THEMA Working Paper n°2021-03
CY Cergy Paris Université, France

Economic distributions, primitive distributions, and demand recovery in Monopolistic Competition

Simon P. Anderson, André de Palma

January 2021

Economic distributions, primitive distributions, and demand recovery in Monopolistic Competition

Simon P. Anderson and André de Palma*

Revised January 2021

Abstract

We link fundamental technological and taste distributions to endogenous economic distributions of prices and firm size (output, profit) generated under monopolistic competition with heterogeneous productivities as per recent Trade and IO models. We new derive properties for monopoly pricing and equivalence properties on demand curvature, profit functions, and marginal revenue, which we use to ensure distributions of cost, price, output, and profit can be matched under monopolistic competition. Demand and one distribution determine the rest. We provide constructive proofs to recover demand and all distributions from just two (e.g., price and cost distributions uncover demand form), and derive consistency conditions that distribution pairs must satisfy. We then extend to include mark-up distributions.

JEL Classification: L13, F12

Keywords: Primitive and economic distributions, monopoly, monopolistic competition, pass-through and demand recovery, mark-up, price and profit dispersion

*Department of Economics, University of Virginia, USA, sa9w@virginia.edu; THEMA, Cergy Paris Université, FRANCE. andre.de-palma@cyu.fr. The first author gratefully acknowledges research funding from the NSF. We thank Michal Fabinger, Isabelle Mejean, Julien Monardo, Maxim Engers, Farid Toubal, James Harrigan, and Ariell Resheff for valuable comments, and seminar participants at Hebrew University, Melbourne University, Stockholm University, KU Leuven, Laval, Vrij Universiteit Amsterdam, CUHK, Hitotsubashi, University of Tokyo, Monash, ANU, and Paris Dauphine. This paper draws on some material from CEPR DP 10748.

1 Introduction

Distributions of economic variables have attracted the interest of economists at least since Pareto (1896). In industrial organization, firm size distributions (measured by output, sales, or profit) have been analyzed. Other studies have looked at the distribution of prices within an industry and across industries (Kaplan and Menzio, 2015, and Hitsch, Hortacsu, and Lin, 2017); recent research has focussed on mark-up distributions (De Loecker, Eeckhout, and Unger, 2020). Firm sizes (profitability or sales, say) within industries are wildly asymmetric, and frequently involve a long-tail of smaller firms (e.g., Anderson, 2006, Gabaix, 2016). Particular distributions – mainly the Pareto and log-normal – seem to fit the data well. Much work in international trade looks at the size distribution of firms (e.g., Melitz, 2003, Eaton, Kortum, and Kramarz, 2011, Head, Mayer, and Thoenig, 2014).

We show how the distributions of these "economic" variables (prices, output, profit, and mark-ups) are jointly determined by the fundamental underlying distributions of tastes and technologies, and we determine the links between the various distributions. We link the economic ones to each other and to the primitive (cost) distribution and consumer tastes. More surprisingly, the primitives can be uncovered from the observed economic distributions.

The idea of linking demand to distributions is analyzed in two recent papers which complement the present study. Mrázová, Neary, and Parenti (2020) study the relations between equilibrium distributions of sales and relative mark-ups and the (primitive) productivity distribution for a cleverly parametrized demand form. They are mainly interested in when distributions are in the same ("self-reflecting") class (e.g., when both productivity and sales are log-normal or Pareto). They provide some empirical analysis of log-normal and Pareto distributions. In parallel, Anderson and de Palma (2020) start with the special (and central) case of the CES and the "Pareto circle" that all relevant distributions are Pareto if one is, and they extend to find the distribution classes associated to other key distribution forms.[1] They extend the analysis to generalized logit-CES demand forms, which are not covered by the MNP parameterization. These papers provide useful schematic links, but they do not address *general* demand functions nor how *arbitrary* distribution shapes can be used to recover demand nor which distribution combinations are consistent with the monopolistic competition model.

We start by deploying a general monopolistic competition model with a continuum of firms (see Thisse and Ushchev, 2018, for a review of this literature).[2] We focus on productivity (cost) differences across firms, in line with much recent work in Trade, although theoretical IO models have almost exclusively looked at symmetric settings. The Trade literature is mainly based on CES demand while we take general demand functions as our starting point. We show how the demand function delivers a mark-up function, and then we show our key converse result that the mark-up (or "pass-through" function of Weyl and Fabinger, 2013) determines the form of the demand function. We next engage these results and analogous ones with constructive proofs to show how, and under which conditions, cost and price distributions suffice to determine the shape of the economic profit and output distributions and the demand form. Along broader lines, we determine when and how any two elements (e.g., two distributions) suffice to deliver all the missing pieces.

We contribute several results to the theory of pass-through and monopoly more generally, and they apply to the monopolistic competition model. The first is that a continuously differentiable and strictly monotonic mark-up with strictly positive pass-through implies a strictly (-1)-concave demand (henceforth "the demand property"). Conversely, we show, by integrating back, how to construct the demand function from the mark-up function. Second, the counterpart to Hotelling's Lemma for monopolistic competition shows that profit is strictly convex in

[1]They also introduce heterogeneous product qualities to break the Pareto circle for the CES.

[2]Ironically, Chamberlin (1933) is best remembered for his symmetric monopolistic competition analysis. Yet he went to great length to point out that he believed asymmetry to be the norm, and that symmetry was a very restrictive assumption.

unit cost and its derivative is the inverse marginal revenue under the demand property. The third relation is between optimal price and optimized profit: this profit is strictly decreasing in this price under the demand property. We show the equivalence of the following properties: strictly (-1)-concave demand; strictly increasing mark-up function; strictly decreasing marginal revenue.

We use these results to uncover consistency conditions on distributions to be compatible with monopolistic competition under the demand property. First, we determine how the properties of demand and a single distribution (such as cost, price, output, or profit) feed through into the properties of all the other distributions. In particular, under the demand property, a twice continuously differentiable (henceforth $\mathcal{C}^2$) and strictly monotonic distribution implies that the other distributions are also $\mathcal{C}^2$ and strictly monotonic. Our next contribution is perhaps more surprising because it works in the other direction. In particular, any pair of $\mathcal{C}^2$ and strictly monotonic distributions (sometimes under some restrictions which we uncover) suffices to determine the demand form as well as the other distributions. We reconstruct demand from the distribution pairs. For example, if both the profit distribution and the price distribution are $\mathcal{C}^2$ and strictly monotonic then there exists a strictly (-1)-concave demand function which renders them consistent with the monopolistic competition model. We deliver the demand function, which is unique up to a positive multiplicative shift. That is, net demand is tied down. We also consider mark-up distributions in the penultimate section.

The next Section provides the back-drop and road-map to the first part of the paper.

2 The Model: overview

A continuum of firms produce substitute goods. Each has constant unit production costs, c, but these differ across firms, with support $[\underline{c}, \bar{c}]$, where $\underline{c} \geq 0$. With a continuum of firms, each firm effectively faces a monopoly problem where the price choice is independent of the actions of rivals. We allow for a general demand formulation.

Assumption 1 *Suppose that demand for a firm charging p*

$$y = h(p), \tag{1}$$

is *a positive, strictly decreasing, strictly* (-1)*-concave, and* $\mathcal{C}^2$ *function on its support* $[\underline{c}, \infty)$, *with* $h(0) > \bar{c}$.

This is equivalent to $\frac{1}{h(.)}$ strictly convex, and is a minimal condition ensuring profit strict quasi-concavity: see Caplin and Nalebuff (1991) and Anderson, de Palma, and Thisse (1992, p.164) for more on ρ-concave functions and see Weyl and Fabinger (2013) for the properties of pass-through as a function of demand curvature. We suppress the impact of other firms' actions on demand, which would be expressed as aggregate variables in the individual demand function. Under monopolistic competition with a continuum of firms, each firm's individual action has no measurable impact on the aggregate variables (for example, the "price index" in the CES model, or the Logit denominator). Because we look at the cross-section relation between equilibrium distributions, the actions of other firms are held constant across the comparison, and therefore are not changing.

Our focus is on cost and the endogenous economic variables: price, profit, and output. We show that A1 implies strictly monotonic and continuously differentiable relations between any pair of these variables. Our first set of results is that knowing any one of these relations suffices to determine all the others. In Section 3 we show the properties that A1 implies for equilibrium mark-ups and equilibrium profit functions. We also show how these properties imply A1. We thus determine a set of equivalence results for the demand system. The properties extend the theory of monopolistic competition (and of monopoly). One key relation involves strict monotonicity: A1 implies that equilibrium profits strictly increase with equilibrium output and strictly decrease with equilibrium prices, etc. We also show that A1 implies that each of the variables we track can be written as a $\mathcal{C}^1$ function of any of the others. This characterization enables us to back out the equilibrium distribution relations.

These equilibrium relations are written in the following form. Let z denote the fraction of firms with profit below some level π. Given the strict monotonic relations between variables, the same set of firms have output *below* some corresponding level y, and these same firms have costs and prices *above* corresponding levels c and p. Thus the firms with costs strictly higher than some value c are the same ones that have prices strictly higher than p, an output strictly below y and a profit strictly below π, where the specific values satisfy $\pi = (p - c)\,h\,(p)$, where $h\,(p) = y$ and the mark-up $(p - c)$ satisfies the first-order condition (4) below. Writing $F_C\,(c)$ as the cumulative cost distribution function, etc., gives the following key ranking property:[3]

$$1 - F_C\,(c) = 1 - F_P\,(p) = F_Y\,(y) = F_\Pi\,(\pi) = z. \tag{2}$$

In Section 4, our first results are to show how the primitives (demand and cost distribution) feed through to the endogenous economic distributions and variables. When A1 (or an equivalent statement) holds, we show that if any one distribution is strictly increasing and $\mathcal{C}^2$, then each other distribution is strictly increasing and $\mathcal{C}^2$. We then go in the other direction and use the equivalence results from Section 3 to determine the restrictions on equilibrium distributions implied by the monopolistic competition model under A1, and to determine the demand shape from any pair of distributions (F_C, F_P, F_Y, and F_Π). If two distributions are both strictly increasing and $\mathcal{C}^2$, then we find conditions upon them for them to be rationalized by a demand function satisfying A1. When the conditions are met, each other distribution is strictly increasing and $\mathcal{C}^2$. Our proofs are constructive: we derive the relations between distributions and primitives.

We shall primarily work with distributions that are strictly increasing and $\mathcal{C}^2$. Section 4.2

[3] Mark-ups will be treated separately because they are not necessarily monotone increasing in c.

amends the analysis to when this is not the case.

3 Demands, mark-ups, and profits

Here we demonstrate the equivalence between properties of these three variables under A1. In Section 5, we shall determine how to recover demand (and other distributions) from any pair of distributions, and what restrictions on distributions (if any) must be obeyed in order to satisfy A1 (that demand is strictly (-1)-concave and $\mathcal{C}^2$). To do so, we use the results of this Section. The next sub-section and Proposition 1 delivers the properties implied by A1 on the economic variables (output, profit, and mark-ups/prices); the two sub-sections after deliver the converse relations.

In what follows, whenever we use a "prime" symbol on a variable, we shall understand the function to be continuously differentiable ($\mathcal{C}^1$). Hence a strictly positive or negative sign for a derivative denotes that the variable is $\mathcal{C}^1$ and strictly monotonic.

3.1 Demand to mark-ups and profits

The profit for a firm with per unit cost c is $\pi = (p-c)\,h(p) = mh(m+c)$, where $m = p - c$ is its mark-up. We will make extensive use of the following result on demand:

Lemma 1 *A $\mathcal{C}^2$ function $h(u)$ is strictly (-1)-concave, i.e. iff $[h(u)/h'(u)]' > -1$.*

This result follows because strictly (-1)-concavity is by definition that $1/h(u)$ is strictly convex. (Equivalently, $h(u)\,h''(u) - 2\,[h'(u)]^2 < 0$.) With a continuum of firms (monopolistic competition), the equilibrium mark-up satisfies the first-order condition

$$m = -\frac{h(m+c)}{h'(m+c)}, \qquad c \in [\underline{c}, \bar{c}]\,. \tag{3}$$

The solution to (3), denoted $\mu(c)$, is uniquely determined (and strictly positive) when the RHS of (3) has slope less than one, as is implied by A1 and Lemma 1. It constitutes a maximum to profit because profit is rising for all $m < \mu(c)$ and falling for all $m < \mu(c)$.

Let the value of equilibrium demand be $h^*(c) = h(\mu(c) + c)$, and $h^*(c)$ is a strictly decreasing and $\mathcal{C}^1$ function under A1 (as shown in the proof of Proposition 1(iii) below). Call the equilibrium profit $\pi^*(c) = \mu(c) h^*(c)$. Applying the envelope theorem to the profit function $\pi = (p - c) h(p)$ implies that $\pi^{*\prime}(c) = -h^*(c) < 0$. Because $h^*(c)$ is $\mathcal{C}^1$ and strictly decreasing, $\pi^*(c)$ is $\mathcal{C}^2$ and is strictly convex.

Lemma 2 *Under A1, the equilibrium profit function,* $\pi^*(c) = \mu(c) h^*(c)$*, is strictly convex and* $\mathcal{C}^2$ *with* $\pi^{*\prime}(c) = -h^*(c) < 0$.

This is the monopolistic competition (and monopoly) counterpart to Hotelling's Lemma for competitive firms (that the derivative of profit with respect to price is minus demand). While the result is straightforward, we do not know any statement of it for monopolistic competition.

Because it specifies optimal output as a function of marginal cost, $h^*(c)$ is the inverse marginal revenue curve. We substantiate in the next Proposition its slope and continuity.

Proposition 1 *Under A1: (i) the equilibrium mark-up,* $\mu(c) > 0$ *is the unique* $\mathcal{C}^1$ *solution to (3), with* $\mu'(c) > -1$*; (ii) the equilibrium price,* $p(c) > c$*, is a* $\mathcal{C}^1$ *function with* $p'(c) > 0$*; (iii) the associated equilibrium demand,* $h^*(c) \equiv h(\mu(c) + c)$*, is* $\mathcal{C}^1$ *with* $h^{*\prime}(c) < 0$.

Proof. Applying the implicit function theorem to (3) shows that

$$\mu'(c) = \frac{-\left[h(\mu + c)/h'(\mu + c)\right]'}{1 + \left[h(\mu + c)/h'(\mu + c)\right]'} > -1, \tag{4}$$

where the denominator is strictly positive under A1 by Lemma 1. Continuity of $\mu'(c)$ implies equilibrium price is $\mathcal{C}^1$. Because $\mu'(c) > -1$, equilibrium price is strictly increasing in cost.

Because $h^*(c) = h(\mu(c) + c)$ denotes the value of $h(.)$ under the profit-maximizing mark-up, $h^*(c)$ is strictly increasing:

$$dh^*(c)/dc = (\mu'(c) + 1) h'(\mu(c) + c) < 0, \tag{5}$$

given that $\mu'(c) > -1$. ■

Notice that the property $\mu'(c) > -1$ is just the standard property that price never goes down as costs increase. Because $h^*(c)$ is a continuous and strictly decreasing function, marginal revenue (which is $h^{*-1}(c)$) is also continuous and strictly decreasing.

The key implication of Proposition 1 and Lemma 2 is that we can rely on strictly monotonic relations between variables, which is crucial in twinning strictly monotonic distributions (as we do below). The demand assumption A1 (that demand is strictly (-1)-concave and $\mathcal{C}^2$) implies various properties, which we can group under two headings. First, the equilibrium price increases with cost. Second, the equilibrium profit, $\pi^*(c)$, is strictly convex and its derivative is the equilibrium demand, which in turn is strictly decreasing in cost. Recognizing that equilibrium demand is simply inverse marginal revenue, then we know that marginal revenue is strictly decreasing. In the next two sub-sections, we show equivalence of these statements by showing first that the mark-up property implies A1, and second that the marginal revenue property implies A1.

3.2 From mark-ups to demand (and demand reconstruction)

Here we show how mark-up $\mu(c)$ (with $\mu'(c) > -1$) can be used to find the associated equilibrium demand and demand function, $h(p)$. Equivalently, we can start with a $\mathcal{C}^1$ and strictly increasing relation between equilibrium price and cost, $p(c)$. Our converse result to Proposition 1 indicates how the mark-up function $\mu(c)$ determines the form of inverse marginal revenue, $h^*(c)$, and hence determines the form of $h(p)$.

Proposition 2 *Consider any positive mark-up function* $\mu(c) > 0$ *for* $c \in [\underline{c}, \bar{c}]$ *with* $\mu'(c) > -1$. *(Equivalently, consider an equilibrium pricing function* $p(c) > c$ *with* $p'(c) > 0$.*) Then there exists an equilibrium output function* $h^*(c)$ *with* $h^{*\prime}(c) < 0$, $c \in [\underline{c}, \bar{c}]$ *and given by (7), which is unique up to a positive multiplicative factor. The associated primitive demand function,* $h(p)$ *(unique up to a positive multiplicative factor) is given by (8) and satisfies A1 on its support* $[\mu(\underline{c}) + \underline{c}, \mu(\bar{c}) + \bar{c}]$.

Proof. First note from (3) and (5) that

$$\frac{dh^*(c)/dc}{h^*(c)} = \frac{(\mu'(c)+1)\,h'(\mu(c)+c)}{h(\mu(c)+c)} = -\frac{\mu'(c)+1}{\mu(c)} \equiv g(c) < 0, \tag{6}$$

because $\mu'(c) > -1$ by assumption. Thus $[\ln h^*(c)]' = g(c)$, and so $\ln\left(\frac{h^*(c)}{h^*(\underline{c})}\right) = \int_{\underline{c}}^{c} g(v)\,dv$, or

$$h^*(c) = h^*(\underline{c}) \exp\left(\int_{\underline{c}}^{c} g(v)\,dv\right), \quad c \geq \underline{c}, \tag{7}$$

which determines $h^*(c)$ up to the positive factor $h^*(\underline{c})$; it is strictly decreasing because $g(c) < 0$.

We can now use the output function, $h^*(c)$ (which is inverse marginal revenue), to back out the demand function, $h(m+c)$, via the following steps. First, define $u \equiv p(c) = \mu(c) + c$, which is strictly increasing because $\mu'(c) + 1 > 0$, so the inverse function $p^{-1}(\cdot)$ is strictly increasing. Now, $h(u) = h^*(p^{-1}(u))$ and thus the function $h(\cdot)$ is recovered on the support $u \in [\mu(\underline{c}) + \underline{c}, \mu(\bar{c}) + \bar{c}]$ (cf. Proposition 1). Using (7) with $h(u) = h^*(p^{-1}(u))$,

$$h(u) = h^*(\underline{c}) \exp\left(\int_{\underline{c}}^{p^{-1}(u)} g(v)\,dv\right), \tag{8}$$

and so we recover the pricing first-order condition (3):

$$\frac{h(u)}{h'(u)} = \frac{1}{g(p^{-1}(u))\,[p^{-1}(u)]'} = \frac{p'(c)}{-\frac{\mu'(c)+1}{\mu(c)}} = -\mu(c) < 0, \tag{9}$$

where first and second step follow from (6) with $u = p(c)$ and the last step follows because $p'(c) = \mu'(c) + 1$. Thus, since $h(u) = h(\mu(c) + c) = h^*(c)$. So,

$$\left[\frac{h(u)}{h'(u)}\right]' = -\frac{\mu'(c)}{\mu'(c) + 1} > -1, \tag{10}$$

and so $h(u)$ is strictly (-1)-concave (using Lemma 1). Note that $h(.)$ is $\mathcal{C}^2$ because $\mu(.)$ was assumed differentiable. ■

Recalling that $\mu(c) = p(c) - c$ for $c \in [\underline{c}, \bar{c}]$, the restriction used in the Lemma ($\mu'(c) > -1$) is that $p'(c) > 0$ so that *any* arbitrary (differentiable) increasing price function of costs can be associated to a unique demand function that could generate it (up to the multiplicative factor).

The reason that demand is only determined up to a positive factor is simply that multiplying demand by a positive constant does not change the optimal mark-up (when marginal costs are constant, as here). The mark-up function can only determine the demand shape, but not its scale. The steps in the proof are readily confirmed for the $\rho-$linear example given at the end of this Section.

The results so far indicate that knowing either of $\mu(c)$ or $h(.)$ suffices to determine the other and $h^*(c)$ (up to constants in the first case). Furthermore, knowing $h^*(c)$ determines $\pi^*(c)$ too, by integrating back (see Lemma 2). This set of results constitutes a strong characterization result for monopoly pass-through (see Weyl and Fabinger, 2013, for the state of the art, which deeply engages ρ-concave functions). The next sub-section takes the characterization further by working back from the condition that marginal revenue is strictly decreasing.

Notice that the function $h(\cdot)$ is tied down only on the support corresponding to the domain on which we have information about the equilibrium mark-up value in the market. Outside that support, we know only that $h(\cdot)$ must be consistent with the maximizer $\mu(c)$, which restricts the shape of $h(\cdot)$ to be not "too" convex.

3.3 Strictly decreasing MR implies strictly (-1)-concave demand

First note that $h^*(c)$ is strictly decreasing if and only if marginal revenue, $h^{*-1}(y) \equiv MR(y) > 0$, is strictly decreasing, with both $\mathcal{C}^1$. This is because these are inverse functions. Next, integrating $MR(y)$ yields total revenue, $TR(y)$, which is therefore $\mathcal{C}^2$ (and it is strictly quasi-concave, and monotone increasing for $MR(y) > 0$). Inverse demand, $p(y)$, is then $TR(y)/y$, and this is a $\mathcal{C}^2$ function. Inverting it yields $h(p)$ as a $\mathcal{C}^2$ function. It remains to show that $h(p)$ is strictly (-1)-concave. The next result concludes the issue.

Proposition 3 *If inverse marginal revenue, $h^*(c) > 0$, is strictly decreasing and $\mathcal{C}^1$, then demand, $h(p)$ satisfies A1, and can be recovered from $h^*(c)$ up to a constant.*

Proof. First note that $h(p)$ is strictly (-1)-concave if and only if $h''h - 2(h')^2 < 0$. Write the inverse demand as $p(y)$ so that $h'(p) = \frac{1}{p'(y)}$ and $h''(p) = -\frac{p''(y)}{(p'(y))^3}$. Then the strict (-1)-concavity condition we are to show becomes:

$$p''y + 2p' < 0. \tag{11}$$

Now we want to find $p(y)$, using the steps explained before the Proposition. Let $MR(y)$ denote $h^{*-1}(c)$, i.e., marginal revenue. So then Total Revenue, $TR(y)$ is the integral of $MR(y)$ and equilibrium inverse demand, $p(y)$, is

$$p(y) = \frac{TR(y)}{y} = \frac{\int_0^y MR(u)\,du}{y} = \frac{\int_0^y c(u)\,du}{y},$$

and its inverse is $h(p)$: notice that the demand $h(p)$ is only determined up to a constant (from the step where $MR(.)$ is integrated). Hence $p'(y) = \frac{yc(y) - \int_0^y c(u)du}{y^2}$ and $p''(y) = \frac{c'(y)y^2 - 2\left(yc(y) - \int_0^y c(u)du\right)}{y^3}$. Using these expressions in (11) gives $MR'(y) = p''(y)y + 2p'(y) < 0$ (where we used $c'(y) < 0$, i.e. marginal revenue slopes down). ■

Intuitively, one can always add a rectangular hyperbola to any inverse demand (the rectangular hyperbola has a zero Marginal Revenue) and get the same Marginal Revenue function.

3.4 Equilibrium prices and profits, and output and profit

Another new characterization result (used in Theorem 6) concerns the properties of equilibrium profit when written as a function of equilibrium price, $\tilde{p}$. Call this relationship $\tilde{\pi}(\tilde{p})$. Inserting the mark-up first-order condition (3) into the profit function $\pi(p) = (p-c)\,h(p)$ gives the desired relation as $\tilde{\pi}(\tilde{p}) = -h^2(\tilde{p})/h'(\tilde{p})$. This relation defines a strictly decreasing and continuous function if and only if $h^2(p)/h'(p)$ is strictly increasing. But this is the condition for $1/h(p)$ to be strictly convex: equivalently, this is A1.

Lemma 3 *Equilibrium profit as a function of equilibrium price, $\tilde{\pi}(\tilde{p}) = -h^2(\tilde{p})/h'(\tilde{p})$, is a strictly decreasing and continuous function if and only if A1 holds.*

The intuition for the relation between equilibrium profit and equilibrium price is as follows. Suppose that some price is optimally chosen on the demand curve $h(p)$. Then it must be that the price satisfies the condition that marginal revenue equals marginal cost, with marginal revenue downward-sloping locally: as shown above, marginal revenue is strictly decreasing if and only if demand satisfies A1. The optimal profit is continuously decreasing with the optimal price because both are driven in a continuous way by costs: higher costs entail both higher prices and lower profits, as Proposition 1 and Lemma 2 attest.

Analogously (for completeness), we can describe the relation between equilibrium output and equilibrium profit. Denote the former $\hat{y}$ and denote the relation $\hat{\pi}(\hat{y})$. This is quickest to derive using the inverse demand, $p(y)$, for which the standard $MR = MC$ condition writes as $p'(\hat{y})\,\hat{y} + p(\hat{y}) - c = 0$, and hence $\hat{\pi}(\hat{y}) = -p'(\hat{y})\,\hat{y}^2$. Then: $\hat{\pi}'(\hat{y}) = -\hat{y}\left(2p'(\hat{y}) + p''(\hat{y})\,\hat{y}\right)$. Noting that $p'(\hat{y}) = 1/h'(\tilde{p})$ and $p''(\hat{y}) = -h''(\tilde{p})/(h'(\tilde{p}))^3$, this gives $\hat{\pi}'(\hat{y}) = -\frac{h(\tilde{p})}{(h'(\tilde{p}))^3}\left\{2(h'(\tilde{p}))^2 - h''(\tilde{p})\,h(\tilde{p})\right\}$, which is strictly negative if and only if A1 holds.

Lemma 4 *Equilibrium profit as a function of equilibrium output, $\hat{\pi}(\hat{y})$, is a strictly increasing and continuous function if and only if A1 holds.*

The intuition again comes from thinking about higher costs delivering both lower output and lower profit, so that these variables move together.

3.5 Decreasing or increasing mark-ups

Some characterization results rely on a delineation of the degree of curvature of demand.

Corollary 1 *Under A1, if demand is strictly log-concave (resp. strictly log-convex), higher cost firms have lower (resp. higher) equilibrium markups $\mu'(c) < 0$, (resp. $\mu'(c) > 0$). Equivalently, $p'(c) \in (0,1)$ for strictly log-concave demand, and $p'(c) > 1$ for strictly log-convex demand. Conversely, $h(p)$ is strictly log-convex if $\mu'(c) > 0$ and strictly log-concave if $\mu'(c) < 0$.*

Proof. First, the numerator of (4), $-[h(\mu+c)/h'(\mu+c)]'$ is (weakly) positive for h log-convex and (weakly) negative for h log-concave. The last result follows from (9). ■

In the log-concave case, low-cost firms use their advantage in both mark-up and output dimensions. Under log-convexity, low-cost firms exploit the opportunity to capitalize on much larger demand by setting small mark-ups. In both cases though, as per Proposition 1, profits are higher with lower costs. For $h(\cdot)$ strictly log-concave, $\mu'(c) < 0$, so firms with higher costs have lower mark-ups in the cross-section of firm types (price pass-through is less than 100%). They also have lower equilibrium outputs. The only demand function with constant (absolute) mark-up is the exponential (associated to the Logit), which has $h(\cdot)$ log-linear in p (i.e., $h(p) \propto \exp\left(\frac{-p}{\sigma}\right)$ where σ is a positive constant), and so $\frac{h(m+c)}{h'(m+c)}$ is constant. When $h(\cdot)$ is strictly log-convex, the mark-up *increases* with c, so cost pass-through is greater than 100%, which is a hall-mark of CES demands, which have constant elasticity and hence constant relative mark-up (and $\mu'(c) > 0$) so a 1% cost rise causes a 1% equilibrium price rise.

These properties indicate properties of the price distribution relative to the cost distribution. The price distribution is a *compression* of the cost distribution when h is log-concave, and a *magnification* when h is log-convex, in the simple sense that prices are closer together (or, respectively, farther apart) than costs. The border case (Logit / log-linear demand) has constant mark-ups, so the price distribution mirrors the cost one.[4]

3.6 An illustrative example: $\rho-$linear demand

An important special case is when demand is $\rho-$linear (which means that h^{ρ} is linear):

$$h(p) = (1 + \rho(k - p))^{1/\rho}, \tag{12}$$

where k is a constant satisfying $1 + \rho(k - c) > 0$, and $\rho > -1$ as required for A1. Then

$$\mu(c) = \frac{1 + \rho(k - c)}{1 + \rho} > 0, \tag{13}$$

which is linear in c, with $\mu'(c) > -1$, since $1 + \rho > 0$. Moreover, $\mu'(c) < 0$ if $h(p)$ is log-concave, for $-1 < \rho < 0$, and $\mu'(c) > 0$ if $h(p)$ is log-convex for $\rho > 0$. For $\rho = 1$ demand is linear and the standard property is apparent that mark-ups fall fifty cents on the dollar with cost. Log-linearity is $\rho = 0$ (note that $\lim_{\rho\to 0} h(.) = \exp(k - p))$ and delivers a constant mark-up (see Anderson and de Palma, 2020). A constant elasticity of demand (which the CES model delivers) results from the parameter restriction $\rho = -1/k \in (-1, 0)$ and $h(p) \propto p^{-1/\rho}$.[5]

For $\rho-$linear demands, equilibrium demand is $h^*(c) = \left(\frac{1+\rho(k-c)}{1+\rho}\right)^{1/\rho}$ and then (by (6)), $\frac{dh^*(c)/dc}{h^*(c)} = \frac{-1}{1+\rho(k-c)} = -\frac{\mu'(c)+1}{\mu(c)} < 0$. Notice that $h^*(c)$ is also ρ-linear.

[4] One parameterized example is the flexible CES-Logit demand function model introduced in Anderson and de Palma (2020): $h(p) = kp^{b-1}\exp\left(\frac{(p^b-1)/b}{\mu}\right)$ where $k > 0$ is constant for monopolistic competition and $\mu > 0$ is a measure of product differentiation. The CES corresponds to the limit $b \to 0^+$, while the Logit corresponds to the limit $b \to 1^-$. From the f.o.c. $\left(1 - \frac{c}{p}\right)p^b + (1-b)\mu = 1$, $\mu_i'(c) < 0$ for $b < 1$ (strict log-concavity) and $\mu_i'(c) > 0$ for $b > 1$ (strict log-convexity), while $\frac{d(p_i/c)}{dc} > 0$ iff $b < 0$ $b \neq 0$.

[5] Unfortunately, the symbol ρ is also traditionally used to parameterize the CES function: call the CES version ρ^c ($\in (0,1)$ for substitutes, which we consider), and then the current usage says that $\rho = \rho^c - 1$.

From (12) and (13) we have $\pi^* (c) = \mu (c) h^* (c) = \left(\frac{1+\rho(k-c)}{1+\rho}\right)^{(1+\rho)/\rho}$, which is indeed decreasing in c, and convex for $\rho > -1$, as anticipated (Lemma 2). Finally, the expression for $\tilde{\pi} (\tilde{p})$ is $(1 + \rho (k - \tilde{p}))^{(1+\rho)/\rho}$, decreasing in $\tilde{p}$ for $\rho > -1$, which concurs with Lemma 3.

4 Equilibrium distributions

The relations above in Section 3.5 already determine some links between the equilibrium price distribution, the cost distribution, and the demand. We now show how the other economic distributions are determined and linked in the model. That is, how is one distribution "passed through" to the others via the demand function and the corresponding equilibrium links between variables shown in Section 3.

We make extensive use of the following standard result. We assume throughout the paper that all distributions are absolutely continuous and strictly increasing on their domains.

Lemma 5 *Consider two distributions $F_{X_1} (x_1)$ and $F_{X_2} (x_2)$, which are absolutely continuous and strictly increasing on their respective domains. Let x_1 and x_2 be related by a monotone function $x_1 = \xi (x_2)$. Then $F_{X_2} (x_2) = F_{X_1} (\xi (x_2))$ for $\xi (.)$ increasing, and $F_{X_2} (x_2) = 1 - F_{X_1} (\xi (x_2))$ for $\xi (.)$ decreasing.*

Proof. For $\xi (.)$ increasing, $F_{X_1} (x_1) = \Pr (X_1 < x_1) = \Pr (\xi (X_2) < x_1) = \Pr \left(X_2 < \xi^{-1} (x_1)\right)$ $= F_{X_2} \left(\xi^{-1} (x_1)\right)$. Equivalently, $F_{X_2} (x_2) = F_{X_1} (\xi (x_2))$. For $\xi' (.) < 0$, $F_{X_1} (x_1) = \Pr (\xi (X_2) < x_1) =$ $\Pr \left(X_2 > \xi^{-1} (x_1)\right) = 1 - F_{X_2} \left(\xi^{-1} (x_1)\right)$; equivalently, $F_{X_2} (x_2) = 1 - F_{X_1} (\xi (x_2))$. ■

We can now turn to the equilibrium analysis.

4.1 Monopolistically competitive equilibrium

Figure 1 illustrates. The upper right panel gives the demand curve, from which we determine the corresponding marginal revenue function. The latter is the key to finding the output distribution from the cost distribution. Notice that $h^*(c)$ defined above determines the equilibrium output (for a firm with per unit cost c) as a function of its cost. As earlier noted, the inverse function, $c = h^{*-1}(y)$ therefore traces out the marginal revenue curve.

Figure 1: Construction of marginal revenue, output, and price from demand, and cost distribution to price and output distribution

The distribution of costs is given in the upper left panel. The negative linear relation between the cost and output distributions is given in the lower left panel: as noted in Proposition 1, higher costs are associated to lower outputs. Therefore, the $z\%$ of firms with costs below c are the $z\%$ of firms with output above $y = h^*(c)$. We hence choose some arbitrary level $z \in (0,1)$ (see (2)). This means that all firm types with cost levels above $c(z) = F_C^{-1}(1-z)$ are the firms with outputs and profits below y and π. That is, $1 - F_C(c) = F_Y(h^*(c)) \; (= z)$. The lower right panel therefore connects this relation as the output distribution, $F_Y(y)$. (Notice that in the above argument, only the marginal revenue curve was used from the demand side: as we show later in Section 5, the cost and output distribution determine the marginal revenue, but we then need to integrate up to find demand).

Figure 1 also provides the information to determine the price distribution. The upper right panel gives the vertical distance between the marginal revenue and demand, which is the mark-up (which can be expressed as $\mu(c)$), and is thus the vertical shift between cost and price distributions in the upper left panel. It can be constructed simply from the information in the top two panels[6] by drawing across the demand price associated to a marginal revenue - marginal cost intersection. We could also draw in the mark-up distribution in the upper left panel, but have avoided the extra clutter here. Notice that (as drawn) the price and cost distributions diverge, as is consistent with Proposition 1 for increasing $\mu(c)$, i.e., log-concave demand.

[6] Hence we were able to give results on the relationships between cost and price distributions in Section 3.5 without reference to the output distribution.

In summary, the marginal revenue curve $h^{*-1}(y)$ together with the cost distribution tie down the output distribution (and conversely, by reversing the analysis). The demand function then finds the price distribution, and therefore relates price and output distributions.

One relation that is missing in the Figure is the profit distribution. But, as Proposition 1 shows, analogous arguments apply: $\pi^*(c)$ is a decreasing function and so the relation $1 - F_C(c) = F_\Pi(\pi^*(c))$ $(= z)$ can be used to construct the profit distribution.

The following result establishes the existence of a unique equilibrium for the monopolistic competition model. Consequently, equilibrium distributions are tied down from the primitives on costs and demand.

Theorem 1 *Let there be a continuum of firms, with demand (1) satisfying A1. Let F_C be strictly increasing and $\mathcal{C}^2$ on its support. Then the distributions F_P, F_Y, and F_Π are strictly increasing and $\mathcal{C}^2$ on their supports and given by $F_P(p) = F_C(c(p))$; $F_Y(y) = 1 - F_C(h^{*-1}(y))$; and $F_\Pi(\pi) = 1 - F_C(\pi^{*-1}(\pi))$, where $c(p)$ inverts $p(c)$, $h^{*-1}(y)$ inverts $h^*(c)$, and $\pi^{*-1}(\pi)$ inverts $\pi^*(c)$.*

Proof. Let $p(c)$ denote the equilibrium price for a firm with cost c; from (4) we have $\mu'(c) > -1$ so that $p(c)$ is strictly increasing, and define the inverse relation as $c(p)$, which is strictly increasing. The relation $p(c)$ (and hence its inverse) is determined from $h(.)$ by Proposition 1.

Given F_C, then $F_P(p)$ is determined by $F_P(p) = F_C(c(p))$. Next, consider $F_Y(y)$. By result (5) we know that output $y = h^*(c)$ is a monotonic decreasing function, and so (by Lemma 5) the fraction of firms with output below $y = h^*(c)$ is the fraction of firms with cost above c, so $F_Y(h^*(c)) = 1 - F_C(c)$, or indeed

$$F_Y(y) = \Pr(h^*(C) < y) = \Pr\left(C > h^{*-1}(y)\right) = 1 - F_C\left(h^{*-1}(y)\right). \tag{14}$$

Finally, by Lemma 2 we know that profit $\pi^*(c) = \mu(c)\, h^*(c)$ is a strictly decreasing function, and so the fraction of firms with profit below $\pi^*(c)$ is the fraction of firms with costs above c, so $F_\Pi(\pi^*(c)) = 1 - F_C(c)$. That is

$$F_\Pi(\pi) = \Pr(\Pi < \pi) = \Pr(\pi^*(C) < \pi) = \Pr\left(C > \pi^{*-1}(\pi)\right) = 1 - F_C\left(\pi^{*-1}(\pi)\right). \qquad (15)$$

■

The key relations underlying the twinning of distributions are the strictly monotonic relations between cost, output, profit, and price (see Proposition 1). A specific cost distribution generates specific output, profit, and price distributions. Conversely, as we show in the next result, this output, profit, or price distribution could only have been generated from the initial cost distribution.

Researchers often impose specific demand functions (such as CES, or logit). Here we forge the (potentially testable) empirical links that are imposed by so doing: Theorem 1 shows that when a specific functional form is imposed for h (as is done in most of the literature), then all the relevant distributions can be found from $F_C(c)$. Furthermore, all distributions can be found from any one of them.

Theorem 2 *Let there be a continuum of firms with demand (1) satisfying A1. Consider the set of 3 distributions, $\{F_P, F_Y, F_\Pi\}$. Suppose that demand and any one distribution is known and is strictly increasing and $\mathcal{C}^2$ on its support. Then F_C and all other distributions in the set are explicitly recovered and all are strictly increasing and $\mathcal{C}^2$ functions on their supports.*

Proof. Consider F_P. Then $F_C(c) = F_P(p(c))$, where $p(c)$ is the equilibrium price relation, which we showed in Proposition 1 to be $\mathcal{C}^2$, and both the other distributions are determined from the steps in the proof of Theorem 1 earlier.

Next start with F_Y. Because $h(p)$ is strictly decreasing, then F_P is determined by $F_P(p) = 1 - F_Y(h(p))$ and is $\mathcal{C}^2$. By the argument above, F_C is then determined, and hence so is $F_\Pi(\pi)$.

Finally, start with F_Π. By Proposition 1 we know that profit $\pi^*(c) = \mu(c) h^*(c)$ is a strictly decreasing function. Therefore $F_C(c)$ is recovered from $F_C(c) = 1 - F_\Pi(\pi^*(c))$ and is $\mathcal{C}^2$. From Theorem 1, F_P is recovered, and so is F_Y. ∎

The Theorem says that for any (-1)-concave demand function and any economic distribution, there is only one cost distribution that is consistent with them. The other economic distributions are likewise pinned down.

Later on we turn our attention to pairs of distributions that are not be consistent with the monopolistic competition model. That is, which pairs would indicate violation of the model? Conversely, for admissible pairs, we show how the implicit demand function is uncovered.

4.2 Atoms and gaps

Any gaps in a distribution's support will correspond to gaps in supports of the other distributions; the analysis above applies piecewise on the interior of the supports. Likewise, mass-points in the interior of the support pose no problem because they correspond to mass points in the other distributions. We therefore now discuss relaxing the assumptions made in the last two Theorems. In addition to the two main issues with distributions; gaps in the support, and spikes, we address failures of (-1)-concavity on the demand side.

If $F_C(c)$ has an atom (meaning a positive measure of firms with the same cost), then $F(p)$ and the other two economic distributions have corresponding atoms of the same size. Likewise, if $F_C(c)$ has a gap, then the three economic distributions have corresponding gaps.

If $h(p)$ has a kink down at some price, while $F_C(c)$ remains continuous, then $F_P(p)$ and $F_Y(y)$ have atoms corresponding to the kink (a range of costs are associated to the same price and output) while the profit distribution remains continuous.

If $h(p)$ is not (-1)-concave over some range, the corresponding marginal revenue curve slopes up. As a function of c, equilibrium price jumps down (and equilibrium output jumps up) so that $F_P(p)$ and $F_Y(y)$ have corresponding gaps, while $F_\Pi(\pi)$ does not.

Conversely, $F_P(p)$ and $F_Y(y)$ have gaps, while $F_C(c)$ does not, then $h(p)$ is not (-1)-concave over some part of the intervening range, etc. Therefore, such behavior of the distributions can still be consistent with monopolistic competition, although not under A1 and a continuous $F_C(c)$.

4.3 Density elasticity relations

There are clean and useful conditions that relate the elasticities of equilibrium densities.[7] We will use them below both to indicate contradictions that reject the model and also to link the shapes of densities. These simple formulae show which other elasticities connect the densities, and they are all different aspects of the demand side. For example, the profit density elasticity is related to the cost density elasticity via the elasticities of profit and (inverse) marginal revenue (with respect to unit cost, c), both of which are derived from the demand form.

The fundamental relations are derived from the following elasticity lemma.

Lemma 6 *Consider two distributions $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$, which are absolutely continuous and strictly increasing on their respective domains. Let x_1 and x_2 be related by a monotone function $x_1 = \xi(x_2)$. Then we have the elasticity relation between densities as:*

$$\eta_{f_{X_2}} = \eta_{f_{X_1}} \eta_\xi + \eta_{\xi'},$$

where $\eta_{\xi'}$ is the elasticity of $\xi'(x_2)$ and $\eta_{f_{X_2}}$ is the elasticity of f_{X_2}: $\eta_{f_{X_2}} = x_2 f'_{X_2}(x_2) / f_{X_2}(x_2)$, etc.

[7] Elasticities of distributions (or survivor functions) are also readily calculated from (2).

Proof. For $\xi(.)$ increasing, from Lemma 5, we have $F_{X_2}(x_2) = F_{X_1}(\xi(x_2))$. Differentiation yields $f_{X_2}(x_2) = f_{X_1}(\xi(x_2))\xi'(x_2)$; differentiating again gives $f'_{X_2}(x_2) = f'_{X_1}(\xi(x_2))(\xi'(x_2))^2 + f_{X_1}(\xi(x_2))\xi''(x_2)$. Dividing through the second expression by the first and multiplying both sides by x_2 delivers the expression given in the Lemma from

$$x_2\frac{f'_{X_2}(x_2)}{f_{X_2}(x_2)} = \xi(x_2)\frac{f'_{X_1}(\xi(x_2))}{f_{X_1}(\xi(x_2))}\frac{\xi'(x_2)}{\xi(x_2)}x_2 + \frac{\xi''(x_2)}{\xi'(x_2)}x_2.$$

For $\xi(.)$ decreasing, from Lemma 5, we have $F_{X_2}(x_2) = 1 - F_{X_1}(\xi(x_2))$. Differentiating, $f_{X_2}(x_2) = -f_{X_1}(\xi(x_2))\xi'(x_2)$; and the steps above then again imply the expression given. ■

A1 imposes several restrictions on the various demand-side elasticities that appear in the density elasticity relations below. In particular, $\eta_h < -1$ is the property that demand must be elastic at equilibrium in a monopolistic competition setting, mirroring the standard monopoly property. Furthermore, $\eta_{h^*} < 0$ is the property that marginal revenue slopes down. The elasticity of the demand curve slope, $\eta_{h'} = \frac{h"p}{h'}$, has the sign of $-h$" and so is positive for concave demand, and negative for convex demand. The elasticity of the output function slope, $\eta_{h^{*\prime}}$, involves third derivatives of demand, though notable benchmarks are that it is zero for linear demand (because marginal revenue is linear) and for constant elasticity. Finally, the elasticity of maximized profit (with respect to c), η_π, is particularly interesting. Write this as

$$\eta_\pi = \frac{\pi'(c)}{\pi(c)}c = -\frac{ch^*(c)}{\mu(c)h^*(c)} = -\frac{c}{\mu(c)} = -\frac{1}{\ell - 1} < 0. \tag{16}$$

The third expression is the ratio of total cost to total profit;[8] the fourth one is the reciprocal of mark-up over cost; the last one writes this in terms of relative mark-up $\ell = p/c$ which we examine in Section 6.

We describe the other more interesting elasticity relations in the sequel.

[8] Else $\eta_\pi = -\frac{TC}{TR-TC} = \frac{-1}{(TR-TC)-1}$.

5 Rationalizability of distributions via demand

An old question in consumer theory is whether a demand system can be generated from a set of underlying preferences (see Antonelli, 1896, and the discussion in Mas-Collel, Whinston, and Green, 1995, pp. 70-75). Here we look at whether any arbitrary pair of economic/primitive distributions (F_C, F_P, F_Y, F_Π) could be consistent with the monopolistic competition model with demand satisfying A1. Surprisingly, for 4 of the possible pairs of distributions, the answer is affirmative, so that the model places no restrictions (above the $\mathcal{C}^2$ assumption we retain for simplicity). For the other two, we derive the conditions the distributions must satisfy, and in all cases we can recover the implied demand function.

5.1 Deriving demand and all distributions from price and cost ones

We now determine demand when there are strict monotone relations between two variables. Suppose first that price and cost distributions, F_P and F_C, are known and are strictly monotonic. Because mark-ups are necessarily positive, it must be that the price distribution first-order stochastically dominates the cost one. However, we will show that this is the *only* restriction on the distributions. The demand function will ensure that they are compatible, though the only restriction on it is that it be strictly (-1)-concave.

Because price strictly increases with cost, the price and cost distributions are matched: the fraction of firms with costs below some level c equals the fraction of firms with prices below the price charged by a firm with cost c. This enables us to back out the corresponding mark-up function $\mu(c)$ and then access Proposition 2.

Theorem 3 *Let the cost and price distributions, F_C and F_P be two arbitrary strictly increasing and $\mathcal{C}^2$ functions on their supports with $F_C(c) > F_P(c)$. Then there exists a strictly (-1)-concave demand function (unique up to a positive multiplicative factor) that rationalizes these* *distributions in the monopolistic competition model.*

Proof. Consider a distribution of costs, F_C and a distribution of prices, F_P satisfying $F_C(c) > F_P(c)$ (so that the price distribution is right of the cost one: note that $F_C(c) > F_P(c) = 0$ for c below the lower bound of the support of the price distribution). We wish to find a demand function satisfying A1. Define $p(c) = F_P^{-1}(F_C(c))$, which is a strictly increasing function. It satisfies $p(c) > c$ given that $F_C(c) > F_P(c)$. Then Proposition 2 implies that there exists an $h(.)$ satisfying A1 up to a positive multiplicative factor. ■

We can then determine the other economic relations (see also Theorem 1):

Corollary 2 *Let the cost and price distributions, F_C and F_P be two arbitrary strictly increasing and $\mathcal{C}^2$ functions on their supports with $F_C(c) > F_P(c)$. Then the mark-up function $\mu(c)$ (with $\mu'(c) > 0$) is found from (17); inverse marginal revenue is found from (7) and the demand function is given from (8), up to a positive multiplicative factor, $h^*(\underline{c})$. The output and profit distributions are determined, up to $h^*(\underline{c})$, by (14) and (15).*

Proof. We can write the price-cost margin, as a function of c, as

$$\mu(c) = F_P^{-1}(F_C(c)) - c, \tag{17}$$

with $\mu(c) > 0$ because $F_C(c) > F_P(c)$ and so $\mu'(c) > -1$. Hence a unique such mark-up function $\mu(c)$ exists given the cost and price distributions. With the function $\mu(c)$ thus determined, we can invoke Proposition 2 to uncover the equilibrium demand function $h^*(\cdot)$ (unique up to a positive multiplicative factor) as given by (6) and (7), and the demand function is given from (8). By Proposition 2, this demand function satisfies A1, as postulated. ■

From (17) we write

$$\mu'(c) = \frac{f_C(c)}{f_P(p)} - 1, \tag{18}$$

which shows that $\mu'(c) > 0$ iff $f_C(c) > f_P(p)$. As we know from Corollary 1, log-convex demand begets increasing mark-ups. Thus the equilibrium prices "spread out" vis-a-vis the costs, and hence engender a more spread price density than cost density when we take the price induced from a given cost. Conversely, log-concave demand delivers a decreasing mark-up and so prices tend to pile up, meaning the price density is more peaked than the cost density.

The idea behind Corollary above is as follows. Given the first key property that prices rise with costs, we know that the $z\%$ of firms with cost below c are the $z\%$ of firms with an equilibrium price below p. This links the mark-up and the cost level, so we can use Proposition 2 to uncover the demand form and equilibrium output of the zth percentile firm, due to the second key property that equilibrium output is a decreasing function of cost. We hence uncover the output distribution. The profit distribution then follows immediately from knowing the output and mark-up distributions. The latter two distributions are only determined up to a positive factor because the mark-up function is consistent with any multiple of the demand (under the maintained hypothesis of constant returns to scale).

The construction of the demand function is illustrated in Figure 1. The only restriction we use here is that the cost distribution first-order stochastically dominates the price one. Given this property, any pair of ($\mathcal{C}^2$) price and cost functions is consistent with the monopolistic competition model. In the next section, we show that the price and output distributions are restricted if they are to be consistent.

5.2 Price and output distributions

Now suppose that price and output distributions, F_P and F_Y, are known.

Theorem 4 *Let the price and output distributions, F_P and F_Y be two arbitrary strictly increasing and $\mathcal{C}^2$ functions on their supports. Then there exists a unique strictly (-1)-concave demand function, $h(p) = F_Y^{-1}(1 - F_P(p))$, that rationalizes these distributions in the monop-*

olistic competition model if and only if $f_P(p)/f_Y(y)$ is strictly decreasing in p.

Proof. From the two distributions $y = F_Y^{-1}(1 - F_P(p)) = h(p)$ is the unique candidate demand function. While this is decreasing in p, as desired, we also require that the function $F_Y^{-1}(1 - F_P(p))$ is strictly (-1)-concave to be consistent with the monopolistic competition model. This condition holds if and only if $f_P(p)/f_Y(h(p))$ is strictly decreasing in p. ■

The other distributions and relations are determined analogously to Corollary 2. If the implied demand shape does not satisfy the (-1)-concavity condition, the purported demand relation would not have a downward-sloping marginal revenue curve everywhere, and any price-output pair with an upward sloping marginal revenue could not be consistent with profit maximization by a firm.

The condition that $f_P(p)/f_Y(y)$ be strictly decreasing rules out various combinations. For example, if $f_p(p)$ is increasing (locally, say), then we cannot have $f_Y(y)$ (locally) increasing too. But both decreasing is fully consistent. Indeed, the required consistency condition is $f_P'(p) f_Y(y) - f_Y'(y) f_P(p) h'(p) < 0$, or $\eta_P < \eta_Y \eta_h$, so the price density elasticity should be negative if the output one is positive. Conversely, if the price density elasticity is positive then the output one should be negative. If such necessary (empirically testable) conditions do not hold the market cannot be described by the proposed monopolistic competition approach.

The density elasticity relation between price and output is given by applying Lemma 6 to $F_P(p) = 1 - F_Y(h(p))$:

$$\eta_{f_P} = \eta_{f_Y} \eta_h + \eta_{h'}. \tag{19}$$

The elasticity of the demand slope has shown up elsewhere in pricing formulae (e.g., in Helpman and Krugman, 1985). On the RHS, the demand elasticity, η_h, is negative, while the slope elasticity $\eta_{h'} = \frac{h"p}{h'}$ is positive for concave demand and negative for convex demand. For linear demand we have a benchmark that the price and output density elasticities have *opposite* signs.

Concave demand implies that decreasing output density drives increasing price density. For convex demand, increasing price density drives decreasing output density. To interpret the negative relation in the benchmark, recall that the low price firms are the high output ones, so we are looking at opposite ends of the distributions/densities effectively. Think about an increasing price density. Then there are more firms with higher prices: translating to the output density, there are more firms with lower outputs.

5.3 Cost and output distributions

Although price and output distributions are jointly restricted, surprisingly, cost and output distributions are not. Suppose that F_C and F_Y are known.

Theorem 5 *Let the cost and output distributions, F_C and F_Y be two arbitrary strictly increasing and $\mathcal{C}^2$ functions on their supports. Then there exists a strictly (-1)-concave demand function (unique up to a positive constant) that rationalizes these distributions in the monopolistic competition model.*

Proof. From the two distributions $y = F_Y^{-1}(1 - F_C(c)) = h^*(c)$ is the candidate function for optimized demand. The only restriction is that it slope down, which is satisfied, and that it be continuous, which is also immediately satisfied. Hence it is rationalizable, and we can use Proposition 3 to back up to the implied demand function, $h(p)$, which is therefore determined up to a positive constant. ■

The defining relation for elasticity densities for this pair is $F_Y(h^*(c)) = 1 - F_C(c)$. Then[9]

$$\eta_{f_Y}\eta_{h^*} = \eta_{f_C} - \eta_{h^{*\prime}}.$$

[9] Write the density relation $-h^{*\prime}(c) f_Y(h^*(c)) = f_C(c)$ in log form: the elasticity relation follows directly.

This is directly comparable to the price-output relation (19) (namely $\eta_{f_Y}\eta_h = \eta_{f_P} - \eta_{h'}$). Drawing on that analysis, a linear marginal revenue is a useful benchmark,[10] for which output and cost densities necessarily go in opposite directions. Constant elasticity of demand is just like for price-output, given that the parameters are the same for both cases.

5.4 Deriving demand from price and profit distributions

We now use Proposition 2 to find a unique demand function satisfying A1 from *any* pair of distributions. This is quite a surprising result. For example, there exists a demand function that squares Pareto distributions for both prices and profits, or normal and log-normal, etc. All other distributions are then determined.

Theorem 6 *Let the price and profit distributions, F_P and F_Π, be two arbitrary strictly increasing and $\mathcal{C}^2$ functions on their supports. Then there exists a strictly (-1)-concave demand function (unique up to a positive constant) that rationalizes these distributions in the monopolistic competition model.*

Proof. From (2), first write $1-F_P(p) = F_\Pi(\pi) = z$. Then we can write $\pi = F_\Pi^{-1}(1 - F_P(p)) = h(p)\mu(p) \equiv \tilde{\pi}(p)$, where we recall that $\tilde{\pi}(p)$ denotes the relation between the observed maximized profit level observed and the corresponding maximizing price. As shown in Lemma 3, $\tilde{\pi}(p) = -h^2(p)/h'(p)$. Integrating $(1/h(p))' = \frac{1}{\tilde{\pi}(p)}$ gives:

$$h(p) = \frac{1}{\int_{\underline{p}}^{p} \frac{dr}{F_\Pi^{-1}(1-F_P(r))} + k}. \tag{20}$$

[10] This comes from linear demand, but is not limited to that – we can add a rectangular hyperbola to demand and still get a linear marginal revenue.

This determines the demand form up to the positive constant $k = 1/h\left(\underline{p}\right)$ (in the position in the above formula): (20) is $\mathcal{C}^2$ and decreasing in p. Furthermore,

$$\left(\frac{1}{h(p)}\right)' = \frac{1}{F_{\Pi}^{-1}(1 - F_P(p))},$$

which is strictly increasing because both distributions are strictly increasing. That is $1/h(p)$ is convex and so, equivalently, $h(p)$ is (-1)-concave.

By Theorem 2 all the other distributions are determined. ■

Therefore, the price and profit distributions define the function (20) and the resulting demand function satisfies A1 *without any further restrictions*. This means, for example, that a decreasing price density is consistent with an increasing profit density (very many high profit firms and yet very few high price ones). The underlying cost distribution along with demand is what renders these features compatible. As regards the constant k, knowing the demand level at any one point ties down the whole demand function.

We have just shown that there are no restrictions on price and profit distribution shapes, though we have restrictions on some other pairs of distribution functions that can be combined in order to be consistent with the monopolistic competition model.

5.5 Cost and profit distributions

This is another case where monopolistic competition restricts the distribution pair.

Theorem 7 *Let the cost and profit distributions, F_C and F_{Π} be two arbitrary strictly increasing and $\mathcal{C}^2$ functions on their supports. Then there exists a demand function $h(p)$ satisfying A1(unique up to a constant) that rationalizes these distributions in the monopolistic competition model if and only if $f_C(c)/f_{\Pi(\pi^*(c))}$ is strictly increasing in c, with $\pi^*(c) = F_{\Pi}^{-1}(1 - F_C(c))$.*

Proof. From the two distributions, $\pi^*(c) = F_{\Pi}^{-1}(1 - F_C(c))$ is the candidate profit function. This is decreasing in c, as desired, but it also needs to be strictly convex, by Lemma 2, in order to be consistent with the monopolistic competition model. The convexity condition is that $h^*(c) = f_C(c)/f_{\Pi(\pi^*(c))}$ is strictly increasing in c. Using Proposition 3, there exists a demand function $h(p)$ satisfying A1, which is unique up to a constant. ■

Applying Lemma 6 to the case $1 - F_C(c) = F_\Pi(\pi^*(c))$, we get:

$$\eta_{f_\Pi}\eta_\Pi = \eta_{f_C} - \eta_{h^*}.$$

Recall that $\pi^{*\prime}(c) = -h^*(< 0)$ and $h^{*\prime} < 0$, so that $\eta_\Pi < 0$ and $\eta_{h^*} < 0$. If the profit density is increasing, then the cost density is necessarily decreasing, but the reverse is not true: a strong enough decreasing cost density is needed for an increasing profit density. Conversely, $\eta_{f_C} > 0 \Rightarrow \eta_{f_\Pi} < 0$: an increasing cost density implies a decreasing profit one.

Finally, distribution elasticities uncover relations. From $1 - F_C(c) = F_\Pi(\pi^*(c))$, we can write $\frac{f_C}{1-F_C} + \frac{f_\pi}{F_\pi}\pi'(c) = 0$, which in elasticity form (recalling $\eta_\Pi = \frac{-1}{\ell-1}$ from (16)) becomes

$$\eta_{F_\Pi} = -(\ell - 1)\,\eta_{S_C} \tag{21}$$

where the subscript S_C denotes the survivor function of costs and the corresponding elasticity $\eta_{S_C} < 0$. Thus the profit distribution is more elastic the bigger the relative mark-up, ℓ, as the survivor cost parlays into more profit distribution response. (21) also indicates how mark-ups can be estimated directly from the two elasticities.

5.6 Output and profit distributions

The final case returns to no restrictions.

Theorem 8 *Let the output and profit distributions, F_Y and F_Π be two arbitrary strictly increasing and $\mathcal{C}^2$ functions on their supports. Then there exists a strictly (-1)-concave demand function (unique up to a constant) that rationalizes these distributions in the monopolistic competition model. This unique net demand function, and the other distributions, are determined* *explicitly in the proof.*

The proof is in the Appendix (and illustrated in the ρ-linear demand example below). It is based on the relation, $\Psi(z)$, between the counter z and the cost level (or any economic variable): $\Psi(z) = \int_0^z \left[F_\Pi^{-1}(r)\right]' / F_Y^{-1}(r)\, dr = \bar{c} - c$ (see (25)).

What the Theorem ties down is net demand (inverse demand minus cost): if both inverse demand and cost shift by the same amount then equilibrium quantity (output) and mark-up are unaffected, so profit is unchanged too. Thus output and profit distributions tie down the shape of the net inverse demand and the shape of the other distributions, but not the inverse demand curve height. As we saw above, price and cost distributions alone do not tie down the demand scale, and nor do price and profit distributions. But the other pairs of distribution combinations fully determine the demand function and all distributions.

5.7 Examples

We illustrate the Theorems above with distributions that generate ρ-linear demand. Details are in the Appendix.

Recovering ρ-linear demand. *Suppose that* $F_Y(y) = \frac{(1+\rho)y^\rho - 1}{\rho}$, $y \in \left[\frac{1}{(1+\rho)^{1/\rho}}, 1\right]$, *and* $F_\Pi(\pi) = \frac{(1+\rho)\pi^{\rho/(1+\rho)} - 1}{\rho}$, $\pi \in \left[\frac{1}{(1+\rho)^{(1+\rho)/\rho}}, 1\right]$, *with* $\rho > -1$. *Then demand is* ρ*-linear (see (12) with* $k = \bar{c}$*) and the cost distribution is uniform.*

Uniform costs give a useful benchmark for some important properties relating cost and profit distributions. For the example above, we have $f_\Pi(\pi) = \pi^{-1/(1+\rho)}$, so that the density of the profit distribution is decreasing, despite the underlying cost distribution that generates it being flat. This property indicates how profit density "piles up" at the low end. The output density shape is also interesting. For linear demand ($\rho = 1$) it is clearly flat – equilibrium quantity is a linear function of cost. For convex demand ($\rho < 1$) it is decreasing, but for concave demand it is *increasing*, despite the property just noted that the profit density is decreasing. This suggests that (for concave demand), a decreasing output density requires an increasing cost density, which *a fortiori* entails a decreasing profit density. The Appendix also illustrates that knowing the profit and cost distributions ties down the full model, using the same parameters, and gives the steps involved.

6 Mark-up distribution

Recent work has delivered mark-up distributions from several different methodologies, most notably the recent production function approach. We now analyze how mark-up distributions interact with the other ones, and how they help retrieve demand. Thus far in the analysis, via A1 and the subsequent Lemmas it entails, variables are always either positively or negatively linked (e.g. prices and costs, or prices and outputs respectively). Mark-ups though either go up or down with the other variables depending on the degree of concavity of demand. This is true for both absolute and relative mark-ups, which we consider in turn. As we show, this means that each pair of a mark-up distribution and another distribution and can yield two solutions, depending on whether or not costs are fully passed through.

6.1 Absolute mark-ups

Our first result with this distribution, $F_M(m)$ with $m = p - c$, extends and modifies Theorem 2. We claim that knowing the demand function and $F_M(m)$ suffices to tie down the other distributions if the demand is either strictly log-concave or strictly log-convex (see Corollary 1). We divide the analysis into two cases, depending on the log-concavity or log-convexity of $h(.)$.[11] Under log-concavity, we know that $\mu(c)$ is decreasing and cost pass-through is less than 100%. This entails higher (per unit) mark-ups at firms with lower costs, so that $z = F_M(m) = 1 - F_C(c)$. Rewriting, we recover the cost distribution from the relation

[11] The mixed case is discussed briefly below, with more in the CEPR version of the paper.

$F_C(c) = 1 - F_M(\mu(c))$, and we can then find all the other distributions from the relations in Theorem 2.

On the other hand, $\mu(c)$ is increasing when $h(.)$ is log-convex, and cost pass-through is more than 100%. Then higher (per unit) mark-ups are set at lower firm outputs, and then we recover the cost distribution (and hence all others) from $F_C(c) = F_M(\mu(c))$.

When the demand has both log-concave and log-convex segments, then some values of mark-up could have been delivered by two (or more) values of c. Thus $\mu(c)$ is not invertible and so the cost distribution cannot be recovered without supplementary information (at least on the domain for which $\mu(c)$ is not invertible and if $f_M(m) > 0$ for such values).

Likewise, knowing $F_M(m)$ and one other distribution does not tell us all other distributions and demand without further qualification. If we knew in addition that $h(.)$ were log-concave (resp. log-convex), then we can pin down the demand form, and the other distributions using analogues to Theorems 3, 4, 6, and we discuss the procedure at greater length in the next sub-section. However, without knowing a priori which side of log-linearity (the Logit) $h(.)$ falls, we get two candidate solutions. Indeed, if $[\ln(h)]'$ changed sign over its domain then the $h(.)$ cannot be tied down.

We now turn to relative mark-ups and give a fuller treatment since these are more commonly derived empirically.

6.2 Relative mark-ups

A growing recent literature (see e.g. De Loecker, Eeckhout, and Unger, 2020), has been estimating markups from a production function approach. The mark-ups estimated are not the absolute mark-ups with which we started the paper ($m = p - c$), but a unit-free version. This has been expressed in various ways, like the Lerner index $\frac{p-c}{p}$, or else $\frac{p-c}{c}$, or, most commonly, by $\ell = \frac{p}{c} \geq 1$. The other variants can all be expressed in terms of ℓ, and we retain this last version, which we term relative mark-up.

We first recall the classic Inverse Elasticity Rule (IER), which applies to our monopolistic competition formulation, which writes the Lerner index as

$$\frac{(p-c)}{p}=\frac{1}{\varepsilon},$$

with $\varepsilon \equiv -\frac{ph'(p)}{h(p)}$, the elasticity of demand (in absolute terms) and $\varepsilon > 1$ so firms produce where demand is elastic. From the IER we have the equilibrium relation $\ell = \frac{\varepsilon}{\varepsilon-1} > 0$, which is decreasing in ε with support $(1,\infty)$ and range $(1,\infty)$ (see e.g., Melitz, 2018).

If we know the demand form $h(p)$ then we know the corresponding expressions for ε and ℓ. If we know the equilibrium prices we know the equilibrium mark-up relation $\ell(p)$ of those equilibrium prices too. We distinguish two cases.

First, Marshall's Second Law of Demand (henceforth M2L) holds iff ε is increasing in p, equivalently, ε is decreasing with output, $y = h(p)$. Then $\ell'(p) < 0$. Second, the converse to M2L holds iff ε is decreasing in p and $\ell'(p) > 0$.[12] The CES forms the boundary case in which ε is constant, and so too is then equilibrium $\ell(p)$. Note that log-concavity of $h(p)$ implies M2L and the Converse Law implies log-convexity of demand.[13]

We briefly discuss relative pass-through as measured from the price and cost distributions. Recall $F_C(c) = F_P(p)$ and so $\frac{dp}{dc} = \frac{f_C(c)}{f_P(p)}$ and $\mu'(c) > 0$ (log-convexity) entails $f_C(c) > f_P(p)$, which we can interpret as the cost density driving more spread in the price density. The Converse Law, being a stronger property, entails a stronger condition. Indeed, since the Converse Law implies increasing $\ell(c)$, then it implies $p'(c) > \ell$ and hence $cf_C(c) > pf_P(p)$. Equivalently, the elasticity of the cost distribution should exceed that of the price distribution. M2L implies the opposite elasticity relation, which in turn is implied by the condition $f_C(c) < f_P(p)$ for

[12] We restrict attention to when the derivative of ε is monotone, so we rule out cases which switch between the Law and its converse.

[13] That is, $\left[\frac{h'(p)}{h(p)}\right]' < 0$ (log-concavity) $\Rightarrow \left[\frac{ph'(p)}{h(p)}\right]' < 0$ (M2L) because $\left[\frac{ph'(p)}{h(p)}\right]' = p\left[\frac{h'(p)}{h(p)}\right]' + \frac{h'(p)}{h(p)}$.

$\mu'(c) < 0$ (log-concavity).

In the sequel, we treat the case when M2L holds; the converse case follows analogously but flips the distributional relations as indicated below. When M2L holds, $\ell'(p) < 0$, knowing the price distribution ties down the distribution of ℓ from the relation $F_L(\ell(p)) = 1 - F_P(p)$.[14] Conversely the mark-up distribution ties down the price distribution. We can leverage this argument to provide the equilibrium relations with the other distributions. For example, because prices rise with costs we have $F_C(c) = 1 - F_L(\ell(c))$ (and we expand below on the relation $\ell(c)$, which implies $\ell'(c) < 0$ when M2L holds); because equilibrium profits fall with equilibrium prices by Lemma 3 we have $F_\Pi(\pi) = F_L(\ell(p(\pi)))$ as big mark-ups are associated to big profits when M2L holds (and conversely under the Converse Law).

Therefore *knowing the demand form and the mark-up distribution delivers all the other distributions too.* Conversely*, any distribution along with demand form delivers the mark-up distribution.* This means that Theorem 2 extends to include the mark-up distribution when M2L applies or when its converse applies.

We now turn to the question of demand recoverability from the mark-up distribution and one other. Our existing results enable us to find the demand (assuming it obeys M2L). We illustrate with the cost distribution. As noted above, $\ell(c) = F_L^{-1}(1 - F_C(c))$, with $\ell'(c) = -\frac{f_C(c)}{f_L(\ell)}$ is negative, as required. Rewrite this as $p(c) = cF_L^{-1}(1 - F_C(c))$, from which we have $\mu(c) = c\left(F_L^{-1}(1 - F_C(c)) - 1\right)$, so that we recover the absolute mark-up function. Now, as shown in Proposition 1, from the function $\mu(c)$ we can recover the demand function up to a multiplicative factor. However, we need $\mu'(c) > -1$ for A1 to hold and to apply Proposition 1. The required condition is

$$\ell f_L(\ell) > c f_C(c),$$

which can equivalently be written in elasticity form with the interpretation that the elasticity

[14] When the converse law holds, $F_L(\ell(p)) = F_P(p)$.

of the relative mark-up distribution should exceed that of the cost survivor distribution. If the Converse Law holds, $p(c) = cF_L^{-1}(F_C(c))$ and necessarily $p'(c) > 0$ without conditions. Hence $\mu(c) = p - c = c\left(F_L^{-1}(F_C(c)) - 1\right)$ and Proposition 1 then delivers the demand form from $\mu(c)$.[15] The next result summarizes.

Proposition 4 *Let $F_L(\ell)$ and $F_C(c)$ be given. If $\ell f_L(\ell) > cf_C(c)$ there exists a demand satisfying Marshall's Second Law (M2L), which is unique up to a positive factor. There always exists a demand (unique up to a positive factor) satisfying the Converse Law.*

Demand recovery from the price distribution follows a similar procedure, and also accesses our key recovery result in Proposition 1, but yet with some surprise, so these are not completely sibling results. Under M2L, we have $\ell(p) = F_L^{-1}(1 - F_P(p))$, with $\ell'(p) = -\frac{f_P(p)}{f_L(\ell)} < 0$, as the mark-up which is recovered empirically from the two distributions. Rewrite this in terms of the supporting cost $c(p)$ as $c(p) = p/\left(F_L^{-1}(1 - F_P(p))\right)$ and we require this is increasing in p for A1 to hold and price to rise with cost, and this also implies that the relation $c(p)$ is invertible so the corresponding $p(c)$ is increasing. This condition necessarily holds because $c'(p) > 0$ (it has the sign of $\ell f_L + pf_P$). Therefore there is no restriction on the distributions for this case (contrast the analogous cost case). To deploy Proposition 1 we need $\mu(c) = p(c) - c$ to have derivative greater than -1, or $p'(c) > 0$, which we have just argued to be true, and so demand is recovered. For demand to satisfy the Converse Law, we have the mark-up delivered from the two distributions as $\ell(p) = F_L^{-1}(F_P(p))$, with $\ell'(p) = \frac{f_P(p)}{f_L(\ell)} > 0$. Then the supporting cost is $c(p) = p/F_L^{-1}(F_P(p))$ and we need this increasing for the same reason as above. Note that $c'(p) > 0$ again implies the desired invertibility of $p(c)$ and that Proposition 1 can be applied. The condition for $c'(p) > 0$ is $\ell f_L > pf_P$ (so the elasticity of the mark-up distribution should exceed the elasticity of the survivor function of the price distribution). We summarize as:

[15] Note that $\mu'(c) = (\ell - 1) + c\frac{f_C(c)}{f_L(\ell)} > 0$, so the demand is necessarily log-convex.

Proposition 5 *Let $F_L(\ell)$ and $F_P(p)$ be given. If $\ell f_L(\ell) > p f_P(p)$ there exists a demand satisfying the Converse Law, which is unique up to a positive factor. There always exists a demand (unique up to a positive factor) satisfying Marshall's Second Law.*

It is interesting here that the price and cost distributions deliver restrictions in opposite cases for demand. Pairing the mark-up distribution to the output and profit distributions also bear some similar conclusions. In both cases, there is always a solution under the Converse Law without restriction; but the M2L case has (similar) restrictions for both.[16]

7 Conclusions

The basic ideas here are simple. Market performance depends on the economic fundamentals of tastes and technologies, and how these interact in the market-place. The fundamental distribution of tastes and technologies feeds through the economic process to generate the endogenous distribution of economic variables, such as prices, outputs, and profits. It is the assumption of the monopolistically competitive market structure that delivers the clean and tractable feed-through from fundamental distributions to performance distributions.[17]

As we show, any pair of the (endogenous) economic distributions can be reverse engineered to back out the model's primitives. If two distributions can be estimated from a data-set, then they can be checked with respect to the consistency conditions of the model. If so, demand can be recovered and compared to the commonly-used forms (like CES and Logit). The empirical density elasticities also yield relations that can be evaluated in the light of the model.

We have focused on demand recovery. Surprisingly, demand can be recovered just from profit and price distributions (for example). We show what restrictions on the distributions

[16] The restrictions come from the condition $p'(c) > 0$. In elasticity form these are respectively $\eta_{F_L} + \eta_{F_Y}\eta_{\Pi} < 0$ (recall that $\eta_{\Pi} = \frac{-1}{\ell - 1} < 0$ from (16)) and $\eta_{F_L} + \eta_{F_Y}\eta_Y < 0$, with $\eta_Y = \frac{c h^{*\prime}(c)}{h^*(c)} < 0$.

[17] An oligopoly analysis would be hugely more cumbersome because then firms' types would be realized draws of costs from an underlying distribution and we would need to track outcomes across all possible draw combinations.

the model entails, and we provide constructive proofs to find demand. We have chosen to present the details for all distribution pairs because each pair yields different patterns in terms of the distribution restrictions, underlying demand construction, or constants not identified in demand.

The simplest case is output and profit, for which the distributions immediately deliver demand, although there is a restriction on densities consistent with the model. Cost and price distributions deliver a mark-up function, from which we can use our new result (Proposition 2, which goes in the converse direction from Weyl and Fabinger, 2013) on deriving demand from pass-through. Two other cases - output and profit - entail a sort of cost pass-through, and we first deliver new pass-through results on these before again going in the reverse direction and deriving demand from the distribution pairs. Both proofs involve our constructive result (Proposition 3) how to find demand from marginal revenue: the common ground between output and profit is determined from our new result (Lemma 2) on the link that $\pi^{*\prime}(c) = -y$. However, while the cost-profit distribution pair involves a restriction (from the convexity of $\pi^*(c)$), the cost-output distribution pair is not restricted (as we show in the proof). The last two distribution pairs are also not restricted, but need separate proofs. These pairs are profit-price and profit-output, for which variable pairs we also deliver new results showing how their optimized values vary together (they are orchestrated by underlying cost variation).

The distribution of relative mark-ups is also interesting. For each distribution with which the mark-up distribution is paired, we find two cases for demand, depending on whether demand obeys Marshall's Second Law or its converse, and we find either two solutions or just one alone. Interestingly, the assured solution can be in either demand class depending on the distribution.

One future research direction is to investigate more the inheritance properties of distributions both theoretically and empirically. For example, curvature properties such as ρ-concavity translate from one distribution to another. Also, the moments of different economics distributions are related through the economic relations: for example, the modes of the various distributions follow a simple relation via the elasticity analysis.

Finally, we here considered only one dimensional heterogeneity across firms. Multi-dimensional heterogeneity could also be analyzed with a generalized version of the methods described here. For example, suppose that firms differed with respect to both product quality (v) and cost (c) according to a joint fundamental distribution $F(v, c)$, and we wrote demand as a function $h(v - p)$.[18] Naturally, firms with higher v and lower c would have higher equilibrium profit and output. Then the set of firms with profit above any particular level of profit would be those below some critical locus $c(v)$. Likewise for those with the highest outputs. However, the trade-off would be different for output and profit criteria. The economic fundamentals $F(v, c)$ and $h(v - p)$ would determine equilibrium distributions of the economics variables (profit, price, output, etc.). Now, as long as the iso-profit and iso-output loci (to take one pair as an example) satisfy a monotonicity condition, then we can determine the primitive distribution $F(v, c)$ from the profit or output distributions by varying costs and quality across the feasible space and then matching the distributions to uncover it. In this way, we can extend the conceptual idea expounded here. It remains to determine what restrictions on primitives are needed to ensure full invertibility. Notice that we would need demand and (at least) two other distributions to determine the primitive distribution (and hence all the others.) Conversely, we would need at least three economic distributions to find the demand and the whole system. More generally, the informational requirements would increase with the number of dimensions of heterogeneity of primitive variables.

[18] Anderson and de Palma (2020) look at a one-dimensional version of this model for logit-CES with cost depending monotonically on quality.

References

[1] Anderson, Chris (2006). The Long Tail: Why the Future of Business Is Selling Less of More. New York: Hyperion.

[2] Anderson, S. and A. de Palma (2001). Product Diversity in Asymmetric Oligopoly: Is the Quality of Consumer Goods too Low? Journal of Industrial Economics, 49, 113-135.

[3] Anderson, Simon P. and André de Palma (2015). Economic distributions and primitive distributions in monopolistic competition, CEPR Discussion Papers, 10748.

[4] Anderson, Simon P. and André de Palma (2020). Decoupling the CES distribution circle with quality and beyond: equilibrium distributions and the CES-Logit nexus. Economic Journal, 130(628), 911-936.

[5] Anderson, Simon P., André de Palma, and Jacques François Thisse (1992). Discrete Choice Theory of Product Differentiation. Cambridge, MA: MIT Press.

[6] Antonelli G.B. (1886). Sulla teoria matematica dell'economia politica, Edizioni Fochetto, Pisa.

[7] Caplin, Andrew, and Barry Nalebuff (1991). Aggregation and imperfect competition: On the existence of equilibrium. Econometrica, 59(1), 25–59.

[8] Chamberlin, Edward (1933). Theory of Monopolistic Competition. Cambridge, MA: Harvard University Press.

[9] De Loecker, Jan Eeckhout, and Gabriel Unger (2020). The Rise of Market Power and the Macroeconomic Implications. Quarterly Journal of Economics, 135(2), 561-644.

[10] Eaton, Jonathan, Samuel Kortum, and Francis Kramarz (2011). An anatomy of international trade: Evidence from French firms. Econometrica 79(5), 1453-1498.

[11] Gabaix, Xavier (2016). Power Laws in Economics: An Introduction. Journal of Economic Perspectives. 30 (1), 185-206.

[12] Head, Keith, Thierry Mayer, and Mathias Thoenig (2014). Welfare and trade without Pareto. American Economic Review 104(5), 310-16.

[13] Hitsch, Günter J., Ali Hortacsu, and Xiliang Lin (2017). Prices and Promotions in U.S. Retail Markets: Evidence from Big Data. Booth Discussion Paper 17-18, U Chicago.

[14] Kaplan, Greg and Guido Menzio (2015). The Morphology Of Price Dispersion. International Economic Review, 56, 1165-1206.

[15] Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green (1995). Microeconomic Theory. Oxford University Press.

[16] Melitz, Marc J. (2003) The impact of trade on intra-industry reallocations and aggregate industry productivity. Econometrica 71(6), 1695-1725.

[17] Melitz, Marc J. (2018) Competitive Effects of Trade: Theory and Measurement. Review of World Economics, 154 (1), 1-13.

[18] Mrázová, Monika, J. Peter Neary, and Mathieu Parenti (2020). Sales and Markup Dispersion: Theory and Empirics. Econometrica, forthcoming.

[19] Pareto, Vilfredo (1965). "La Courbe de la Répartition de la Richesse" (Original in 1896): Oeuvres Complètes de Vilfredo. Ed. Busino G. Pareto. Geneva: Librairie Droz, 1-5.

[20] Thisse Jacques-François and Philip Ushchev (2018) Monopolistic competition without apology. Handbook of Game Theory and Industrial Organization, Volume I, 1-93.

[21] Weyl, Glen, and Michal Fabinger (2013). Pass-Through as an Economic Tool: Principles of Incidence under Imperfect Competition. Journal of Political Economy, 121(3), 528-583.

Appendix

Proof of Theorem 8

The assumption that F_Y and F_Π are $\mathcal{C}^1$ distributions means that we can invert them and write each of them as a function of the counter z. Both output and profit are increasing functions of cost, c. Therefore we can match the distributions: the firms with the highest $z\%$ of the costs are those with the lowest $z\%$ of the outputs and profits. Furthermore, because the distribution functions are differentiable, then z is a differentiable function of the underlying cost, and we can invert it. Call this inverted relation $c(z)$, with $c'(z) < 0$.

Choose some arbitrary level $z \in (0,1)$ such that $1 - F_C(c) = F_Y(y) = F_\Pi(\pi) = z$. Then the firms with cost levels above $c(z) = F_C^{-1}(1-z)$ are the firms with outputs and profits below y and π. For this proof, we introduce z as an argument into the various outcome variables to track the dependence of the variables on the level of $z(c) = 1 - F_C(c)$. However, the expression for $F_C(c)$ is not known at this point.

Write $y(z) = F_Y^{-1}(z)$ and equilibrium demand is

$$h^*(c) = y(z(c)) = F_Y^{-1}(1 - F_C(c)) > 0. \tag{22}$$

Because $\pi^*(z(c)) = m(z(c))\, y(z(c)) = F_\Pi^{-1}(z(c))$, then

$$m(z(c)) = \frac{F_\Pi^{-1}(z(c))}{F_Y^{-1}(z(c))} = \mu(c), \tag{23}$$

and equilibrium profit is $\pi^*(z((c))) = \mu(z(c))\, h^*(z(c)) = F_\Pi^{-1}(z(c))$.

From Lemma 2 we have $\pi^{*\prime}(z(c)) = -h^*(z(c))$, so the relation between the counter z and the cost level c with $z(\underline{c}) = 1, z(\overline{c}) = 0$ is $dz/dc = -h^*(c)/[\pi^*(z(c))]'$, and hence

$$z'(c) = -\frac{F_Y^{-1}(z(c))}{\left[F_\Pi^{-1}(z(c))\right]'} < 0. \tag{24}$$

Thus $\Psi(z) = -\int_{\bar{c}}^{c} dv = \bar{c} - c$,or $c(z) = \bar{c} - \Psi(z)$, where $\Psi(z)$ is the key transformation between z and c :

$$\Psi(z) = \int_0^z \frac{\left[F_\Pi^{-1}(r)\right]'}{F_Y^{-1}(r)} dr, \tag{25}$$

with $\Psi(0) = 0$, $\Psi(1) = \bar{c} - \underline{c}$. Because $\Psi'(z) = \frac{\left[F_\Pi^{-1}(z)\right]'}{F_Y^{-1}(z)} > 0$, the required relation between z and c is $z(c) = \Psi^{-1}(\bar{c} - c)$.

Since $p = h(\mu(c) + c)$, the inverse demand is $p = \frac{F_\Pi^{-1}(z(c))}{F_Y^{-1}(z(c))} + c = \frac{F_\Pi^{-1}\left(\Psi^{-1}(\bar{c}-c)\right)}{F_Y^{-1}(\Psi^{-1}(\bar{c}-c))} + c$. This makes clear that a shift up in all costs by Δ and a corresponding shift up in the inverse demand by Δ (so the support of the cost distribution shifts up by Δ, i.e., $\bar{c}$ becomes $\bar{c} + \Delta$) keeps both the firm's output choice and mark-up constant so output and profit are not changed. This means that these two distributions can only pin down *net* (inverse) demand.

This allows us to uncover the distribution of cost, which is thus given by

$$F_C(c) = 1 - z(c) = 1 - \Psi^{-1}(\bar{c} - c). \tag{26}$$

The remaining unknowns can be backed out now knowing $z(c)$: equilibrium demand is $h^*(c) = F_Y^{-1}(\Psi^{-1}(\bar{c} - c))$ from (22). Therefore, since $h^*(c)$ is strictly decreasing in c, by Proposition 3 we can claim there exists a demand function $h(p)$ which satisfies A1 up to a constant. Note that this is is exactly the property that the equation (22) delivers because $\Psi^{-1}(\bar{c} - c)$ is strictly decreasing in c (from (26)).

Finally, the mark-up function is recovered from $\mu(c) = \frac{F_\Pi^{-1}\left(\Psi^{-1}(\bar{c}-c)\right)}{F_Y^{-1}(\Psi^{-1}(\bar{c}-c))}$ from (23).

Details for ρ-linear examples

First suppose that we know $F_Y(y) = \frac{(1+\rho)y^\rho - 1}{\rho}$, $y \in \left[\frac{1}{(1+\rho)^{1/\rho}}, 1\right]$, *and* $F_\Pi(\pi) = \frac{(1+\rho)\pi^{\rho/(1+\rho)} - 1}{\rho}$, $\pi \in \left[\frac{1}{(1+\rho)^{(1+\rho)/\rho}}, 1\right]$, *with* $\rho > -1$.

Hence $F_Y^{-1}(z) = \left(\frac{\rho z + 1}{1+\rho}\right)^{1/\rho}$ and $F_\Pi^{-1}(z) = \left(\frac{\rho z + 1}{1+\rho}\right)^{(1+\rho)/\rho}$. By (23), the ratio of these two yields the mark-up, $m(z) = \frac{\rho z + 1}{1+\rho} > 0$. Because $\left[F_\Pi^{-1}(z)\right]' = \left(\frac{\rho z + 1}{1+\rho}\right)^{1/\rho}$, we can write $\Psi(z) =$

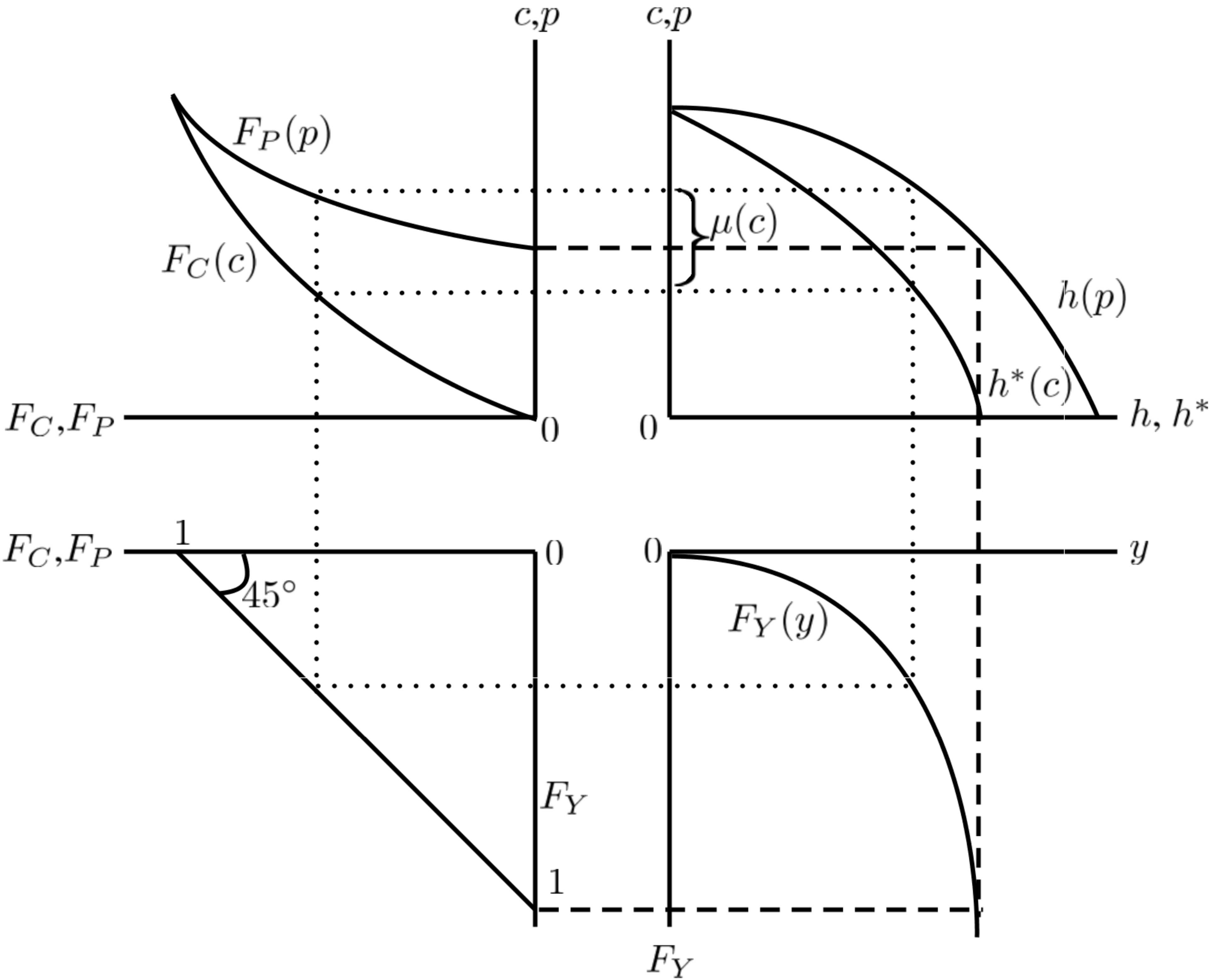
$\int_0^z \frac{\left[F_\Pi^{-1}(r)\right]'}{F_Y^{-1}(r)} dr = z = \bar{c} - c$, so $c(z) = \bar{c} - z$ $(c'(z) = -1)$. Now, $F_C(c) = 1 - \Psi^{-1}(\bar{c} - c) = c - \underline{c}$ (Uniform cost. Hence $\mu(c) = \frac{\rho(\bar{c}-c)+1}{1+\rho}$ (from (23)). Then $y(c) = F_Y^{-1}(z(c)) = \left(\frac{\rho(\bar{c}-c)+1}{1+\rho}\right)^{1/\rho}$, and $h^*(c) = y(c)$. We now want to find the associated demand, $h(p)$. We use the fact that $p = \mu(c) + c = \frac{1+c+\rho\bar{c}}{1+\rho}$ to write $h(p) = (1 + \rho(\bar{c} - p))^{1/\rho}$, which is therefore a $\rho-$linear demand function with the parameter k set at $k = \bar{c}$, and $\rho > -1$ implies $h(.)$ is (-1)-concave.

Note that $y(\bar{c}) = \left(\frac{1}{1+\rho}\right)^{1/\rho}$, as verified by the upper bound, $\bar{c}$, while the lower bound condition $\underline{c} = \bar{c} - 1$ implies that $y(\underline{c}) = 1$, so costs are uniformly distributed on $[\underline{c}, \bar{c}]$. Lastly, $\lim_{\rho\to 0} y(c) = \exp(\bar{c} - c)$ gives the logit equilibrium demand.

Suppose now that it is known that $F_C(c) = c$ *for* $c \in [0,1]$ *and (as above)* $F_\Pi(\pi) = \frac{(1+\rho)\pi^{\rho/(1+\rho)}-1}{\rho}$.

We first write $\pi^*(c)$ to find $h^*(c) = -\pi^{*\prime}(c)$. Matching the distribution levels, $1 - c = \frac{(1+\rho)\pi^{\rho/(1+\rho)}-1}{\rho}$, or $\pi^*(c) = \left(\frac{\rho(1-c)+1}{1+\rho}\right)^{(1+\rho)/\rho}$ and hence the support of the profit function is $\pi \in \left[\frac{1}{(1+\rho)^{(1+\rho)/\rho}}, 1\right]$. Hence $y(c) = h^*(c) = \left(\frac{\rho(1-c)+1}{1+\rho}\right)^{1/\rho}$, so both output and profit are power functions. Then we use $c = 1 - F_Y(y)$ with $F_Y(y) = \frac{(1+\rho)y^\rho - 1}{\rho}$ to get , $\mu(c) = \frac{\pi^*(c)}{h^*(c)} = \left(\frac{\rho(1-c)+1}{1+\rho}\right) = [h^*(c)]^\rho$ (consistent with: $y(c)^\rho = \left(\frac{\rho(1-c)+1}{1+\rho}\right)$). Now use $p = \mu(c) + c$ to find $h(p) = (1 + \rho(1-p))^{1/\rho}$ and hence the (ρ-linear) demand form is tied down, including the value of the constant ($k = 1$: see (12), and consistent with the specification $\bar{c} = 1$).

Finally, we use (17) to write $\mu'(c) = \frac{f_C(c)}{f_P(p)} - 1$; so $f_P(p) = 1 + \rho$ and $F_P(p) = (1+\rho)(p-1)$ for $p \in \left[1, \frac{2+\rho}{1+\rho}\right]$.



c,p
c,p
$F_P(p)$
$\mu(c)$
$F_C(c)$
$h(p)$
$h^*(c)$
F_C, F_P
0
0
h, h^*
1
F_C, F_P
0
0
y
45°
$F_Y(y)$
F_Y
1
F_Y