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Abstract In this paper, we propose two spatial power indices in political games, taking into account ideological preferences of players. To do this, we develop an explanatory spatial model linked to the asymmetry Deegan-Pakel index introduced by Rapoport & Golan [Rapoport, A., Golan, E., 1985. Assessment of political power in the israeli knesset. American Political Science Review 79 (3), 673-692], which is the original Deegan-Packel index readjusted for measuring power according to the spatial preferences of players in real political games. In addition to extending such a readjustment for the original Johnston index — transforming it concomitantly into the Johnston spatial power index — this paper presents both the general versions of these two spatial indices, and their axiomatic characterizations through new axioms such as *the vetoer property* and others mainly inspired from Lorenzo-Freire et al. [Lorenzo-Freire, S., Alonso-Meijide, J. M., Casas-Méndez, B., Fiestras-Janeiro, M. G., 2007. Characterizations of the Deegan-Packel and johnston power indices. European Journal of Operational Research 177 (1), 431-444].

Keywords Game theory · Spatial voting games · Deegan-Packel spatial power index · Johnston spatial power index · Axiomatic characterizations · Political games.

JEL classification. C71, D71

One of the main shortcomings mentioned almost in all publications on power indices is the fact that well-known indices do not take into account the preferences of agents.

⇝ Fuad Aleskerov, *Power indices taking into account agents' preferences*

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1 Introduction and Background

In game theory, voting systems are widely represented in the abstract way by simple games. There are many methods of testing the fairness of a voting system, or determining the influence of each participant (or voter) in the decision-making process via a voting system. Power indice are relevant tools to measure the influence or voting power of each voter, and one common way to measure the power of a voter in such a process, is to determine the number of times that player changes a losing coalition into a winning one. However, classical power indices are not efficient to predict the coalitions that will be formed because the formation of coalitions they consider does not depend on the affinities of players at stake such as habits, outlooks or even ideological preferences. In fact, these classical indices (also called a priori power indices) are more relevant when it comes to measure the fairness of constitutional power in the voting system — to design voting rules in a decision making system, or speaking like Shenoy (1982), to answer questions like "Is this given decision fair? — Does it distribute power equitably?" This process is what is usually behind the concept of power indices, which in their a priori context (or classical forms), assume that voters can form coalitions symmetrically, meaning there is no prior information that a player a can prefer to join player b rather than player c. In fact, experiences from real-world case studies indicate that some coalitions of voters are more likely to form than others. We can explain it by the fact that affinities among the players, play a central role in the coalition formation process.

In this paper, we focus on *political games* which are simple games together with the ideological preferences of the players. Thus, in order to take these ideological preferences into account in measuring the power of each player, Owen (1971) proposed for the first time, a modification of the Shapley value — a nonsymmetric generalization of the Shapley-Shubik power index for political games also known as the Owen-Shapley spatial power index. Shapley (1977), building on the findings of Owen, pursued his investigation in terms of both viewpoint and technique, which in turn inspired Shenoy (1982) in a similar way, to propose a nonsymmetric generalization of the Banzhaf power index for political games that we call the Banzhaf spatial power index. The next nonsymmetric generalization in political games — the Deegan-Packel spatial power index — was taken in another approach introduced by Rapoport & Golan (1985). In this present paper, we propose a more general version of this latter spatial index, followed by its first axiomatic characterizations in spatial models. To date and to the best of our knowledge, no one has published any spatial generalization of the Johnston power index in political games. This paper is a first attempt to do so by following the same model as Rapoport & Golan (1985) did for the Deegan-Packel spatial index. Furthermore, we also give a more general version of this newly spatial version of the Johnston index and characterize it afterwards.

Further Related Literature. For a relatively recent overview on classical power indices for simple games see (Andjiga et al. 2003; Benati & Marzetti 2013), among others. The five widespread power indices in the literature are (Shapley & Shubik 1954; Banzhaf III 1964; Deegan & Packel 1978; Johnston 1978; Holler & Packel 1983). As aforementioned, most of these classical indices mostly measure the power of players independently of the issues at stake, or regardless of the affinities among the players. There are, however, the so-called spatial power indices which attempt to redress these neglects whenever simple games turn into political games through spatial models. The pioneer of such approach is Owen (1971) and he applied his findings upon the eleven-party Israeli Knesset formed in 1965, but his index is more known as the Owen-Shapley spatial index since it is more applied with respect to the Shapley's technique and viewpoint. In fact, the Owen-Shapley spatial index is so far, the most widely spatial index examined and applied in various parliaments (USA, Israel, Japan, Russia, Spain, etc.) — see, (Frank & Shapley 1981; Godfrey 2005a,b; Owen & Shapley 1989; Ono & Muto 1997). However, some significant contributions on spatial power indices in general, were made by (Shapley 1977; Shenoy 1982; Straffin 1977; Rapoport & Golan 1985; Bilal et al. 2001), and more recently have been investigated by (Passarelli & Barr 2007; Alonso-Meijide et al. 2011; Benati & Marzetti 2013; Martin et al. 2017; Grech 2019)[1]. To date, the unique axiomatic characterization of any spatial power index (notably the Owen-Shapley spatial index) was proposed by Peters & Zarzuelo (2017). Our findings are therefore, the first characterizations for the existing Deegan-Packel spatial index and for the new one that we propose — the Johnston spatial index.

[1] To understand more about spatial models of voting, we refer the reader to Enelow & Hinich (1984).

Organization of the paper. Preliminaries on simple games and classical power indices are presented in the next section. Sect. 3 introduces the spatial model and presents spatial probabilities. Our main results are presented both, in Sect. 4 by providing the spatial power indices of the paper, and in Sect. 5 by characterizing all of them. Lastly, Sect. 6 is devoted to concluding comments, and all the proofs are given in the Appendix.

2 Preliminaries

The purpose of this section is twofold, specify the notations and concepts that we will use and, introduce some elements of game theory that we will refer to. First, recall that, a *game* is a real-valued function defined on the subsets of a given nonempty finite set N, which vanishes on the empty set. The elements of N are usually called players.

Notations and Definitions. Throughout this document, we will use the following basic notations. Let $N \equiv \{1, 2, ..., n\}$ denote the finite set of all players indexed by the first n natural numbers. Any nonempty subset of N is called *coalition*. Let 2^N denote the set of all nonempty coalitions. By $|E|$ we mean the cardinality of any set E. Next, we recall the formal definition of simple games, which are particularly attractive in political structures such as parliaments and committees. A simple game can be represented by a pair $(N, \mathcal{W})$, where $\mathcal{W}$ is the set of winning coalitions such that: (i) $\emptyset \notin \mathcal{W}$, $N \in \mathcal{W}$, and (ii) if ($S \in \mathcal{W}$ and $S \subset T$), then $T \in \mathcal{W}$. Any coalition which is not winning is *losing*, and a simple game is said to be *proper* if and only if the complement of every winning coalition is losing. Henceforth, we only consider simple games that are proper.

There are three types of players in simple games, which deserve mentioning in this paper. The first one — a *veto player* — is a member who belongs to every winning coalition. The second one — a *dummy player* — is a player in a simple game who can never, by joining a losing coalition, change it into a winning one. The third one is related to a given winning coalition S, and called *decisive player* for S — it is a player whose by leaving S, makes the remaining coalition losing. We will denote by $\eta(S)$ the set of decisive players in a coalition S, and by $\mathcal{Q}_i(\mathcal{W})$ the set of coalitions for which the player i is decisive. A *minimal winning coalition* is a winning coalition in which all its players are decisive. We denote by $\mathcal{M}(\mathcal{W})$ (respectively $\mathcal{M}_i(\mathcal{W})$) the set of minimal winning coalitions (respectively the set of minimal winning coalitions containing the player i). A *quasi-minimal winning coalition* is a winning coalition in which at least one player is decisive. We denote by $\mathcal{Q}(\mathcal{W})$ the set of all quasi-minimal winning coalitions. More formally, $\mathcal{Q}(\mathcal{W}) := \cup_{i\in N}\mathcal{Q}_i(\mathcal{W})$. In case of no confusion, the sets $\mathcal{Q}(\mathcal{W})$, $\mathcal{Q}_i(\mathcal{W}), \mathcal{M}(\mathcal{W})$ and $\mathcal{M}_i(\mathcal{W})$ will be simply rewritten by $\mathcal{Q}, \mathcal{Q}_i, \mathcal{M}$ and $\mathcal{M}_i$. Using the concepts and tools mentioned so far, we can now introduce the concept of power index as well as many well-known examples.

Power indices are relevant tools for measuring the influence of players on game decisions. A power index is a function Ψ which assigns to each simple game $(N, \mathcal{W})$ a vector $\Psi(N, \mathcal{W}) \in \mathbb{R}^m, m \in \mathbb{N}$, where each component $\Psi_i(N, \mathcal{W})$ provides a measure of power to the player i in the simple game $(N, \mathcal{W})$. As aforementioned, in the literature several classical power indices have been introduced whereby, the five most well-known are the indices of Shapley-Shubik, Banzhaf, Holler-Packel, Deegan-Packel and Johnston. However, in this paper we focus on the two latter ones, which are recalled below.

Deegan-Packel power index (Deegan & Packel 1978) (**DP**). The main idea about this index is to assume that only minimal winning coalitions will emerge, but there are other criteria. Indeed, the Deegan-Packel index denoted by DP is formally given for any simple game $(N, \mathcal{W})$ and any $i \in N$, by:

$$DP_i(N, \mathcal{W}) \quad := \quad \frac{1}{|\mathcal{M}(\mathcal{W})|} \sum_{S\in\mathcal{M}_i(\mathcal{W})} \frac{1}{|S|}.$$

To go deeper into details, the following assumptions are those whereby the DP index is obtained: (i) only minimal winning coalitions will form; (ii) the spoils in each minimal winning coalition is divided equally among all its members; and (iii) each minimal winning coalition has the same probability of forming.

Johnston power index (Johnston 1978) (**J**). This index is quite similar to the DP power index, where in fact instead of considering minimal winning coalitions, we must consider quasi-minimal coalitions. Intuitively, Johnston estimates that the measure of power must depend on the number of decisive players in any coalition. His idea is the following, the larger the number of decisive players in a coalition, the lower the power of any decisive player in that coalition. Furthermore, he suggests that the power in a winning coalition must be shared equally among all its decisive players. Then, relatedly, Johnston index denoted by J is given for any simple game $(N, \mathcal{W})$ and any $i \in N$, by:

$$J_i(N, \mathcal{W}) \quad := \quad \frac{1}{|\mathcal{Q}(\mathcal{W})|} \sum_{S \in \mathcal{Q}_i(\mathcal{W})} \frac{1}{|\eta(S)|},$$

3 The spatial model

This section, representing the groundwork of this paper, concerns what are known as spatial games, based on empirical political games. Roughly speaking, these kinds of games are defined as simple voting games correlated with players spatial preferences in the $m-$ dimensional euclidean space $\mathbb{R}^m$. Arguably, the linear structure of $\mathbb{R}^m$ will help us to capture some intuitive ideas like moderation and extremism, among others. For instance, when $m = 1$, we can think of $\mathbb{R}^1$ as the left to right spectrum of political ideology. Actually, in $\mathbb{R}^m$, each dimension represents a political or ideological parameter, such as left/right in economics policy, or left/right in internationalism policy, or economic growth/environment, etc.

Therefore, spatial games represent in the best way many political games in which each player has her *ideal point*[2] assumed to be, related to the ideological context, her most preferred point in a given subdomain $\mathbb{D}$ of $\mathbb{R}^m$ (voters are represented in this space by ideal points denoting their most preferred positions on the issues). The domain $\mathbb{D}$ is therefore called the *ideological space* associated to the spatial game, and players are identified by their ideal points in $\mathbb{D}$. Furthermore, each player's beliefs are led by euclidean preferences in the sense that, the more a point is close to her ideal point the more she prefers that point. In other words, each player prefers points that are closer to her ideal point to those further away. So far, with respect to spatial power indices, there are very few spatial framework in the literature, but the one we will consider is presented below.

A spatial game $\mathcal{V}$ defined on an m-dimensional space $\mathbb{D} \subset \mathbb{R}^m$, is a triplet $(N, \mathcal{W}, Q_N)$, satisfying the two following items: (i) $(N, \mathcal{W})$ is a simple game, with a characteristic

[2] Those kinds of points represent ideological descriptions or political profiles of players in the Euclidean space.

function $V : 2^N \longrightarrow \{0,1\}$ such that $V(S) = 1$ if and only if $S \in \mathcal{W}$; and (ii) $Q_N := \{Q^i\}_{i\in N}$, where $Q^i \in \mathbb{D}$ represents the ideal point of player i. We denote the spatial game in the ideological space $\mathbb{D} \subseteq \mathbb{R}^m$ by $\mathcal{V}(\mathbb{D}) := (N, \mathcal{W}, Q_N)$, and $Q_N \in \mathbb{D}^n$ is called a *constellation over* N. Furthermore $\mathcal{G}_s$, $\mathcal{G}_s^N$, and $\mathcal{G}_s(Q_N)$, represent respectively the set of all spatial games, the set of all spatial games on N, and the set of all spatial games with the constellation Q_N.

3.1 Spatial framework

We assume in our model that, there is no restriction about ideal points' positions on the political space. In other words, our space $\mathbb{D}$ is the whole $m-$dimensional space $\mathbb{R}^m$, and for the sake of convenience $\mathcal{V}(\mathbb{R}^m)$ would be simply denoted by $\mathcal{V}$, with $\mathcal{V} \equiv (N, \mathcal{W}, Q_N)$. In addition, players are allowed to have the same ideal points[3]. Since players do not necessarily have the same preferences, a useful tool for measuring the affinity's degree between two players, is the so-called *inter-player distance* defined below.

Inter-player distance. Let N be the player set, Q_N any constellation over N, and $i, j \in N$ two players (not necessarily with distinct ideal points). The *inter-player distance* between i and j denoted by $d(i,j)$, is the euclidean distance between Q^i and Q^j. Formally,

$$d(i,j) \quad := \quad \left(\sum_{k=1}^{m} \left(Q_k^i - Q_k^j\right)^2 \right)^{\frac{1}{2}}. \tag{3A}$$

The notion of inter-player distance allows us to define the following concept.

Mean distance. Let $S \in 2^N$ be any coalition, and $s := |S|$. The mean distance of S denoted by $d(S)$, is formally defined from (3A) as follows,

$$d(S) \; := \; 0 \quad \text{if} \;\; s = 0 \; ; \qquad \text{and} \qquad d(S) \; := \; \frac{1}{h_S} \sum_{\substack{i,j\in S \\ i<j}} d(i,j) \quad \text{otherwise.}$$

The parameter $h_S := s(s-1)/2$, represents the total number of pairs in the coalition S. The mean distance of any coalition may be interpreted as a measure of the internal

[3] In fact, some other spatial approaches (e.g. Shapley (1977), Alonso-Meijide et al. (2011)), do not allow this possibility; however we think that, it is the more natural case to fit the simple game without prior information, since players are all supposed to have basically the same preferences.

cohesiveness among all its members. Thus, and relatedly, we argue that, the more cohesive the coalition, the more likely it will form.

3.2 Spatial probabilities

Following the same approach as our considered classical power indices, we will see in the present paper that, coalitions of players play a decisive role in the computation of the Deegan-Packel and Johnston spatial indices that we will define in the next section. For the spatial probabilities about coalition formation, we will first consider a collection of coalitions $\Gamma \subseteq 2^N$, and then spatially determine the probability of formations of any of those coalitions in Γ. The occurrence probability for any coalition solely depends on the weight of that coalition, which is introduced as follows.

Weight functions on coalitions. Given any decreasing non negative function ϕ defined on $[0, +\infty[$, the weight function denoted by W, is defined as follows:

$$\begin{aligned} W \quad : \quad \Gamma \quad &\longrightarrow \quad [0, +\infty[\\ T \quad &\longmapsto \quad \phi\left(d(T)\right). \end{aligned}$$

As already mentioned, the farther the ideal points of some players, the less likely those players will form a coalition. Thus, in terms of weights, the higher the weight of a coalition is, the more that coalition is likely to form. However, from ϕ and given any $T \in \Gamma$, the weight $W(T)$ only depends on $d(T)$, then the lower the mean distance $d(T)$, the higher the weight $W(T)$.[4]

Let us consider the two particular examples that we will also consider in the next section. For any $T \in \Gamma$,

$$\text{if } \ \phi \text{ is the Inverse function } \ (\text{i.e. } \phi(x) := 1/x)\,, \ \text{ then } W(T) = \frac{1}{d(T)}. \tag{3B}$$

$$\text{if } \ \phi \text{ is rather exponential } \big(\text{i.e. } \phi(x) := e^{-x}\big)\,, \ \text{ then } W(T) = e^{-d(T)}. \tag{3C}$$

[4] Note: such functions ϕ, provide a family of weight functions W_ϕ on Γ. For staters, the reader may consider first, the two examples given below.

Probability of occurrence. Given any weight function W on Γ, we now define the probability of occurrence $P_{\Gamma}^{W}(S)$, for any $S \in \Gamma$ as follows.

$$P_{\Gamma}^{W}(S) \quad := \quad \frac{W(S)}{\sum_{T\in\Gamma} W(T)}. \tag{3D}$$

Thus, $\left\{P_{\Gamma}^{W}(T)\right\}_{T\in\Gamma}$ form a probability distribution on Γ. Intuitively, the equiprobability case will bring us to the assumptions of a priori power indices. So, under which conditions could we obtain equiprobable distribution?

Case of equiprobability. It suffices to have the same non-zero weight for each possible coalition, which would happen whenever at least one of the three following conditions, would be satisfied[5]:

(C0) : all ideal points are the same;

(C1) : all inter-player distances $d(i, j)$ are the same for all $i, j \in N$;

(C2) : all the coalitions $S \in \Gamma$ have the same mean distance $d(S)$.

Remark. From the probability approach about the equiprobability case, the natural and intuitive condition is **(C0)**. Indeed, whenever all players have the same ideal points, there are no longer differences between them, and hence we are brought to classical assumptions of the a priori index. In fact, we will see next that, any spatial power index allowing this hypothesis[6], will straightforwardly coincide with its a priori version whenever the condition **(C0)** occurs.

4 Deegan-Packel and Johnston spatial power indices

This section mainly represents the heart of our work. Let $\mathcal{V} \equiv (N, \mathcal{W}, Q_N)$ be a spatial game. We generalize below, in view of our spatial model, the two classical power indices previously recalled. No need to argue that, by contrast to the other spatial models in the literature such as in Owen (1971), Shapley (1977), Shenoy (1982), in our spatial model, a huge advantage is the fact that we do not need to reduce the dimension of

[5] In those three conditions, **(C0)** and **(C1)** are particular cases of **(C2)**.

[6] Actually, some of our spatial power indices constraint players to have different ideal points one another.

data for computing any spatial power index. Therefore, in our model, we do not lose any information in computations.

4.1 Spatial approaches of the Deegan-Packel power index

Following the same approach as Rapoport & Golan (1985) did, we consider the two first hypotheses of the classical Deegan-Packel index unchanged, what changes now is the third assumption, and we will spatially modify it in many ways.

4.1.1 Distance Deegan-Packel spatial index

For every spatial game $\mathcal{V}$, the (Distance) Deegan-Packel spatial index denoted by $\Psi^{DDP} := \left(\Psi_i^{DDP}\right)_{i\in N}$, is defined for any player i as follows,

$$\Psi_i^{DDP}(\mathcal{V}) \quad := \quad \sum_{S\in\mathcal{M}_i} \frac{1}{s}\, P_{\mathcal{M}}^W(S) \quad = \quad \sum_{S\in\mathcal{M}_i} \frac{1}{s} \frac{1/d(S)}{\sum_{T\in\mathcal{M}} 1/d(T)}, \quad \text{with} \quad s := |S|.$$

Where $\left(P_{\mathcal{M}}^W(S)\right)_{S\in\mathcal{M}}$ is the probability distribution obtained from (3D).

To obtain the classical Deegan-Packel power index, it suffices to make $d(S)$ be the same non zero constant for every coalition $S \in \mathcal{M}$ (condition (C2)). To do this, it suffices that all interplayer distances are all equal to the same value, let's say α_0. Then, we will have,

$$P_{\mathcal{M}}^W(S) \quad = \quad \frac{1/d(S)}{\sum_{T\in\mathcal{M}} 1/d(T)} \quad = \quad \frac{1/\alpha_0}{\sum_{T\in\mathcal{M}} 1/\alpha_0} \quad = \quad \frac{1}{|\mathcal{M}|}.$$

Thus,

$$\Psi_i^{DDP}(\mathcal{V}) \quad = \quad \sum_{S\in\mathcal{M}} \frac{1}{s} \frac{1}{|\mathcal{M}|} \quad = \quad DP_i(N, \mathcal{W}).$$

Remark. If we allow players to have same ideal points, then it could happen that $d(S) = 0$. In that case, $\frac{1}{d(S)}$ would make no sense for computing the Distance Deegan-Packel spatial index. Therefore, we require for the application of this index that, players have different ideal points from one to another. But if it happens that some players have same ideal points, one might consider the spatial power index we propose below.

4.1.2 The Exponential Deegan-Packel spatial index

An alternative to the previous spatial index, is the one that we call the *Exponential Deegan-Pakel spatial index*, allowing players to have same ideal points. In fact, inspired from Bilal et al. (2001), we defined the Exponential Deegan-Packel spatial index denoted $\Psi^{EDP} := \left(\Psi_i^{EDP}\right)_{i\in N}$ as follows: for every spatial game $\mathcal{V}$ and any $i \in N$,

$$\Psi_i^{EDP}(\mathcal{V}) \quad := \quad \sum_{S\in\mathcal{M}_i} \frac{1}{s} \frac{\exp\left(-d(S)\right)}{\sum_{T\in\mathcal{M}} \exp\left(-d(T)\right)} \quad \text{with} \quad s := |S|.$$

This new spatial index is more intuitive when all players have the same ideal points (condition (C0)), which means that players are all symmetric and therefore, the index corresponds to the third hypothesis of the original Deegan-Packel power index, in which our new spatial index coincides, since $d(S) = 0$ implies that $\Psi_i^{EDP}(\mathcal{V}) = DP_i(N, \mathcal{W})$. To obtain the original Deegan-Packel power index, it suffices that players behave symmetrically, whereas, the Distance Deegan-Packel spatial index requires more complex hypotheses to do so, especially in higher dimensions m (where we should have at least $m \geq n - 1$).

4.1.3 Generalized Deegan-Packel spatial index

Basically, the classical Deegan-Packel power index is defined from conditions given in Sect. 2. The two first are retained, but the third one is replaced by the spatial probability of forming coalition with respect to our spatial model. Given any weight function W on $\Gamma := \mathcal{M}$, the probability of forming any coalition $S \in \Gamma$ is given from the probability distribution $\left(P_{\mathcal{M}}^{W}(S)\right)_{S\in\mathcal{M}}$ defined in (3D). Then, we straightforwardly obtain the Generalized Deegan-Packel spatial index denoted $\Psi^{GDP} := \left(\Psi_i^{GDP}\right)_{i\in N}$ and defined for any $i \in N$ as follows:

$$\Psi_i^{GDP}(\mathcal{V}) \quad := \quad \sum_{S\in\mathcal{M}_i} \frac{1}{s} \frac{W(S)}{\sum_{T\in\mathcal{M}} W(T)} \quad \text{where} \quad s := |S|. \qquad (\star)$$

Under condition (C2), we are reduced to the original Deegan-Packel index. On the other hand, from $(\star)$, when $W : T \mapsto 1/d(T)$, we obtain the Distance Deegan-Packel spatial index, while when $W : T \mapsto e^{-d(T)}$ we get the Exponential Deegan-Packel spatial index.

4.2 Spatial approaches of the Johnston power index

Following the two previous approaches, we now define the Johnston spatial index version with respect to our spatial model as follows.

4.2.1 Distance Johnston spatial power index

For every spatial game $\mathcal{V} = (N, \mathcal{W}, Q_N)$, the Distance Johnston spatial power index $\Psi^{DJ} := \left(\Psi_i^{DJ}\right)_{i\in N}$, is given for each player $i \in N$ by,

$$\Psi_i^{DJ}(\mathcal{V}) \quad := \quad \sum_{S\in\mathcal{Q}_i} \frac{1}{|\eta(S)|}\, P_{\mathcal{Q}}^{W}(S) \quad = \quad \sum_{S\in\mathcal{Q}_i} \frac{1}{|\eta(S)|} \frac{1/d(S)}{\sum_{T\in\mathcal{Q}} 1/d(T)}.$$

Where $\left(P_{\mathcal{Q}}^{W}(S)\right)_{S\in\mathcal{Q}}$ is the probability distribution obtained from (3D).

We clearly obtain the original Johnston power index under condition (C2).

4.2.2 The Exponential Johnston spatial index

An alternative spatial version of the Johnston power index from the previous one), which allows players to have the same ideal points, is what we call the *Exponential Johnston spatial index* denoted by $\Psi^{EJ} := \left(\Psi_i^{EJ}\right)_{i\in N}$ and defined as follows. For every m-dimensional spatial game $\mathcal{V}$ the vector $\Psi^{EJ}(\mathcal{V})$ is such that, for each $i \in N$,

$$\Psi_i^{EJ}(\mathcal{V}) \quad := \quad \sum_{S\in\mathcal{Q}_i} \frac{1}{|\eta(S)|} \frac{\exp\left(-d(S)\right)}{\sum_{T\in\mathcal{Q}} \exp\left(-d(T)\right)}.$$

As above, this new spatial index is more intuitive under condition (C0), which means that players are all symmetric and therefore, corresponds to the third hypothesis of the original Johnston power index, in which this new spatial index coincides. To obtain the original Johnston power index, it suffices that players behave symmetrically, which is not the case for the Distance Johnston spatial index.

4.2.3 Generalized Johnston spatial power index

Basically, the Johnston power index is defined as given in Sect. 2. Now, we focus on the last assumption, the one related to the coalition formation. Below is its general form with respect to our spatial model. Given any weight function W on $\Gamma := \mathcal{Q}$, the probability of forming any coalition $S \in \Gamma$ is $P_{\Gamma}^{W}(S)$, as defined in (3D) . Once done, we straightforwardly obtain the Generalized Johnston spatial power index denoted $\Psi^{GJ} := \left(\Psi_i^{GJ}\right)_{i\in N}$ which is given for each $i \in N$ by:

$$\Psi_i^{GJ}(\mathcal{V}) \quad := \quad \sum_{S\in\mathcal{Q}_i} \frac{1}{|\eta(S)|} \frac{W(S)}{\sum_{T\in\mathcal{Q}} W(T)}. \qquad (\star\star)$$

Under condition (C2), we are reduced to the original Johnston index. On the other hand, from $(\star\star)$, when $W : T \mapsto 1/d(T)$, we obtain the Distance Johnston spatial index, while when $W : T \mapsto e^{-d(T)}$ we get the Exponential Johnston spatial index.

5 Axiomatic characterizations

We first need to define some essential notions.

Unanimous spatial games. Let $\mathcal{V} \equiv (N, \mathcal{W}, Q_N)$ be a spatial game; $\mathcal{V}$ is said to be unanimous if it has a unique minimal winning coalition. Thus, there exist $S \in 2^N$ such that $\mathcal{M}(\mathcal{W}) = \{S\}$, and $\mathcal{V}$ would be henceforth denoted by $\mathcal{V}_S \equiv (N, \mathcal{W}_S, Q_N)$ with $\mathcal{M}(\mathcal{W}_S) = \{S\}$.

Mergeable spatial games. Let $\mathcal{V}_1 \equiv (N, \mathcal{W}_1, Q_N)$ and $\mathcal{V}_2 \equiv (N, \mathcal{W}_2, Q_N)$ be two spatial games. Then, $\mathcal{V}_1, \mathcal{V}_2$, are mergeable if, in addition to have the same constellation, they satisfy the following property:

$$\text{if } \ S_1 \in \mathcal{M}(\mathcal{W}_1) \text{ and } \ S_2 \in \mathcal{M}_2(\mathcal{W}_2), \text{ then } S_1 \nsubseteq S_2.$$

This definition says that both games have no overlap in the sense that no minimal winning coalition in one game can be winning in the other.

Before stating the axioms, we need to define the operations $\vee$ and $\wedge$ as follows. Let N be a player set, $Q_N \in \mathbb{R}^m$, and $\mathcal{V}_1, \mathcal{V}_2 \in \mathcal{G}_s(Q_N)$, we have $\mathcal{V}_1 \vee \mathcal{V}_2$, $\mathcal{V}_1 \wedge \mathcal{V}_2 \in \mathcal{G}_s(Q_N)$ where:

$$\mathcal{W}(\mathcal{V}_1 \vee \mathcal{V}_2) = \mathcal{W}(\mathcal{V}_1) \cup \mathcal{W}(\mathcal{V}_2) \quad \text{and} \quad \mathcal{W}(\mathcal{V}_1 \wedge \mathcal{V}_2) = \mathcal{W}(\mathcal{V}_1) \cap \mathcal{W}(\mathcal{V}_2).$$

Here are our first axioms.

A1. (*Efficiency*) Let $\mathcal{V} \equiv (N, \mathcal{W}, Q_N)$ be a spatial game.

$$\sum_{i \in N} \Psi_i(\mathcal{V}) = 1.$$

This appealing property, together with both axioms **A2 - A3** below, will appear in each forthcoming characterization.

A2. (*Strong Dummy property*) A player i is dummy if and only if $\Psi_i(\mathcal{V}) = 0$.

This property ensures that only dummy players have a null power. Some spatial power indices like the Owen-Shapley index (Shapley 1977), do not satisfy this axiom.

A3. (*Vetoer property*) If $i, j \in N$ are veto players, then $\Psi_i(\mathcal{V}) = \Psi_j(\mathcal{V})$.

This new axiom that we propose, says that any veto player in a spatial game has the same power, which is a desirable property since without all veto players, a coalition can never win. Throughout this section, we will consider this axiom rather than the standard *symmetric axiom* given by Shapley (1953).[7]

Our new axiom *Vetoer property* along with both the *Strong Dummy property* and *Efficiency*, are crucial for all our three characterizations in this paper (since they appear in each of our characterizations). Therefore, they will be maintained for the remainder of this section. For the sake of convenience, we posit $\Delta(\mathcal{V}) := \sum_{T \in \mathcal{M}(\mathcal{W})} W(T)$.

A4. (*DP-mergeability*) Let $\mathcal{V}_1 \equiv (N, \mathcal{W}_1, Q_N)$ and $\mathcal{V}_2 \equiv (N, \mathcal{W}_2, Q_N)$ be two mergeable spatial games. Then,

$$\Psi_i(\mathcal{V}_1 \vee \mathcal{V}_2) = \frac{1}{\Delta(\mathcal{V}_1) + \Delta(\mathcal{V}_2)} \left[\Delta(\mathcal{V}_1)\Psi_i(\mathcal{V}_1) + \Delta(\mathcal{V}_2)\Psi_i(\mathcal{V}_2)\right].$$

This is similar in spirit to Deegan & Packel (1978), p 27. It requires that power in the merged game must be the weighted mean of powers in the component games, with respect to $\Delta(\mathcal{V}_1)/[\Delta(\mathcal{V}_1) + \Delta(\mathcal{V}_2)]$ and $\Delta(\mathcal{V}_2)/[\Delta(\mathcal{V}_1) + \Delta(\mathcal{V}_2)]$ as weights.

Here is our first axiomatic characterization.

[7] Our new axiom weakens the standard symmetric property proposed by Shapley (1953) and we argue that we can replace the symmetric axiom by the Vetoer axiom in a large number of axiomatic characterizations, without many changes in their corresponding proofs.

Theorem 1 *The unique spatial power index satisfying the four axioms **A1 - A4** is the Generalized Deegan-Packel spatial power index* Ψ^{GDP}.

The second axiomatic characterization requires the next axiom.

A5. (*DP-minimal monotonicity*) Let $\mathcal{V}_1 \equiv (N, \mathcal{W}_1, Q_N)$ and $\mathcal{V}_2 \equiv (N, \mathcal{W}_2, Q_N)$ be two spatial games. Then,

$$\Psi_i(\mathcal{V}_1) \times \Delta(\mathcal{V}_1) \qquad \geq \qquad \Psi_i(\mathcal{V}_2) \times \Delta(\mathcal{V}_2),$$

for all player $i \in N$, such that $\mathcal{M}_i(\mathcal{W}_1) \subseteq \mathcal{M}_j(\mathcal{W}_2)$.

This axiom is similar to the strong monotonicity proposed in Young (1985), p 69. Here is our second axiomatic characterization.

Theorem 2 *The unique spatial power index satisfying the four axioms **A1 - A3**, **A5** is the Generalized Deegan-Packel spatial power index* Ψ^{GDP}.

It is worth mentioning that, when the function weight W is given by, $W : T \longmapsto 1/d(T)$, the two theorems above become the first axiomatic characterizations (to be best of our knowledge) of the Deegan-Packel spatial index proposed by Rapoport & Golan (1985).

Now, the next axiom is mainly inspired from Lorenzo-Freire et al. (2007). Recall that for all $S \in 2^N, \mathcal{V}_S \equiv (N, \mathcal{W}_S, Q_N)$, with $\mathcal{M}(\mathcal{W}_S) := \{S\}$.

A6. (*J-mergeability*) Let $\mathcal{V} \equiv (N, \mathcal{W}, Q_N)$ be any spatial game such that $\mathcal{M}(\mathcal{W}) := \{S_1, \ldots, S_m\}$. Let $M := \{1, \ldots, m\}$, and for all $S \in 2^N$, we posit, $\mathcal{Q}(S, \mathcal{W}) := \{T \in \mathcal{Q}(\mathcal{W}) : \eta(T) = S\}$. Then, for each $i \in N$,

$$\Psi_i(\mathcal{V}) \quad = \quad \frac{1}{\sum_{T \in Q(\mathcal{W})} W(T)} \sum_{S \in \mathcal{F}} \sum_{R \in Q(S, \mathcal{W})} W(R) \Psi_i(\mathcal{V}_S).$$

Where $\mathcal{F} := \left\{ \bigcap_{j \in R} S_j : \bigcap_{j \in R} S_j \neq \varnothing, R \subseteq M \right\}$.

Below is the first characterization of the Generalized Johnston spatial power index.

Theorem 3 *The unique spatial power index satisfying the four axioms **A1 - A3**, **A6** is the Generalized Johnston spatial power index* Ψ^{GJ}.

6 Concluding Comments

Although the Owen-Shapley index is the most known spatial power index in the literature, the Banzhaf spatial power index proposed by Shenoy, can perform better in some real case studies as Rapoport & Golan (1985) have known. This suggests that the widespread Owen-Shapley spatial index can be defeated by other spatial indices in some cases — including the ones presented in this paper. On another hand, when we need to obtain the original power indices from the spatial ones, our indices perform better. In effect, the restriction of the Owen-Shapley index upon its original form is "quite hard": it requires $n = m$ for coinciding with the classic case. However, whenever n is bigger than 3, the computation is overwhelming. So for more than four players, we get stuck on how we could restrict to the classic case; and to overcome this latter problem, some scholars usually use some dimensionality reduction techniques (but at the cost of lost of data). However, and in contrast to the Owen-Shapley index, our spatial indices easily coincide with their original forms, since it suffices to restrict all the ideal points to the same point (which basically means that all voters have symmetric behaviors — a fundamental assumption to the original power indices' concept, as we already know).

Our paper might reply to the observation raised by Fuad Aleskerov in his article *Power indices taking into account agents' preferences*, which says that: "One of the main shortcomings mentioned almost in all publications on power indices is the fact that well-known indices do not take into account the preferences of agents". Indeed, our paper develops two alternatives power indices which consider the preferences of the players — the Deegan-Packel spatial index and the Johnston spatial index — but we argue that a large amount of classical power indices can be generalized in our spatial model, a straightforward example (among others), is the *Andjiga-Berg index* (Andjiga et al., 2003 p. 120). In fact, we argue that, through the spatial model detailed in the present paper, we are able to consistently generalize, using the same hypotheses, the most classical power indices so that we will henceforth obtain a uniform family of spatial power indices. Another spatial index which deserves mentioning is the generalization of the Shapley-Subik power index, proposed by Alonso-Meijide et al. (2011) and called "Distance index". They proposed it along with some properties in the goal of characterizing, but without doing it. We argue that, with respect to our methodology of generalization using weight functions W, we can also generalize their index, but a similar axiomatic characterization is not guaranteed so far.

Apart from the six axioms presented in the paper, one might also investigate many other relevant and "desirable" properties that our spatial power indices could ideally satisfy, such as the *null player property* proposed by (Alonso-Meijide et al. 2011) or the *rotational invariance axiom* defined in Peters & Zarzuelo (2017). Obviously, our results can be seen as the beginning of a sequence of future researches in which we will deeply investigate about the properties of these spatial indices. Lastly, and empirically speaking, applications of our spatial power indices to real case studies (such as any parliament of any country in the European Union, or others), are beyond the scope of this paper, however, there are devoted to future works. Furthermore, the fundamental conception of power is not discussed in this paper. Thus, whether some indices are P-power or I-power[8], we just wanted to consider the preferences of voters through their ideal points to some of the existing power indices, and we did it.

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Appendix: Proofs

The next lemma is useful to demonstrate our theorems. We will also need to refer to the assumption (H0) specified below.

Lemma 1 *Let N be a player set, and $\mathcal{V}_1, \mathcal{V}_2 \in \mathcal{G}_s(Q_N)$ be two mergeable games. Then,*

1. $\mathcal{M}(\mathcal{W}_1 \vee \mathcal{W}_2) = \mathcal{M}(\mathcal{W}_1) \bigcup \mathcal{M}(\mathcal{W}_2)$, *and* $\mathcal{M}(\mathcal{W}_1) \bigcap \mathcal{M}(\mathcal{W}_2) = \emptyset$.
2. $\mathcal{M}_i(\mathcal{W}_1 \vee \mathcal{W}_2) = \mathcal{M}_i(\mathcal{W}_1) \bigcup \mathcal{M}_i(\mathcal{W}_2)$, *and* $\mathcal{M}_i(\mathcal{W}_1) \bigcap \mathcal{M}_i(\mathcal{W}_2) = \emptyset$, *for all* $i \in N$.
3. $\Delta(\mathcal{V}_1 \vee \mathcal{V}_2) = \Delta(\mathcal{V}_1) + \Delta(\mathcal{V}_2)$.

Proof. Clearly obvious by definitions.

[8] See more about P-Power and I-power in Felsenthal et al. (1998)

Assumption (H0): $\mathcal{V} \equiv (N, \mathcal{W}, Q_N)$ is a spatial game such that $\mathcal{M}(\mathcal{W}) := \{S_1, \ldots, S_m\}$, $M := \{1, \ldots, m\}$, and for all $p \in M, \mathcal{V}_p \equiv (N, \mathcal{W}_p, Q_N)$, with $\mathcal{M}(\mathcal{W}_p) := \{S_p\}$.

Proof of Theorem 1.

Existence : obviously, Ψ^{GDP} satisfies the three axioms **A1 - A3**. For the last one **A4**, let $\mathcal{V}_1, \mathcal{V}_2$ be two spatial games, then with $i \in N$ we have :

$$\begin{aligned}
\Psi_i^{GDP}(\mathcal{V}_1 \vee \mathcal{V}_2) &:= \sum_{S \in \mathcal{M}_i(\mathcal{W}_1 \vee \mathcal{W}_2)} \frac{1}{|S|} \frac{W(S)}{\Delta(\mathcal{V}_1 \vee \mathcal{V}_2)} = \frac{1}{\Delta(\mathcal{V}_1 \vee \mathcal{V}_2)} \sum_{S \in \mathcal{M}_i(\mathcal{W}_1 \vee \mathcal{W}_2)} \frac{W(S)}{|S|} \\
&\underset{(*)}{=} \frac{1}{\Delta(\mathcal{V}_1 \vee \mathcal{V}_2)} \left[\sum_{S \in \mathcal{M}_i(\mathcal{W}_1)} \frac{W(S)}{|S|} + \sum_{S \in \mathcal{M}_i(\mathcal{W}_2)} \frac{W(S)}{|S|} \right] \\
&\underset{(\dagger)}{=} \frac{1}{\Delta(\mathcal{V}_1 \vee \mathcal{V}_2)} \left[\Delta(\mathcal{V}_1) \Psi_i^{GDP}(\mathcal{V}_1) + \Delta(\mathcal{V}_2) \Psi_i^{GDP}(\mathcal{V}_2) \right].
\end{aligned}$$

Where $(*)$ is by Lemma 1, and $(\dagger)$ by definition of Ψ^{GDP}. Therefore **A4** is satisfied.

Uniqueness : let ϕ be a spatial power index satisfying all the four axioms **A1 - A4**. Let $\mathcal{V}$ be a spatial game from assumption (H0), then by the three axioms **A1 - A3**, we have for any $i \in N$,

$$\phi_i(\mathcal{V}_p) = \begin{cases} \frac{1}{|S_p|} & \text{if } \quad i \in S_p \\ 0 & \text{otherwise.} \end{cases} \tag{E1}$$

Thus, axiom **A4** implies that,

$$\phi_i(\mathcal{V}) = \phi_i(\mathcal{V}_1 \vee \mathcal{V}_2 \vee \cdots \vee \mathcal{V}_m) = \frac{1}{\Delta(\mathcal{V})} \sum_{p \in M} \Delta(\mathcal{V}_p) \phi_i(\mathcal{V}_p).$$

$$\text{Then,} \quad \phi_i(\mathcal{V}) \underset{(*)}{=} \frac{1}{\Delta(\mathcal{V})} \sum_{p \in M} W(S_p) \phi_i(\mathcal{V}_p) \underset{\text{(E1)}}{=} \frac{1}{\Delta(\mathcal{V})} \sum_{S \in \mathcal{M}_i(\mathcal{W})} \frac{W(S)}{|S|} = \Psi_i^{GDP}(\mathcal{V}).$$

Where $(*)$ is by definition of $\Delta(\mathcal{V}_p)$. □

Proof of Theorem 2.

Existence : clearly, Ψ^{GDP} satisfies the three axioms **A1 - A3**. Now, let $\mathcal{V}_1, \mathcal{V}_2$ be two spatial games and $i \in N$ be such that $\mathcal{M}_i(\mathcal{W}_1) \subseteq \mathcal{M}_i(\mathcal{W}_2)$, then:

$$\Psi_i^{GDP}(\mathcal{V}_1) = \sum_{S\in\mathcal{M}_i(\mathcal{W}_1)} \frac{1}{|S|}\frac{W(S)}{\Delta(\mathcal{V}_1)}.$$

Furthermore, $$\Psi_i^{GDP}(\mathcal{V}_2) := \sum_{S\in\mathcal{M}_i(\mathcal{W}_2)} \frac{1}{|S|}\frac{W(S)}{\Delta(\mathcal{V}_2)}$$
$$= \frac{1}{\Delta(\mathcal{V}_2)}\left[\sum_{S\in\mathcal{M}_i(\mathcal{W}_1)} \frac{W(S)}{|S|} + \sum_{S\in\mathcal{M}_i(\mathcal{W}_2)\setminus\mathcal{M}_i(\mathcal{W}_1)} \frac{W(S)}{|S|}\right].$$

Thus, $\Psi_i^{GDP}(\mathcal{V}_2)\times\Delta(\mathcal{V}_2) \geq \Psi_i^{GDP}(\mathcal{V}_1)\times\Delta(\mathcal{V}_1)$. Therefore, **A5** is satisfied.

Uniqueness : let ϕ be a spatial power index satisfying the four axioms of the theorem.

The uniqueness of ϕ will be proved by induction.

Let $\mathcal{V}\in\mathcal{G}_s(Q_N)$ with $|\mathcal{M}(\mathcal{W})| = 1$, then there is $S\in 2^N$ such that $\mathcal{M}(\mathcal{W}) := \{S\}$. Clearly, the three axioms **A1 - A3** imply that $\phi_i(\mathcal{V}) = \begin{cases} \frac{1}{|S|} & \text{if } i\in S \\ 0 & \text{otherwise} \end{cases}$.

Then, the solution ϕ is unique when $|\mathcal{M}(\mathcal{W})| = 1$.

Next, assume the uniqueness of ϕ whenever $|\mathcal{M}(\mathcal{W})| \leq m-1$ with $m > 1$, and let $\mathcal{V}$ be any spatial game with $|\mathcal{M}(\mathcal{W})| = m$. We posit $\mathcal{M}(\mathcal{W}) := \{S_1, S_2, \ldots, S_m\}$. Let R be the set of veto players $\left(i.e\ R := \bigcap_{S\in\mathcal{W}} S\right)$. Consider for all $i\in N$ with $i\notin R$, the spatial game $\mathcal{V}_i \equiv (N, \mathcal{W}_i, Q_N)$ where $\mathcal{M}(\mathcal{W}_i) := \{S\in\mathcal{M}(\mathcal{W}) : i\in S\}$. Given that $\mathcal{M}_i(\mathcal{W}_i) = \mathcal{M}_i(\mathcal{W})$, by axiom **A5** we obtain $\phi_i(\mathcal{V})\times\Delta(\mathcal{V}) = \phi_i(\mathcal{V}_i)\times\Delta(\mathcal{V}_i)$. Therefore, for all $i\notin R$, it holds that: $\phi_i(\mathcal{V}) = \frac{\Delta(\mathcal{V}_i)}{\Delta(\mathcal{V})}\phi_i(\mathcal{V}_i)$. However, for all $i\notin R$, $|\mathcal{M}(\mathcal{W}_i)| < m$, and then, by the induction basis, $\phi_i(\mathcal{V}_i)$ is unique. That is to say, $\phi_i(\mathcal{V})$ is unique for all $i\notin R$.

Henceforth, it remains to show the uniqueness of $\phi_i(\mathcal{V})$ when $i\in R$. Indeed, through axiom **A3**, there exists a constant c, such that for all $i\in R, \phi_i(\mathcal{V}) = c$. Since $\phi_i(\mathcal{V})$ is unique for all $i\notin R$, axiom **A1** enforces the uniqueness of c. □

Proof of Theorem 3. For this proof, we consider a game $\mathcal{V}$ as given in assumption (H0).

Existence : clearly, Ψ^{GJ} satisfies the three axioms **A1 - A3**. Thus, to show that it also satisfies axiom **A6**, let us consider for all $S \in 2^N$, the unanimous spatial game $\mathcal{V}_S \equiv (N, \mathcal{W}_S, Q_N)$. Hence, by definition of such a game, we have for each $i \in N$:

$$\Psi_i^{GJ}(\mathcal{V}_S) = \begin{cases} \frac{1}{|S|} & \text{if} \quad i \in S \\ 0 & \text{otherwise} \quad . \end{cases} \tag{E2}$$

Next, take any spatial game $\mathcal{V}$ and any player $i \in N$. We have by definition of Ψ^{GJ},

$$\left[\sum_{T \in Q} W(T)\right] \Psi_i^{GJ}(\mathcal{V}) = \sum_{R \in Q_i(\mathcal{W})} \frac{W(R)}{|\eta(R)|} = \sum_{S \in \mathcal{F}} \sum_{R \in Q, \eta(R)=S} W(R) \Psi_i^{GJ}(\mathcal{V}_S)$$

$$\text{Thus,} \quad \left[\sum_{T \in Q} W(T)\right] \Psi_i^{GJ}(\mathcal{V}) \underset{\text{(E2)}}{=\!=} \sum_{S \in \mathcal{F}} \Psi_i^{GJ}(\mathcal{V}_S) \sum_{R \in Q_i(S, \mathcal{W})} W(R).$$

Hence, $\Psi_i(\mathcal{V}) = \sum_{S \in \mathcal{F}} \frac{\sum_{R \in Q(S,\mathcal{W})} W(R)}{\sum_{T \in Q(\mathcal{W})} W(T)} \Psi_i^{GJ}(\mathcal{V}_S)$, which means that Ψ^{GJ} satisfies **A6**.

Uniqueness : let ϕ be any spatial index satisfying the four axioms in the theorem. Consider $S \in 2^N$ and the unanimous game $\mathcal{V}_S$, then by the axioms **A1 - A3**, we have :

$$\text{for all } i \in N, \quad \phi_i(\mathcal{V}_S) = \begin{cases} \frac{1}{|S|} & \text{if} \quad i \in S \\ 0 & \text{otherwise} \end{cases} = \Psi_i^{GJ}(\mathcal{V}_S).$$

Now, let $\mathcal{V}$ be any spatial game, by axiom **A6** we have:

$$\text{for all } i \in N, \quad \phi_i(\mathcal{V}) = \sum_{S \in \mathcal{F}} \frac{\sum_{R \in Q(S,\mathcal{W})} W(R)}{\sum_{T \in Q(\mathcal{W})} W(T)} \phi_i(\mathcal{V}_S).$$

However, for all $i \in N$ $\phi_i(\mathcal{V}_S) = \Psi_i^{GJ}(\mathcal{V}_S)$, which implies that $\phi_i(\mathcal{V}) = \Psi_i^{GJ}(\mathcal{V})$. □

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