
				3	2	0.362	0.499	0.688	0.794
				2	3	0.349	0.549	0.773	0.868
				3	3	0.232	0.415	0.563	0.669
	2	1	0	-	-	0.065	0.069	0.083	0.11
			1	0	-	0.099	0.116	0.166	0.22
				1	-	0.233	0.29	0.402	0.497
				2	-	0.26	0.331	0.457	0.553
				3	-	0.144	0.194	0.279	0.346
			2	0	0	0.108	0.127	0.221	0.297
				1	0	0.242	0.268	0.445	0.566
				0	1	0.245	0.327	0.533	0.668
				2	0	0.252	0.302	0.48	0.596
				1	1	0.437	0.556	0.794	0.894
				0	2	0.266	0.377	0.619	0.757
				3	0	0.139	0.174	0.29	0.382
				2	1	0.443	0.564	0.799	0.896
				1	2	0.452	0.621	0.859	0.94
				0	3	0.161	0.245	0.441	0.576
				3	1	0.266	0.358	0.574	0.712
				2	2	0.456	0.635	0.864	0.942
				1	3	0.301	0.469	0.723	0.847
				3	2	0.257	0.411	0.649	0.777
				2	3	0.265	0.471	0.707	0.828
				3	3	0.166	0.342	0.512	0.633
	1	2	0	-	-	0.056	0.06	0.076	0.103
			1	0	-	0.078	0.092	0.137	0.185
				1	-	0.188	0.241	0.348	0.444
				2	-	0.221	0.29	0.414	0.515
				3	-	0.12	0.168	0.257	0.328
			2	0	0	0.078	0.094	0.177	0.247
				1	0	0.176	0.198	0.363	0.484
				0	1	0.191	0.268	0.471	0.616
				2	0	0.182	0.226	0.404	0.526
				1	1	0.356	0.476	0.739	0.86
				0	2	0.225	0.328	0.577	0.723
				3	0	0.103	0.133	0.243	0.339
				2	1	0.358	0.489	0.756	0.873
				1	2	0.395	0.567	0.83	0.928
				0	3	0.132	0.214	0.406	0.543
				3	1	0.2	0.297	0.541	0.698
				2	2	0.387	0.583	0.841	0.931

DGP			regression			$T =$			
b	q_1	q_2	b	q_1	q_2	15	20	30	50
2	1	2	2	1	3	0.255	0.421	0.683	0.82
				3	2	0.199	0.364	0.634	0.771
				2	3	0.212	0.418	0.673	0.803
				3	3	0.13	0.3	0.483	0.612
	0	3	0	-	-	0.056	0.06	0.075	0.104
			1	0	-	0.074	0.089	0.128	0.175
				1	-	0.215	0.272	0.387	0.487
				2	-	0.271	0.356	0.494	0.598
				3	-	0.135	0.197	0.301	0.381
			2	0	0	0.069	0.083	0.158	0.217
				1	0	0.157	0.179	0.34	0.457
				0	1	0.191	0.27	0.463	0.613
				2	0	0.158	0.209	0.396	0.528
				1	1	0.339	0.459	0.731	0.862
				0	2	0.238	0.34	0.597	0.745
				3	0	0.087	0.12	0.25	0.361
				2	1	0.342	0.49	0.781	0.899
				1	2	0.361	0.539	0.828	0.928
				0	3	0.124	0.204	0.401	0.542
				3	1	0.187	0.299	0.583	0.75
				2	2	0.335	0.559	0.849	0.94
				1	3	0.191	0.357	0.631	0.79
				3	2	0.132	0.326	0.644	0.787
				2	3	0.13	0.349	0.645	0.784
				3	3	0.074	0.239	0.457	0.597
	3	1	0	-	-	0.064	0.068	0.082	0.109
			1	0	-	0.105	0.123	0.174	0.229
				1	-	0.247	0.305	0.418	0.513
				2	-	0.277	0.345	0.47	0.567
				3	-	0.153	0.2	0.284	0.351
			2	0	0	0.115	0.136	0.235	0.314
				1	0	0.27	0.296	0.475	0.593
				0	1	0.26	0.343	0.551	0.684
				2	0	0.28	0.331	0.509	0.625
				1	1	0.46	0.576	0.811	0.906
				0	2	0.283	0.397	0.636	0.766
				3	0	0.154	0.188	0.303	0.393
				2	1	0.473	0.592	0.816	0.906
				1	2	0.478	0.648	0.871	0.948
				0	3	0.178	0.263	0.459	0.593
				3	1	0.289	0.38	0.588	0.719
				2	2	0.49	0.661	0.875	0.947
				1	3	0.323	0.49	0.739	0.86
				3	2	0.28	0.434	0.658	0.781
				2	3	0.285	0.49	0.722	0.84
				3	3	0.187	0.357	0.524	0.64
	2	2	0	-	-	0.059	0.064	0.078	0.105
			1	0	-	0.084	0.098	0.146	0.198
				1	-	0.197	0.251	0.359	0.455
				2	-	0.225	0.295	0.42	0.516
				3	-	0.127	0.173	0.258	0.326
			2	0	0	0.086	0.102	0.191	0.264
				1	0	0.195	0.219	0.387	0.508
				0	1	0.206	0.285	0.486	0.629
				2	0	0.204	0.25	0.426	0.545
				1	1	0.38	0.498	0.753	0.868
				0	2	0.237	0.344	0.59	0.732
				3	0	0.114	0.145	0.256	0.347

Table 15: Empirical rejection rates of the null of no cointegration at the 5% nominal level when the alternative is true, $m = 3$ (cont.)

DGP			regression			$T =$			
b	q_1	q_2	b	q_1	q_2	15	20	30	50
2	2	2		2	1	0.384	0.509	0.768	0.877
				1	2	0.417	0.588	0.84	0.93
				0	3	0.147	0.228	0.421	0.556
				3	1	0.223	0.315	0.546	0.696
				2	2	0.421	0.61	0.853	0.935
				1	3	0.282	0.446	0.702	0.83
				3	2	0.23	0.391	0.643	0.773
				2	3	0.248	0.452	0.695	0.817
				3	3	0.158	0.331	0.507	0.628
	1	3	0	-	-	0.058	0.062	0.076	0.101
			1	0	-	0.073	0.086	0.129	0.176
				1	-	0.184	0.238	0.346	0.441
				2	-	0.219	0.29	0.418	0.516
				3	-	0.116	0.167	0.257	0.33
			2	0	0	0.071	0.085	0.164	0.233
				1	0	0.164	0.188	0.35	0.473
				0	1	0.186	0.262	0.459	0.605
				2	0	0.174	0.215	0.391	0.514
				1	1	0.341	0.462	0.725	0.852
				0	2	0.214	0.318	0.569	0.719
				3	0	0.095	0.126	0.236	0.333
				2	1	0.344	0.473	0.75	0.87
				1	2	0.383	0.555	0.821	0.921
				0	3	0.129	0.21	0.4	0.54
				3	1	0.19	0.291	0.54	0.7
				2	2	0.371	0.571	0.837	0.929
				1	3	0.249	0.413	0.678	0.815
				3	2	0.184	0.355	0.628	0.77
				2	3	0.202	0.413	0.673	0.801
				3	3	0.126	0.299	0.483	0.609
	3	2	0	-	-	0.061	0.066	0.081	0.107
			1	0	-	0.089	0.103	0.154	0.206
				1	-	0.209	0.263	0.367	0.463
				2	-	0.233	0.299	0.427	0.52
				3	-	0.133	0.176	0.257	0.327
			2	0	0	0.093	0.112	0.203	0.279
				1	0	0.216	0.239	0.411	0.533
				0	1	0.22	0.301	0.5	0.638
				2	0	0.224	0.27	0.444	0.562
				1	1	0.398	0.515	0.764	0.875
				0	2	0.249	0.359	0.596	0.737
				3	0	0.126	0.155	0.265	0.358
				2	1	0.405	0.529	0.779	0.883
				1	2	0.436	0.607	0.849	0.936
				0	3	0.16	0.24	0.434	0.564
				3	1	0.244	0.334	0.556	0.705
				2	2	0.446	0.629	0.862	0.938
				1	3	0.301	0.464	0.714	0.84
				3	2	0.251	0.408	0.65	0.777
				2	3	0.27	0.472	0.71	0.829
				3	3	0.176	0.349	0.517	0.636
	2	3	0	-	-	0.057	0.061	0.076	0.102
			1	0	-	0.076	0.089	0.136	0.185
				1	-	0.181	0.234	0.345	0.441
				2	-	0.212	0.279	0.408	0.505
				3	-	0.118	0.165	0.247	0.317
			2	0	0	0.073	0.089	0.171	0.244
2	2	3	2	1	0	0.179	0.202	0.361	0.482
				0	1	0.19	0.267	0.463	0.61
				2	0	0.189	0.231	0.404	0.524
				1	1	0.362	0.477	0.736	0.854
				0	2	0.226	0.329	0.575	0.721
				3	0	0.103	0.135	0.247	0.338
				2	1	0.357	0.483	0.753	0.868
				1	2	0.399	0.573	0.827	0.923
				0	3	0.142	0.223	0.413	0.549
				3	1	0.209	0.303	0.535	0.691
				2	2	0.395	0.592	0.841	0.931
				1	3	0.274	0.436	0.695	0.823
				3	2	0.211	0.377	0.633	0.766
				2	3	0.239	0.443	0.69	0.815
				3	3	0.154	0.329	0.506	0.624
	3	3	0	-	-	0.06	0.064	0.079	0.104
			1	0	-	0.081	0.098	0.148	0.196
				1	-	0.182	0.236	0.347	0.442
				2	-	0.211	0.278	0.406	0.502
				3	-	0.119	0.162	0.244	0.313
			2	0	0	0.083	0.098	0.177	0.251
				1	0	0.194	0.214	0.376	0.498
				0	1	0.2	0.278	0.477	0.614
				2	0	0.207	0.25	0.421	0.54
				1	1	0.37	0.491	0.747	0.863
				0	2	0.231	0.335	0.575	0.72
				3	0	0.111	0.139	0.246	0.341
				2	1	0.377	0.504	0.762	0.872
				1	2	0.415	0.583	0.831	0.926
				0	3	0.149	0.235	0.422	0.548
				3	1	0.223	0.316	0.539	0.69
				2	2	0.42	0.608	0.847	0.932
				1	3	0.286	0.446	0.701	0.828
				3	2	0.232	0.395	0.635	0.765
				2	3	0.26	0.46	0.697	0.823
				3	3	0.172	0.343	0.512	0.628

Table 16: Empirical rejection rates of the null of no cointegration at the 5% nominal level when the alternative is true, $m = 3$ (cont.)

D Estimating the trend component in the case of models estimated in first difference

When estimating a model in first difference, a siginificant intercept is translated into a trend component for the annual estimates in level, thus also for the dissagregated estimates. Estimating model (dO) yields predicted first difference values :

$$\Delta\hat{Y} = \begin{pmatrix}\mathbf{1}_{T-1} & \Delta X\end{pmatrix}\begin{pmatrix}\hat{\mu}\\ \hat{\alpha}\end{pmatrix}$$

where μ is the intercept and α the vector of slope coefficients. The equation implies that the predicted level values are defined up to an initial value $\hat{Y}_1$. We set the latter to the initial observed annual value Y_1, therefore :

$$\begin{cases}\hat{Y}_1 & = & Y_1\\ \hat{Y}_t & = & \hat{Y}_{t-1} + \hat{\mu} + (X_t - X_{t-1})'\hat{\alpha} \quad \text{for} \quad t = 2, ..., T\end{cases}$$

or equivalently,

$$\begin{cases}\hat{Y}_1 & = & Y_1\\ \hat{Y}_t & = & Y_1 + (t-1)\hat{\mu} + (X_t - X_1)'\hat{\alpha} \quad \text{for} \quad t = 2, ..., T\end{cases} \tag{12}$$

which we write matricially

$$\hat{Y} = \begin{pmatrix}\mathbf{1}_T & \mathbf{T}(T) & X\end{pmatrix}\begin{pmatrix}Y_1 - \hat{\mu} - X_1'\hat{\alpha}\\ \hat{\mu}\\ \hat{\alpha}\end{pmatrix}$$

where $\mathbf{T}(T) = \begin{pmatrix}1 & \ldots & T\end{pmatrix}'$. The presence of an annual trend component implies also that the predicted values at the disaggregated level also have a trend component:

$$\hat{y}_t \quad = \quad \hat{y}_{t-1} + (g(t) - g(t-1))\,\hat{\mu} + (x_t - x_{t-1})'\hat{\alpha} \tag{13}$$

for $t = 1, \ldots fT$, where $g(t)$ is a quarterly trend component, i.e for a constant ν

$$g(t) = g(t-1) + \nu \ \text{ for } \ t \in \{2, ...fT\}$$

therefore

$$g(t) = g(1) + (t-1)\nu \ \text{ for } \ t \in \{2, ...fT\}. \tag{14}$$

By backward induction we can rewrite (13) up to an initial value $\hat{y}_1$.

$$\hat{y}_t \quad = \quad \hat{y}_1 + (g(t) - g(1))\,\hat{\mu} + (x_t - x_1)'\hat{\alpha}$$

therefore

$$\hat{y}_t \quad = \quad \hat{y}_1 + (t-1)\nu\hat{\mu} + (x_t - x_1)'\hat{\alpha}. \tag{15}$$

The first difference of the annual aggregation must sum up to the first difference of the annual estimates, i.e. in the case of a disaggregation to a higher frequency f:

$$\sum_{s=f(t-1)+1}^{ft} \hat{y}_s - \sum_{s=f(t-2)+1}^{f(t-1)} \hat{y}_s = \hat{Y}_t - \hat{Y}_{t-1} \ \text{ for } \ t \in \{2, ..., T\}.$$

It implies for $t = 2$:

$$\begin{aligned}
\hat{\mu}\nu\left(\sum_{s=f+1}^{2f}(s-1)-\sum_{s=1}^{f}(s-1)\right)+\left(\sum_{s=f+1}^{2f}x_s-\sum_{s=1}^{f}x_s\right)'\hat{\alpha} &= \hat{Y}_2-\hat{Y}_1 \\
\hat{\mu}\nu\left(\sum_{s=f}^{2f-1}s-\sum_{s=0}^{f-1}s\right)+(X_2-X_1)'\hat{\alpha} &= Y_1+\hat{\mu}+\hat{\alpha}(X_2-X_1)-Y_1
\end{aligned}$$

therefore

$$\nu=\left(\sum_{s=f}^{2f-1}s-\sum_{s=0}^{f-1}s\right)^{-1}=\left(\sum_{s=0}^{2f-1}s-2\sum_{s=0}^{f-1}s\right)^{-1}=\left(\frac{(2f-1)2f}{2}-2\frac{(f-1)f}{2}\right)^{-1}=f^{-2}.$$

The higher frequency estimates become, for $t = 1,\ldots fT$:

$$\hat{y}_t = \hat{y}_1+\hat{\mu}\frac{t-1}{f^2}+(x_t-x_1)'\hat{\alpha}. \tag{16}$$

Now we want the first annual aggregation to equal the first annual observation Y_1, therefore:

$$\begin{aligned}
\sum_{s=1}^{f}\hat{y}_1+\hat{\mu}\sum_{s=1}^{f}\frac{s-1}{f^2}+\hat{\sum}_{s=1}^{f}(x_t-x_1)'\hat{\alpha} &= Y_1 \\
f(\hat{y}_1-x_1'\hat{\alpha})-\frac{f-1}{2f}\hat{\mu}+X_1'\hat{\alpha} &= Y_1 \\
\hat{y}_1 &= \frac{1}{f}\left(Y_1-X_1'\hat{\alpha}-\frac{f-1}{2f}\hat{\mu}\right)+x_1'\hat{\alpha}.
\end{aligned}$$

The quarterly estimates $\hat{y}_t$ when the model is estimated in first difference are therefore :

$$\hat{y}_t=\frac{Y_1}{f}+\hat{\mu}\frac{t-0.5(f+1)}{f^2}+\left(x_t-\frac{X_1}{f}\right)'\hat{\alpha} \quad \forall t\in\{1,...,fT\}. \tag{17}$$

E Predictive performance : by DGP

DGP				$T=$							
m	b	q_1	q_2	15	20	25	30	35	40	45	50
1	0	-	-	1.18	1.03	1.02	1.01	1.01	1.01	1	1
	1	0	-	1.36	1.3	1.18	1.14	1.12	1.11	1.1	1.09
		1	-	1.01	0.96	0.89	0.86	0.84	0.83	0.81	0.8
	2	0	0	1.32	1.26	1.18	1.16	1.15	1.13	1.12	1.12
		1	0	1.06	1.06	1	0.96	0.92	0.92	0.89	0.87
		0	1	0.96	0.94	0.87	0.84	0.82	0.83	0.8	0.79
		1	1	1.42	0.96	0.89	0.87	0.85	0.85	0.82	0.81
2	0	-	-	2.15	1.54	1.07	1.03	1.02	1.01	1.01	1.01
	1	0	-	2.16	1.64	1.48	1.23	1.18	1.16	1.14	1.12
		1	-	1.69	1.24	1.13	0.98	0.92	0.92	0.87	0.85
		2	-	1.42	1.26	1.06	0.95	0.89	0.88	0.83	0.82
	2	0	0	2.33	1.56	1.34	1.21	1.18	1.16	1.14	1.13
		1	0	1.67	1.26	1.19	1.05	1	1	0.96	0.92
		0	1	1.59	1.14	1.07	0.94	0.9	0.9	0.86	0.83
		2	0	1.5	1.28	1.2	1.04	0.98	0.98	0.94	0.9
		1	1	1.68	1.34	1.03	0.94	0.9	0.89	0.85	0.83
		0	2	1.4	1.07	1.02	0.92	0.86	0.87	0.83	0.79
		2	1	1.58	1.3	1.04	0.96	0.91	0.9	0.86	0.84
		1	2	1.63	1.39	0.99	0.9	0.86	0.85	0.82	0.81
		2	2	1.6	1.28	1	0.93	0.88	0.87	0.83	0.82
3	0	-	-	3.38	2.08	1.21	1.12	1.04	1.03	1.02	1.01
	1	0	-	6.31	2.79	1.72	1.42	1.26	1.25	1.19	1.15
		1	-	3.21	1.79	1.51	1.23	1.03	1.03	0.96	0.91
		2	-	2.78	1.65	1.38	1.21	1	1.01	0.94	0.88
		3	-	2.45	1.59	1.48	1.21	0.99	1.01	0.93	0.87
	2	0	0	4.36	4.03	1.59	1.39	1.25	1.23	1.17	1.15
		1	0	3.18	2.08	1.46	1.27	1.1	1.13	1.05	0.98
		0	1	3	2.15	1.29	1.16	1	1.01	0.94	0.89
		2	0	2.5	2.12	1.47	1.3	1.09	1.15	1.04	0.97
		1	1	3	1.88	1.31	1.1	0.98	0.96	0.91	0.87
		0	2	2.4	1.93	1.26	1.14	0.98	0.99	0.92	0.86
		3	0	2.3	2.12	1.5	1.32	1.09	1.16	1.04	0.98
		2	1	3.05	1.97	1.31	1.11	0.99	0.98	0.91	0.88
		1	2	2.99	1.92	1.29	1.05	0.95	0.93	0.87	0.84
		0	3	2.08	1.84	1.28	1.16	0.97	1	0.91	0.85
		3	1	2.76	1.87	1.29	1.11	0.99	1	0.92	0.89
		2	2	3.24	1.98	1.31	1.07	0.96	0.94	0.88	0.86
		1	3	2.88	1.79	1.28	1.04	0.92	0.92	0.86	0.83
		3	2	2.76	1.94	1.3	1.07	0.96	0.95	0.89	0.87
		2	3	3.24	1.93	1.29	1.05	0.94	0.92	0.86	0.85
		3	3	2.86	1.95	1.27	1.05	0.94	0.93	0.88	0.85

Table 17: Predictive performance : RMSE($\hat{\mathbf{y}}$) / RMSE($\hat{\mathbf{y}}_{\text{diff}}$)