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# Approval Voting & Majority Judgment in Weighted Representative Democracy

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**Abstract** Pivato and Soh [Pivato, M., Soh, A., 2020. Weighted representative democracy. Journal of Mathematical Economics 88 (2020) 52–63] proposed a new system of democratic representation whereby any individual can choose any legislator as her representative and different legislators can represent different numbers of individuals, concomitantly determining their weights in the legislature. For such legislatures, we consider other voting rules, namely, the *Weighted Approval Voting rule* and *Weighted Majority Judgment rule*. We show that if the size of the electorate is large, then with very high probability, the decisions made by the legislature will be the same as the decisions that would have been reached by a direct democracy, as decided by the corresponding simple (unweighted) voting rules.

**Keywords** Social Choice · Ideal direct democracy · Representative democracy · Multioption decisions · Weighted Approval Voting · Weighted Majority Judgment.

**JEL classification.** D71, D72

*[A legislature] . . . should be an exact portrait, in miniature, of the people at large, as it should feel, reason and act like them".*

$\rightsquigarrow$ John Adams, 1776

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## 1 Introduction

Very recently, we have been confronted with some historical events, including the climate change issue, the Brexit issue, the Covid 19 pandemic and its consequences, and even protests against police brutality and institutional racism. From these recent events, one might argue that representative democracy is in crisis and the rise of populism reflects a loss of trust and respect for democratic institutions; stodgy, traditional political parties have been cast aside by voters in favor of demagogues who pander to fear, ignorance and racism, offering simplistic solutions to complex problems. Evidently, the current institutions of representative democracy have failed to satisfy these voters or to retain their trust. Indeed, these institutions often seem sclerotic, dysfunctional, and unresponsive.

These problems are related to decision-making procedures, and one might argue that the real will of the voting population may provide valuable insights into the answers. But, this assumes that the whole voting population is well informed about each individual issue, both in the short run and in the long run. In other words, the decisions that must be applied in society would ideally be those made by the whole population via a direct vote, under condition that voters are well informed about each issue and its implications in the short run and long-run future — in short, that voters are experts on each issue. This process, with all the previous requirements, is what we henceforth call *"ideal" direct democracy*. Due to the impracticability of such an ideal direct democracy (since no one can be an expert on all problems), the systems widely used instead are traditional representative democracies. However, the limitations and ineffectiveness of the latter have been amply demonstrated by many scholars (Dionne 2004; Cain et al. 2006; Craig 2018).

It is worth mentioning that, faced with the aforementioned public choice issues, a great number of hybrid possibilities between traditional representative democracies and what we have called ideal direct democracy do exist and were proposed more than half a century ago. For example, Miller (1969) argued in favor of direct referendums and delegations by proxy of giving voting rights to others, supported by using modern computer and communications technology to replace traditional systems of representative government. He was inspired by Kafoglis (1968) on *participatory democracies*, and many years later, Tullock

(1992) pushed forward this idea. By contrast, a new system called *weighted representative democracy* has been proposed very recently by Pivato and Soh (2020) to replace present representative democracies, but without necessarily using advanced technologies (its mechanism is detailed and discussed at length in the next section). However, whether the population itself is to decide on issues or not, another relevant aspect (and it is a serious one) is the voting rule that must be adopted to make final decisions on issues.

Although direct democracy is the best system to represent the will of the population, its performance depends on the voting rule to be used. There are many methods that we can consider, depending on what desirable properties we put forward. In practice, the majority rule is the rule used in most instances of democratic decision-making. For instance, most presidential elections use the majority rule or some variants. However, this voting rule has well-known unpleasant properties in problems with more than two alternatives. Fortunately, there are many other voting rules in social choice theory[1] that avoid the shortcomings of the majority rule on multioption problems. However, politics is the art of compromise, as it is often said; and in that spirit, we will therefore advocate for multioption problems, the use of two appealing voting rules that are well suited to this context, viz., Approval Voting and Majority Judgment (their mechanisms will be detailed in Sects. 3 and 4, respectively). Roughly speaking, the first — Approval Voting, formally elaborated by Brams and Fishburn (1978) — is a single and practicable voting rule that ameliorates, if not solves, serious multioption problems. The second rule — Majority Judgment, proposed by Balinski and Laraki (2007, 2011) nearly 30 years after Approval Voting — gives voters more freedom of expression than Approval Voting does, but does so at the cost of simplicity. In fact, both are similar in many ways and have stimulated much interest in the social choice community.

In this paper, some could be surprise by our use of Approval Voting and Majority Judment since these rules are usually proposed in representative democracy for choosing representatives, not to decide on issues. Actually, in practice, as raised by Antony Macé , in parliaments, sequential binary votes are usually used (see ********************** Agendas, for example) and then, what makes the use of Approval Voting or Majority Judgment more appealing? Or what may justify the use of those rules in that way? A

1 Actually, most of these rules coincide with the majority rule on dichotomous problems.

straightforward answer is that: Approval Voting is good for finding compromise solutions (according to Steven Brams), which is arguably a good thing and is of central importance in governance today; whereas Majority Judgement is good in situations where voters have "shared standards", and it complies with the four conditions of Arrow's theorem because it is based on judgments and not on rankings.

*My Contribution.* The intent of this paper is to show, provided that Approval Voting or Majority Judgment is used, that the weighted representative democracy proposed by Pivato and Soh (2020) almost perfectly approximates, in large populations, the decisions made through an imaginary ideal direct democracy. After a clear statement of the problem in Sect. 2, the two sections that follow are devoted to our main contributions. First, we compare in Sect. 3 the decisions of legislatures that use Weighted Approval Voting (the weighted version of traditional Approval Voting) to the decisions of an imaginary ideal direct democracy using traditional Approval Voting. We show that, as the population size becomes large, these decisions converge under some realistic statistical assumptions (Theorem 2). Next, we undertake a similar comparison in Sect. 4, where we consider the Weighted Majority Judgment rule in the legislature while the imaginary ideal direct democracy uses the traditional Majority Judgment rule. At the end of this second comparison, a similar result to the previous one, related to the Majority Judgment rule, is demonstrated (Theorem 3). Finally, in Sect. 5 we give several concluding remarks.

*Related Literature.* Dissatisfaction with conventional representative democratic institutions has also motivated other ambitious proposals, such as *participatory democracy* (Kafoglis 1968; Pateman 2012), *deliberative democracy* (Green-Armytage 2005; Pivato 2009; Leib 2010; Fishkin 2011), *interactive democracy* (Gould 2014; Brill 2019), e-democracy (Grönlund 2003; Coleman and Norris 2005; Petrik 2009), and others such as *proxy voting* and *liquid democracy*, as mentioned below.

Based on a voting system suggested by Tullock (1967) (1967:145–146), *voting by proxy* was introduced and developed by Alger (2006). He also used the term "ideal direct democracy" to mean "the ideal performance of a costless direct democracy with well-informed voters". Like us, he compared his model to the ideal direct democracy. The weight of each legislator in Alger's system and other systems *based on proxy voting* (Green-Armytage

2015; Cohensius et al. 2017) and on *liquid democracy* (Paulin 2014; Blum and Zuber 2016; Christoff and Grossi 2017a,b; Brill and Talmon 2018; Kahng et al. 2018) is directly proportional to the number of voters she represents; this idea can be traced back to early proposals by Sterne (1871, p. 62), Tullock (1967 Ch. 10 and 1992), Miller (1969), Chamberlin and Courant (1983) and more recently Laslier (2017). Along the same lines, our system follows the idea of Pivato and Soh (2020), where, in contrast to all aforementioned systems, the weight of a legislator is determined not only by the *number* of voters she represents but also by a weighting factor that measures how effectively she represents them.

Some recent papers have considered the idea of taking the decisions of a small assembly as approximating those of the whole population via direct democracy; among them, those that are most closely related to our work are Skowron (2015), Abramowitz and Mattei (2019), and Meir et al. (2020). In addition to the improvement of the weights in our models as previously mentioned, another ingredient in our work is that the legislature's decisions must reflect those of the population, under the fundamental hypothesis that the voters are all well informed about the long-term implications of the policies (both for themselves and for their communities).

## 2 Model and Basic Results

This section summarizes (Pivato and Soh 2020) which is crucial for understanding the rest of this paper.[2] In addition, it clearly establishes the problem we resolve throughout the paper. Let us first address the main features of weighted representation democracies.

### 2.1 Weighted Representation

By weighted representation, we mean a representative democracy in which the representatives have weights obtained from elections by a particular process (summarized below) and in which the aim is for the representative assembly to reach the same decisions on policy questions as the whole electorate would have reached by direct votes in an objective way.

[2] For further details, we refer the reader to (Pivato and Soh 2020, Sects. 2 and 3).

*Terminology and Notation.* Throughout the paper, the expression *population* or *electorate* designates a *society* of voters, whereas the expression *legislature* means the set of all representatives of voters in the society. Usually, voters (resp., representatives) in society are designated by $i$ (resp, $\ell$), and for greater clarity of the variables, we usually use the term *individual* (resp., *legislator*). Therefore, as society is the set of all individuals, it is denoted by $\mathcal{I}$, and similarly, the legislature is denoted by $\mathcal{L}$.[3]

Following the same logic, we frequently consider *decision problems* that are indexed by natural numbers $t = 1, 2, 3...$, and each problem $t$ involves a finite set of *options*, denoted by $\mathcal{O}^t$, where the options are designated by the variable $\theta$; the goal is to select the *collective outcome* from this set.

*Formation of the Legislature.* We posit a legislature with a fixed number $L$ of seats. Elections work as follows:

Step 1. A set of $M$ candidates present themselves for possible election ($M \geq L$).

Step 2. Each individual selects one candidate as her "representative".

Step 3. The $M - L$ candidates who received the *smallest* number of votes are eliminated, and thereafter, all the individuals who voted for these eliminated candidates must repeat Step 2 and select one of the $L$ retained candidates.

After this process, the $L$ most popular candidates form the legislature, and each individual has selected one of the $L$ legislators as her representative.

*The Individuals and Their Legislators.* Let $\mathcal{I}$ be the set of all individuals, and set $I := |\mathcal{I}| < \infty$. From $\mathcal{I}$, we run elections with respect to the legislature formed above and obtain the set of legislators $\mathcal{L}$; thus, $L := |\mathcal{L}| < \infty$. Each individual $i$ in $\mathcal{I}$ has therefore chosen some member $\ell$ of $\mathcal{L}$ as her representative. For each $i \in \mathcal{I}$, let $\widetilde{\mathbf{p}}^i := (\widetilde{p}^i_\ell)_{\ell\in\mathcal{L}}$ be a random vector in $[0, 1]^L$. For any $\ell \in \mathcal{L}$ the random variable $\widetilde{p}^i_\ell$ represents the *probability of agreement* between $i$ and $\ell$ on as-yet undetermined future policy questions. The random vectors $\{\widetilde{\mathbf{p}}^i\}_{i\in\mathcal{I}}$ thus describe the relationship between the electorate $\mathcal{I}$ and the legislature $\mathcal{L}$. Let $\rho$ be any probability distribution on $[0, 1]^L$. Below is our first assumption.

[3] More generally, we use the following notational conventions: calligraphic letters ($\mathcal{I}, \mathcal{L}, \ldots$) represent sets; elements of these sets are normally indicated by the corresponding lower-case italic letters ($i, \ell, \ldots$), while their cardinalities are normally indicated by the corresponding upper-case italic letters ($I, L, \ldots$). Boldface letters ($\mathbf{p}, \mathbf{q}, \ldots$) represent vectors. We indicate random variables with a tilde ($\widetilde{a}, \widetilde{b}, \ldots$) and possible values of these random variables as the same letters *without* a tilde ($a, b, \ldots$).

(A) $\{\widetilde{\mathbf{p}}^i\}_{i\in\mathcal{I}}$ are independent, identically distributed random variables drawn from $\rho$.

As $\rho$ is not made precise, for any individual $i$ and different legislators $\ell$ and $\ell'$, the random variables $\widetilde{p}^i_\ell$ and $\widetilde{p}^i_{\ell'}$ may be correlated or not. Moreover, the marginal projections of $\rho$ onto different coordinates $\ell$ and $\ell'$ could be different. We make only the following assumption about $\rho$:

(B) For all distinct $\ell, \ell' \in \mathcal{L}$, $\rho\big\{\mathbf{p} \in [0,1]^L \; ; \; p_\ell = p_{\ell'}\big\} = 0$.

This ensures that each individual has different probabilities of agreement with any two different legislators. Each individual $i$ then has as her representative the legislator $\ell$ whose $\widetilde{p}^i_\ell$ is maximal (i.e., $i$ is an $\ell-$*supporter*). Now, let $\widetilde{\mathcal{I}}_\ell$ be the set of $\ell-$supporters for any legislator $\ell$, and let $\widetilde{p}_\ell$ be the *average probability of agreement* between $\ell$ and her supporters on future policy questions. Formally, $\widetilde{p}_\ell$ can be defined as follows:

$$\widetilde{p}_\ell \quad := \quad \frac{1}{|\widetilde{\mathcal{I}}_\ell|}\sum_{i\in\widetilde{\mathcal{I}}_\ell}\widetilde{p}^i_\ell. \tag{2A}$$

Each legislator $\ell$ has a weight $\widetilde{w}_\ell$ in the legislature, which is given by

$$\widetilde{w}_\ell \quad := \quad (2\widetilde{p}_\ell - 1)\,|\widetilde{\mathcal{I}}_\ell|. \tag{2B}$$

As we can see, this is the number of $\ell$-supporters, multiplied by a factor $(2\widetilde{p}_\ell \ - \ 1)$ that depends on the probability of agreement with all the $\ell-$supporters.

*Voting Rule in the Legislature.* All the features of the legislature are then well defined, except for the voting rule that will be used in the legislature. John Adams argued that a legislature "*should be an exact portrait, in miniature, of the people at large, as it should think, feel, reason, and act like them*". Therefore, we will consider that the legislature uses the weighted version of the same voting rule as the one that would have been used by the electorate itself in a direct democracy. For example, in the case of referendums by *simple majority rule*, we have in the legislature the *weighted majority rule*; in the case of direct votes with the *Approval Voting rule*, we have the *Weighted Approval Voting rule* (defined in the next section); in the case of direct votes with the *Majority Judgment rule*, we have the *Weighted Majority Judgment rule* (defined in Sect. 4); and so forth.

*Some Important Features.*

- The advantages of such weighted representations are explained in (Pivato and Soh 2020, Sects. 2 and 6). In addition, and related to the previously mentioned references, many comparisons with some existing representation systems, such as proportional representation and regional representation, are provided.
- By electing legislators in the way described, our model of representative democracy is resistant to strategic votes and behaviors, as explained in (Pivato and Soh 2020, Sect. 6).
- For more information about estimating the probability of agreement, we refer the reader to Pivato and Soh (2020), as we have treated those random variables as exogenous. Indeed, for all legislators $\ell$, the average probability of agreement $\widetilde{p}_\ell$ can be computed after the legislature formation process. Pivato and Soh 2020, Sect. 6 details two methods — an *incentive-compatible mechanism*, and an *empirical method based on surveys and forecasts* — to accurately estimate the values of $\widetilde{p}_\ell$ and $\widetilde{w}_\ell$.

All the hypotheses made so far will be maintained throughout the paper and, along with them, the following *Central Question*:

**Central Question.** *Assume the above formation of the legislature where each individual chooses her representative, followed by hypotheses (A) and (B). Under what conditions, will the decisions made by the legislature $\mathcal{L}$ using a (weighted) voting rule usually be the same as those that would have been made by the whole electorate $\mathcal{I}$ via a direct democracy using the same (unweighted) voting rule?*

When we talk about the decision made by the electorate, we are speaking counterfactually about the decision that the electorate *would have made* if voters had invested the effort to develop a well-informed, objective and carefully considered opinion on these decisions. In this sense, the legislature $\mathcal{L}$ would be "ideally representative" of its society $\mathcal{I}$. A quick answer to this question for simple binary decisions is provided in the next subsection.

## 2.2 Binary Decisions

*Policy Problems.* Suppose the electorate $\mathcal{I}$ will face a sequence of as-yet undetermined future binary questions indexed by natural numbers $t = 1, 2, 3, \ldots$. Then, for each of these binary questions, an individual can have either a "positive" or "negative" opinion. Since there are only two options, the simplest way to determine the collective opinion is to use the *simple majority rule*. In doing so, we would consider the simple majority rule in the hypothetical referendum, whereas in the legislature, we would use the weighted majority rule where each legislator $\ell$ would have the weight $\widetilde{w}_\ell$ given by (2B). The choice of this voting rule is also appealing because of May's (1952) theorem, which says that simple majority rule is the only dichotomous voting rule that is *neutral*, *anonymous*, *decisive*, and *monotone* — four requirements that are highly appealing in situations where there is no obvious asymmetry between two options or between different individuals. Taking into consideration all the above hypotheses, we then need to determine what assumption we have to consider in answering our Central Question above. Let us start with some formalization.

For all $t \in \mathbb{N}$, all $i \in \mathcal{I}$, and all $\ell \in \mathcal{L}$, let $\widetilde{b}_i^t, \widetilde{b}_\ell^t \in \{\pm 1\}$ denote the opinions of individual $i$ and legislator $\ell$, respectively, on question $t$ (with $-1$ and $+1$ representing "negative" and "positive" opinions, resp.). For all $\ell \in \mathcal{L}$ and all $i \in \widetilde{\mathcal{I}}_\ell$, let $\widetilde{a}_{i,\ell}^t := \widetilde{b}_i^t \cdot \widetilde{b}_\ell^t$. This represents the presence or absence of *agreement* between $i$ and $\ell$ on issue $t$: we then have $\widetilde{a}_{i,\ell}^t = 1$ if $\widetilde{b}_i^t = \widetilde{b}_\ell^t$ (i.e., $i$ and $\ell$ agree on $t$), and $\widetilde{a}_{i,\ell}^t = -1$ if $\widetilde{b}_i^t = -\widetilde{b}_\ell^t$ (i.e., $i$ and $\ell$ disagree on $t$). Since we cannot anticipate any binary questions in advance, we will regard $\{\widetilde{b}_i^t;\ \ i \in \mathcal{I}, t \in \mathbb{N}\}$ and $\{\widetilde{b}_\ell^t;\ \ \ell \in \mathcal{L}\ t \in \mathbb{N}\}$ as random variables. Thus, $\{\widetilde{a}_{i,\ell}^t;\ \ i \in \mathcal{I}, \ell \in \mathcal{L}, t \in \mathbb{N}\}$ are also random variables.

We now make the following key assumption:

(C) For all $\ell \in \mathcal{L}$ and all $i \in \widetilde{\mathcal{I}}_\ell$, $\{\widetilde{a}_{i,\ell}^t\}_{t \in \mathbb{N}}$ is a sequence of identically distributed random variables. For any $t \in \mathbb{N}$, $\text{Prob}\left[\widetilde{a}_{i,\ell}^t = 1 \;\middle|\; \widetilde{p}_\ell^i\right] = \widetilde{p}_\ell^i$. Furthermore, for any $t \in \mathbb{N}$, the variables $\{\widetilde{a}_{i,\ell}^t\}_{i \in \widetilde{\mathcal{I}}_\ell}$ are independent.

Note that for any $i$ in $\widetilde{\mathcal{I}}_\ell$, we do *not* suppose that the variables $\{\widetilde{a}_{i,\ell}^t\}_{t \in \mathbb{N}}$ are independent. We allow the possibility of correlations between an individual's opinions over different decisions.

*Legislative Votes vs. Popular Referenda.* For all $t \in \mathbb{N}$, we have already supposed that the legislature $\mathcal{L}$ decides policy problem $t$ by *weighted majority vote*; then, the legislature's decision, denoted by $\widetilde{D}^t_{\mathcal{L}}$, can be viewed as follows:

$$\widetilde{D}^t_{\mathcal{L}} \;=\; \mathrm{sign}[\widetilde{S}^t_{\mathcal{L}}], \quad \text{where} \quad \widetilde{S}^t_{\mathcal{L}} \;:=\; \sum_{\ell\in\mathcal{L}} \widetilde{w}_\ell\, \widetilde{b}^t_\ell. \tag{2C}$$

As we can see, $\widetilde{D}^t_{\mathcal{L}} \in \{-1, 0, 1\}$, where $-1, 0$, and $+1$ stand for "negative", "undecided" (a perfect tie), and "positive" opinions, respectively. On the other hand, let $\widetilde{D}^t_{\mathcal{I}} \in \{-1, 0, 1\}$ denote the outcome of a (hypothetical) simple majority vote in the population $\mathcal{I}$ as a whole. Then,

$$\widetilde{D}^t_{\mathcal{I}} \;=\; \mathrm{sign}[\widetilde{S}^t_{\mathcal{I}}], \quad \text{where} \quad \widetilde{S}^t_{\mathcal{I}} \;:=\; \sum_{i\in\mathcal{I}} \widetilde{b}^t_i. \tag{2D}$$

Now, it remains to show how we will obtain $\widetilde{D}^t_{\mathcal{L}} = \widetilde{D}^t_{\mathcal{I}}$ for all binary questions $t$. The last crucial assumption is as follows:

(D) There exists $\epsilon > 0$ such that $\lim\limits_{I\to\infty} \mathrm{Prob}\left[|\widetilde{S}^t_{\mathcal{I}}/I| > \epsilon\right] = 1$ (for all $t \in \mathbb{N}$).

This means that nearly tied votes in large populations are very unlikely to occur.

Let $P(I) := \mathrm{Prob}\left[\widetilde{D}^t_{\mathcal{L}} = \widetilde{D}^t_{\mathcal{I}}\right]$ indicate the probability that the legislature's decision agrees with the decision that would have been reached via a popular referendum. The value $P(I)$ is independent of the problem $t$ and depends only on the population size $I$. Below is the main result of (Pivato and Soh 2020, Sect. 3).

**Theorem 1** *Given assumptions* (A) – (D), $\lim\limits_{I\to\infty} P(I) \;=\; 1.$

This result answers to our Central Question above — in a sufficiently large population, the outcome of a weighted majority vote on binary decisions in the legislature is highly likely to agree with the outcome of a popular referendum on the same decisions. In this sense, the *legislature is "ideally representative" of its society with respect to any binary question.*

The main limitation of Theorem 1 is that we cannot use it in the case of questions with more than two options. This is addressed in the next two sections. Regarding such questions, there are many rules we can consider, depending on the structures of both the questions and the individual preferences. These include the following:

- If the decisions involve *non-interconnected options* and the individual preference orders are all strict, then an answer to our Central Question is already given in Pivato and Soh (2020) by the *preference aggregation model* (theorems 2 and 3).

– If the decisions can involve *interconnected options* and the individual preferences are all logically consistent, then an answer to our Central Question is given in Pivato and Soh (2020) by the *judgment aggregation model* (theorems 4 and 5).

The first item above is mostly related to "preference systems" and the second to "view systems".[4] In this paper, by contrast, we answer our Central Question with types of "approval systems" and "judgment systems". More specifically, we answer any question on the assumption that each individual gives her evaluation of every option of the question. Therefore, we consider two methods, each taking all the individual evaluations of every option and returning the option with the best evaluation as the collective outcome. These two methods are Approval Voting and Majority Judgment, which are investigated in the next two sections.

## 3 Approval Voting

There are many real social issues in which we ask people to make choices on ballots (such as some polls, for example) in such a way that any individual can tick as many options as she believes useful. For "simple questions", we can easily do this in very large populations. By contrast, a problem will appear if the questions become more specific or complex and particular knowledge is required to truly determine the best choices. To overcome this problem, we can roughly apply the same process, restricted to the representatives of the electorate. Once again, the hope is that their final decisions will reflect the decision that all of the individuals would have collectively selected if they had voted, provided that all of them would have been well informed about those issues. In such a context, there exists a very practical method that best fits the previous process, known as the Approval Voting rule.

*Standard Approval Voting.* Brams and Fishburn (1978) were the first to formally and thoroughly analyze Approval Voting (AV). Informally, given any problem $t$ with a finite set $\mathcal{O}^t$ of options from which we want to choose an outcome, AV works as follows: "each individual approves of some options, and the option with the most approvals is chosen".

[4] For more information, see the beginnings of (Pivato and Soh 2020, Sects. 4 and 5).

AV is a single-winner voting system where each individual may approve any number of options, and the outcome is the most approved option.[5] This rule, which is as simple to implement as the plurality rule, overcomes several drawbacks of plurality and other rules. Mueller (2003) writes,

> "Beyond whatever advantages it possesses in discouraging strategic behavior, however, Approval Voting deserves serious attention as a possible substitute for the plurality and majority rule–runoff rules because of its superior performance, as judged by the Condorcet or utilitarian efficiency criteria, and greater simplicity than the Hare, Coombs, Borda, and to some extent majority rule–runoff procedures".

AV as defined so far remains valid when all votes are equal. By contrast, in the legislature, some legislators may have different weights, leading to a corresponding rule that we call Weighted Approval Voting.

*Weighted Approval Voting.* In the legislature, the voting rule works as follows:

(i) Every legislator $\ell$ approves one or many options.
(ii) For each option $\theta \in \mathcal{O}^t$, we attribute a score corresponding to the sum of the weights of the legislators who approve $\theta$.
(iii) The *legislature's outcome* is then the option with the highest score (or that determined by a tie-breaking rule, if needed).

*Policy Problems.* Suppose the electorate $\mathcal{I}$ will face a sequence of as-yet undetermined future policy problems indexed by natural numbers $t = 1, 2, 3, \ldots$. The objective of each problem $t$ is to choose an outcome from the finite set $\mathcal{O}^t$ of possible options related to problem $t$, about which each individual has to (dis)approve some option(s). Let us fix some problem $t$; thus, any individual $i$ has an *approval view* $\widetilde{\mathbf{b}}_i^t := (\widetilde{b}_i^\theta)_{\theta \in \mathcal{O}^t} \in \{\pm 1\}^{\mathcal{O}^t}$ about problem $t$, such that the random variable $\widetilde{b}_i^\theta \in \{\pm 1\}$ represents the *opinion of individual* $i$ *on option* $\theta$, where $+1$ encodes "approve" and $-1$ encodes "disapprove" (as mentioned above, we take this to be what individual $i$'s carefully considered opinions *would be* if she were well-informed). Hence, for all $t \in \mathbb{N}$, $\left(\widetilde{\mathbf{b}}_i^t\right)_{i \in \mathcal{I}}$ is the *electorate approval view profile* of problem $t$.

[5] Further investigations related to Approval Voting rule, are provided in (Brams et al. 2007; Laslier et al. 2010)

Furthermore, for all $\ell \in \mathcal{L}$, all $i \in \widetilde{\mathcal{I}}_\ell$, all $t \in \mathbb{N}$ and all $\theta \in \mathcal{O}^t$, let $\widetilde{a}^\theta_{i,\ell} := \widetilde{b}^\theta_\ell \cdot \widetilde{b}^\theta_i$. This represents the presence or absence of *agreement* between $i$ and her representative $\ell$ about the option $\theta \in \mathcal{O}^t$. Thus, $\widetilde{a}^\theta_{i,\ell} = 1$ if $\widetilde{b}^\theta_i = \widetilde{b}^\theta_\ell$ (i.e., $i$ and $\ell$ agree on $\theta$), and $\widetilde{a}^\theta_{i,\ell} = -1$ if $\widetilde{b}^\theta_i \neq \widetilde{b}^\theta_\ell$ (i.e., $i$ and $\ell$ disagree on $\theta$). Since $\{\widetilde{b}^\theta_i;\ i \in \mathcal{I}, t \in \mathbb{N}, \theta \in \mathcal{O}^t\}$ and $\{\widetilde{b}^\theta_\ell;\ \ell \in \mathcal{L}, t \in \mathbb{N}, \theta \in \mathcal{O}^t\}$ are random variables, $\{\widetilde{a}^\theta_{i,\ell};\ \ i \in \mathcal{I},\ \ell \in \mathcal{L},\ t \in \mathbb{N},\ \theta \in \mathcal{O}^t\}$ are also random variables.

We will make the following key assumption:

($\mathrm{C}_A$) Letting $t \in \mathbb{N}$ be any decision problem, for all $\ell \in \mathcal{L}$ and all $i \in \widetilde{\mathcal{I}}_\ell$, $\{\widetilde{a}^\theta_{i,\ell}; \theta \in \mathcal{O}^t\}$ is a set of identically distributed random variables. For any $\theta \in \mathcal{O}^t$, $\mathrm{Prob}\left[\widetilde{a}^\theta_{i,\ell} = 1 \;\middle|\; \widetilde{p}^i_\ell\right] = \widetilde{p}^i_\ell$. Furthermore, for any $\ell \in \mathcal{L}$ and $\theta \in \mathcal{O}^t$, the variables $\{\widetilde{a}^\theta_{i,\ell}\}_{i \in \widetilde{\mathcal{I}}_\ell}$ are independent.

Note that for a given $i$ in $\widetilde{\mathcal{I}}_\ell$, we do *not* suppose that the variables $\{\widetilde{a}^\theta_{i,\ell}\}_{\theta \in \mathcal{O}^t}$ are independent. We allow the possibility of correlations between an individual's approvals over different options. However, we *do* assume that the approvals of different individuals in $\widetilde{\mathcal{I}}_\ell$ are independent (not necessarily identically distributed) random variables for any particular option $\theta$ related to any problem $t$.

One might misunderstand the expression $\mathrm{Prob}\left[\widetilde{a}^\theta_{i,\ell} = 1 \;\middle|\; \widetilde{p}^i_\ell\right] = \widetilde{p}^i_\ell$ in ($\mathrm{C}_A$). Note that the vector $\widetilde{\mathbf{p}}^i$ has already been realized at the time of the election. Thus, ($\mathrm{C}_A$) says that, for any possible realization $p^i_\ell \in [0,1]$, it holds that $\mathrm{Prob}\left[\widetilde{a}^\theta_{i,\ell} = 1 \;\middle|\; \widetilde{p}^i_\ell = p^i_\ell\right] = p^i_\ell$.

*Legislative Approvals vs. Popular Approvals.* For any fixed problem $t \in \mathbb{N}$, we suppose that the legislature decides policy problem $t$ by *Weighted Approval Voting*, where the weight $\widetilde{w}_\ell$ of legislator $\ell$ is given by formula (2B). Formally, let $\widetilde{D}^t_{\mathcal{L}} \in \mathcal{O}^t$ denote the legislature's decision, and let $\widetilde{\mathbf{S}}^t_{\mathcal{L}} := (\widetilde{S}^\theta_{\mathcal{L}})_{\theta \in \mathcal{O}^t} \in [0,1]^{\mathcal{O}^t}$ be the *legislature view*, where $\widetilde{S}^\theta_{\mathcal{L}}$ represents the *legislative score* of option $\theta$. Then,

$$\widetilde{D}^t_{\mathcal{L}} \;:=\; \arg\max\{\widetilde{S}^\theta_{\mathcal{L}}, \theta \in \mathcal{O}^t\}, \quad \text{where } \widetilde{S}^\theta_{\mathcal{L}} \;:=\; \sum_{\ell \in \mathcal{L}} \widetilde{w}_\ell \widetilde{b}^\theta_\ell \;\text{ for all } \theta \in \mathcal{O}^t. \tag{3A}$$

The value of $\widetilde{D}^t_{\mathcal{L}}$ is unique in large populations as a consequence of assumption ($\mathrm{D}_A$) below and proposition 1 in the Appendix. ($\widetilde{S}^\theta_{\mathcal{L}}$ and $\widetilde{D}^t_{\mathcal{L}}$ are random variables because both the legislators' approval views $\widetilde{\mathbf{b}}^t_\ell$ and their weights $\widetilde{w}_\ell$ are random.)

Similarly, let $\widetilde{D}^t_{\mathcal{I}} \in \mathcal{O}^t$ denote the outcome of (hypothetical) AV in the population as a whole, with $\widetilde{\mathbf{S}}^t_{\mathcal{I}} := (\widetilde{S}^\theta_{\mathcal{I}})_{\theta\in\mathcal{O}^t} \in [0,1]^{\mathcal{O}^t}$ representing the *average popular view* on problem $t$. Then,

$$\widetilde{D}^t_{\mathcal{I}} \;:=\; \arg\max\{\widetilde{S}^\theta_{\mathcal{I}}, \theta \in \mathcal{O}^t\}, \qquad \text{where } \; \widetilde{S}^\theta_{\mathcal{I}} \;:=\; \sum_{i\in\mathcal{I}} \widetilde{b}^\theta_i \;\text{ for all } \theta \in \mathcal{O}^t. \tag{3B}$$

As above, the value of $\widetilde{D}^t_{\mathcal{I}}$ is unique in large populations by assumption ($\mathrm{D}_A$) given below.

($\mathrm{D}_A$) There exists $\epsilon > 0$ such that, for all $t \in \mathbb{N}$ and all distinct $\theta_1, \theta_2 \in \mathcal{O}^t$,

$$\lim_{I\to\infty} \mathrm{Prob}\left[\left|\widetilde{S}^{\theta_1}_{\mathcal{I}}/I - \widetilde{S}^{\theta_2}_{\mathcal{I}}/I\right| > \epsilon\right] = 1.$$

This says that, in large populations, near ties in the numbers of approvals between any pair of options are extremely unlikely.

Let $P_A(I) := \mathrm{Prob}\left[\widetilde{D}^t_{\mathcal{L}} = \widetilde{D}^t_{\mathcal{I}}\right]$ — this is the probability that the legislature's decision (by Weighted Approval Voting) agrees with the decision that would have been reached through a direct democracy by the standard (unweighted) AV rule. This probability depends on the population size $I$, since when $I$ increases, the sum in (3B) involves more terms; likewise, the expressions (2A) and (2B) defining the weight $\widetilde{w}_\ell$ of each legislator involve more terms, which may indirectly affect the value of the sum in (3A).

Here is the first result of this paper.

**Theorem 2** *Given assumptions* (A), (B), ($\mathrm{C}_A$), *and* ($\mathrm{D}_A$), $\lim_{I\to\infty} P_A(I) \;=\; 1$.

This gives an answer to our Central Question by saying that, in a sufficiently large population, the outcome of a Weighted Approval Voting rule on any multioption problem in the legislature is highly likely to agree with the outcome of standard AV on the same problem by the whole electorate. In this sense, *the legislature is "ideally representative" of its society with respect to any multioption problem.*

*Main Limitation of Approval Voting.* Although AV is very simple and has very good properties, its main limitation is that the options approved by an individual do not always yield the same satisfaction to that individual, since she is only allowed to approve of them or not. As a consequence, individuals cannot fully express the strength of their opinions.

Fortunately, one way to overcome this limitation is to consider a more expressive rule than AV, which leads us to the Majority Judgment rule defined in the next section.

## 4 Majority Judgment

This section is concerned with Majority Judgment, another alternative method to majority rule, which overcomes a problem raised by Tullock (1959):

> ...every voter simply indicates his preference, and the preference of the majority of the voters is carried out. The defect, and it is a serious one, of this procedure is that it ignores the various intensities of the desires of the voters. A man who is passionately opposed to a given measure and a man who does not much care but is slightly in favor of it are weighted equally.

Let us start by considering the following: given any decision problem $t$ with a finite set of options $\mathcal{O}^t$ from which the electorate wants to select an outcome, in this section, the process of selecting follows certain patterns. To wit, an individual $i$ evaluates each option on the basis of some grades (e.g., letter grades $A, B, C, D \ldots$) in a finite totally ordered set of scales $\mathcal{G}^t$. For all problem $t$, let $\mathcal{G}^t$ be the collection of grades used to evaluate the options in $\mathcal{O}^t$. More formally, for all problems $t$, let $M_t \in \mathbb{N}, M_t \geq 2$ and $\mathcal{M}^t := \{1, 2, \ldots, M_t\}$, define $\mathcal{G}^t := \{G_k, k \in \mathcal{M}^t\}$, and define on $\mathcal{G}^t$ the total order $\succ$ as the strict binary relation on $\mathcal{G}^t$ such that $G_j \succ G_k$ if $j > k$.

*Standard Majority Judgment.* Majority judgment (MJ) was proposed by Michel Balinski and Rida Laraki in 2006. This rule is based on judgments and not on rankings; an individual gives a judgment to every option based on a language of common grades that are measured on an ordinal scale (e.g., Reject, Poor, Acceptable, Good, Very Good, Excellent). The question now is not how to transform many individual preferences into a single collective preference, but rather how to determine a collective evaluation from individual evaluations (sometimes by means of tie-breaking rules). Hence, "comparing" is replaced by a new paradigm, "evaluating"; likewise, "voting" is replaced by "judging" (Balinski and Laraki 2014). The mechanism is informally described as follows:

(i) An individual evaluates each option on $\mathcal{O}^t$ using the grades given in $\mathcal{G}^t$.

(ii) For each option $\theta \in \mathcal{O}^t$, we determine the collective grade of $\theta$ as the *median grade* of $\theta$, and we denote it by $G_\theta \in \mathcal{G}^t$ (the median grade $G_\theta$ is defined here as the highest grade such that a majority of individuals have assigned to $\theta$ a grade of at least $G_\theta$, and a majority of individuals have assigned to $\theta$ a grade of at most $G_\theta$).

(iii) The collective outcome chosen is the one with the highest median grade (possibly decided by means of a tie-breaking rule).

The formalization is not relevant to our model,[6] and if the individuals do not have the same weights (for instance, in weighted representative assemblies), then we must take all the weights into consideration; with them, we will describe the mechanism in the legislature, which we call *Weighted Majority Judgment.*

*Weighted Majority Judgment.* In the legislature, the voting rule's mechanism is described as follows:

(i) A legislator $\ell$ evaluates each option on $\mathcal{O}^t$ by assigning to it a grade $G^\theta_{k_\ell}$ with respect to $\mathcal{G}^t$, weighted by her weight $w_\ell$ ($k_\ell \in \mathcal{M}^t$).

(ii) For each option $\theta \in \mathcal{O}^t$ and grade $G_k$, we set $\Delta^\theta_k$ as the sum of the weights of all the legislators $\ell$ who attribute $G_k$ to option $\theta$ (i.e., $G^\theta_{k_\ell} = G_k$).

(iii) For each option $\theta \in \mathcal{O}^t$, the collective grade of $\theta$ is the *weighted median grade* $G_{k_0} \in \mathcal{G}^t$ of $\theta$, as follows:[7]

$$\frac{1}{2}\sum_{k\in\mathcal{M}^t}\Delta^\theta_k < \Delta^\theta_1 \quad \text{and in this case} \quad k_0 = 1,$$

or

$$k_0 > 1 \quad \text{and satisfies} \quad \Delta^\theta_1 + \ldots + \Delta^\theta_{k_0-1} \leq \frac{1}{2}\sum_{k\in\mathcal{M}^t}\Delta^\theta_k < \Delta^\theta_1 + \ldots + \Delta^\theta_{k_0-1} + \Delta^\theta_{k_0}.$$

(iv) The legislature's choice is that with the highest median grade (possibly decided by means of a tie-breaking rule).

*Policy Problems.* Suppose the electorate $\mathcal{I}$ will face a sequence of as-yet undetermined future policy problems indexed by natural numbers $t = 1, 2, 3, \ldots$. The goal of each problem $t$ is to select, by the MJ rule, an outcome from the finite set $\mathcal{O}^t$ of possible

[6] We refer the reader to chapters 1, 2, 7 of Balinski and Laraki (2011) for details and many practical examples.

[7] The median grade $G_{k_0}$ determined in this way is always unique.

options related to problem $t$, where every individual has to evaluate each option of $t$ using the grades $\mathcal{G}^t$. The rule allows any individual $i$ to judge each option $\theta \in \mathcal{O}^t$ (on the basis of the grades in $\mathcal{G}^t$), and the collective outcome on $\mathcal{O}^t$ is the option with the highest median grade. Let $J_i^\theta \in \mathcal{G}^t$ be $i$'s grade for option $\theta$, and let $\mathbf{J}_i^t := \left(J_i^\theta\right)_{\theta \in \mathcal{O}^t}$ be her *judgment view* on problem $t$. Since we cannot predict the opinions of any individual $i \in \mathcal{I}$ in advance, we will model the judgment of $i$ as the random variable $\widetilde{\mathbf{J}}_i^t$. Furthermore, considering the legislature $\mathcal{L}$ provided by the legislature formation rule in Sect. 2, whereby each legislator $\ell$ has her weight $\widetilde{w}_\ell$ given by formula (2B), we also denote by $\widetilde{\mathbf{J}}_\ell^t$ and $(\widetilde{\mathbf{J}}_\ell^t)_{\ell \in \mathcal{L}}$ the analogies in the legislature of the random variables $\widetilde{\mathbf{J}}_i^t$ and $(\widetilde{\mathbf{J}}_i^t)_{i \in \mathcal{I}}$, respectively. Henceforth, we consider all the aforementioned random variables as our inputs. Thus, to answer our Central Question, we will transform these inputs into the inputs of our basic model from Sect. 2 as shown below.

*Adaptation to the Basic Model.* For each option $\theta \in \mathcal{O}^t$ and $k \in \mathcal{M}^t$, one might ask the question "*Is the grade that you give $\theta$ at least as good as $G_k$?*" (Henceforth, this question is denoted by $q_{(\theta,k)}$). The questions $\{q_{(\theta,k)}; \theta \in \mathcal{O}^t, k \in \mathcal{M}^t\}$ transform individual $i'$s evaluation of the options into a "view" in a judgment aggregation problem. For any pair $(\theta, k) \in (\mathcal{O}^t, \mathcal{M}^t)$, we represent the answer to question $q_{(\theta,k)}$ by the random variable $\widetilde{b}^{\theta,k} \in \{\pm 1\}$ such that $\widetilde{b}^{\theta,k} = -1$ when the judgment of $\theta$ is lower than grade $G_k$ and $\widetilde{b}^{\theta,k} = 1$ otherwise. Taking this into account, for all $\theta \in \mathcal{O}^t$ and $i \in \mathcal{I}$, the grade $\widetilde{J}_i^\theta$ given to $\theta$ by $i$ can be represented by the random vector $\widetilde{\mathbf{b}}_i^\theta := (\widetilde{b}_i^{\theta,k})_k \in \{\pm 1\}^{M_t}$ defined by $\widetilde{b}_i^{\theta,k} = -1$ if $G_k \succ \widetilde{J}_i^\theta$ and $\widetilde{b}_i^{\theta,k} = 1$ otherwise; thus, $\widetilde{J}_i^\theta = \max\{G_k \in \mathcal{G} : \widetilde{b}_i^{\theta,k} = 1\}$. Note that $\widetilde{b}_i^{\theta,k} \geq \widetilde{b}_i^{\theta,k+1}$; for instance, the worst grade $G_{M_t}$ is represented by the vector $(1, -1, -1, \ldots, -1)$, while the best grade $G_1$ is represented by $(1, \ldots, 1, 1)$. Now, for all $\theta \in \mathcal{O}^t$ and $k \in \mathcal{M}^t$, let $\widetilde{a}_{i,\ell}^{\theta,k} := \widetilde{b}_\ell^{\theta,k} \cdot \widetilde{b}_i^{\theta,k}$. This represents the presence or absence of agreement between $i$ and $\ell$ about question $q_{(\theta,k)}$. Thus,

$$\widetilde{a}_{i,\ell}^{\theta,k} := \begin{cases} \;\;1 \text{ if } i \text{ and } \ell \text{ agree about } q_{(\theta,k)}; \\ -1 \text{ if } i \text{ and } \ell \text{ disagree about } q_{(\theta,k)}. \end{cases}$$

For all $\ell \in \mathcal{L}, i \in \widetilde{\mathcal{I}}_\ell$, the variables $\{\widetilde{b}_i^{\theta,k};\; \theta \in \mathcal{O}^t,\, k \in \mathcal{M}^t\}$ and $\{\widetilde{b}_\ell^{\theta,k};\; \theta \in \mathcal{O}^t,\, k \in \mathcal{M}^t\}$ are random; thus, $\{\widetilde{a}_{i,\ell}^{\theta,k};\; \theta \in \mathcal{O}^t,\, k \in \mathcal{M}^t\}$ are also random variables.

We will make the following key assumption:

($C_M$) Letting $t \in \mathbb{N}$ be any decision problem, for all $\ell \in \mathcal{L}$ and all $i \in \widetilde{\mathcal{I}}_\ell$, $\{\widetilde{a}_{i,\ell}^{\theta,k};\ \theta \in \mathcal{O}^t, k \in \mathcal{M}^t\}$ is a set of identically distributed random variables. For any $\theta \in \mathcal{O}^t$ and $k \in \mathcal{M}^t$, Prob $\left[\widetilde{a}_{i,\ell}^{\theta,k} = 1 \;\middle|\; \widetilde{p}_\ell^i\right] = \widetilde{p}_\ell^i$. Furthermore, for any $\ell \in \mathcal{L}$, $k \in \mathcal{M}^t$ and $\theta \in \mathcal{O}^t$, the variables $\{\widetilde{a}_{i,\ell}^{\theta,k}\}_{i \in \widetilde{\mathcal{I}}_\ell}$ are independent.

Once again, note that for a given $i$ in $\widetilde{\mathcal{I}}_\ell$, we do *not* suppose that the variables $\{\widetilde{a}_{i,\ell}^{\theta,k}\}_{\theta \in \mathcal{O}^t}$ are independent. Likewise, we do not assume that $\{\widetilde{a}_{i,\ell}^{\theta,k}\}_{k \in \mathcal{M}^t}$ are independent (indeed, they cannot be because of the aforementioned strict order on $\mathcal{G}^t$).

*Legislative Judgment vs. Popular Judgment.* For any fixed problem $t \in \mathbb{N}$, we suppose that the legislature decides policy problem $t$ by Weighted Majority Judgment, where the weight $\widetilde{w}_\ell$ of legislator $\ell$ is given by formula (2B). Consider the popular judgment view profile to be $(\widetilde{\mathbf{J}}_i^t)_{i \in \mathcal{I}}$, with each $\widetilde{\mathbf{J}}_i^t := (\widetilde{J}_i^\theta)_{\theta \in \mathcal{O}^t}$ and $\widetilde{J}_i^\theta$ as the view $\widetilde{\mathbf{b}}_i^\theta := (\widetilde{b}_i^{\theta,k})_{k \in \mathcal{M}_t}$, which is a random vector. The *popular judgment* over $\mathcal{O}^t$ is denoted by $\widetilde{\mathbf{J}}_{\mathcal{I}}^t := (\widetilde{J}_{\mathcal{I}}^\theta)_{\theta \in \mathcal{O}^t}$, where $\widetilde{J}_{\mathcal{I}}^\theta$ is represented by the *popular view* $\widetilde{\mathbf{b}}_{\mathcal{I}}^\theta := (\widetilde{b}_{\mathcal{I}}^{\theta,k})$ defined by:

$$\widetilde{b}_{\mathcal{I}}^{\theta,k} \;:=\; \operatorname{sign}\left(\widetilde{S}_{\mathcal{I}}^{\theta,k}\right)\;; \quad \text{where} \quad \widetilde{S}_{\mathcal{I}}^{\theta,k} \;:=\; \sum_{i \in \mathcal{I}} \widetilde{b}_i^{\theta,k}. \tag{4A}$$

Here, $\widetilde{b}_{\mathcal{I}}^{\theta,k} \in \{-1, 0, 1\}$, but we will at first suppose that $\widetilde{b}_{\mathcal{I}}^{\theta,k} \in \{-1, 1\}$ for the sake of clarity, and because in large populations, it is very unlikely that the case $\widetilde{b}_{\mathcal{I}}^{\theta,k} = 0$ will happen. Then, for each option $\theta$, the *popular grade* $\widetilde{J}_{\mathcal{I}}^\theta$ satisfies $\widetilde{J}_{\mathcal{I}}^\theta = \max\left\{G_k \in \mathcal{G}^t : \widetilde{b}_{\mathcal{I}}^{\theta,k} = 1, k \in \mathcal{M}^t\right\}$, and we denote the *popular decision* on $\mathcal{O}^t$ by $\widetilde{D}_{\mathcal{I}}^t \in \arg\max\left\{\widetilde{J}_{\mathcal{I}}^\theta, \theta \in \mathcal{O}^t\right\}$.

On the other hand, the legislature judgment view profile is $(\widetilde{\mathbf{J}}_\ell^t)_{\ell \in \mathcal{L}}$, where each $\widetilde{\mathbf{J}}_\ell^t := (\widetilde{J}_\ell^\theta)_{\theta \in \mathcal{O}^t}$ and $\widetilde{J}_\ell^\theta$ are seen as the view $\widetilde{\mathbf{b}}_\ell^\theta = (\widetilde{b}_\ell^{\theta,k})_k$. The *legislature judgment* on $\mathcal{O}^t$ is denoted by $\widetilde{\mathbf{J}}_{\mathcal{L}}^t := (\widetilde{J}_{\mathcal{L}}^\theta)_{\theta \in \mathcal{O}^t}$, where $\widetilde{J}_{\mathcal{L}}^\theta$ is represented by the *legislature view* $\widetilde{\mathbf{b}}_{\mathcal{L}}^\theta := (\widetilde{b}_{\mathcal{L}}^{\theta,k})_k$ defined by:

$$\widetilde{b}_{\mathcal{L}}^{\theta,k} \;:=\; \operatorname{sign}\left(\widetilde{S}_{\mathcal{L}}^{\theta,k}\right)\;; \quad \text{where} \quad \widetilde{S}_{\mathcal{L}}^{\theta,k} \;:=\; \sum_{l \in \mathcal{L}} \widetilde{w}_\ell \widetilde{b}_\ell^{\theta,k}. \tag{4B}$$

As above, we will assume $\widetilde{b}_{\mathcal{L}}^{\theta,k} \in \{-1, 1\}$. For each option $\theta$, the *legislature grade* $\widetilde{J}_{\mathcal{L}}^\theta$ satisfies $\widetilde{J}_{\mathcal{L}}^\theta = \max\left\{G_k \in \mathcal{G}^t : \widetilde{b}_{\mathcal{L}}^{\theta,k} = 1, k \in \mathcal{M}^t\right\}$, and we denote the *legislature decision* on $\mathcal{O}^t$ by $\widetilde{D}_{\mathcal{L}}^t \in \arg\max\left\{\widetilde{J}_{\mathcal{L}}^\theta, \theta \in \mathcal{O}^t\right\}$. Now, we require the following assumption:

($D_M$) There exists $\epsilon > 0$ such that, for all $t \in \mathbb{N}$ and $k \in \mathcal{M}^t$ and all distinct $\theta_1, \theta_2 \in \mathcal{O}^t$,

$$\lim_{I\to\infty} \text{Prob}\left[\left|\widetilde{S}_{\mathcal{I}}^{\theta_1,k}/I - \widetilde{S}_{\mathcal{I}}^{\theta_2,k}/I\right| > \epsilon\right] = 1.$$

This says that, in large populations, near ties of judgment percentages in the population between any pair of options are extremely unlikely.

For all problems $t$ and all $\theta \in \mathcal{O}^t$, $\text{Prob}\left[\widetilde{J}_{\mathcal{L}}^{\theta} = \widetilde{J}_{\mathcal{I}}^{\theta}\right]$ is the probability that the legislature's decision (by Weighted Majority Judgment) agrees with the grades (on every option of problem $t$) that would have been reached through a "popular referendum by Majority Judgment rule". This probability depends on the population size $I$ only in the sense that when $I$ increases, the sum in (4A) involves more terms; likewise, the expressions (2A) and (2B) defining the weight $\widetilde{w}_{\ell}$ of each legislator $l \in \mathcal{L}$ involve more terms, which may indirectly affect the value of the sum in (4B). Below is our intermediate result.

**Lemma 1** *Given assumptions* (A)*,* (B)*,* ($C_M$)*, and* ($D_M$)*, it follows that* $\lim_{I\to\infty} \text{Prob}\left[\widetilde{J}_{\mathcal{L}}^{\theta} = \widetilde{J}_{\mathcal{I}}^{\theta}\right] = 1$*, for all* $\theta \in \mathcal{O}^t$ *and all* $t \in \mathbb{N}$.

At this stage, what remains is to show, for any problem $t$, that $\widetilde{D}_{\mathcal{L}}^t = \widetilde{D}_{\mathcal{I}}^t$ holds with very high probability. The remaining part of this section is focused on this issue.

*Tie-Breaking Rule.* For any problem $t$, more than one option could have the same median grade, so we must consider some tie-breaking rule.[8] Fortunately, our model provides a natural tie breaking rule, which we call the *majority margin tie breaking rule*, that is defined in both the electorate and the legislature. Informally, for any $\theta \in \mathcal{O}^t$ and any grade $G_k, k \in \mathcal{M}^t$, the majority margin is proportional to the percentage of individuals in the electorate whose collective opinion on question $q_{(\theta,k)}$ is enacted. We denote it by $\widetilde{M_{\mathcal{I}}}(\theta, k)$ and formally define it as:

$$\widetilde{M_{\mathcal{I}}}(\theta, k) \quad := \quad \frac{1}{I}\left|\widetilde{S}_{\mathcal{I}}^{\theta,k}\right|.$$

Thus, the larger the majority margin $\widetilde{M_{\mathcal{I}}}(\theta, k)$ is, the higher the collective agreement degree on $q_{(\theta,k)}$. Similarly, in the legislature, for any $\theta \in \mathcal{O}^t$ and $k \in \mathcal{M}^t$, the notion of

[8] Balinski and Laraki (2011) define some concepts of tie-breaking rules: majority value, majority gauge (see p. 6–9), upper tie-breaking rule, lower tie-breaking rule, difference tie-breaking rule (chapter 14, p. 244–246). However, we consider the natural tie-breaking rule obtained from our own model which, among the previous tie-breaking rules, is more close to the *difference tie-breaking rule*.

the *legislature majority margin* on $\theta$ with respect to $G_k$, denoted by $\widetilde{M}_{\mathcal{L}}(\theta, k)$, is formally defined as follows:

$$\widetilde{M}_{\mathcal{L}}(\theta, k) \quad := \quad \frac{1}{I}\left|\widetilde{S}_{\mathcal{L}}^{\theta,k}\right|.$$

Taking these new notions into account, we will describe the mechanism of our tie-breaking rule. First, for all $k \in \mathcal{M}^t$, let $\mathcal{O}_k^t$ be the set of options for which the median grade is $G_k$;[9] we then continue with the two following steps:

Step 1. Assume $G_{k_0}, k_0 \in \mathcal{M}^t$ as the maximal median grade among all options in $\mathcal{O}^t$. Then, $\mathcal{O}_{k_0}^t$ is the set of options in $\mathcal{O}^t$ for which the popular median grade is maximal.

Step 2. The collective outcome is then the option in $\mathcal{O}_{k_0}^t$ with the greatest majority margin with respect to the maximal median grade $G_{k_0}$.[10]

To end this section, recall that for any option $\theta \in \mathcal{O}^t$, its popular median grade $\widetilde{J}_{\mathcal{I}}^{\theta}$ is the one such that $\widetilde{J}_{\mathcal{I}}^{\theta} = \max\left\{G_k : \widetilde{b}_{\mathcal{I}}^{\theta,k} = 1, k \in \mathcal{M}^t\right\}$, while its legislature median grade $\widetilde{J}_{\mathcal{L}}^{\theta}$ is the one such that $\widetilde{J}_{\mathcal{L}}^{\theta} = \max\left\{G_k : \widetilde{b}_{\mathcal{L}}^{\theta,k} = 1, k \in \mathcal{M}^t\right\}$. For all $t$, let $P_M^t(I) :=$ Prob $\left[\widetilde{D}_{\mathcal{L}}^t = \widetilde{D}_{\mathcal{I}}^t\right]$ represent the probability that the legislature's decision (by Weighted Majority Judgment) agrees with the decision that would have been reached through direct democracy by the standard (unweighted) MJ rule. Below is the main result of this section.

**Theorem 3** *Given assumptions* (A), (B), (C$_M$), *and* (D$_M$), $\lim\limits_{I\to\infty} P_M^t(I) = 1$ *for all* $t \in \mathbb{N}$.

Thus, if the size of the electorate is very large, then with very high probability, the decisions obtained in the legislature through a Weighted Majority Judgment will be the same as those that would have been reached by a popular referendum. In this sense, *the legislature is "ideally representative" of its electorate with respect to any multioption problem.*

## 5 Summary and Concluding remarks

In view of weighted representative democracy, we have answered our Central Question by applying in this paper either Approval Voting or Majority Judgment. As a result, in

[9] Then, the collection $\{\mathcal{O}_k^t; k \in \mathcal{M}^t\}$ recovers the whole set $\mathcal{O}^t$.

[10] The case with more than one such option in $\mathcal{O}_{k_0}^t$ is rule out by assumption (D$_M$). Even without assumption (D$_M$), any of those greatest options in $\mathcal{O}_{k_0}^t$ would produce the same satisfaction, and they would be equivalent (in practice, such a case would be extremely unlikely in large populations).

large populations, with very high probability in the limit as the population size grows to infinity, the decisions made by the legislature turn out to be the same as those that the individuals themselves would have made via direct democracy, provided that they were well informed and carefully reasoned through their decisions for both the mid-term and long-term future. This result was obtained on the one hand by Theorem 2 for Approval Voting, and by Theorem 3 for Majority Judgment.

Unlike most research, which addresses in isolation either the voting system used to select legislators or the voting rules within the legislature itself, both are taken into account in our models. In addition, similar to Alger (2006), citeskowron2015we and citeCoffman16, in our paper, we measure the performance of the legislature by comparing its decisions with the outcomes that would be reached by the whole population via direct votes. We also abstract from several issues. First, we do not assume that individuals understand (or care about) the technical details of policy issues enough for direct democracy to be practical. Second, we assume that each candidate truthfully reveals her own ideology and opinions during the election campaign. These are obviously idealizing assumptions, but it is beyond the scope of this article to model these issues in more detail.

The goal of this paper was to demonstrate the effectiveness of Approval Voting and Majority Judgment in the highlighted weighted representative democracy. For both Approval Voting and Majority Judgment, there is a large literature about the pros and cons of these methods for eliciting and aggregating collective outcomes. However, in both cases, individuals do not have rankings in mind, and rankings do not usually allow an individual to express an equal evaluation of the options. It is beyond the scope of this paper to summarize this literature or choose the best of these two methods; we argue only that our weighted representative democracy can be applied with a very wide number of voting rules. The dynamic of the applications of other existing voting rules within a weighted representative democracy is a fascinating topic for future work (especially as Rida Laraki and Antonin Macé suggested, investigate on other rules whereby we might reply negatively about the Central Question of the paper, and there are many approaches such as STV, Borda, Condorcet, and even utilitarianism which deserve investigating).

Finally, one can raise some relevant questions like: (i) If we are aiming for a representative body, why make the legislature $\mathcal{L}$ the plurality winners. They could all be from the same party or faction and reflect the same interests? (ii) Would we force the voters who did not vote for any member of $\mathcal{L}$ to select just one? What if they liked nobody or found several acceptable? A straightforward answer to these questions (as also suggested Steven Brams) is the use of Approval Voting for the choice of $\mathcal{L}$[11] — and AV also makes sense for the weighting of its members — as well as its use by those members, like it has been done in Sect. 3. We have treated the candidates themselves as exogenous — we have not modelled the process by which citizens choose to present themselves as candidates in the first place. Clearly, the performance of our proposed system depends on the menu of candidates available to the voters. Although all the previous issues are beyond the scope of this paper, they are important points for the effectiveness of our models summarized as follows: "*Show how votes can be aggregated so that voting in a representative democracy mirrors the outcome in the direct democracy of a Greek city-state*".

## Appendix: Proofs

The proofs in this paper depend on some results from Pivato and Soh (2020). Thus, in some parts of this appendix, we will refer the reader to Pivato and Soh (2020) when the proof is exactly the same. Here are the overall steps of our proofs: the proof of Theorem 1 is already given in Pivato and Soh (2020), and the first part of the Appendix concerns the proof of Theorem 2, which is mainly based on Proposition 1 stated below; the rest of the Appendix concerns the proof steps of Lemma 1 and Theorem 3, which are mainly based on Proposition 2 stated below. Now, we begin with the proof of Theorem 2.

Let $t \in \mathbb{N}$, $\ell \in \mathcal{L}$, and $\theta \in \mathcal{O}^t$. Let $\widetilde{s}^{\theta}_{\mathcal{I}} := \widetilde{S}^{\theta}_{\mathcal{I}}/I$ and $\widetilde{s}^{\theta}_{\mathcal{L}} := \widetilde{S}^{\theta}_{\mathcal{L}}/I$. From Equations (3A) and (3B), it is clear that Theorem 2 follows from the next result.

**Proposition 1** *Assume* (A)*,* (B) *and* (C$_A$)*. For any* $\epsilon > 0$*,* $\lim\limits_{I\to\infty} \text{Prob}\left[|\widetilde{s}^{\theta}_{\mathcal{I}} - \widetilde{s}^{\theta}_{\mathcal{L}}| > \epsilon\right] = 0$.

[11] Indeed, (Brams et al. 2019) propose modifcations of Approval Voting in order to elect multiple winners.

*Proof (Proof of Proposition 1.)* This proof follows the same path as the proof of Proposition 6 from Pivato and Soh (2020). Below are some details, in which we first note that

$$\widetilde{s}^{\theta}_{\mathcal{I}} = \frac{1}{I}\sum_{i\in\mathcal{I}}\widetilde{b}^{\theta}_{i} \;=\; \frac{1}{I}\sum_{\ell\in\mathcal{L}}\sum_{i\in\widetilde{\mathcal{I}}_\ell}\widetilde{b}^{\theta}_{i} \;=\; \frac{1}{I}\sum_{\ell\in\mathcal{L}}\widetilde{b}^{\theta}_{\ell}\sum_{i\in\widetilde{\mathcal{I}}_\ell}\widetilde{b}^{\theta}_{\ell}\,\widetilde{b}^{\theta}_{i} \;=\; \frac{1}{I}\sum_{\ell\in\mathcal{L}}\widetilde{b}^{\theta}_{\ell}\sum_{i\in\widetilde{\mathcal{I}}_\ell}\widetilde{a}^{\theta}_{i,\ell}.$$

$$\text{Thus,}\quad \widetilde{s}^{\theta}_{\mathcal{L}} - \widetilde{s}^{\theta}_{\mathcal{I}} = \frac{1}{I}\sum_{\ell\in\mathcal{L}}\widetilde{b}^{\theta}_{\ell}\,\widetilde{w}_\ell - \frac{1}{I}\sum_{\ell\in\mathcal{L}}\widetilde{b}^{\theta}_{\ell}\sum_{i\in\widetilde{\mathcal{I}}_\ell}\widetilde{a}^{\theta}_{i,\ell} \;=\; \frac{1}{I}\sum_{\ell\in\mathcal{L}}\widetilde{b}^{\theta}_{\ell}\left(\widetilde{w}_\ell - \sum_{i\in\widetilde{\mathcal{I}}_\ell}\widetilde{a}^{\theta}_{i,\ell}\right).$$

$$\text{Thus,}\quad \left|\widetilde{s}^{\theta}_{\mathcal{L}} - \widetilde{s}^{\theta}_{\mathcal{I}}\right| \le \sum_{\ell\in\mathcal{L}}\frac{1}{I}\left|\widetilde{w}_\ell - \sum_{i\in\widetilde{\mathcal{I}}_\ell}\widetilde{a}^{\theta}_{i,\ell}\right|, \tag{A1}$$

because $\left|\widetilde{b}^{\theta}_{\ell}\right| = 1$ for all $\ell\in\mathcal{L}$. For all $\ell\in\mathcal{L}$ and all $i\in\widetilde{\mathcal{I}}_\ell$, let $\widetilde{a}^{i}_{\ell} := 2\widetilde{p}^{i}_{\ell} - 1$. Then,

$$\widetilde{w}_\ell \underset{(*)}{=} (2\widetilde{p}_\ell - 1)\,|\widetilde{\mathcal{I}}_\ell| \underset{(\dagger)}{=} 2\sum_{i\in\widetilde{\mathcal{I}}_\ell}\widetilde{p}^{i}_{\ell} - |\widetilde{\mathcal{I}}_\ell| = \sum_{i\in\widetilde{\mathcal{I}}_\ell}(2\widetilde{p}^{i}_{\ell} - 1) = \sum_{i\in\widetilde{\mathcal{I}}_\ell}\widetilde{a}^{i}_{\ell}, \tag{A2}$$

where $(*)$ is obtained by defining formula (2B) and $(\dagger)$ is obtained by (2A). Let $\widetilde{I}_\ell := |\widetilde{\mathcal{I}}_\ell|$; then,

$$\frac{1}{I}\left|\widetilde{w}_\ell - \sum_{i\in\widetilde{\mathcal{I}}_\ell}\widetilde{a}^{\theta}_{i,\ell}\right| \underset{(*)}{=} \frac{1}{I}\left|\sum_{i\in\widetilde{\mathcal{I}}_\ell}\left(\widetilde{a}^{i}_{\ell} - \widetilde{a}^{\theta}_{i,\ell}\right)\right| \underset{(\dagger)}{\le} \left|\frac{1}{\widetilde{I}_\ell}\sum_{i\in\widetilde{\mathcal{I}}_\ell}\left(\widetilde{a}^{i}_{\ell} - \widetilde{a}^{\theta}_{i,\ell}\right)\right|, \tag{A3}$$

where $(*)$ is obtained by (A2) and $(\dagger)$ holds because $\widetilde{I}_\ell \le I$. At this point, it remains to control the size of the right-hand side in inequality (A3).

Let $\delta > 0$ and $\mu^*_\ell := \overline{\mu}_\ell - \delta$, where $\overline{\mu}_\ell := \rho\{\mathbf{p}\in[0,1]^{\mathcal{L}};\ p_\ell > p_m \text{ for all } m\in\mathcal{L}\setminus\{\ell\}\}$. Then, we obtain the following claim from Pivato and Soh (2020):

*Claim 1* For all $\ell\in\mathcal{L}$, $\lim_{I\to\infty} \text{Prob}\left[\widetilde{I}_\ell \le \mu^*_\ell\, I\right] = 0$.

Now fix some $\epsilon > 0$, as in the theorem statement; then, the two claims below hold.

*Claim 2* For all $\ell\in\mathcal{L}$, $\lim_{I\to\infty} \text{Prob}\left[\left|\frac{1}{\widetilde{I}_\ell}\sum_{i\in\widetilde{\mathcal{I}}_\ell}\left(\widetilde{a}^{i}_{\ell} - \widetilde{a}^{\theta}_{i,\ell}\right)\right| > \frac{\epsilon}{L} \;\middle|\; \widetilde{I}_\ell > \mu^*_\ell\, I\right] = 0$.

*Proof* Notice first that the following holds by assumption ($\mathrm{C}_A$): for every $i\in\widetilde{\mathcal{I}}_\ell$, the random variable $\widetilde{a}^{\theta}_{i,\ell}$ takes the values $+1$ (with probability $\widetilde{p}^{i}_{\ell}$) and $-1$ (with probability $1-\widetilde{p}^{i}_{\ell}$). Thus, $\mathbb{E}\left[\widetilde{a}^{\theta}_{i,\ell} \;\middle|\; \widetilde{p}^{i}_{\ell}\right] = 2\widetilde{p}^{i}_{\ell} - 1 = \widetilde{a}^{i}_{\ell}$; i.e., $\mathbb{E}\left[\widetilde{a}^{\theta}_{i,\ell} - \widetilde{a}^{i}_{\ell} \;\middle|\; \widetilde{p}^{i}_{\ell}\right] = 0$ for every possible realization of $\widetilde{p}^{i}_{\ell}$. Then, $\mathbb{E}[\widetilde{a}^{\theta}_{i,\ell} - \widetilde{a}^{i}_{\ell}] = 0$. On the other hand, the random variables $\{\widetilde{a}^{i}_{\ell} - \widetilde{a}^{\theta}_{i,\ell}\}_{i\in\widetilde{\mathcal{I}}_\ell}$ are independent, and since $\widetilde{a}^{i}_{\ell}\in[-1,1]$, it follows that $\widetilde{a}^{\theta}_{i,\ell} - \widetilde{a}^{i}_{\ell}\in[-2,2]$.

Now, let some constant $J > \mu_\ell^* I$. Then, Hoeffding's inequality (1963) implies that

$$\text{Prob}\left[\left|\frac{1}{\widetilde{I}_\ell}\sum_{i\in\widetilde{\mathcal{I}}_\ell}\left(\widetilde{a}_\ell^i-\widetilde{a}_{i,\ell}^\theta\right)\right|>\frac{\epsilon}{L}\;\middle|\;\widetilde{I}_\ell=J\right]\;\leq\;2\exp\left(-\frac{J\,(\epsilon/L)^2}{8}\right)\;\leq\;2\exp\left(-\frac{\mu_\ell^*\,I\,\epsilon^2}{8\,L^2}\right).$$

These inequalities hold for any $J > \mu_\ell^* I$. Thus,

$$\text{Prob}\left[\left|\frac{1}{\widetilde{I}_\ell}\sum_{i\in\widetilde{\mathcal{I}}_\ell}\left(\widetilde{a}_\ell^i-\widetilde{a}_{i,\ell}^\theta\right)\right|>\frac{\epsilon}{L}\;\middle|\;\widetilde{I}_\ell>\mu_\ell^*\,I\right]\quad\leq\quad2\exp\left(-\frac{\mu_\ell^*\,I\,\epsilon^2}{8\,L^2}\right)\;\xrightarrow[I\to\infty]{}\;0,$$

as claimed.

*Claim 3* For all $\ell \in \mathcal{L}$, $\lim\limits_{I\to\infty}\text{Prob}\left[\frac{1}{I}\left|\widetilde{w}_\ell-\sum_{i\in\widetilde{\mathcal{I}}_\ell}\widetilde{a}_{i,\ell}^\theta\right|>\frac{\epsilon}{L}\right]=0.$

*Proof*

$$\begin{aligned}
&\text{Prob}\left[\frac{1}{I}\left|\widetilde{w}_\ell-\sum_{i\in\widetilde{\mathcal{I}}_\ell}\widetilde{a}_{i,\ell}^\theta\right|>\frac{\epsilon}{L}\right]\;\underset{(\diamond)}{\leq}\;\text{Prob}\left[\left|\frac{1}{\widetilde{I}_\ell}\sum_{i\in\widetilde{\mathcal{I}}_\ell}\left(\widetilde{a}_\ell^i-\widetilde{a}_{i,\ell}^\theta\right)\right|>\frac{\epsilon}{L}\right]\\
&=\text{Prob}\left[\left|\frac{1}{\widetilde{I}_\ell}\sum_{i\in\widetilde{\mathcal{I}}_\ell}\left(\widetilde{a}_\ell^i-\widetilde{a}_{i,\ell}^\theta\right)\right|>\frac{\epsilon}{L}\;\middle|\;\widetilde{I}_\ell>\mu_\ell^*\,I\right]\times\text{Prob}\left[\widetilde{I}_\ell>\mu_\ell^*\,I\right]\\
&\qquad+\text{Prob}\left[\left|\frac{1}{\widetilde{I}_\ell}\sum_{i\in\widetilde{\mathcal{I}}_\ell}\left(\widetilde{a}_\ell^i-\widetilde{a}_{i,\ell}^\theta\right)\right|>\frac{\epsilon}{L}\;\middle|\;\widetilde{I}_\ell\leq\mu_\ell^*\,I\right]\times\text{Prob}\left[\widetilde{I}_\ell\leq\mu_\ell^*\,I\right]\\
&\underset{(*)}{\leq}\text{Prob}\left[\left|\frac{1}{\widetilde{I}_\ell}\sum_{i\in\widetilde{\mathcal{I}}_\ell}\left(\widetilde{a}_\ell^i-\widetilde{a}_{i,\ell}^\theta\right)\right|>\frac{\epsilon}{L}\;\middle|\;\widetilde{I}_\ell>\mu_\ell^*\,I\right]+\text{Prob}\left[\widetilde{I}_\ell\leq\mu_\ell^*\,I\right],
\end{aligned}$$

where $(\diamond)$ is obtained by inequality (A3), while $(*)$ holds because probabilities are at most 1. Thus, by Claims 1 and 2, the desired result holds.

Now, by (A1), $\left|\widetilde{s}_{\mathcal{L}}^\theta-\widetilde{s}_{\mathcal{I}}^\theta\right|>\epsilon$ implies that $\frac{1}{I}\left|\widetilde{w}_\ell-\sum_{i\in\widetilde{\mathcal{I}}_\ell}\widetilde{a}_{i,\ell}^\theta\right|>\frac{\epsilon}{L}$ for some $\ell\in\mathcal{L}$. Thus,

$$\text{Prob}\left[\left|\widetilde{s}_{\mathcal{L}}^\theta-\widetilde{s}_{\mathcal{I}}^\theta\right|>\epsilon\right]\quad\leq\quad\text{Prob}\left[\frac{1}{I}\left|\widetilde{w}_\ell-\sum_{i\in\widetilde{\mathcal{I}}_\ell}\widetilde{a}_{i,\ell}^\theta\right|>\frac{\epsilon}{L},\text{ for some }\ell\in\mathcal{L}\right]\;\xrightarrow[I\to\infty]{}\;0$$

by Claim 3 because $L$ is fixed.

*Proof (Proof of Theorem 2.)* By assumption $(D_A)$, there exists $\epsilon > 0$ such that, for all $t \in \mathbb{N}$ and all distinct $\theta_1, \theta_2 \in \mathcal{O}^t$, $\lim\limits_{I\to\infty}\text{Prob}\left[\left|\widetilde{S}_{\mathcal{I}}^{\theta_1}/I-\widetilde{S}_{\mathcal{I}}^{\theta_2}/I\right|>\epsilon\right]=1$. Meanwhile, by assumptions $(A), (B)$, and $(C_A)$, Proposition 1 implies that for any $\epsilon > 0$,

$\lim_{I\to\infty} \text{Prob}\left[|\widetilde{s}^{\theta}_{\mathcal{I}} - \widetilde{s}^{\theta}_{\mathcal{L}}| > \epsilon\right] = 0$. Therefore, by joining the two previous statements, we obtain $\lim_{I\to\infty} P_A(I) = 1$.

We now turn to the proof of Theorem 3, which is straightforwardly obtained by combining the majority margin tie-breaking rule with the result of Lemma 1. Therefore, it suffices to prove Lemma 1, which is clearly obtained from Proposition 2 stated below.

Let $t \in \mathbb{N}$, $k \in \mathcal{M}^t$ and $\theta \in \mathcal{O}^t$. Let $\widetilde{s}^{\theta,k}_{\mathcal{I}} := \widetilde{S}^{\theta,k}_{\mathcal{I}}/I$ and $\widetilde{s}^{\theta,k}_{\mathcal{L}} := \widetilde{S}^{\theta,k}_{\mathcal{L}}/I$. From Equations (4A) and (4B), Lemma 1 follows from the next Proposition.

**Proposition 2** *By* (A), (B) *and* ($\text{C}_M$). *For any* $\epsilon > 0$, $\lim_{I\to\infty} \text{Prob}\left[|\widetilde{s}^{\theta,k}_{\mathcal{I}} - \widetilde{s}^{\theta,k}_{\mathcal{L}}| > \epsilon\right] = 0$.

In fact, Proposition 2, combined with assumption ($\text{D}_M$), proves Lemma 1. Since Propositions 1 and 2 are quite similar, the proof of Proposition 2 follows the same pattern as the proof of Proposition 1. Analogously, as we have proven Theorem 2 by assumption ($\text{D}_A$) and Proposition 1, we prove Lemma 1 by assumption ($\text{D}_M$) and Proposition 2.

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