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Testing for cointegration with structural changes in very small sample

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Abstract

This article proposes an adaptation of existing tests of cointegration with endogenous structural changes to very small sample sizes. Size-corrected critical values for both testing cointegration with endogenous structural breaks and testing structural breaks in the parameters in a cointegration model are computed in this context. We show that the power of such a testing procedure is satisfying in sample sizes smaller than fifty observations. This is of interest for macroeconometric studies of emerging economies for which the data history is usually not long enough to apply conventional methods. When the serial correlation is low, we find the tests to be powerful for even less than thirty observations. A combined procedure of testing for cointegration and structural change allows us to improve the power of testing cointegration in very small sample sizes while staying agnostic about the underlying data generating processes. An example using the Chinese data finds a cointegration relationship with two structural breaks between the national household consumption expenditures, the retail sales of consumer goods, and the investment in fixed assets during the last four decades.

JEL classification codes: C32, E17

Keywords: Time series, cointegration, structural change, very small sample, emerging economies

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1 Introduction

The lack of official annual reporting from emerging economies has prevented economists to obtain measures of macroeconomic fundamentals for periods before the last decades of the 20th century. These data are nonetheless crucial for determining long-term relationships among them or estimating the ones implied by the macroeconomic theory. Estimating these long-term relationships is useful for macroeconomic analyses or forecasting. However, they ought not to stay the same over time and can be subject to structural changes which are out of the scope of the theory. For example, a new international trade agreement will impact the magnitude and the composition of trade flows, as well as consumption expenditures and private investment in the long term. The correlations between the macroeconomic variables will then change but won't disappear, and failing to take the structural changes into account will lead to spurious regressions. In order to maintain the long-term relationships, it is necessary to allow for structural breaks in the parameters. To this end, Gregory and Hansen (1996) test for cointegration with a structural change at an unknown date and compute asymptotical critical values for rejecting the null of no cointegration against the alternative of cointegration in the presence of a structural break in the coefficients. The procedure is an extension of conventional two-step tests of cointegration by testing the stationarity of the residuals of a regression in the vein of Engle and Granger (1987) and Phillips and Ouliaris (1990). It is fairly easy to implement and its results are directly interpretable, which makes it useful for both researchers and practitioners. Many extensions have been developed, such as more general cases by allowing multiple structural break dates (Hatemi-j (2008) for two breaks and Maki (2012) for an unknown number of breaks). The finite sample performance of such tests is usually found to be good for a sample size of at least a hundred observations, but emerging and developing economies' official data hardly have as long a history. In particular, considering annual data across three or four decades means applying the test procedure to less than fifty observations. For example, current account balance data are not observed before 1980 for 60 various countries (advanced, emerging, or developing ones) which constitute 18% of total world trade[1] and we can expect the same issue about other time series of interest. In the particular case of the Chinese national statistics, most quarterly data that could be used in annual aggregation to link them to annual national accounts start around the

[1]Source: World Trade Organisation. The most prominent component is China, which counts for 12% of total world trade.

1990s[2]. Yet size distortion can be severe in very small sample, especially in the case of residual-based test statistics for which the rate of convergence is high, which would lead to an over rejection of the null hypothesis.

In this paper, we extend the test of Gregory and Hansen (1996) to two endogenous structural breaks in very small sample (less than fifty observations). We compute size-corrected critical values and assess the performance of the test in terms of power for very small sample sizes. We also compare competing structural break dates selection criteria as well as the power of the test against different alternative hypotheses of structural break models. A combined procedure of testing for cointegration and structural breaks in the parameters is then proposed to maximize the detection of cointegration with structural breaks. Finally, we illustrate the testing procedure by applying it to the official Chinese macroeconomic data.

2 Estimation procedure

We generalize the test of cointegration of Gregory and Hansen (1996) by allowing two endogenous structural breaks. Let us denote the observed data $(\mathbf{Y}, \mathbf{X})$ where every column vector is a I(1) process, $\mathbf{Y}$ is of dimension $T \times 1$ and $\mathbf{X}$ is $T \times m$. A general model of b structural breaks in the cointegrating relationship can be written :

$$\mathbf{Y} = \begin{pmatrix} \mathbf{1} & \mathbf{X} \end{pmatrix} \begin{pmatrix} \mu \\ \beta \end{pmatrix} + \sum_{i=1}^{b} \mathbf{B}_i \begin{pmatrix} \mathbf{1} & \mathbf{X} \end{pmatrix} \begin{pmatrix} \mu_i \\ \beta_i \end{pmatrix} + \epsilon \tag{1}$$

where $\mathbf{1}$ is a vector of ones of length T, μ and $\{\mu_i\}_{0<i\leq b}$ scalars, β and $\{\beta_i\}_{0<i\leq b}$ the vector of coefficients of length m. $\mathbf{B}_i$ is the matrix of structural break dummy variables $\{d_{k\ell}\}_{0<k,\ell\leq T}$ where $d_{k\ell} = 1$ if $k = \ell$ and j is equal or greater to t_i the time period of the i-th break, or 0 otherwise. ϵ is the vector $T \times 1$ of residuals. We distinguish 4 cases :

- no cointegration : $\epsilon \sim I(1)$
- cointegration with no structural change (model o): $\epsilon \sim I(0)$, $\mu_i = 0$ and $\beta_i = \mathbf{0}\ \ \forall i$
- cointegration with structural breaks in the constant (model c): $\epsilon \sim I(0)$ and $\beta_i = \mathbf{0}\ \ \forall i$
- cointegration with structural breaks in the constant and the slope coefficient (model cs): $\epsilon \sim I(0)$

[2]Source: National Bureau of Statistics of China. For example, retail sales data are available from 1984, net export of goods from 1995, government expenditures, and investment in fixed assets from 1998

where $\mathbf{0}$ is a vector of zeroes of length m. By writing (1) in a compact form $\mathbf{Y} = \mathbf{Z}\Gamma + \epsilon$, we obtain the estimated residuals $\hat{\epsilon} = \mathbf{Y} - \mathbf{Z}\hat{\Gamma}$ where $\hat{\Gamma} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}\mathbf{Y}$, then test the unit root for $\hat{\epsilon}$. We only consider the Augmented Dickey-Fuller (ADF) version of the test for its relative efficiency in very small sample, in which we select the minimal number of lags for which the residuals of the ADF model are not serially correlated. This yields the test statistic $ADF(\hat{\epsilon})$. The selected break dates are the ones that minimize the test statistic as in Gregory and Hansen (1996) or Hatemi-j (2008). Therefore the test statistic of cointegration with b structural breaks is:

$$ADF^* = \inf_{\{t_i\}_{0<i\leq b}} ADF(\hat{\epsilon}). \tag{2}$$

An alternative method of model selection could be to choose the break dates that minimize the sum of squares of residuals $\hat{\epsilon}'\hat{\epsilon}$, as in Carrion-i Silvestre and Sansó (2006) or Maki (2012) in cointegration tests, or more conventional endogenous structural break tests in a stationary setting such as Bai and Perron (1998) and their extensions. These types of procedures are computationally lighter and more oriented towards model fitting. The test statistic would then be

$$ADF^*_{SSR} = ADF(\epsilon^*) \text{ where } \epsilon^{*\prime}\epsilon^* = \inf_{\{t_i\}_{0<i\leq b}} \hat{\epsilon}'\hat{\epsilon}. \tag{3}$$

3 Asymptotic distributions

Gregory and Hansen (1996) give expressions for the limit distribution of the ADF^*-test statistics for the case with one endogenous structural break, by deriving the asymptotic distributions of the Z_t^* test statistics based on the Phillips (1987) portmanteau test, and the ADF^* test is expected to have the same asymptotic properties. We provide the extended expressions for the case with two structural breaks[3]. In an analogous way of Gregory and Hansen (1996)[4], we obtain the expression of the limiting distributions under the null to the case of up to two partial structural breaks as follows:

$$Z_t^* \to_d \inf_{\tau} \frac{\int_0^1 W_\tau dW_\tau}{\left[\int_0^1 W_\tau^2\right]^{\frac{1}{2}} \left[1 + \kappa_\tau' \mathbf{D}_\tau \kappa_\tau\right]^{\frac{1}{2}}}$$

[3] Hatemi-j (2008) provide a less detailed expression of the case with two breaks.

[4] See Gregory and Hansen (1996) for the precise expressions of the functions W_1 and W_2 of Brownian motions.

where

$$W_\tau(r) = W_1(r) - \int_0^1 W_1 W_{2\tau}' \left[\int_0^1 W_{2\tau} W_{2\tau}'\right]^{-1} W_{2\tau}(r)$$

$$\kappa_\tau = \left[\int_0^1 W_{2\tau} W_{2\tau}'\right]^{-1} \int_0^1 W_{2\tau} W_1$$

and $W_{2\tau}(r)$ and $\mathbf{D}_\tau$ depend on the model.

For model C, if $b = 1$, then $\tau = \tau_1$ and

$$W_{2\tau} = [1, \phi_{\tau_1}(r), W_2'(r)]'$$

where $\phi_{\tau_i}(r) = \mathbb{1}\{r \geq \tau_i\}$ and

$$\mathbf{D}_\tau = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{pmatrix},$$

If $b = 2$, then $\tau = \{\{\tau_1, \tau_2\}, \tau_1 < \tau_2\}$ and

$$W_{2\tau} = [1, \phi_{\tau_1}(r), \phi_{\tau_2}(r), W_2'(r)]'$$

and

$$\mathbf{D}_\tau = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{pmatrix},$$

For model CS, if $b = 1$, then $\tau = \tau_1$ and

$$W_{2\tau} = [1, \phi_{\tau_1}(r), W_2'(r), W_2'(r)\phi_{\tau_1}(r)]'$$

and

$$\mathbf{D}_\tau = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m & (1-\tau_1)\mathbf{I}_m \\ \mathbf{0} & (1-\tau_1)\mathbf{I}_m' & (1-\tau_1)\mathbf{I}_m \end{pmatrix},$$

which is exactly model 4 of Gregory and Hansen (1996).

If $b = 2$, then $\tau = \{\{\tau_1, \tau_2\}, \tau_1 < \tau_2\}$, and

$$W_{2\tau} = [1, \phi_{\tau_1}(r), \phi_{\tau_2}(r), W_2'(r), W_2'(r)\phi_{\tau_1}(r), W_2'(r)\phi_{\tau_2}(r)]'$$

and

$$\mathbf{D}_\tau = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m & (1-\tau_1)\mathbf{I}_m & (1-\tau_2)\mathbf{I}_m \\ \mathbf{0} & (1-\tau_1)\mathbf{I}_m & (1-\tau_1)\mathbf{I}_m & (1-\tau_2)\mathbf{I}_m \\ \mathbf{0} & (1-\tau_2)\mathbf{I}_m & (1-\tau_2)\mathbf{I}_m & (1-\tau_2)\mathbf{I}_m \end{pmatrix}.$$

Carrion-i Silvestre and Sansó (2006) derive the limit distribution of the ADF^*_{SSR}-test statistics.

4 Critical values

The limit distributions are not expressed in closed form. Approximate critical values for the test can be obtained by using simulations following MacKinnon (1991)'s surface response methodology. Gregory and Hansen (1996) compute the test statistics on replications of simulated data of sample sizes $T = 50$, 100, 150, 200, 250, 300 under the null hypothesis, then they estimate the quantiles of their distribution as functions of the sample size. Asymptotic critical values are then obtained by making the sample size tend to infinity and critical values for any sample size can be directly calculated using the estimated function of sample size. However residual-based tests of cointegration have a high rate of convergence with respect to sample size for the distribution of the test statistics, which makes the critical values very sensitive to sample size, especially for the very small ones. In our case, we need to apply the test to data of sample size varying from $T = 15$ to 50 which are out of the scope of the surface response functions estimated by Gregory and Hansen (1996). In consequence, we use the same procedure on a wider array of sample sizes, particularly by adding smaller ones, to estimate size-corrected critical values for very small sample sizes. 10 000 replications of time series of size $T = 12$, ...20, 25, ...50, 100, 200, 500, 1 000 are replicated under the null hypothesis (unit root in the residuals of a linear model without structural breaks), with the following parametrization:

$$\begin{cases} Y_t = \mu + \mathbf{X}_t'\beta + U_t, & U_t = U_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathrm{NID}(0,1) \\ \mathbf{X}_t = \theta + \mathbf{V}_t, & \mathbf{V}_t = \mathbf{V}_{t-1} + \eta_t, \quad \eta_t \sim \mathrm{NID}(0,\Sigma) \end{cases} \tag{4}$$

where $t = 1, \dots, T$, the values of the parameters are drawn from the uniform distributions: $\mu, \beta \sim \mathcal{U}[-4; 4]$ and $\theta, \Sigma \sim \mathcal{U}[0.5; 4]$. The choice of parametrization is discussed in the next section and does not matter in this one because under the null hypothesis the residuals follow a random walk. We apply the test of Gregory and Hansen (1996) to every replication and compute the cumulative distribution function for each sample size, then quantiles of the distribution are retrieved. Following MacKinnon (1991), a polynomial of $1/T$'s is fitted by OLS for each q-th quantile of the simulated distribution of the test statistics, number of regressors m and of structural breakpoints b and type of regression model M:

$$Crt(T, q, m, b, M) = \psi_\infty + \sum_{k=1}^{K} \psi_k T^{-k} + error$$

where the order K of polynomial is selected by minimizing the Akaike information criterion (AIC), with a maximum order $K_{max} = 6$. The critical value for the test for any sample size T can then be computed: asymptotical critical values are obtained by reporting the intercept of the regression ψ_∞ and finite sample critical values are calculated by applying the function to a chosen sample size T. The estimated functions for the fifth percentile are reported in Appendix B. Table 1 reports the critical values for the Gregory and Hansen (1996) ADF^* test of the null of no cointegration, both asymptotic and size-corrected for small sample sizes. We naturally obtain the same asymptotic critical values (with more or less 0.05 difference in absolute value) as Gregory and Hansen (1996), where they consider $m = 1, ...4$ and $b = 1$. As expected, the critical values for any finite sample are more negative when the sample size decreases and the number of regressors and/or structural breaks increases.

Figure 1 plots the estimated surface response functions for the test with $m = 1$ and

			$T =$				
# regressors	# breaks	model	15	20	30	50	∞
$m = 1$		o	-3.95	-3.87	-3.68	-3.53	-3.33
	$b = 1$	c	-5.85	-5.73	-5.37	-5.08	-4.62
		cs	-6.25	-6.08	-5.72	-5.4	-4.96
	$b = 2$	c	-8.16	-7.11	-6.9	-6.04	-5.21
		cs	-8.87	-7.81	-7.47	-6.67	-5.94
$m = 2$		o	-4.48	-4.41	-4.2	-3.99	-3.75
	$b = 1$	c	-6.38	-6.22	-5.85	-5.51	-4.97
		cs	-7.17	-6.93	-6.48	-6.11	-5.55
	$b = 2$	c	-8.69	-7.55	-7.3	-6.41	-5.53
		cs	-10.47	-9.1	-8.36	-7.65	-6.9
$m = 3$		o	-5.01	-4.87	-4.65	-4.43	-4.1
	$b = 1$	c	-6.91	-6.67	-6.27	-5.87	-5.3
		cs	-8.13	-7.7	-7.14	-6.72	-6.09
	$b = 2$	c	-9.31	-8.04	-7.7	-6.76	-5.88
		cs	-17.67	-10.74	-9.36	-8.53	-7.67

Note: model o: no structural break, model c: change in the intercept, model cs: change in the intercept and slope coefficient(s).

Table 1: 5% size adjusted approximate critical values for the ADF^* test of no cointegration with endogenous structural break

$b = 1$. We observe a sharp decrease in the critical values as the sample size decreases below $T = 40$. Such a drop towards more negative values would not have been highlighted if only sample sizes higher than 50 were to be considered as they are in the literature. Even if we were to use the estimated surface responses in Gregory and Hansen (1996) or Hatemi-j (2008), we would face an aggravating size distortion as we are to apply the test for a smaller sample size than $T = 50$.

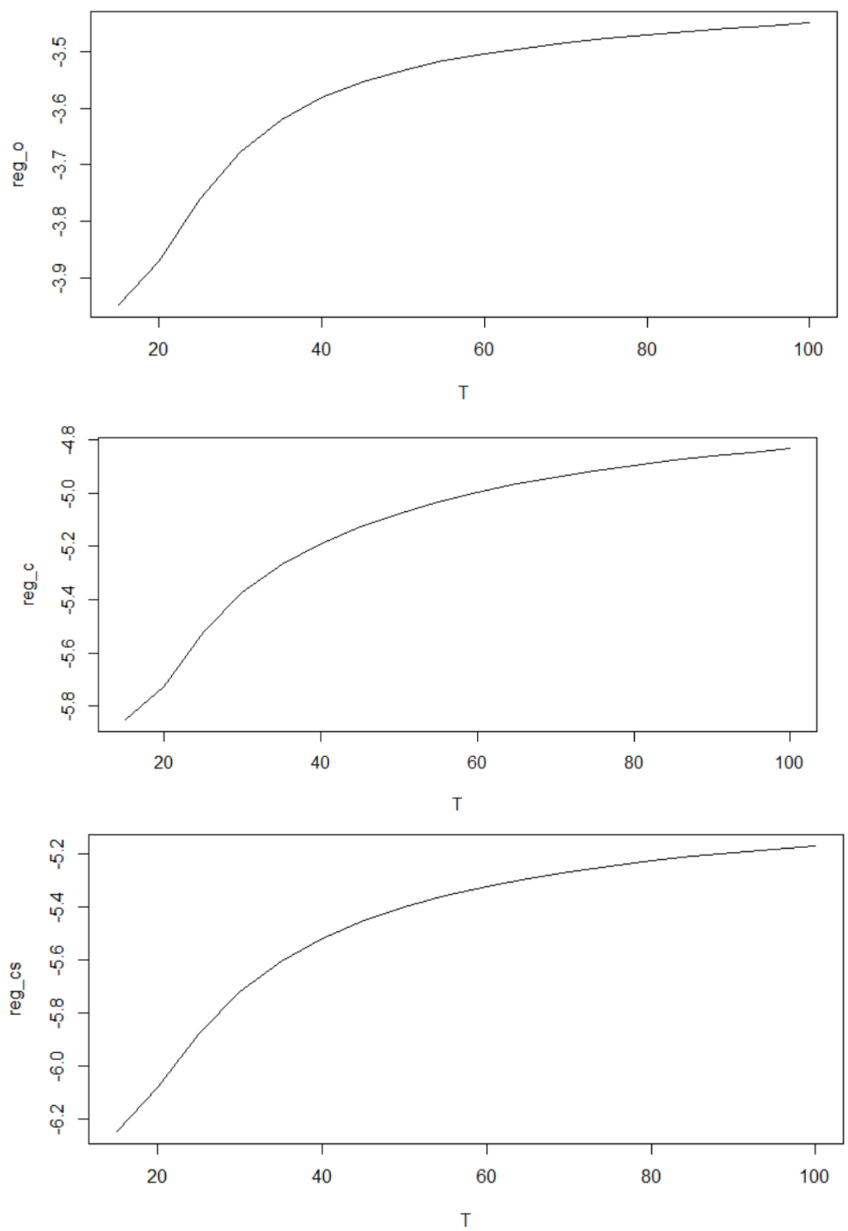

Figure 1: Size-corrected 5% critical values for the test of Gregory and Hansen (1996) estimated by surface response, by regression model, $m = 1$ and $b = 1$

We apply the same procedure to estimate the size-corrected values for the ADF^*_{SSR} test which selects the breakdates by minimizing the sum of square of residuals instead of the test statistic and are reported in Table 2[5]. The test shows to provide less negative

[5] We omit the case $b = 2$ for ADF^*_{SSR} as we only report it for comparison and focus on the ADF^* for the rest of the chapter.

values for the critical values. Therefore ADF^*_{SSR} is more liberal at rejecting the null hypothesis than ADF^*.

# regressors	# breaks	model	$T=$ 15	20	30	50	∞
$m=1$	$b=1$	c	-5.23	-5.15	-4.93	-4.68	-4.43
		cs	-5.59	-5.46	-5.22	-5	-4.67
$m=2$	$b=1$	c	-5.73	-5.6	-5.37	-5.12	-4.76
		cs	-6.34	-6.16	-5.91	-5.67	-5.28
$m=3$	$b=1$	c	-6.16	-6.05	-5.8	-5.46	-5.09
		cs	-7.11	-6.85	-6.54	-6.26	-5.77

Note: model c: change in the intercept, model cs: change in the intercept and slope coefficient(s).

Table 2: 5% size adjusted approximate critical values for the ADF^*_{SSR} test of no cointegration with endogenous structural break

The next section assesses the performance of the test in terms of power, using the computed size-corrected critical values.

5 Performance of the size-corrected test of cointegration

We expect that the closer the regression model fits the true model, the more powerful the test will be. This true model should be more easily detected when the data have more available observations, the structural change easier to detect, and the omitted variables in the regression less impacting. Therefore, we expect the test to have more power when the sample size is larger, the structural change parameters more significant[6], and finally when the serial correlation is smaller.

10 000 replications of time series of size $T = 12$, ...20, 25, ...50, 100, 200, 500, 1 000 under the alternative of cointegration with or without a structural break are simulated,

[6]The structural change would also be more easily estimated if there are enough observations in each regime. In our small sample setting, the location of the structural break date can be critical, since a date too close to the bounds of the sample would imply a very small number of observations for some of the regimes at the extremities of the observed period.

parametrized as follows:

$$\begin{cases} Y_t = \mu + \mathbf{X}_t'\beta + \sum_{i=1}^{b}(\mu_i + \mathbf{X}_t'\beta_i)\mathbb{1}_{t\geq t_i} + U_t, & U_t = \rho U_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \text{NID}(0,1) \\ \mathbf{X}_t = \theta + \mathbf{V}_t, & \mathbf{V}_t = \mathbf{V}_{t-1} + \eta_t, \quad \eta_t \sim \text{NID}(0,\Sigma) \end{cases} \tag{5}$$

We choose an agnostic parametrization for the data generating process under the alternative. The standard deviation of ε_t is normalized to unity. We draw $\mu, \beta \sim \mathcal{U}[-4;4]^{m+1}$ and for $1 \leq i \leq b$, $(\mu_i, \beta_i) \sim \mathcal{U}[-4;4]^{m+1}$ to allow for various levels of significance for the parameters of the linear relationship[7], $\theta \sim \mathcal{U}[0.5;4]$ and Σ is a diagonal matrix which diagonal elements are drawn in $\mathcal{U}[0.5;4]$ to allow for various levels of determination of $\mathbf{Y}$ by $\mathbf{X}$. The drawn breakdates are in the interval of possible estimated breakdates, i.e. following Gregory and Hansen (1996) $t_i = [\![\tau_i T]\!]$ where $\tau_i \sim \mathcal{U}[0.15;0.85]$, and $\tau_i < \tau_j$ if $i < j$. We also allow for various levels of serial correlation $\rho \sim \mathcal{U}[0;0.9]$. Consistently with the models previously detailed, we set $\mu_i = 0$ and $\beta_i = \mathbf{0}$ for model o, or $\beta_i = \mathbf{0}$ for model c.

Gregory and Hansen (1996) assess the power of test for $\rho = 0$ and $\rho = 0.5$, sample sizes $T = 50$ and $T = 100$, one structural break $b = 1$ and one related series $m = 1$. When there is no serial correlation ($\rho = 0$), they find a good power of around 80%. When $\rho = 0.5$, this same level of power for the test is maintained when $T = 100$ only and the true model has either the same type of structural change as the alternative or no structural change at all. They find the power to drop by 30 to 50 percentage points when $T = 50$ or when the alternative doesn't consider a structural change affecting the true model. Hatemi-j (2008) finds similar results[8] for $b = 2$. Table 3 reports the power of the test by adding smaller sample sizes than the literature and using size-corrected critical values. For high sample sizes ($T \geq 500$), the power of the test converges naturally to a systemic rejection of the null, except when not considering a structural change when there is actually one, or considering only a change in the constant when there is also a change in the slope coefficients. More generally we recover the usual result that the unit root test loses power against structural changes in the parameters, but we also show that the test of cointegration with endogenous structural break is only powerful in large sample sizes against less general alternatives. With around 80% of power for $T = 100$ and between 50% and 80% for $T = 50$, we share the same results as the literature. For smaller sample sizes, we observe that power decreases as the sample size decreases, however at a significantly different rate depending on the choice of the regression model with respect to the true data generating process. Figure 2 shows the

[7] Appendix C shows that the power of the test is not affected by the parametrization of μ or β

[8] However only the case of no structural change under the true model is considered.

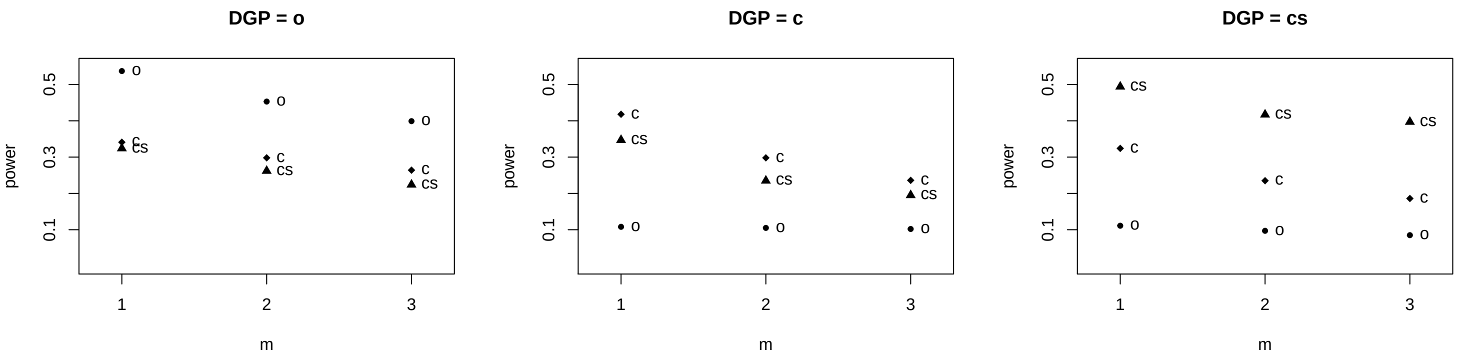

Figure 2: Empirical rejection rates (ERFs) for ADF^* of the null of no cointegration at the 5% nominal level when the alternative is true, by regression model, $T = 30$ and $b = 1$

power of the test for $T = 30$ and $b = 1$ by regression model, as a function of the the type of true model and m the dimension of $\mathbf{X}$. It shows that for a given regression model, power decreases as m increases. However, for a given m, the power of the test is best when the regression model is the same as the true model, the highest being notably when the true DGP has no structural change (left panel), followed by a change in the intercept and slope coefficients (right panel) then finally by a change in the intercept only (center panel), with 54%, 49% and 42% for $m = 1$ respectively. From there, the power uniformly decreases with misspecification. When the true model doesn't have any structural change, considering one in the regression decreases the power by half on average, with a more severe drop in the case of a structural break in the intercept and the slope coefficients. When the true model has a break in the intercept, considering in addition a break in the slope coefficients mildly decreases power, but not considering a structural change at all takes out all the power of the test. When the true model has a break in the intercept and the slope coefficients, only considering a break in the intercept severely decreases the power, and not considering a structural change takes it out completely. All this indicates that overspecifying the model in comparison to the true model mildly decreases the power of the test when there is a structural break in the true model, and more severely where there is actually none, but it does not prevent convergence to a systematic rejection of the null in large sample; underspecifying the model however severely decreases it when there is a structural change in the true model, or even wipes it out when failing to consider a structural change when there is actually one, in addition to preventing a convergence to a systematic rejection of the null in large sample sizes, as shown in Table 3. Model specification therefore seems critical in order to detect cointegration in small sample. In consequence, one could use a model selection criterion based on model specification to improve the power of the test. Such is the spirit of the tests of endogenous structural break that select the breakpoints by minimizing the sum of squares of residuals, as in Bai and Perron (1998) for the stationary framework. In order to test for cointegration, the ADF^*_{SSR} test statistic minimizes the sum of squares of residuals as in Carrion-i Silvestre and Sansó (2006) or Maki (2012), instead of minimizing the test statistic. Table 4 reports the power of the test ADF^*_{SSR} for each model under the alternative and the data generating process. It shows qualitatively the same results as ADF^*. Quantitatively though, ADF^*_{SSR} uniformly rejects the null of no cointegration less often than ADF^* when the alternative is true. Both having a well-specified model and rejecting the null as much as possible are therefore important to improve the power of the test. Since ADF^* uniformly outperforms ADF^*_{SSR} in terms of power, we only consider the former in the rest of the chapter.

We expect the test to reject more frequently the null the further the true model is from the null, i.e. the further the serial correlation is from unity. Tables 5 and 6 report the power of the ADF^* test for different value of the serial correlation ρ in the data generating process. There is significant power gains for small values of ρ. When ρ is between 0 and 0.2, the probability of rejecting the null when the alternative is true is over 90% for model O, 80% for model C, and 70% for model CS, for $T \geq 30$ when the true model is the same as the regression model. Such power for the test was previously reached only when $T \geq 100$ if we don't discriminate by the value of ρ. As before, the power of the test decreases when T decreases, m or b increases, and with over-specification. Under-specifying the model still severely decreases the power of the test.

Investigating the estimation bias of the serial correlation can provide helpful insights to justify the previous results. Figure 3 shows the average estimated serial correlation in the case with $m = 1$ and $b = 1$ (Appendix D reports the average difference between the value of the true parameter ρ and its estimated value $\hat{\rho}$ for the all cases with $b = 1$). It shows the estimation bias of the serial correlation to be more negative as the sample size decreases. When the model is correctly or over-specified, the bias is on average negative and worsens with over-specification, but tends to zero as sample size increases. This means that the probability to reject the null is higher when correctly considering a structural change. When the model is underspecified, the bias is positive when the true serial correlation is small and negative when it is high for very small sample sizes which means that underspecification translates into a significantly positive serial correlation, which tends to the unit root as sample size increases. This is consistent with our

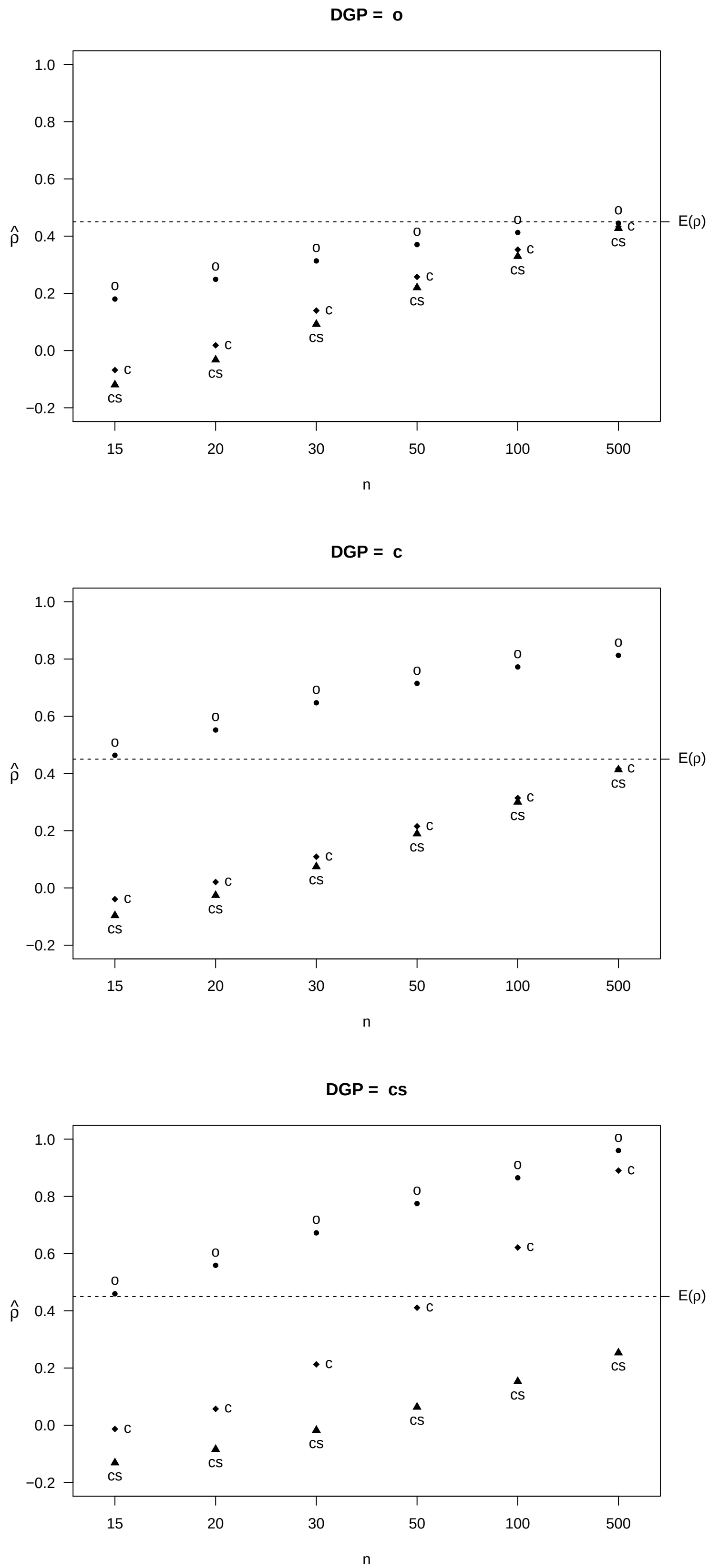

Figure 3: Estimated serial correlation, $m = 1$ and $b = 1$

	m	DGP	regression model : o, $T=$						regression model : c, $T=$						regression model : cs, $T=$					
			15	20	30	50	100	500	15	20	30	50	100	500	15	20	30	50	100	500
	1	o	0.2	0.31	0.54	0.75	0.92	1	0.12	0.18	0.34	0.59	0.83	1	0.11	0.18	0.32	0.58	0.82	1
		c	0.07	0.08	0.11	0.21	0.39	0.63	0.09	0.14	0.42	0.66	0.85	1	0.10	0.14	0.35	0.63	0.83	1
		cs	0.09	0.10	0.11	0.13	0.17	0.22	0.12	0.19	0.32	0.44	0.51	0.52	0.14	0.23	0.49	0.84	0.94	1
	2	o	0.17	0.24	0.45	0.70	0.89	1	0.11	0.16	0.30	0.55	0.81	1	0.1	0.14	0.26	0.51	0.78	1
$b=1$		c	0.06	0.07	0.10	0.22	0.42	0.66	0.08	0.11	0.3	0.62	0.83	1	0.09	0.12	0.24	0.56	0.78	1
		cs	0.07	0.08	0.10	0.11	0.12	0.14	0.1	0.14	0.24	0.36	0.41	0.4	0.14	0.22	0.42	0.88	0.96	1
	3	o	0.12	0.20	0.40	0.64	0.87	1	0.09	0.14	0.26	0.52	0.78	1	0.07	0.12	0.22	0.46	0.75	1
		c	0.06	0.07	0.10	0.20	0.43	0.7	0.06	0.09	0.24	0.58	0.8	1	0.07	0.10	0.20	0.49	0.75	1
		cs	0.07	0.07	0.08	0.10	0.10	0.11	0.08	0.11	0.19	0.32	0.37	0.32	0.13	0.21	0.4	0.86	0.98	1
			regression model : o, $T=$						regression model : c, $T=$						regression model : cs, $T=$					
	m	DGP	15	20	30	50	100	500	15	20	30	50	100	500	15	20	30	50	100	500
	1	o	0.20	0.31	0.54	0.75	0.92	1	0.05	0.15	0.19	0.56	0.77	1	0.06	0.13	0.17	0.5	0.74	1
		c	0.07	0.08	0.11	0.21	0.39	0.63	0.04	0.11	0.17	0.7	0.82	1	0.06	0.13	0.17	0.62	0.78	1
		cs	0.09	0.10	0.11	0.13	0.17	0.22	0.05	0.13	0.13	0.38	0.42	0.45	0.10	0.23	0.36	0.88	0.96	1
	2	o	0.17	0.24	0.45	0.7	0.89	1	0.05	0.15	0.17	0.53	0.75	1	0.05	0.11	0.17	0.4	0.71	0.99
$b=2$		c	0.06	0.07	0.10	0.22	0.42	0.66	0.03	0.08	0.15	0.66	0.81	1	0.06	0.12	0.18	0.47	0.76	0.99
		cs	0.07	0.08	0.10	0.11	0.12	0.14	0.04	0.09	0.1	0.29	0.33	0.34	0.12	0.26	0.40	0.82	0.99	1
	3	o	0.12	0.20	0.40	0.64	0.87	1	0.04	0.12	0.16	0.50	0.74	1	0.15	0.08	0.14	0.33	0.67	0.98
		c	0.06	0.07	0.10	0.20	0.43	0.7	0.03	0.07	0.13	0.62	0.79	1	0.15	0.09	0.17	0.36	0.71	0.98
		cs	0.07	0.07	0.08	0.10	0.10	0.11	0.03	0.08	0.07	0.25	0.29	0.30	0.16	0.20	0.40	0.73	0.99	1

Table 3: ERFs for ADF^* of the null of no cointegration at the 5% nominal level when the alternative is true

	m	DGP	regression model : o, $T=$						regression model : c, $T=$						regression model : cs, $T=$					
			15	20	30	50	100	500	15	20	30	50	100	500	15	20	30	50	100	500
	1	o	0.2	0.31	0.54	0.75	0.92	1	0.12	0.19	0.36	0.63	0.86	1	0.10	0.16	0.34	0.60	0.84	1
		c	0.07	0.08	0.11	0.21	0.39	0.63	0.09	0.14	0.30	0.58	0.84	1	0.08	0.12	0.26	0.54	0.81	1
		cs	0.09	0.10	0.11	0.13	0.17	0.22	0.10	0.14	0.19	0.26	0.34	0.40	0.08	0.11	0.25	0.52	0.78	1
	2	o	0.17	0.24	0.45	0.7	0.89	1	0.10	0.16	0.31	0.57	0.83	1	0.09	0.12	0.26	0.52	0.8	1
$b=1$		c	0.06	0.07	0.10	0.22	0.42	0.66	0.08	0.11	0.24	0.52	0.81	1	0.07	0.09	0.20	0.45	0.77	1
		cs	0.07	0.08	0.10	0.11	0.12	0.14	0.08	0.11	0.14	0.20	0.23	0.27	0.06	0.07	0.19	0.42	0.73	1
	3	o	0.12	0.2	0.4	0.64	0.87	1	0.09	0.13	0.27	0.54	0.81	1	0.06	0.10	0.20	0.45	0.75	1
		c	0.06	0.07	0.10	0.20	0.43	0.70	0.07	0.10	0.20	0.48	0.78	1	0.05	0.08	0.15	0.37	0.71	1
		cs	0.07	0.07	0.08	0.10	0.10	0.11	0.07	0.09	0.12	0.17	0.19	0.20	0.04	0.06	0.13	0.34	0.67	0.99

Table 4: ERFs for ADF^*_{SSR} of the null of no cointegration at the 5% nominal level when the alternative is true

m	DGP	ρ	regression model : o, $T=$						regression model : c, $T=$						regression model : cs, $T=$					
			15	20	30	50	100	500	15	20	30	50	100	500	15	20	30	50	100	500
1	o	$[0,0.2]$	0.38	0.62	0.92	0.99	1	1	0.21	0.34	0.71	0.98	1	1	0.2	0.34	0.68	0.98	1	1
		$]0.2,0.4]$	0.25	0.39	0.77	0.98	1	1	0.12	0.22	0.45	0.87	1	1	0.12	0.22	0.43	0.85	1	1
		$]0.4,0.6]$	0.15	0.22	0.46	0.88	1	1	0.1	0.13	0.22	0.54	0.98	1	0.09	0.12	0.21	0.51	0.98	1
		$]0.6,0.8]$	0.08	0.12	0.21	0.44	0.92	1	0.06	0.08	0.11	0.23	0.67	1	0.06	0.08	0.1	0.22	0.6	1
		$]0.8,0.9]$	0.06	0.07	0.1	0.15	0.43	1	0.07	0.08	0.08	0.1	0.2	1	0.07	0.06	0.06	0.1	0.18	1
	c	$[0,0.2]$	0.07	0.05	0.07	0.11	0.12	0.12	0.13	0.22	0.81	1	1	1	0.16	0.25	0.68	1	1	1
		$]0.2,0.4]$	0.09	0.09	0.14	0.26	0.37	0.45	0.1	0.18	0.59	0.94	1	1	0.12	0.18	0.48	0.92	1	1
		$]0.4,0.6]$	0.07	0.09	0.14	0.31	0.56	0.77	0.08	0.11	0.3	0.69	1	1	0.08	0.11	0.24	0.63	0.99	1
		$]0.6,0.8]$	0.06	0.07	0.1	0.22	0.56	0.96	0.06	0.09	0.14	0.28	0.72	1	0.07	0.08	0.12	0.25	0.65	1
		$]0.8,0.9]$	0.05	0.07	0.08	0.09	0.31	1	0.05	0.07	0.08	0.11	0.22	1	0.07	0.06	0.07	0.1	0.19	1
	cs	$[0,0.2]$	0.1	0.1	0.11	0.11	0.12	0.12	0.15	0.24	0.39	0.5	0.51	0.45	0.23	0.36	0.73	1	1	1
		$]0.2,0.4]$	0.09	0.11	0.12	0.12	0.16	0.16	0.13	0.22	0.39	0.48	0.56	0.48	0.15	0.29	0.62	0.98	1	1
		$]0.4,0.6]$	0.08	0.1	0.12	0.15	0.19	0.22	0.11	0.18	0.32	0.46	0.56	0.53	0.11	0.18	0.45	0.89	1	1
		$]0.6,0.8]$	0.08	0.09	0.1	0.14	0.19	0.3	0.09	0.15	0.24	0.38	0.5	0.59	0.09	0.14	0.31	0.68	0.91	1
		$]0.8,0.9]$	0.08	0.08	0.07	0.09	0.16	0.35	0.08	0.1	0.2	0.3	0.35	0.61	0.08	0.1	0.23	0.48	0.65	1
2	o	$[0,0.2]$	0.31	0.49	0.86	0.99	1	1	0.19	0.32	0.64	0.98	1	1	0.16	0.27	0.58	0.96	1	1
		$]0.2,0.4]$	0.2	0.31	0.64	0.96	1	1	0.13	0.19	0.39	0.81	1	1	0.12	0.16	0.33	0.75	1	1
		$]0.4,0.6]$	0.14	0.17	0.35	0.79	0.99	1	0.09	0.12	0.19	0.47	0.97	1	0.08	0.1	0.15	0.41	0.93	1
		$]0.6,0.8]$	0.09	0.1	0.15	0.34	0.84	1	0.06	0.07	0.1	0.18	0.58	1	0.07	0.06	0.09	0.16	0.5	1
		$]0.8,0.9]$	0.06	0.06	0.07	0.12	0.35	1	0.05	0.06	0.05	0.1	0.19	1	0.05	0.05	0.05	0.09	0.14	1
	c	$[0,0.2]$	0.06	0.06	0.08	0.12	0.16	0.16	0.1	0.16	0.58	1	1	1	0.14	0.21	0.48	0.99	1	1
		$]0.2,0.4]$	0.08	0.09	0.14	0.28	0.45	0.49	0.08	0.13	0.4	0.91	1	1	0.1	0.12	0.3	0.83	1	1
		$]0.4,0.6]$	0.08	0.08	0.13	0.31	0.58	0.84	0.07	0.1	0.22	0.6	0.98	1	0.08	0.1	0.16	0.48	0.94	1
		$]0.6,0.8]$	0.06	0.07	0.09	0.18	0.56	0.98	0.06	0.07	0.1	0.22	0.63	1	0.06	0.06	0.09	0.16	0.51	1
		$]0.8,0.9]$	0.05	0.05	0.06	0.09	0.26	1	0.05	0.05	0.06	0.11	0.18	1	0.04	0.05	0.05	0.09	0.12	0.99
	cs	$[0,0.2]$	0.07	0.09	0.1	0.1	0.1	0.12	0.11	0.17	0.26	0.39	0.4	0.35	0.2	0.33	0.6	1	1	1
		$]0.2,0.4]$	0.08	0.08	0.1	0.1	0.12	0.12	0.11	0.17	0.27	0.37	0.42	0.4	0.16	0.25	0.49	0.99	1	1
		$]0.4,0.6]$	0.06	0.07	0.1	0.11	0.14	0.13	0.1	0.13	0.25	0.37	0.44	0.4	0.14	0.2	0.4	0.92	1	1
		$]0.6,0.8]$	0.07	0.08	0.09	0.12	0.14	0.16	0.08	0.12	0.2	0.34	0.42	0.42	0.1	0.16	0.29	0.75	0.95	1
		$]0.8,0.9]$	0.06	0.06	0.09	0.09	0.09	0.18	0.07	0.09	0.17	0.28	0.33	0.43	0.09	0.14	0.21	0.57	0.79	1
3	o	$[0,0.2]$	0.22	0.42	0.78	0.97	1	1	0.16	0.28	0.56	0.96	1	1	0.12	0.23	0.47	0.91	1	1
		$]0.2,0.4]$	0.16	0.26	0.54	0.92	1	1	0.1	0.17	0.32	0.76	1	1	0.08	0.14	0.27	0.66	1	1
		$]0.4,0.6]$	0.09	0.15	0.28	0.67	0.99	1	0.08	0.1	0.17	0.44	0.94	1	0.07	0.09	0.14	0.33	0.88	1
		$]0.6,0.8]$	0.06	0.08	0.13	0.26	0.79	1	0.06	0.07	0.08	0.17	0.53	1	0.06	0.06	0.08	0.14	0.44	1
		$]0.8,0.9]$	0.05	0.05	0.08	0.1	0.26	1	0.05	0.04	0.07	0.07	0.16	0.99	0.04	0.05	0.05	0.07	0.13	0.98
	c	$[0,0.2]$	0.04	0.05	0.08	0.13	0.19	0.21	0.08	0.14	0.44	0.99	1	1	0.12	0.19	0.38	0.95	1	1
		$]0.2,0.4]$	0.06	0.08	0.14	0.29	0.49	0.59	0.06	0.11	0.31	0.87	1	1	0.07	0.12	0.24	0.73	1	1
		$]0.4,0.6]$	0.06	0.08	0.13	0.29	0.62	0.87	0.06	0.09	0.18	0.52	0.96	1	0.06	0.08	0.14	0.35	0.9	1
		$]0.6,0.8]$	0.06	0.06	0.08	0.16	0.54	0.99	0.06	0.06	0.08	0.19	0.56	1	0.05	0.05	0.08	0.14	0.44	1
		$]0.8,0.9]$	0.04	0.06	0.06	0.09	0.2	1	0.04	0.05	0.07	0.08	0.16	0.99	0.04	0.05	0.05	0.07	0.13	0.97
	cs	$[0,0.2]$	0.07	0.07	0.1	0.09	0.09	0.1	0.1	0.13	0.2	0.34	0.36	0.31	0.19	0.3	0.55	0.99	1	1
		$]0.2,0.4]$	0.06	0.08	0.08	0.09	0.11	0.1	0.08	0.13	0.2	0.33	0.38	0.31	0.15	0.23	0.46	0.96	1	1
		$]0.4,0.6]$	0.07	0.08	0.08	0.1	0.1	0.12	0.08	0.12	0.19	0.33	0.38	0.3	0.12	0.18	0.38	0.91	1	1
		$]0.6,0.8]$	0.06	0.06	0.08	0.1	0.11	0.13	0.08	0.1	0.18	0.31	0.37	0.35	0.1	0.16	0.29	0.73	0.97	1
		$]0.8,0.9]$	0.07	0.07	0.08	0.09	0.1	0.14	0.07	0.07	0.15	0.27	0.34	0.38	0.08	0.13	0.2	0.58	0.86	1

Table 5: ERFs of the null of no cointegration at the 5% nominal level when the alternative is true, $b=1$

previous results where an underspecified cointegration fails to systematically reject the absence of cointegration in large sample.

6 Testing for misspecification and selecting cointegrated regression models

The previous section showed that there is a significant loss of power in small sample when the regression model isn't well specified. More precisely, overspecifying the regression model can prevent cointegration to be detected in small sample (especially when considering a structural break when there is actually none), and underspecifying it prevents it in all sample sizes. Testing for a well-specified model can therefore help to increase the power of the test. Bai and Perron (1998) test an endogenous structural change in stationary linear relationships. If we were to use such a test, we would do so on a model where the unit root in the residuals has already been rejected. Adapted to our setting, it is equivalent to a test of joint significance of the structural break parameters, i.e. $\mu_i = 0$ and $\beta_i = \mathbf{0}$, $1 \leq i \leq b$ in equation (1). In matricial form, the null hypothesis can be rewritten $\mathbf{R}\Gamma = 0$. In presence of serially correlated residuals, the robust Wald test of the null hypothesis of no structural break in the cointegrating relationship is given by:

$$W = \frac{T-(b+1)k-(m+1)}{b} \times \hat{\Gamma}'\mathbf{R}'(\mathbf{R}T\hat{V}(\hat{\Gamma})\mathbf{R}')^{-1}\mathbf{R}\hat{\Gamma} \tag{6}$$

where k is the number of structural break parameters ($k = 1$ for model "c" and $k = m+1$ for model "cs") and $\hat{V}(\hat{\Gamma})$ the estimator of the matrix of variance-covariance of the estimated residuals $\hat{\epsilon}$.

To bypass size distortion and non-monotonic power due to persistent errors, we compute a version of the estimator using residuals obtained both under the null and the alternative hypotheses, as detailed in Kejriwal and Perron (2010)) :

$$W^* = \frac{1}{\hat{\sigma}^2}\frac{T-(b+1)k-(m+1)}{b} \times \hat{\Gamma}'\mathbf{R}'(\mathbf{R}T(X'X)^{-1}\mathbf{R}')^{-1}\mathbf{R}\hat{\Gamma}. \tag{7}$$

The consistent estimate $\hat{\sigma}^2$ of σ^2_ϵ is given by:

$$\hat{\sigma}^2 = T^{-1}\sum_{t=1}^{T}\tilde{\epsilon}_t^2 + 2T^{-1}\sum_{j=1}^{T-1} w(j/\hat{h})\sum_{t=j+1}^{T}\tilde{\epsilon}_t\tilde{\epsilon}_{t-j} \tag{8}$$

where $\tilde{\epsilon}_t$ are the residuals obtained by imposing the null hypothesis, w the quadratic spectral kernel function[9] with the estimate of the bandwidth given by $\hat{h} = 1.3221[4T\hat{\rho}^2/(1-\hat{\rho})^4]^{1/5}$, with $\hat{\rho} = \sum_{t=2}^{T} \hat{\epsilon}_t \hat{\epsilon}_{t-1} / \sum_{t=2}^{T} \hat{\epsilon}_{t-1}^2$ the empirical serial correlation under the alternative hypothesis. Table 8 reports the empirical rejection rates at a 5% level of the null model O against the alternative of model C or CS and of the null of model C against the alternative of model CS, using the critical values of Bai and Perron (1998) (see Appendix A). For our sample sizes of interest, $T \leq 100$, the power of the test is very low, with at most 30% for $T = 100$, $b = 1$ and $m = 1$. We also observe the null to be always rejected when it is true. This indicates the empirical size in finite sample to be significantly lower than the nominal size, which means that the test is too conservative in rejecting the null hypothesis. Therefore, we propose to use the simulated distribution of the modified Wald test statistic under the null to correct for size distortion in small sample. Table 9 reports the power of the test using the size-corrected critical values for small sample sizes at a 5% level. We naturally recover the nominal size when the null is true (values in parenthesis) while significantly gaining power, especially when testing for a structural break in the slope coefficients where the probability of rejecting the null when the alternative is true is more than 50% in sample size smaller than $T = 50$ for $m = 1$ and than $T = 100$ for $m > 1$. It is important to note that in the case of the test of model "c" against model "cs", the null is rejected more often when the true model doesn't have any structural break than when the null is true. This indicates that when there is actually no structural change in the parameters, the test may conclude to a presence of a structural break in the slope coefficients. One should therefore only test the joint significance of the slope coefficients after rejecting the joint insignificance of both the intercept and the slope coefficients.

We can upgrade our procedure for testing cointegration by testing the null of no cointegration against cointegration in three different cases: without structural change, with a change in the intercept, and with a change in the intercept and the slope parameters.

1. If the null is never rejected, we conclude to the absence of cointegration.

2. If the null is rejected in only one of the cases, we conclude to cointegration in the relevant case.

3. If the null is rejected in more than one case, we test for each of the candidate

[9] $w(x) = \frac{25}{12\pi^2 x^2}\left(\frac{\sin(6\pi x/5)}{6\pi x/5} - \cos(6\pi x/5)\right)$, see Andrews (1991).

models the joint significance of the structural change parameters. The retained model is the one where the joint insignificance of the parameters is rejected. If it is not rejected, the most parsimonious cointegration model is selected.

Table 7 reports the frequency of selection of each cointegration model for each data generating process. We observed that as the sample size increases the true model is more systematically selected, as we could have expected. This is especially noticeable when there is either a structural change in the intercept and the slope coefficient or none at all. When there is a structural change in the intercept only, the rate at which it becomes successfully selected as the sample size increases is slower. For our small sample sizes of interest, we observe significantly different behavior in terms of cointegration model detection. When the DGP is model O, it is quite more often successfully selected, and models C and CS are rarely selected. In contrast with Table 3 where cointegration with structural change was detected at a fair rate for the same DGP, this indicates that an overspecified would succeed in detecting cointegration. When the DGP is model C, it is also the most retained, although at a lesser successful rate than in the case of model O. On the other hand, when the DGP is model C, the rate of detection of the true model is lower in small sample and an underspecified model is selected instead at a significant rate, but the total frequency of detection of cointegration is higher. Table 10 reports the power of the test of cointegration with or without structural change using the composite methodology of considering all cointegration models detailed in this paper. By doing so, we make sure that at least the true model is considered. In consequence, we naturally obtain empirical rejection rates higher than the ones recovered previously by matching the regression model to the true model. Particularly for one of our sample sizes of interest, the power of the test is at a satisfactory 50% for $T = 30$ and $b = 1$. The gain in power is notably significant when there is a structural change in the true model with a 9 to 13 percentage points increase against a 5 to 7 increase when there is actually no structural change. This indicates that when there is no structural change in the data generating process, the power of the test of cointegration without structural change is already good and there is little gain in considering a structural change, even though sometimes a piece-wise linear model can fit a linear model in a very small sample. However, when there is a structural change in the true model, considering alternative models to the unknown underlying one has a fair chance of fitting a cointegration relationship. In our simulations, this gain is roughly the same when the true model is model C or model CS. Since the latter shows a better power in itself (see Table 3), it reaches the best power for the composite methodology in the end. Detailing with respect to the value of ρ (see Table

			regression model : c, $T =$						regression model : cs, $T =$					
m	DGP	ρ	15	20	30	50	100	500	15	20	30	50	100	500
1	o	$[0, 0.2]$	0.08	0.26	0.44	0.97	1	1	0.09	0.24	0.4	0.94	1	1
		$]0.2, 0.4]$	0.06	0.17	0.22	0.81	1	1	0.07	0.16	0.22	0.72	1	1
		$]0.4, 0.6]$	0.05	0.11	0.1	0.48	0.92	1	0.05	0.09	0.1	0.39	0.87	1
		$]0.6, 0.8]$	0.03	0.08	0.05	0.21	0.47	1	0.04	0.07	0.05	0.16	0.4	1
		$]0.8, 0.9]$	0.04	0.07	0.03	0.1	0.13	0.99	0.03	0.06	0.03	0.08	0.11	0.96
	c	$[0, 0.2]$	0.06	0.16	0.33	1	1	1	0.13	0.25	0.36	0.99	1	1
		$]0.2, 0.4]$	0.04	0.12	0.23	0.95	1	1	0.05	0.13	0.21	0.87	1	1
		$]0.4, 0.6]$	0.05	0.09	0.13	0.7	0.97	1	0.04	0.09	0.12	0.58	0.94	1
		$]0.6, 0.8]$	0.03	0.07	0.07	0.38	0.62	1	0.03	0.06	0.06	0.27	0.53	1
		$]0.8, 0.9]$	0.02	0.08	0.04	0.18	0.21	0.99	0.03	0.06	0.04	0.13	0.17	0.96
	cs	$[0, 0.2]$	0.05	0.14	0.16	0.41	0.44	0.44	0.17	0.35	0.54	1	1	1
		$]0.2, 0.4]$	0.05	0.13	0.14	0.41	0.45	0 .43	0.1	0.26	0.43	0.98	1	1
		$]0.4, 0.6]$	0.05	0.12	0.14	0.39	0.43	0.48	0.07	0.19	0.32	0.9	1	1
		$]0.6, 0.8]$	0.04	0.1	0.11	0.36	0.41	0.45	0.06	0.16	0.23	0.77	0.94	1
		$]0.8, 0.9]$	0.03	0.12	0.1	0.29	0.33	0.46	0.04	0.13	0.18	0.6	0.8	0.96
2	o	$[0, 0.2]$	0.09	0.29	0.4	0.96	1	1	0.07	0.2	0.36	0.84	1	1
		$]0.2, 0.4]$	0.05	0.18	0.2	0.76	1	1	0.06	0.12	0.2	0.54	0.99	1
		$]0.4, 0.6]$	0.04	0.11	0.09	0.43	0.9	1	0.04	0.1	0.1	0.27	0.81	1
		$]0.6, 0.8]$	0.04	0.07	0.05	0.18	0.43	1	0.04	0.06	0.07	0.1	0.35	1
		$]0.8, 0.9]$	0.04	0.07	0.02	0.1	0.13	0.99	0.04	0.05	0.04	0.06	0.1	0.92
	c	$[0, 0.2]$	0.03	0.11	0.3	1	1	1	0.1	0.24	0.38	0.87	1	1
		$]0.2, 0.4]$	0.03	0.1	0.19	0.91	1	1	0.06	0.14	0.22	0.68	1	1
		$]0.4, 0.6]$	0.03	0.08	0.1	0.65	0.94	1	0.04	0.08	0.11	0.36	0.88	1
		$]0.6, 0.8]$	0.02	0.07	0.06	0.31	0.58	1	0.04	0.07	0.08	0.16	0.47	1
		$]0.8, 0.9]$	0.03	0.05	0.04	0.18	0.19	0.99	0.04	0.05	0.06	0.09	0.15	0.93
	cs	$[0, 0.2]$	0.04	0.1	0.1	0.3	0.34	0.32	0.17	0.36	0.54	0.94	1	1
		$]0.2, 0.4]$	0.04	0.09	0.1	0.28	0.33	0.34	0.12	0.3	0.45	0.9	1	1
		$]0.4, 0.6]$	0.04	0.09	0.1	0.31	0.34	0.33	0.11	0.23	0.37	0.83	1	1
		$]0.6, 0.8]$	0.03	0.09	0.09	0.29	0.32	0.34	0.09	0.2	0.32	0.71	0.98	1
		$]0.8, 0.9]$	0.04	0.08	0.09	0.27	0.32	0.36	0.07	0.17	0.24	0.6	0.94	1
3	o	$[0, 0.2]$	0.06	0.22	0.38	0.93	1	1	0.16	0.12	0.29	0.73	1	1
		$]0.2, 0.4]$	0.05	0.14	0.19	0.7	1	1	0.17	0.08	0.17	0.43	0.97	1
		$]0.4, 0.6]$	0.04	0.09	0.1	0.38	0.86	1	0.14	0.07	0.1	0.2	0.7	1
		$]0.6, 0.8]$	0.04	0.07	0.04	0.17	0.37	1	0.14	0.05	0.06	0.09	0.25	1
		$]0.8, 0.9]$	0.04	0.04	0.03	0.09	0.11	0.96	0.13	0.06	0.05	0.05	0.08	0.87
	c	$[0, 0.2]$	0.02	0.07	0.26	0.99	1	1	0.16	0.18	0.37	0.73	1	1
		$]0.2, 0.4]$	0.03	0.07	0.16	0.88	1	1	0.15	0.08	0.18	0.5	0.98	1
		$]0.4, 0.6]$	0.03	0.07	0.09	0.56	0.93	1	0.14	0.06	0.1	0.24	0.8	1
		$]0.6, 0.8]$	0.03	0.06	0.05	0.28	0.52	1	0.15	0.06	0.07	0.12	0.34	1
		$]0.8, 0.9]$	0.02	0.06	0.04	0.12	0.17	0.96	0.13	0.05	0.06	0.06	0.12	0.86
	cs	$[0, 0.2]$	0.03	0.08	0.07	0.24	0.27	0.28	0.19	0.29	0.5	0.84	1	1
		$]0.2, 0.4]$	0.04	0.08	0.08	0.26	0.3	0.31	0.17	0.23	0.47	0.79	1	1
		$]0.4, 0.6]$	0.03	0.07	0.07	0.25	0.29	0.30	0.16	0.18	0.36	0.73	1	1
		$]0.6, 0.8]$	0.03	0.08	0.07	0.23	0.29	0.31	0.13	0.16	0.32	0.66	0.99	1
		$]0.8, 0.9]$	0.02	0.05	0.06	0.25	0.28	0.32	0.14	0.11	0.27	0.55	0.96	1

Table 6: ERFs of the null of no cointegration at the 5% nominal level when the alternative is true, $b = 2$

m	DGP	selected cointegration model	b = 1, T = 15	20	30	50	100	500	b = 2, T = 15	20	30	50	100	500
1	o	o	0.16	0.3	0.51	0.68	0.84	0.93	0.16	0.3	0.53	0.7	0.85	0.93
		c	0.06	0.06	0.05	0.06	0.04	0.03	0.04	0.09	0.07	0.12	0.05	0.01
		cs	0.03	0.03	0.03	0.04	0.04	0.04	0.04	0.05	0.04	0.08	0.08	0.06
	c	o	0.05	0.07	0.1	0.17	0.3	0.27	0.06	0.07	0.1	0.14	0.21	0.22
		c	0.07	0.11	0.36	0.54	0.59	0.72	0.04	0.1	0.16	0.6	0.66	0.76
		cs	0.05	0.06	0.04	0.03	0.01	0.01	0.05	0.08	0.07	0.04	0.03	0.03
	cs	o	0.07	0.09	0.09	0.09	0.1	0.07	0.07	0.09	0.1	0.08	0.05	0.02
		c	0.09	0.14	0.23	0.23	0.13	0.03	0.04	0.11	0.1	0.22	0.09	0.01
		cs	0.09	0.13	0.3	0.57	0.73	0.91	0.08	0.17	0.27	0.62	0.83	0.97
2	o	o	0.14	0.23	0.44	0.63	0.81	0.92	0.14	0.23	0.45	0.66	0.82	0.92
		c	0.06	0.05	0.05	0.06	0.05	0.03	0.04	0.11	0.08	0.15	0.06	0.01
		cs	0.04	0.03	0.03	0.05	0.05	0.05	0.04	0.05	0.05	0.05	0.1	0.07
	c	o	0.05	0.07	0.1	0.17	0.32	0.34	0.05	0.07	0.1	0.17	0.27	0.30
		c	0.06	0.08	0.24	0.49	0.55	0.65	0.03	0.08	0.13	0.55	0.59	0.68
		cs	0.06	0.06	0.06	0.03	0.02	0.01	0.05	0.09	0.09	0.03	0.04	0.02
	cs	o	0.06	0.07	0.09	0.08	0.07	0.04	0.06	0.07	0.09	0.09	0.05	0.01
		c	0.07	0.1	0.18	0.23	0.15	0.02	0.04	0.08	0.08	0.23	0.14	0.01
		cs	0.11	0.16	0.28	0.6	0.75	0.94	0.1	0.21	0.33	0.56	0.8	0.98
3	o	o	0.1	0.19	0.39	0.58	0.78	0.92	0.1	0.2	0.39	0.61	0.79	0.92
		c	0.05	0.05	0.04	0.06	0.04	0.02	0.03	0.1	0.09	0.18	0.08	0.00
		cs	0.04	0.04	0.04	0.05	0.06	0.06	0.14	0.04	0.05	0.04	0.10	0.08
	c	o	0.04	0.06	0.1	0.17	0.34	0.4	0.04	0.06	0.1	0.18	0.32	0.38
		c	0.05	0.07	0.18	0.45	0.5	0.59	0.02	0.06	0.11	0.51	0.55	0.60
		cs	0.05	0.06	0.07	0.03	0.02	0.01	0.14	0.07	0.1	0.02	0.04	0.02
	cs	o	0.05	0.07	0.08	0.07	0.06	0.03	0.05	0.07	0.08	0.08	0.06	0.01
		c	0.06	0.08	0.14	0.23	0.17	0.03	0.02	0.07	0.06	0.2	0.17	0.01
		cs	0.11	0.16	0.29	0.59	0.75	0.94	0.15	0.17	0.34	0.52	0.76	0.98

Table 7: Frequency of cointegration models selected by the composite methodology

		b = 1																	
		H0 : model "o" vs H1 : model "c"						H0 : model "o" vs H1 : model "cs"						H0 : model "c" vs H1 : model "cs"					
		$T =$						$T =$						$T =$					
m	DGP	15	20	30	50	100	500	15	20	30	50	100	500	15	20	30	50	100	500
1	o	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	c	0	0	0.06	0.18	0.3	0.58	0	0	0.02	0.1	0.24	0.51	0	0	0	0	0	0
	cs	0	0	0.04	0.06	0.05	0.04	0	0	0.06	0.27	0.47	0.76	0	0	0.04	0.12	0.28	0.65
2	o	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	c	0	0	0.02	0.09	0.22	0.49	0	0	0	0.02	0.12	0.37	0	0	0	0	0	0
	cs	0	0	0.01	0.03	0.02	0.02	0	0	0	0.12	0.41	0.77	0	0	0.01	0.07	0.24	0.65
3	o	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	c	0	0	0	0.04	0.15	0.42	0	0	0	0	0.05	0.27	0	0	0	0	0	0
	cs	0	0	0	0.02	0.02	0.01	0	0	0	0.04	0.31	0.75	0	0	0	0.03	0.17	0.62

		b = 2																	
		H0 : model "o" vs H1 : model "c"						H0 : model "o" vs H1 : model "cs"						H0 : model "c" vs H1 : model "cs"					
		$T =$						$T =$						$T =$					
m	DGP	15	20	30	50	100	500	15	20	30	50	100	500	15	20	30	50	100	500
1	o	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	c	0	0	0	0.03	0.17	0.43	0	0	0	0	0.08	0.35	0	0	0	0	0	0
	cs	0	0	0	0.02	0.02	0.01	0	0	0	0.01	0.2	0.72	0	0	0	0.01	0.09	0.52
2	o	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	c	0	0	0	0.01	0.09	0.34	0	0	0	0	0.01	0.21	0	0	0	0	0	0
	cs	0	0	0	0.01	0.01	0	0	0	0	0	0.06	0.71	0	0	0	0	0.03	0.47
3	o	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	c	0	0	0	0	0.04	0.27	0	0	0	0	0	0.12	0	0	0	0	0	0
	cs	0	0	0	0	0	0	0	0	0	0	0.01	0.64	0	0	0	0	0.01	0.4

Table 8: ERFs of the null of no structural break in the cointegrating relationship at the 5% nominal level, critical values from Bai and Perron (1998)

		$b=1$																	
		H0 : model "o" vs H1 : model "c"						H0 : model "o" vs H1 : model "cs"						H0 : model "c" vs H1 : model "cs"					
		$T=$						$T=$						$T=$					
m	DGP	15	20	30	50	100	500	15	20	30	50	100	500	15	20	30	50	100	500
1	o	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	0.1	0.11	0.13	0.18	0.26	0.4
	c	0.15	0.19	0.3	0.41	0.5	0.72	0.09	0.12	0.22	0.33	0.44	0.68	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)
	cs	0.13	0.15	0.19	0.17	0.14	0.11	0.17	0.23	0.38	0.54	0.68	0.88	0.15	0.21	0.35	0.57	0.76	0.96
2	o	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	0.11	0.09	0.1	0.15	0.23	0.36
	c	0.1	0.12	0.2	0.32	0.44	0.65	0.04	0.06	0.1	0.19	0.31	0.56	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)
	cs	0.09	0.11	0.14	0.12	0.09	0.05	0.09	0.13	0.22	0.43	0.64	0.88	0.11	0.14	0.21	0.44	0.73	0.96
3	o	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	0.11	0.1	0.11	0.13	0.21	0.34
	c	0.08	0.09	0.16	0.26	0.37	0.59	0.03	0.03	0.06	0.1	0.21	0.45	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)
	cs	0.07	0.08	0.13	0.11	0.08	0.03	0.05	0.07	0.14	0.28	0.58	0.87	0.09	0.1	0.16	0.31	0.66	0.95

		$b=2$																	
		H0 : model "o" vs H1 : model "c"						H0 : model "o" vs H1 : model "cs"						H0 : model "c" vs H1 : model "cs"					
		$T=$						$T=$						$T=$					
m	DGP	15	20	30	50	100	500	15	20	30	50	100	500	15	20	30	50	100	500
1	o	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	0.16	0.19	0.23	0.3	0.46	0.67
	c	0.06	0.08	0.14	0.31	0.45	0.65	0.05	0.06	0.1	0.23	0.4	0.6	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)
	cs	0.07	0.09	0.12	0.16	0.11	0.04	0.09	0.12	0.17	0.35	0.66	0.89	0.19	0.26	0.35	0.56	0.87	0.98
2	o	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	0.15	0.17	0.19	0.22	0.41	0.68
	c	0.05	0.06	0.09	0.19	0.32	0.53	0.03	0.03	0.06	0.11	0.23	0.45	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)
	cs	0.06	0.07	0.1	0.12	0.07	0.02	0.06	0.08	0.1	0.18	0.52	0.9	0.15	0.18	0.23	0.34	0.74	0.99
3	o	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	-	0.15	0.15	0.19	0.33	0.63
	c	0.04	0.05	0.07	0.12	0.24	0.45	0.03	0.03	0.04	0.06	0.14	0.34	-	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)
	cs	0.05	0.06	0.08	0.1	0.06	0.01	0.01	0.05	0.08	0.1	0.36	0.89	-	0.15	0.18	0.23	0.56	0.98

Table 9: ERFs of the null of no structural break in the cointegrating relationship at the 5% nominal level, size-corrected for finite sample critical values

11) shows that if the serial correlation in the residuals is relatively small ($\rho \leq 0.4$), the probabilities of correctly rejecting the null is higher than 75% for $T \geq 30$, decreasing with the number of breaking parameters.

		$b = 1$, $T =$						$b = 2$, $T =$					
m	DGP	15	20	30	50	100	500	15	20	30	50	100	500
1	o	0.25	0.38	0.59	0.78	0.93	1	0.24	0.44	0.64	0.9	0.98	1
	c	0.17	0.24	0.51	0.74	0.9	1	0.14	0.25	0.33	0.77	0.9	1
	cs	0.24	0.36	0.62	0.88	0.96	1	0.19	0.37	0.48	0.92	0.97	1
2	o	0.24	0.32	0.52	0.74	0.91	1	0.22	0.39	0.58	0.86	0.97	1
	c	0.17	0.21	0.41	0.7	0.89	1	0.13	0.23	0.33	0.75	0.9	1
	cs	0.24	0.33	0.54	0.91	0.97	1	0.19	0.37	0.5	0.87	0.99	1
3	o	0.19	0.29	0.47	0.69	0.88	1	0.28	0.34	0.53	0.83	0.97	1
	c	0.14	0.19	0.36	0.66	0.86	1	0.21	0.2	0.31	0.71	0.9	1
	cs	0.22	0.31	0.51	0.9	0.98	1	0.23	0.31	0.48	0.81	0.99	1

Table 10: ERFs of the null of no cointegration at the 5% nominal level when the alternative of cointegration with or without structural break is true

7 An example

We apply our procedure to Chinese macroeconomic data, more precisely we investigate a stationary linear or piece-wise linear relationship between the gross domestic product, the national retail sales of consumer goods, and investment in fixed assets[10]. We use this exercise as an application for the sake of illustration, as it does not provide much more insights than a simple accounting relationship about how production interacts with consumption and investment in the long term. At the date of writing, the data are available as annual time series[11] from 1952 to 2020 for both the gross domestic product and the retail sales and from 1980 to 2019 for the investment in fixed assets[12]. As the series are measured in current prices, we deflate them by using respectively the annual consumer price index (1951-2020), the retail price index (1951-2019), and the price index for investment in fixed assets (1989-2019). Converting all selected data to

[10]Source: http://www.stats.gov.cn/

[11]Series of higher frequencies such as quarterly or monthly publications are available but only for fewer decades and are not annually revised.

[12]Although national accounts data exist and annually detail the gross domestic products and its components by expenditure, they are not reported in volume. We choose to select the three aforementioned series each have their own price index and can be deflated.

m	DGP	ρ	$b=1, T=$ 15	20	30	50	100	500	$b=2, T=$ 15	20	30	50	100	500
1	o	$[0, 0.2]$	0.43	0.69	0.95	1	1	1	0.13	0.33	0.5	0.98	1	1
		$]0.2, 0.4]$	0.3	0.49	0.81	0.99	1	1	0.11	0.24	0.28	0.83	1	1
		$]0.4, 0.6]$	0.21	0.29	0.53	0.9	1	1	0.08	0.15	0.14	0.53	0.94	1
		$]0.6, 0.8]$	0.14	0.18	0.27	0.51	0.93	1	0.06	0.12	0.08	0.25	0.52	1
		$]0.8, 0.9]$	0.13	0.14	0.17	0.21	0.47	1	0.07	0.1	0.06	0.13	0.17	0.99
	c	$[0, 0.2]$	0.25	0.35	0.88	1	1	1	0.16	0.33	0.48	1	1	1
		$]0.2, 0.4]$	0.2	0.31	0.68	0.97	1	1	0.08	0.2	0.33	0.96	1	1
		$]0.4, 0.6]$	0.16	0.2	0.41	0.81	1	1	0.08	0.14	0.18	0.73	0.98	1
		$]0.6, 0.8]$	0.13	0.17	0.23	0.44	0.85	1	0.05	0.1	0.1	0.42	0.66	1
		$]0.8, 0.9]$	0.11	0.13	0.15	0.19	0.42	1	0.06	0.12	0.06	0.22	0.26	0.99
	cs	$[0, 0.2]$	0.34	0.47	0.82	1	1	1	0.2	0.42	0.59	1	1	1
		$]0.2, 0.4]$	0.27	0.42	0.74	0.99	1	1	0.14	0.34	0.48	0.99	1	1
		$]0.4, 0.6]$	0.22	0.32	0.6	0.93	1	1	0.11	0.27	0.39	0.93	1	1
		$]0.6, 0.8]$	0.18	0.27	0.46	0.77	0.95	1	0.1	0.23	0.29	0.83	0.95	1
		$]0.8, 0.9]$	0.18	0.21	0.35	0.57	0.74	1	0.07	0.22	0.24	0.66	0.82	1
2	o	$[0, 0.2]$	0.38	0.58	0.9	1	1	1	0.14	0.35	0.48	0.96	1	1
		$]0.2, 0.4]$	0.27	0.39	0.71	0.98	1	1	0.1	0.23	0.28	0.78	1	1
		$]0.4, 0.6]$	0.2	0.24	0.42	0.82	1	1	0.08	0.17	0.15	0.47	0.91	1
		$]0.6, 0.8]$	0.15	0.16	0.23	0.41	0.87	1	0.07	0.11	0.1	0.22	0.5	1
		$]0.8, 0.9]$	0.12	0.12	0.13	0.21	0.42	1	0.07	0.1	0.05	0.13	0.18	0.99
	c	$[0, 0.2]$	0.23	0.3	0.71	1	1	1	0.13	0.3	0.5	1	1	1
		$]0.2, 0.4]$	0.19	0.24	0.53	0.95	1	1	0.09	0.2	0.32	0.93	1	1
		$]0.4, 0.6]$	0.16	0.19	0.34	0.74	0.99	1	0.07	0.14	0.18	0.68	0.95	1
		$]0.6, 0.8]$	0.13	0.14	0.19	0.36	0.8	1	0.06	0.12	0.12	0.36	0.63	1
		$]0.8, 0.9]$	0.1	0.11	0.12	0.19	0.36	1	0.06	0.09	0.08	0.22	0.25	0.99
	cs	$[0, 0.2]$	0.3	0.44	0.7	1	1	1	0.2	0.41	0.58	0.95	1	1
		$]0.2, 0.4]$	0.27	0.37	0.62	0.99	1	1	0.15	0.35	0.5	0.92	1	1
		$]0.4, 0.6]$	0.22	0.31	0.53	0.95	1	1	0.14	0.29	0.42	0.87	1	1
		$]0.6, 0.8]$	0.19	0.27	0.43	0.81	0.97	1	0.11	0.26	0.37	0.79	0.99	1
		$]0.8, 0.9]$	0.17	0.22	0.33	0.67	0.83	1	0.11	0.22	0.3	0.67	0.94	1
3	o	$[0, 0.2]$	0.31	0.52	0.83	1	1	1	0.22	0.28	0.46	0.94	1	1
		$]0.2, 0.4]$	0.21	0.35	0.61	0.95	1	1	0.21	0.19	0.26	0.74	1	1
		$]0.4, 0.6]$	0.16	0.23	0.36	0.73	1	1	0.17	0.14	0.16	0.42	0.88	1
		$]0.6, 0.8]$	0.13	0.15	0.21	0.35	0.82	1	0.18	0.11	0.09	0.21	0.42	1
		$]0.8, 0.9]$	0.11	0.11	0.14	0.16	0.34	1	0.16	0.08	0.07	0.12	0.15	0.97
	c	$[0, 0.2]$	0.19	0.29	0.61	0.99	1	1	0.18	0.23	0.48	0.99	1	1
		$]0.2, 0.4]$	0.15	0.22	0.44	0.92	1	1	0.17	0.14	0.27	0.89	1	1
		$]0.4, 0.6]$	0.14	0.18	0.3	0.65	0.99	1	0.17	0.12	0.16	0.59	0.94	1
		$]0.6, 0.8]$	0.12	0.13	0.17	0.32	0.75	1	0.18	0.11	0.11	0.32	0.57	1
		$]0.8, 0.9]$	0.09	0.12	0.13	0.16	0.3	1	0.15	0.1	0.08	0.15	0.22	0.96
	cs	$[0, 0.2]$	0.28	0.4	0.63	0.99	1	1	0.21	0.34	0.53	0.88	1	1
		$]0.2, 0.4]$	0.23	0.33	0.56	0.97	1	1	0.2	0.29	0.5	0.83	1	1
		$]0.4, 0.6]$	0.21	0.3	0.5	0.93	1	1	0.19	0.24	0.41	0.79	1	1
		$]0.6, 0.8]$	0.19	0.26	0.41	0.8	0.98	1	0.15	0.23	0.36	0.72	0.99	1
		$]0.8, 0.9]$	0.17	0.21	0.34	0.68	0.89	1	0.16	0.16	0.32	0.64	0.96	1

Table 11: ERFs of the null of no cointegration at the 5% nominal level when the alternative is true

series to volume leaves the concomitant annual data for estimation from 1989 to 2019, i.e. 31 observations. Finally, we apply a logarithmic transformation to account for a scale effect. As a consequence, the estimated slope coefficients measure elasticities. Figure 4 plots the series used for estimation. We observe that the growth rates of the GDP and the retail sales of consumer goods are similar and stable during the considered time period, whereas the investment in fixed assets grows faster until a slowdown in 2010 and decreases from 2016. We test for cointegration with up to two endogenous breaks in the intercept and/or the slope coefficient. Table 12 reports the estimation results. A conventional cointegration test without structural change (model "o")

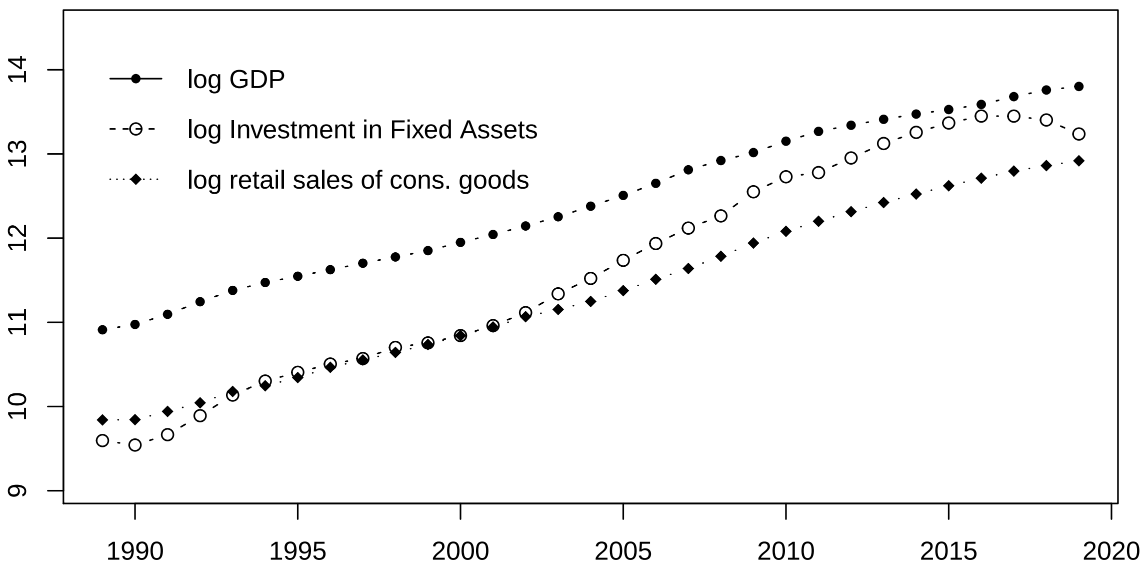

Figure 4: Chinese GDP (by expenditure approach), retail sales of consumer goods and investment in fixed assets (in log volume)

but using size-corrected critical values gives a p-value of 24%, hence would have not rejected the absence of cointegration at a 5% level. Considering one or two endogenous structural breaks in the parameters only rejects the null at a 5% level in the case of model CS with one break ($b = 1$) in 2011, with a p-value of 2.2%. All structural break parameters appear significant[13]. The estimated coefficients indicate that the elasticity

[13]We however didn't account for serial correlation.

regression model	o	c	cs	c	cs
# of breaks	-	1	1	2	2
intercept	2.917	2.247	2.744	2.227	2.794
	(11.547)	(9.394)	(9.026)	(8.924)	(10.209)
retail sales	0.563	0.703	0.502	0.706	0.467
	(5.972)	(9.075)	(4.963)	(8.918)	(4.971)
inv. fixed assets	0.269	0.192	0.346	0.192	0.376
	(3.745)	(3.39)	(4.707)	(3.316)	(5.378)
breakdate t_1		2013	2011	2012	2008
intercept after t_1		-0.117	1.862	-0.015	0.124
		(-4.668)	(3.46)	(-0.356)	(0.153)
retail sales after t_1			0.364		0.677
			(2.94)		(0.698)
inv. fixed assets after t_1			-0.495		-0.659
			(-4.702)		(2.063)
breakdate t_2				2013	2015
intercept after t_2				-0.106	1.325
				(-2.595)	(-0.836)
retail sales after t_2					-0.27
					(-2.56)
inv. fixed assets after t_2					0.153
					(0.557)
ADF^*	-2.718	-4.443	-6.141	-4.681	-6.973
p-value	0.24	0.238	0.022	0.701	0.086

Table 12: Cointegration test: Chinese GDP, retail sales of consumer goods and investment in fixed assets (in log volumes 1989-2019), t-statistics are in parenthesis

of the GDP with respect to the retail sales of consumer goods increases after 2011 while it decreases with the investment in fixed assets. The breakdate in 2011 corresponds to the aftermath of the global economic crisis of 2008 and is consistent with our previous visual inspection. Indeed, the growth of investment in fixed assets is the only one to slow down after the crisis (the growth of the GDP and retail sales don't). Alleviating the rejection threshold to 10% leads us to consider two breaks ($b = 2$) in the intercept and slope parameters in 2008 and 2015, with a p-value of 8.6%. The first breakdate also highlights the effect of the 2008 economic crisis but at an earlier date and more amplified than the first cointegration model, as the estimated coefficients are of the same sign but 1.33 to 1.86 more important. The second breakdate in 2015 indicates a decrease of the elasticity of the GDP with respect to retail sales and an increase with respect to the investment in fixed assets. This indicates attenuation of the effect of the first break and is concomitant with the start of the market turbulences which led to the Shanghai stock market crash in the summer of 2015. This model with two structural breaks however shows low values for the t-statistics associated with the estimated structural break coefficients, especially for the intercepts (t-stats = 0.153 after 2008 and -0.836 after 2015), the retail sales after 2008 (t-stat = 0.698) and the investment in fixed assets after 2015 (t-stat = 0.557). As the other slope coefficients are more significant (t-statistics are bigger than 2), a better-fitted model would be to only consider a break in the coefficients associated with the investment in fixed assets after the first break and with the retail sales after the second break, which is not feasible with our procedure as it considers a structural change in all linear parameters. Considering the poor accuracy of the model with two structural breaks, the best cointegration model would be to take into account one structural change in 2011 in the intercept and the slope coefficients. Such a more parsimonious model would also facilitate the rejection of the unit root in the residuals, therefore increasing the detection of cointegration.

8 Concluding remarks

In this paper, we provide size-corrected critical values for a test of cointegration with endogenous structural breaks, as well of a fairly simple composite procedure to increase the rate of detection of a cointegrated relationship by considering several models of structural change and selecting among them by testing the joint significance of the structural break parameters. We find that the composite procedure is able to correctly reject the null of no cointegration when the alternative of cointegration with or without structural break is true even for very small sample sizes when the model is correctly specified and the serial correlation is low ($\rho \leq 0.4$ if $T = 30$ and $\rho \leq 0.6$ if $T = 50$). We also find that the power of the test is the highest when the model is correctly specified. While underspecification leads to low power in all sample sizes and over-specification decreases power in small samples, considering possibly misspecified models can help to fit alternative cointegrated linear relationships to the small sample data and increase the chance of detecting cointegration. Finer tuning of the model selection process, by allowing for example for a change in a subset of the regressors, could generalize the set of alternative models and increase even more the power of the testing procedure. We however find the serial correlation to be hardly precisely estimated, even less in very small samples in which case the estimation bias can be severe. This implies that the model selection procedure favors the rejection of the unit root at the expense of accuracy. The effects on the predictive performance in very small sample will therefore be investigated.

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A Bai and Perron (1998) critical values

ASYMPTOTIC CRITICAL VALUES OF THE MULTIPLE-BREAK TEST.
THE ENTRIES ARE QUANTILES x SUCH THAT $P(\sup F_{k,q} \le x/q) = \alpha$.

		Number of Breaks, k										
q	α	1	2	3	4	5	6	7	8	9	UDmax	WDmax
1	.90	8.02	7.87	7.07	6.61	6.14	5.74	5.40	5.09	4.81	8.78	9.14
	.95	9.63	8.78	7.85	7.21	6.69	6.23	5.86	5.51	5.20	10.17	10.91
	.975	11.17	9.81	8.52	7.79	7.22	6.70	6.27	5.92	5.56	11.52	12.53
	.99	13.58	10.95	9.37	8.50	7.85	7.21	6.75	6.33	5.98	13.74	15.02
2	.90	11.02	10.48	9.61	8.99	8.50	8.06	7.66	7.32	7.01	11.69	12.33
	.95	12.89	11.60	10.46	9.71	9.12	8.65	8.19	7.79	7.46	13.27	14.19
	.975	14.53	12.64	11.20	10.29	9.69	9.10	8.64	8.18	7.80	14.69	16.04
	.99	16.64	13.78	12.06	11.00	10.28	9.65	9.11	8.66	8.22	16.79	18.11
3	.90	13.43	12.73	11.76	11.04	10.49	10.02	9.59	9.21	8.86	14.05	14.76
	.95	15.37	13.84	12.64	11.83	11.15	10.61	10.14	9.71	9.32	15.80	16.82
	.975	17.17	14.91	13.44	12.49	11.75	11.13	10.62	10.14	9.72	17.36	18.79
	.99	19.25	16.27	14.48	13.40	12.56	11.80	11.22	10.67	10.19	19.38	20.81
4	.90	15.53	14.65	13.63	12.91	12.33	11.79	11.34	10.93	10.55	16.17	16.95
	.95	17.60	15.84	14.63	13.71	12.99	12.42	11.91	11.49	11.04	17.88	19.07
	.975	19.35	16.85	15.44	14.43	13.64	13.01	12.46	11.94	11.49	19.51	20.89
	.99	21.20	18.21	16.43	15.21	14.45	13.70	13.04	12.48	12.02	21.25	22.81
5	.90	17.42	16.45	15.44	14.69	14.05	13.51	13.02	12.59	12.18	17.94	18.85
	.95	19.50	17.60	16.40	15.52	14.79	14.19	13.63	13.16	12.70	19.74	20.95
	.975	21.47	18.75	17.26	16.13	15.40	14.75	14.19	13.66	13.17	21.57	23.04
	.99	23.99	20.18	18.19	17.09	16.14	15.34	14.81	14.26	13.72	24.00	25.46
6	.90	19.38	18.15	17.17	16.39	15.74	15.18	14.63	14.18	13.74	19.92	20.89
	.95	21.59	19.61	18.23	17.27	16.50	15.86	15.29	14.77	14.30	21.90	23.27
	.975	23.73	20.80	19.15	18.07	17.21	16.49	15.84	15.29	14.78	23.83	25.22
	.99	25.95	22.18	20.29	18.93	17.97	17.20	16.54	15.94	15.35	26.07	27.63
7	.90	21.23	19.93	18.75	17.98	17.28	16.69	16.16	15.69	15.24	21.79	22.81
	.95	23.50	21.30	19.83	18.91	18.10	17.43	16.83	16.28	15.79	23.77	25.02
	.975	25.23	22.54	20.85	19.68	18.79	18.03	17.38	16.79	16.31	25.46	26.92
	.99	28.01	24.07	21.89	20.68	19.68	18.81	18.10	17.49	16.96	28.02	29.57
8	.90	22.92	21.56	20.43	19.58	18.84	18.21	17.69	17.19	16.70	23.53	24.55
	.95	25.22	23.03	21.48	20.46	19.66	18.97	18.37	17.80	17.30	25.51	26.83
	.975	27.21	24.20	22.41	21.29	20.39	19.63	18.98	18.34	17.78	27.32	28.98
	.99	29.60	25.66	23.44	22.22	21.22	20.40	19.66	19.03	18.46	29.60	31.32
9	.90	24.75	23.15	21.98	21.12	20.37	19.72	19.13	18.58	18.09	25.19	26.40
	.95	27.08	24.55	23.16	22.08	21.22	20.49	19.90	19.29	18.79	27.28	28.78
	.975	29.13	25.92	24.14	22.97	21.98	21.28	20.59	19.98	19.39	29.20	30.82
	.99	31.66	27.42	25.13	24.01	23.06	22.18	21.35	20.63	19.94	31.72	33.32
10	.90	26.13	24.70	23.48	22.57	21.83	21.16	20.57	20.03	19.55	26.66	27.79
	.95	28.49	26.17	24.59	23.59	22.71	21.93	21.34	20.74	20.17	28.75	30.16
	.975	30.67	27.52	25.69	24.47	23.45	22.71	21.95	21.34	20.79	30.84	32.46
	.99	33.62	29.14	26.90	25.58	24.44	23.49	22.75	22.09	21.47	33.86	35.47

Notes: 1. The test UDmax is defined as $\max_{1 \le k \le 5} \sup_{(\lambda_1,\ldots,\lambda_k) \in \Lambda_\epsilon} F(\lambda_1,\ldots,\lambda_k;q)$ multiplied by q. 2. The test WDmax is given in (9) multiplied by q, and M is chosen to be 5.

B Estimated surface response functions for the 5% critical values

$Crt(T, q, m, b, M)$	Intercept	T^{-1}	T^{-2}	T^{-3}	T^{-4}	T^{-5}	T^{-6}
$Crt(T, 0.05, 1, 0, o)$	-3.33	-16.88	798.01	-30818.40	460634.58	-2279397.87	
$Crt(T, 0.05, 2, 0, o)$	-3.75	-10.25	80.17	-13337.52	302551.21	-1848305.35	
$Crt(T, 0.05, 3, 0, o)$	-4.10	-12.16	-321.05	7197.98	-40759.64		
$Crt(T, 0.05, 1, 1, c)$	-4.62	-13.05	-1399.49	76213.27	-1939275.51	23030362.35	-99593635.82
$Crt(T, 0.05, 2, 1, c)$	-4.97	-28.28	112.01	-3338.30	48647.86		
$Crt(T, 0.05, 3, 1, c)$	-5.30	-40.62	1759.22	-94294.31	2287166.84	-25326822.40	106995759.84
$Crt(T, 0.05, 1, 2, c)$	-5.21	-279.74	35643.95	-1963265.34	49133408.78	-564440298.77	2411884754.22
$Crt(T, 0.05, 2, 2, c)$	-5.53	-287.06	36360.62	-2001591.63	50089808.47	-575604522.50	2460441619.22
$Crt(T, 0.05, 3, 2, c)$	-5.88	-272.48	34788.63	-1938696.30	48860977.69	-564638501.27	2425723155.38
$Crt(T, 0.05, 1, 1, cs)$	-4.96	-20.19	-64.18	-1901.05	45903.20		
$Crt(T, 0.05, 2, 1, cs)$	-5.55	-29.61	205.17	-7483.02	84068.24		
$Crt(T, 0.05, 3, 1, cs)$	-6.09	-13.81	-2439.38	125430.43	-2972990.17	32209309.76	-126768011.40
$Crt(T, 0.05, 1, 2, cs)$	-5.94	-207.45	26499.94	-1492615.06	37796851.87	-437913647.90	1884665034.50
$Crt(T, 0.05, 2, 2, cs)$	-6.90	-57.52	5352.61	-403380.76	12234074.42	-164125034.32	802737634.37
$Crt(T, 0.05, 3, 2, cs)$	-7.67	65.2	-18638.41	1.263981E+6	-4.139881E+7	6.391186E+8	-3.757791E+9

Table 13: Estimated surface response functions, order of polynomial selected by AIC

C Robustness check: parametrization under the alternative

		regression model : o, $T=$								regression model : o, $T=$					
DGP	μ	15	20	30	50	100	500	DGP	β	15	20	30	50	100	500
c	$[-4;-3]$	0.2	0.3	0.55	0.72	0.93	1	cs	$[-4;-3]$	0.22	0.32	0.54	0.73	0.92	1
	$]-3;-2]$	0.2	0.29	0.55	0.74	0.92	1		$]-3;-2]$	0.19	0.32	0.54	0.74	0.92	1
	$]-2;-1]$	0.18	0.32	0.54	0.74	0.91	1		$]-2;-1]$	0.2	0.31	0.54	0.74	0.91	1
	$]-1;0]$	0.18	0.32	0.51	0.76	0.92	1		$]-1;0]$	0.21	0.31	0.53	0.74	0.93	1
	$]0;1]$	0.2	0.32	0.55	0.75	0.9	1		$]0;1]$	0.19	0.3	0.54	0.74	0.92	1
	$]1;2]$	0.21	0.3	0.51	0.77	0.92	1		$]1;2]$	0.17	0.31	0.53	0.77	0.92	1
	$]2;3]$	0.2	0.31	0.54	0.76	0.92	1		$]2;3]$	0.22	0.3	0.55	0.76	0.92	1
	$]3;4]$	0.21	0.32	0.54	0.76	0.92	1		$]3;4]$	0.19	0.3	0.53	0.76	0.91	1
		regression model : c, $T=$								regression model : c, $T=$					
DGP	μ	15	20	30	50	100	500	DGP	β	15	20	30	50	100	500
c	$[-4;-3]$	0.11	0.18	0.34	0.58	0.84	1	cs	$[-4;-3]$	0.11	0.18	0.34	0.59	0.84	1
	$]-3;-2]$	0.13	0.2	0.34	0.61	0.83	1		$]-3;-2]$	0.12	0.18	0.34	0.59	0.83	1
	$]-2;-1]$	0.12	0.17	0.34	0.59	0.82	1		$]-2;-1]$	0.12	0.18	0.36	0.6	0.83	1
	$]-1;0]$	0.11	0.19	0.31	0.58	0.84	1		$]-1;0]$	0.13	0.2	0.33	0.57	0.84	1
	$]0;1]$	0.12	0.18	0.33	0.58	0.82	1		$]0;1]$	0.11	0.18	0.33	0.58	0.83	1
	$]1;2]$	0.12	0.18	0.34	0.6	0.84	1		$]1;2]$	0.11	0.17	0.33	0.6	0.84	1
	$]2;3]$	0.11	0.18	0.36	0.62	0.84	1		$]2;3]$	0.12	0.16	0.36	0.61	0.84	1
	$]3;4]$	0.13	0.18	0.35	0.58	0.84	1		$]3;4]$	0.12	0.2	0.34	0.6	0.83	1
		regression model : cs, $T=$								regression model : cs, $T=$					
DGP	μ	15	20	30	50	100	500	DGP	β	15	20	30	50	100	500
c	$[-4;-3]$	0.11	0.18	0.34	0.55	0.82	1	cs	$[-4;-3]$	0.12	0.18	0.32	0.57	0.82	1
	$]-3;-2]$	0.12	0.19	0.32	0.59	0.82	1		$]-3;-2]$	0.12	0.17	0.32	0.56	0.82	1
	$]-2;-1]$	0.11	0.17	0.33	0.58	0.8	1		$]-2;-1]$	0.12	0.18	0.33	0.59	0.81	1
	$]-1;0]$	0.11	0.18	0.28	0.58	0.82	1		$]-1;0]$	0.11	0.19	0.32	0.56	0.82	1
	$]0;1]$	0.12	0.18	0.31	0.57	0.81	1		$]0;1]$	0.1	0.18	0.32	0.57	0.8	1
	$]1;2]$	0.12	0.17	0.33	0.58	0.82	1		$]1;2]$	0.1	0.18	0.32	0.59	0.82	1
	$]2;3]$	0.1	0.17	0.35	0.59	0.82	1		$]2;3]$	0.13	0.15	0.34	0.58	0.82	1
	$]3;4]$	0.11	0.18	0.33	0.57	0.82	1		$]3;4]$	0.12	0.17	0.32	0.58	0.81	1

Table 14: Empirical rejection rates of the null of no cointegration at the 5% nominal level when the alternative is true, $b=1$, $m=1$

D Average serial correlation bias

			regression model : o, $T =$						regression model : c, $T =$						regression model : cs, $T =$					
m	DGP	ρ	15	20	30	50	100	500	15	20	30	50	100	500	15	20	30	50	100	500
1	o	$[0, 0.2]$	-0.16	-0.11	-0.08	-0.05	-0.02	0	-0.33	-0.27	-0.2	-0.12	-0.07	-0.01	-0.37	-0.31	-0.23	-0.15	-0.08	-0.02
		$]0.2, 0.4]$	-0.21	-0.15	-0.11	-0.06	-0.03	-0.01	-0.41	-0.35	-0.25	-0.16	-0.08	-0.02	-0.46	-0.4	-0.29	-0.19	-0.1	-0.02
		$]0.4, 0.6]$	-0.29	-0.21	-0.13	-0.08	-0.04	-0.01	-0.54	-0.44	-0.31	-0.2	-0.1	-0.02	-0.59	-0.49	-0.36	-0.23	-0.12	-0.03
		$]0.6, 0.8]$	-0.35	-0.26	-0.18	-0.11	-0.05	-0.01	-0.66	-0.55	-0.39	-0.25	-0.12	-0.02	-0.72	-0.61	-0.45	-0.29	-0.15	-0.03
		$]0.8, 0.9]$	-0.41	-0.32	-0.22	-0.12	-0.06	-0.01	-0.77	-0.65	-0.48	-0.3	-0.15	-0.03	-0.84	-0.71	-0.54	-0.34	-0.18	-0.03
	c	$[0, 0.2]$	0.42	0.53	0.62	0.69	0.76	0.8	-0.21	-0.2	-0.19	-0.16	-0.11	-0.05	-0.28	-0.25	-0.22	-0.17	-0.12	-0.05
		$]0.2, 0.4]$	0.15	0.23	0.32	0.39	0.45	0.47	-0.38	-0.36	-0.31	-0.23	-0.15	-0.04	-0.44	-0.4	-0.33	-0.24	-0.16	-0.04
		$]0.4, 0.6]$	-0.07	0.01	0.1	0.16	0.22	0.25	-0.53	-0.47	-0.36	-0.25	-0.14	-0.03	-0.58	-0.5	-0.39	-0.28	-0.16	-0.03
		$]0.6, 0.8]$	-0.25	-0.16	-0.07	0	0.05	0.09	-0.68	-0.57	-0.43	-0.27	-0.14	-0.03	-0.73	-0.62	-0.47	-0.31	-0.16	-0.03
		$]0.8, 0.9]$	-0.37	-0.28	-0.18	-0.09	-0.03	0.02	-0.79	-0.66	-0.5	-0.31	-0.15	-0.03	-0.85	-0.72	-0.55	-0.35	-0.18	-0.03
	cs	$[0, 0.2]$	0.37	0.47	0.58	0.69	0.78	0.88	-0.12	-0.07	0.09	0.29	0.52	0.81	-0.29	-0.28	-0.25	-0.21	-0.15	-0.1
		$]0.2, 0.4]$	0.15	0.25	0.36	0.48	0.57	0.66	-0.33	-0.28	-0.13	0.09	0.3	0.59	-0.45	-0.44	-0.39	-0.33	-0.28	-0.2
		$]0.4, 0.6]$	-0.06	0.05	0.17	0.26	0.36	0.46	-0.52	-0.45	-0.29	-0.11	0.1	0.38	-0.63	-0.57	-0.5	-0.43	-0.35	-0.25
		$]0.6, 0.8]$	-0.24	-0.14	-0.03	0.07	0.16	0.25	-0.7	-0.6	-0.45	-0.26	-0.06	0.18	-0.78	-0.71	-0.62	-0.5	-0.38	-0.24
		$]0.8, 0.9]$	-0.37	-0.29	-0.16	-0.07	0.01	0.11	-0.81	-0.73	-0.56	-0.37	-0.17	0.05	-0.91	-0.81	-0.69	-0.54	-0.37	-0.19
2	o	$[0, 0.2]$	-0.22	-0.16	-0.11	-0.07	-0.04	-0.01	-0.38	-0.32	-0.23	-0.15	-0.08	-0.02	-0.45	-0.39	-0.29	-0.2	-0.11	-0.02
		$]0.2, 0.4]$	-0.3	-0.23	-0.15	-0.09	-0.05	-0.01	-0.49	-0.43	-0.31	-0.19	-0.1	-0.02	-0.58	-0.5	-0.38	-0.25	-0.14	-0.03
		$]0.4, 0.6]$	-0.39	-0.28	-0.19	-0.11	-0.06	-0.01	-0.64	-0.53	-0.38	-0.24	-0.12	-0.03	-0.73	-0.62	-0.46	-0.31	-0.17	-0.04
		$]0.6, 0.8]$	-0.47	-0.37	-0.25	-0.14	-0.07	-0.01	-0.77	-0.65	-0.47	-0.3	-0.15	-0.03	-0.87	-0.75	-0.57	-0.38	-0.2	-0.04
		$]0.8, 0.9]$	-0.55	-0.43	-0.29	-0.18	-0.08	-0.02	-0.88	-0.76	-0.55	-0.36	-0.18	-0.03	-0.99	-0.88	-0.66	-0.45	-0.24	-0.05
	c	$[0, 0.2]$	0.25	0.37	0.5	0.6	0.7	0.78	-0.28	-0.26	-0.21	-0.17	-0.12	-0.05	-0.38	-0.34	-0.27	-0.2	-0.13	-0.06
		$]0.2, 0.4]$	-0.03	0.08	0.21	0.3	0.38	0.45	-0.45	-0.41	-0.33	-0.25	-0.17	-0.05	-0.55	-0.49	-0.39	-0.28	-0.19	-0.05
		$]0.4, 0.6]$	-0.23	-0.12	-0.01	0.09	0.16	0.22	-0.63	-0.54	-0.41	-0.28	-0.16	-0.04	-0.72	-0.62	-0.48	-0.33	-0.19	-0.04
		$]0.6, 0.8]$	-0.41	-0.29	-0.17	-0.06	0.01	0.07	-0.77	-0.66	-0.5	-0.32	-0.16	-0.03	-0.87	-0.76	-0.58	-0.38	-0.2	-0.04
		$]0.8, 0.9]$	-0.54	-0.41	-0.27	-0.15	-0.06	0.01	-0.89	-0.77	-0.57	-0.37	-0.19	-0.03	-1	-0.88	-0.67	-0.46	-0.24	-0.05
	cs	$[0, 0.2]$	0.21	0.33	0.49	0.62	0.75	0.87	-0.2	-0.12	0.06	0.28	0.53	0.82	-0.41	-0.38	-0.31	-0.24	-0.18	-0.11
		$]0.2, 0.4]$	0	0.13	0.27	0.42	0.55	0.67	-0.4	-0.33	-0.14	0.08	0.32	0.61	-0.57	-0.54	-0.46	-0.38	-0.32	-0.25
		$]0.4, 0.6]$	-0.21	-0.08	0.07	0.22	0.34	0.47	-0.61	-0.51	-0.33	-0.12	0.11	0.4	-0.77	-0.7	-0.61	-0.52	-0.43	-0.34
		$]0.6, 0.8]$	-0.4	-0.27	-0.13	0.02	0.14	0.26	-0.78	-0.7	-0.52	-0.3	-0.08	0.2	-0.92	-0.86	-0.75	-0.63	-0.51	-0.35
		$]0.8, 0.9]$	-0.55	-0.4	-0.26	-0.12	0	0.11	-0.91	-0.82	-0.64	-0.43	-0.2	0.06	-1.06	-0.98	-0.84	-0.69	-0.51	-0.29
3	o	$[0, 0.2]$	-0.27	-0.2	-0.14	-0.09	-0.05	-0.01	-0.43	-0.37	-0.27	-0.18	-0.1	-0.02	-0.51	-0.45	-0.35	-0.24	-0.13	-0.03
		$]0.2, 0.4]$	-0.36	-0.28	-0.19	-0.12	-0.06	-0.01	-0.55	-0.48	-0.35	-0.22	-0.12	-0.02	-0.66	-0.58	-0.44	-0.31	-0.17	-0.04
		$]0.4, 0.6]$	-0.45	-0.36	-0.25	-0.15	-0.07	-0.02	-0.7	-0.6	-0.44	-0.28	-0.14	-0.03	-0.82	-0.71	-0.55	-0.37	-0.2	-0.04
		$]0.6, 0.8]$	-0.58	-0.45	-0.31	-0.19	-0.09	-0.02	-0.86	-0.74	-0.55	-0.35	-0.18	-0.03	-0.99	-0.87	-0.68	-0.46	-0.25	-0.05
		$]0.8, 0.9]$	-0.66	-0.55	-0.38	-0.23	-0.11	-0.02	-0.97	-0.86	-0.64	-0.42	-0.22	-0.04	-1.12	-0.99	-0.78	-0.54	-0.3	-0.06
	c	$[0, 0.2]$	0.1	0.25	0.39	0.54	0.65	0.75	-0.33	-0.3	-0.24	-0.18	-0.12	-0.05	-0.47	-0.42	-0.32	-0.22	-0.15	-0.06
		$]0.2, 0.4]$	-0.15	-0.03	0.11	0.23	0.32	0.41	-0.51	-0.46	-0.36	-0.27	-0.17	-0.05	-0.64	-0.57	-0.44	-0.33	-0.2	-0.05
		$]0.4, 0.6]$	-0.35	-0.24	-0.1	0.02	0.12	0.19	-0.68	-0.6	-0.46	-0.31	-0.17	-0.04	-0.81	-0.72	-0.56	-0.38	-0.22	-0.05
		$]0.6, 0.8]$	-0.54	-0.39	-0.25	-0.12	-0.03	0.06	-0.86	-0.74	-0.56	-0.36	-0.19	-0.03	-1	-0.88	-0.69	-0.47	-0.25	-0.05
		$]0.8, 0.9]$	-0.64	-0.53	-0.36	-0.21	-0.09	0	-0.98	-0.88	-0.66	-0.42	-0.22	-0.04	-1.11	-1	-0.79	-0.54	-0.3	-0.06
	cs	$[0, 0.2]$	0.07	0.22	0.39	0.56	0.72	0.86	-0.27	-0.19	0.01	0.24	0.51	0.8	-0.49	-0.46	-0.37	-0.27	-0.19	-0.12
		$]0.2, 0.4]$	-0.15	0.01	0.19	0.37	0.52	0.66	-0.48	-0.38	-0.19	0.04	0.29	0.61	-0.67	-0.62	-0.52	-0.43	-0.34	-0.27
		$]0.4, 0.6]$	-0.33	-0.18	-0.02	0.17	0.31	0.46	-0.67	-0.57	-0.39	-0.16	0.09	0.4	-0.86	-0.79	-0.69	-0.57	-0.48	-0.38
		$]0.6, 0.8]$	-0.53	-0.37	-0.2	-0.04	0.11	0.26	-0.86	-0.77	-0.58	-0.36	-0.11	0.2	-1.05	-0.97	-0.85	-0.7	-0.58	-0.42
		$]0.8, 0.9]$	-0.68	-0.53	-0.34	-0.18	-0.03	0.11	-1	-0.91	-0.7	-0.49	-0.24	0.05	-1.18	-1.1	-0.95	-0.78	-0.61	-0.38

Table 15: Average serial correlation bias ($\hat{\rho}$ - ρ) by ρ, $b = 1$