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The finagle point might not be within the ε-core: a contradiction with Bräuninger's result.

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The finagle point might not be within the ϵ-core: a contradiction with Bräuninger's result.

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Abstract

In this paper, we focus on a result stated by Bräuninger that the finagle point is within the ϵ-core in a spatial voting game with Euclidean individual preferences. Through a counterexample with 7 players, we show that Bräuninger's result is not valid.

Key words : Spatial voting, majority game, dominance, core, finagle.

JEL Classification : C62, C70, D71, D72.

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1 Introduction

In general, there is no Condorcet winner or undominated alternatives in a majority voting game of a Euclidean spatial model with at least two dimensions (Davis et al. [2], Plott [3]). This important result is obviously very negative and many scholars studied why this lack of stability is not verified when empirical observations are considered (Tullock and Brennan [10]). Other solution concepts or approaches have been proposed and among them, we can cite the ϵ-core (Wooders [11], Salant and Godstein [6], Scott and Grofman [7], Tovey [8]) and the finagle point (Wuffle et al. [12]). Indeed, the relationships between these two solution concepts is the purpose of this note. The ϵ-core is the set of alternatives which are not ϵ-dominated knowing that an alternative x ϵ-dominates an alternative y if there exists a majority of voters such that the distance from their ideal point to x is less than the one to y minus ϵ. In other words, if the difference of distance between x and y compared to her ideal point is not sufficient (at least ϵ), then an individual is indifferent between them. ϵ can be viewed as a cost implied by the vote itself (time, acquisition of information and so on). The finagle point is defined as follows by Wuffle et al. [12, p. 348]: "[...] from it, a candidate can, with only minuscule changes in his ideal policy location, find a response to any challenger that will defeat that challenger".

The relationships between the ϵ-core and the finagle point is established by Bräuninger [1] in the following result: if ϵ is such that the ϵ-core is always nonempty, then the finagle point is within the ϵ-core. It is obviously a positive result when two solution concepts lead to the same outcome. However, Tovey [9] shows that Bräuninger's proof has a logical gap but he doesn't show that it is false: "I do not know whether or not the claimed result is true, but I will show that the argument given there is not sufficient to prove it", Tovey [p. 1]. In other words, Bräuninger's result needs a complete proof. Unfortunately, it is shown in this note that such a proof does not exist. Indeed, we present a counter-example with seven individuals in a two-dimensional space where the finagle point is not in the ϵ-core which shows that Bräuninger's result is false.

2 Notation and definitions

Let $N = \{1, 2..., n\}$ be a finite set of voters represented by ideal points Q_1, Q_2,..., Q_n in $\mathbb{R}^2$, the two-dimensional Euclidean policy space. We assume that n is odd. Voters preferences are Euclidean: voter i strictly prefers a position (policy) X to Y if $d(Q_i, X) < d(Q_i, Y)$ with d the classical Euclidean distance. Voter i is indifferent between X and Y if $d(Q_i, X) = d(Q_i, Y)$. A coalition $S \subseteq N$ is winning if it contains at least half of the voters. W is the set of winning coalitions. A position X dominates a position Y via a winning coalition S if every member of S strictly prefers X to Y. X dominates Y if there exists a winning coalition S such that X dominates Y via S. Let $V = (N, W, (Q_i))$ be a spatial voting game. The core, denoted $C(V)$ is the set of undominated positions.

For any coalition S, let $Conv(S)$ be the convex hull of the ideal points on S. Saari [4] shows that $C(V) = \underset{S \in W^m}{\cap} Conv(S)$ with $W^m \subseteq W$ the set of winning coalitions containing exactly $\frac{n+1}{2}$ voters.

We say that X ϵ-dominates Y via a winning coalition S if for every $i \in S$, we have: $d(Q_i, Y) - d(Q_i, X) > \epsilon$. The ϵ-core of a spatial voting game V, denoted $C^\epsilon(V)$ is the set of ϵ-undominated positions.

For every coalition S, let us denote by ϵ-$Conv(S)$, the set of positions that are not ϵ-dominated via S. Then $C^\epsilon(V) = \underset{S \in W^m}{\cap} \epsilon\text{-}Conv(S)$. It is obvious that $Conv(S) \subseteq \epsilon\text{-}Conv(S)$ since an undominated position is also an ϵ-undominated position.

Bräuninger [1] shows that if the value of ϵ is gradually increased, then there is a threshold ϵ^* from which the ϵ-core is non-empty. More generally, when ϵ increases, the ϵ-core becomes larger and can contain any position.

For $X \in \mathbb{R}^2$ and $\tau \geq 0$ let $B(X, \tau) = \{Z : d(X, Z) \leq \tau\}$ be the (closed) circle around X of radius τ. The finagle radius $f(X)$ of position X is the minimum value of τ such that for each $Y \in \mathbb{R}^2$, there exists $Z \in B(X, f(X))$ which is not dominated by Y. A finagle point is a point X with minimal radius $f(X)$.

3 Results

Proposition *Bräuninger [1]: Let f be the finagle point, r_f the minimal finagle radius, and ϵ^* the minimal voting threshold for a majority voting game $V = (N, W, (Q_i))$ so that $C^\epsilon(V) \neq \emptyset$. Then: $\epsilon^* = r_f$ and $f \in C^\epsilon(V)$.*

We want to show that this result is wrong in general and, for this, we present a counterexample in which the finagle point is not within the ϵ-core. Before, we need several technical lemmas.

Lemma 1 *Let $V = (N, W, (Q_i))$ be a spatial voting game and let $S = \{i, j\}$ be a coalition. A position Y belongs to ϵ-$Conv(S)$ if and only if $d(Q_i, Y) + d(Q_j, Y) - d(Q_i, Q_j) - 2\epsilon \leq 0$.*

Proof. $\Rightarrow$) For a given Y, let us show that $Y \in \epsilon\text{-}Conv(S) \Longrightarrow d(Q_i, Y) + d(Q_j, Y) - d(Q_i, Q_j) - 2\epsilon \leq 0$. It is tantamount to show that $d(Q_i, Y) + d(Q_j, Y) - d(Q_i, Q_j) - 2\epsilon > 0 \Longrightarrow Y \notin \epsilon\text{-}Conv(S)$. Through the triangular inequality of distance, we have: $d(Q_i, Q_j) + d(Q_j, Y) \geq d(Q_i, Y)$.

$$\begin{aligned} d(Q_i, Q_j) + d(Q_j, Y) \geq d(Q_i, Y) &\Longrightarrow d(Q_j, Y) \geq d(Q_i, Y) - d(Q_i, Q_j) \\ &\Longrightarrow 2d(Q_j, Y) - 2\epsilon \geq d(Q_i, Y) + d(Q_j, Y) - d(Q_i, Q_j) - 2\epsilon \end{aligned}$$

It follows that $d(Q_i, Y) + d(Q_j, Y) - d(Q_i, Q_j) - 2\epsilon > 0 \Longrightarrow 2d(Q_j, Y) - 2\epsilon > 0$, that is to say that $d(Q_j, Y) > \epsilon$. In the same way, we easily show that $d(Q_i, Y) > \epsilon$. The assumption $d(Q_i, Y) + d(Q_j, Y) - d(Q_i, Q_j) - 2\epsilon > 0$ is equivalent to $(d(Q_i, Y) - \epsilon) + (d(Q_j, Y) - \epsilon) > d(Q_i, Q_j)$ with $d(Q_j, Y) > \epsilon$ and $d(Q_i, Y) > \epsilon$. This means that the circle with center Q_i and radius $d(Q_i, Y) - \epsilon$ meets the circle with center Q_j and radius $d(Q_j, Y) - \epsilon$. Because of strict inequality, each circle goes beyond the border of the other. For any element X within the meeting area, we have: $d(Q_i, X) < d(Q_i, Y) - \epsilon$ and $d(Q_j, X) < d(Q_j, Y) - \epsilon$ i.e. $\epsilon < d(Q_i, Y) - d(Q_i, X)$ and $\epsilon < d(Q_j, Y) - d(Q_j, X)$. Thus, X is ϵ-preferred to Y via the coalition S, hence $Y \notin \epsilon\text{-}Conv(S)$.

$\Leftarrow$) Conversely, for a given Y, we need to show that $d(Q_i, Y) + d(Q_j, Y) - d(Q_i, Q_j) - 2\epsilon \leq 0 \Longrightarrow Y \in \epsilon\text{-}Conv(S)$. It is sufficient to show that $Y \notin \epsilon\text{-}Conv(S) \Longrightarrow d(Q_i, Y) + d(Q_j, Y) - d(Q_i, Q_j) - 2\epsilon > 0$. If $Y \notin \epsilon\text{-}Conv(S)$ then there is X such that $d(Q_i, Y) - d(Q_i, X) > \epsilon$ and $d(Q_j, Y) - d(Q_j, X) > \epsilon$. Adding member to member these two inequalities gives $d(Q_i, Y) + d(Q_j, Y) - (d(Q_i, X) + d(Q_j, X)) - 2\epsilon > 0$. Once again, the triangular inequality allows us to write $d(Q_i, X) + d(Q_j, X) \geq d(Q_i, Q_j)$. Consequently, $-d(Q_i, Q_j) \geq -(d(Q_i, X) + d(Q_j, X))$ then $d(Q_i, Y) + d(Q_j, Y) - d(Q_i, Q_j) - 2\epsilon \geq d(Q_i, Y) + d(Q_j, Y) - (d(Q_i, X) + d(Q_j, X)) - 2\epsilon > 0$ i.e. $d(Q_i, Y) + d(Q_j, Y) - d(Q_i, Q_j) - 2\epsilon > 0$. The two implications provide a proof of the Lemma. ■

We can deduce from the previous Lemma that $\epsilon\text{-}Conv(S)$ boundary is given by the following equation: $d(Q_i, Y) + d(Q_j, Y) - d(Q_i, Q_j) - 2\epsilon = 0$. This lemma holds for any number of dimensions greater than 1. In a two dimensional space, we get an ellipse of foci Q_i and Q_j, Figure 1. Following Saari [5], the term ϵ-ellipse[1] is used.

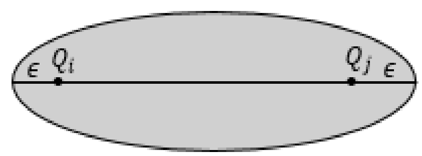

Figure 1

Lemma 2 *Let $V = (N, W, (Q_i))$ be a spatial voting game and let $S = \{i, j\}$ be a coalition. Assume that for every $k \in S \backslash \{i, j\}$, we have, $[Y, Q_k] \cap [Q_i, Q_j] \neq \varnothing$. Position Y belongs to $\epsilon\text{-}Conv(S)$ if and only if Y belongs to $\epsilon\text{-}Conv(\{i, j\})$.*

Proof. $\Rightarrow$) To show that $Y \in \epsilon\text{-}Conv(S) \Longrightarrow Y \in \epsilon\text{-}Conv(\{i, j\})$, we just have to show that $Y \notin \epsilon\text{-}Conv(\{i, j\}) \Longrightarrow Y \notin \epsilon\text{-}Conv(S)$. Given such a Y, we need to find Z such that for every $k \in S$ we have: $d(Q_k, Y) - d(Q_k, Z) > \epsilon$. The assumption $Y \notin \epsilon\text{-}Conv(\{i, j\})$ means that the set of elements that are ϵ-preferred by i and j to Y is not empty. This set of elements is the meeting area of the circle with center Q_i and radius $d(Q_i, Y) - \epsilon$ and the circle with center Q_j and radius $d(Q_j, Y) - \epsilon$, see Figure 2. This area contains a part of the segment line $[Q_i, Q_j]$ and consider X belonging to it. Thus we have: $d(Q_i, Y) - d(Q_i, X) > \epsilon$ and $d(Q_j, Y) - d(Q_j, X) > \epsilon$.

By assumption, we have $[Y, Q_k] \cap [Q_i, Q_j] \neq \varnothing$ for any individual $k \in S \backslash \{i, j\}$. Consider $M \in [Y, Q_k] \cap [Q_i, Q_j]$, thus $M \in [Q_i, X]$ or $M \in [X, Q_j]$. Without loss of generality, assume that $M \in [Q_i, X]$.

By triangular inequality, we have: $d(Q_i, M) + d(M, Y) \geq d(Q_i, Y)$ and $d(Q_k, M) + d(M, X) \geq d(Q_k, X)$. Adding member to member the two inequalities, we obtain $[d(Q_i, M) + d(M, X)] + [d(Q_k, M) + d(M, Y)] \geq d(Q_i, Y) + d(Q_k, X)$ and then $d(Q_i, X) + d(Q_k, Y) \geq d(Q_i, Y) + d(Q_k, X)$. Finally, we have: $d(Q_k, Y) - d(Q_k, X) \geq d(Q_i, Y) - d(Q_i, X)$. From the assumption $d(Q_i, Y) - d(Q_i, X) > \epsilon$ we deduce that $d(Q_k, Y) - d(Q_k, X) > \epsilon$, which proves that voter k ϵ-prefers X to Y.

[1] A nice description of the construction of an ϵ-ellipse is proposed by Saari [5].

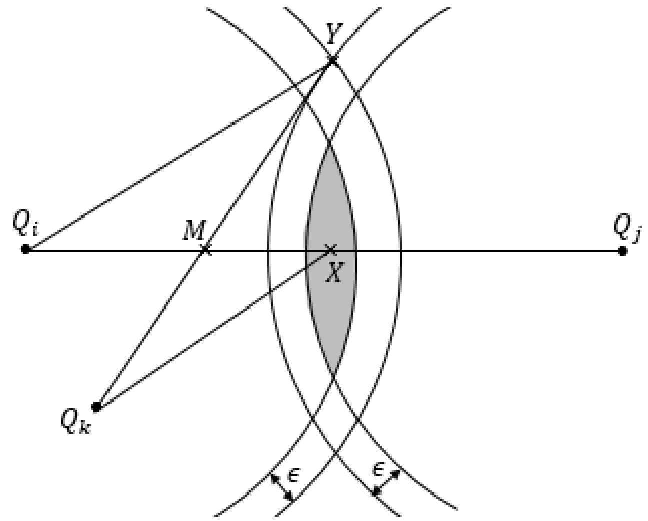

Figure 2

$\Leftarrow$)Conversely, it must be shown that $Y \in \epsilon$-$Conv(\{i,j\}) \Longrightarrow Y \in \epsilon$-$Conv(S)$. If $Y \in \epsilon$-$Conv(\{i,j\})$ then, there is no X such that $d(Q_i,Y)-d(Q_i,X) > \epsilon$ and $d(Q_j,Y)-d(Q_j,X) > \epsilon$. It follows that there is no X such that for every $k \in S$, $d(Q_k,Y)-d(Q_k,X) > \epsilon$ i.e. $Y \in \epsilon$-$Conv(S)$. The proof is complete ■

Lemma 3 *For every spatial voting game $V = (N, W, (Q_i))$, we have $C^{\epsilon^*}(V) \subseteq Conv(N)$.*

Proof. Let $V = (N, W, (Q_i))$ be a spatial voting game, assume that $C^{\epsilon^*}(V) \nsubseteq Conv(N)$ then, there is $A \in C^{\epsilon^*}(V)$ such that $A \notin Conv(N)$. Since $Conv(N)$ is convex, there is a line $(\triangle)$ that strictly separates A to $Conv(N)$. Without loss of generality translate, scale, and rotate so that $(\triangle) : x = 0$ and $A = (-1, 0)$, see Figure 3. The line $(\triangle)$ divides the plan into two disjointed half-planes such that the half-plane containing A is defined by $x < 0$ and the half-plane containing $Conv(N)$ is defined by $x > 0$.

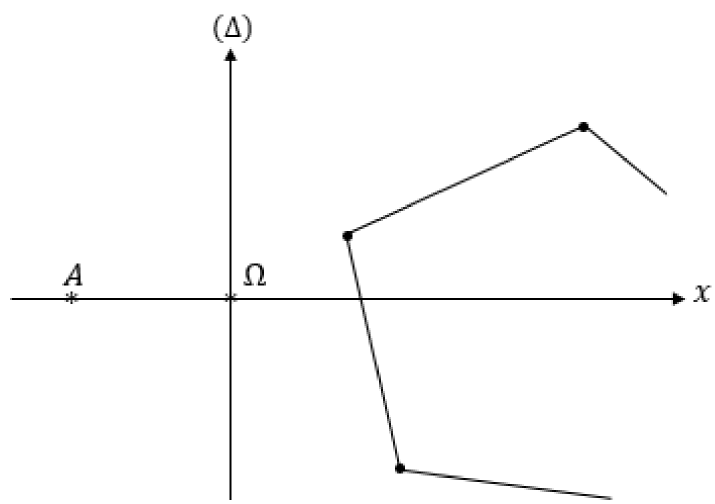

Figure 3

Consider $\Omega = (0,0)$ and $Q_i = (x_i, y_i)$ for any $i \in S \subseteq N$ (thus $x_i > 0$) . We have $d(Q_i, \Omega) = \sqrt{x_i^2 + y_i^2}$ and $d(Q_i, A) = \sqrt{(x_i+1)^2 + y_i^2}$. $x_i > 0 \Longrightarrow \sqrt{(x_i+1)^2 + y_i^2} > \sqrt{x_i^2 + y_i^2} \Longrightarrow$

$d(Q_i, A) > d(Q_i, \Omega) \Longrightarrow d(Q_i, A) - d(Q_i, \Omega) > 0$. Set $\gamma = \frac{1}{2}\min\{d(Q_i, A) - d(Q_i, \Omega) : i \in N\}$, we have $\gamma > 0$, and for every $i \in N$, $d(Q_i, A) - d(Q_i, \Omega) > \gamma$. We know that A is not ϵ^*-dominated, in particular, A is not ϵ^*-dominated by Ω. Therefore, for every winning coalition S, there is $i \in S$ such that $d(Q_i, A) - d(Q_i, \Omega) \leq \epsilon^*$. We deduce that $\gamma < \epsilon^*$ i.e. $\epsilon^{**} > 0$ with $\epsilon^{**} = \epsilon^* - \gamma$

Now, we want to show that $\Omega \in C^{\epsilon^{**}}(V)$, which is equivalent to show that for every winning coalition S, Ω is not ϵ^{**}-dominated via S. In other words, for every X, there is $k \in S$ such that $d(Q_k, \Omega) - d(Q_k, X) \leq \epsilon^{**}$. Consider a position X and let us look at for such a k. We know that $A \in C^{\epsilon^*}(V)$ consequently, there is $i \in S$ such that $d(Q_i, A) - d(Q_i, X) \leq \epsilon^*$. Since $d(Q_i, A) - d(Q_i, \Omega) > \gamma$ we have: $d(Q_i, A) - d(Q_i, X) > d(Q_i, \Omega) - d(Q_i, X) + \gamma$. The inequality $d(Q_i, A) - d(Q_i, X) \leq \epsilon^*$ induced $\epsilon^* > d(Q_i, \Omega) - d(Q_i, X) + \gamma$ i.e. $\epsilon^{**} = \epsilon^* - \gamma > d(Q_i, \Omega) - d(Q_i, X)$. Just take $k = i$ to conclude.

Finally, we have $\Omega \in \epsilon^{**}$-$Conv(S)$ for every winning coalition S, thus $\Omega \in C^{\epsilon^{**}}(V)$ with $\epsilon^{**} = \epsilon^* - \gamma < \epsilon^*$, a contradiction of ϵ^* is the threshold of non-emptiness. ■

Note that this lemma hold for any number of dimensions greater than 1.

Proposition *Let $V = (N, W, (Q_i))$ be a spatial voting game. f does not necessarily belong to $C^{\epsilon^*}(V)$ and r_f is not necessarily equal to ϵ^*.*

This proposal is in contradiction with Bräuninger 's result [1] who shows that $\epsilon^* = r_f$ and $f \in C^{\epsilon^*}(V)$ (proposition 2, page 182). Tovey [9] points out a significant logical gap in Bräuninger 's proof but he does not prove that his proposition is wrong, what we do with a 7-voter example.

Proof. Consider the following graph:

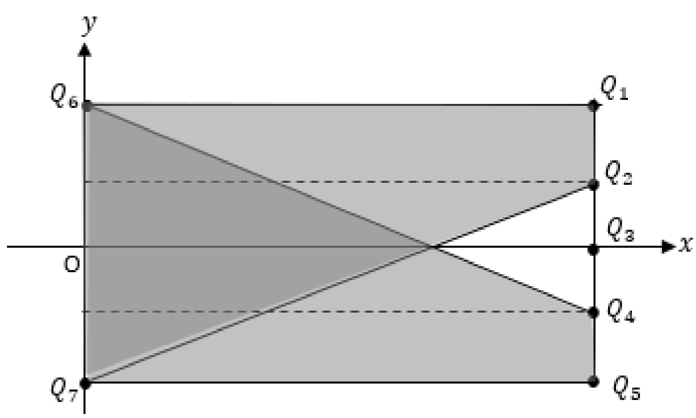

Figure 4

with the following coordinates:

$$Q_1\begin{pmatrix}9\\2\end{pmatrix};Q_2\begin{pmatrix}9\\1\end{pmatrix};Q_3\begin{pmatrix}9\\0\end{pmatrix};Q_4\begin{pmatrix}9\\-1\end{pmatrix};Q_5\begin{pmatrix}9\\-2\end{pmatrix};Q_6\begin{pmatrix}0\\2\end{pmatrix} \text{ and } Q_7\begin{pmatrix}0\\-2\end{pmatrix}$$

First step: we want to show that there exist a point A and $\epsilon = \bar{\epsilon}$ such that $\bigcap_{S \in W^m} \bar{\epsilon}-Conv(S) = A$ considering the following winning coalitions S: $S_1 = \{1,2,3,4\}$, $S_2 = \{2,3,4,5\}$, $S_3 = \{1,2,6,7\}$

and $S_4 = \{4,5,6,7\}$. Notice that the core is empty since $\bigcap_S Conv(S) = \emptyset$ with $S = S_1, S_2, S_3, S_4$. Furthermore, since $Conv(S) \subseteq \epsilon\text{-}Conv(S)$ and since our graph is symmetric, it is obvious that there exists ϵ such that $A \in \bigcap_S \bar{\epsilon} - Conv(S)$ and A is on the x axis. Our purpose is to show that there exists $\epsilon = \bar{\epsilon}$ such that A is unique. Of course, by Lemma 3, A belongs to the rectangle $Q_1Q_5Q_6Q_7$.

Consider first the coalition $S_=\{1,2,6,7\}$.

On the rectangle $Q_1Q_5Q_6Q_7$, we know that the trapeze $Q_1Q_2Q_6Q_7$ that represents $Conv(S_3)$ is included in $\epsilon\text{-}Conv(S_3)$. It remains to find the points of the triangle $Q_2Q_5Q_7$ which belong to $\epsilon\text{-}Conv(S_3)$. Indeed, it is obvious that the intersection between $\epsilon\text{-}Conv(S_3)$ and $\epsilon\text{-}Conv(S_4)$ will be under the line $[Q_2, Q_7]$ and not above.

For every Y belonging to this triangle, we have $[Y, Q_1] \cap [Q_2, Q_7] \neq \varnothing$ and $[Y, Q_6] \cap [Q_2, Q_7] \neq \varnothing$. The conditions of Lemma 2 are satisfied, it follows that: $Y \in \epsilon\text{-}Conv(S_3)$ if and only if $Y \in \epsilon\text{-}Conv(\{2,7\})$. According to Lemma 1, $\epsilon\text{-}Conv(\{2,7\})$ is the ϵ-ellipse of foci Q_2 and Q_7: we obtain the following Figure 5.

In the same way, the positions above the line $[Q_4, Q_6]$ which are not dominated via the coalition S_4 are in the ϵ-ellipse of foci Q_4 and Q_6. Denote M_ϵ the intersection of $\epsilon\text{-}Conv(\{2,7\})$ and $\epsilon\text{-}Conv(\{4,6\})$. By symmetry, it is obvious that M_ϵ is on the x axis and tends to Q_3 when ϵ increases.

Consider now the two coalitions S_1 and S_2: by Lemma 1 and Lemma 2, $\epsilon\text{-}Conv(S_1)$ and $\epsilon\text{-}Conv(S_2)$ are simply the ϵ-ellipses of foci Q_1 and Q_4 for S_1 and Q_2 and Q_5 for S_2. Denote N_ϵ the intersection of $\epsilon\text{-}Conv(\{1,4\})$ and $\epsilon\text{-}Conv(\{2,5\})$. By symmetry, it is obvious that N_ϵ is on the x axis and tends to 0 when ϵ increases.

Since, when ϵ increases, M_ϵ goes from $M\,(6,0)$ to Q_3 and N_ϵ goes from Q_3 to O. Note that M is the meeting point of (Q_4Q_6) and (Q_2Q_7), see Figure 6. There exists $\epsilon = \bar{\epsilon}$ such that $M_{\bar{\epsilon}} = N_{\bar{\epsilon}}$ and this point is $A\,(x_A, 0)$. The first step is proved.

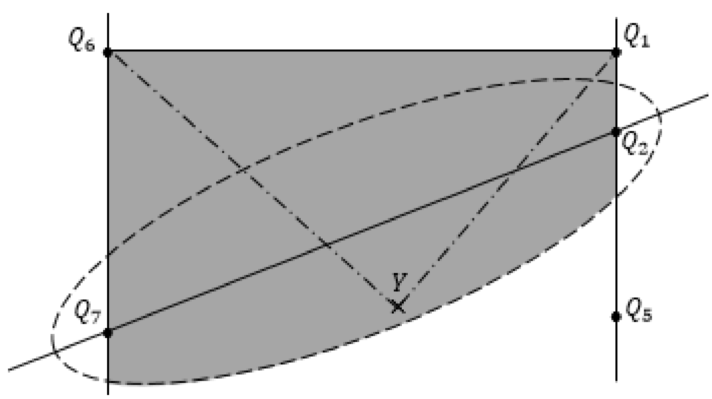

Figure 5

Second step: we want to show that A is not the finagle point.

Failing to find the exact values of x_A and $\bar{\epsilon}$, we must provide the upper bound values. Let $H\,(x,0)$ be a point such that $H \in [M, Q_3]$. According to Lemma 1, $H \in \epsilon\text{-}Conv(\{2,7\})$ if

$d(Q_2,H)+d(Q_7,H)-d(Q_2,Q_7)-2\epsilon \leq 0$. We obtain the following relation: $R_1(\epsilon;x) : \sqrt{x^2-18x+82}+\sqrt{x^2+4}-3\sqrt{10}-2\epsilon \leq 0$. In the same way, we have $H \in \epsilon\text{-}Conv(\{1,4\})$ if $R_2(\epsilon;x) : \sqrt{x^2-18x+85}+\sqrt{x^2-18x+82}-3-2\epsilon \leq 0$.

Note that $R_1(\epsilon;x)$ allow us to know if $H \in \epsilon\text{-}Conv(S_3)$ and $R_2(\epsilon;x)$ corresponds to $\epsilon\text{-}Conv(S_1)$. By symmetry, we can conclude for S_2 and S_4. If there is H that does not check any of $R_1(\epsilon;x)$ and $R_2(\epsilon;x)$ then $\epsilon < \bar{\epsilon}$. Moreover, if there is H that checks $R_1(\epsilon;x)$ and $R_2(\epsilon;x)$ then $\epsilon \geq \bar{\epsilon}$.

For $\epsilon = 0.14$ and $H(8.35;0)$, we have : $R_1(0.14;8.35) \approx 0,012$ and $R_2(0.14;8.35) \approx 0,015$, H does not check any of $R_1(\epsilon;x)$ and $R_2(\epsilon;x)$, we can conclude that $0.14 < \bar{\epsilon}$.

For $\epsilon = 0.15$, we have: $R_1(0.15;8.35) \approx -0,007 \leq 0$ and $R_2(0.15;8.35) \approx -0.004 \leq 0$, H checks $R_1(\epsilon;x)$ and $R_2(\epsilon;x)$ we can conclude that $0.15 \geq \bar{\epsilon}$. Consider now the point $G(0;8.4)$, we have: $R_1(0.15;8.4) \approx 0.01 > 0$, it follows that $A \in [M,G]$. We have: $d(A,Q_3) \geq d(G,Q_3)$ and $0.15 \geq \bar{\epsilon}$.

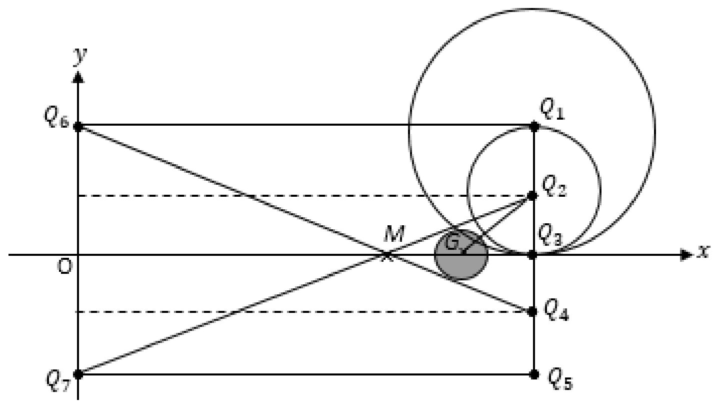

Figure 6

We want to show that it is not possible to move G in a ball of radius 0.15 to counter Q_3, see Figure 6. Consequently, it is not possible to move A in a ball of radius $\bar{\epsilon} \leq 0.15$ to counter Q_3 since A is further from Q_1, Q_2, Q_3, Q_4 and Q_5 to G. A simple verification shows that the voters 1, 2, 3, 4 and 5 prefer Q_3 to G. Indeed, $d(G,Q_1) = \sqrt{(9-8.4)^2+2^2} \approx 2.08 > 2 = d(Q_3,Q_1)$ and $d(G,Q_2) \approx 1.16 > 1 = d(Q_3,Q_2)$. This proves that voters 1 and 2 prefer Q_3 to G. By symmetry, we have the same conclusion for voters 4 and 5. The case of voter 3 is obvious.

Moreover, $d(G,Q_2) - \epsilon \approx 1.16 - 0.15 = 1.01 > 1 = d(Q_2,Q_3)$, this proves that when we move G directly from 0.15 to Q_2, we remain too far from Q_2 (voter 2 always prefers Q_3 whatever the position of G within the circle of center G and radius 0.15). It is also the case for voters 3 and 4. In other words, voters 1,2, 3, 4, and 5 prefer Q_3 to G and among them, the individuals 2, 3 and 4 ϵ-prefer Q_3 to G.

It follows that the only winning coalition that can counter Q_3 is $\{1,5,6,7\}$. A necessary condition is to find a point F belonging to the circle with center G and radius 0.15 such that $d(F,Q_1) \leq d(Q_1,Q_3) = 2$ and $d(F,Q_5) \leq d(Q_3,Q_5) = 2$. Under these conditions, the best position that is closer to $Q_1$and Q_5 at the same time is $F = (8.55,0)$. However, $d(Q_1,F) = d(Q_5,F) = \sqrt{(9-8.55)^2+2^2} = 2.05 > 2$. Thus, it is not possible to move G in a ball of radius 0.15 to counter

Q_3 and, consequently, it is not possible to move A in a ball of radius $\bar{\epsilon}$ to counter Q_3, A is not the finagle point, which proves the second step.

Third step: we want to show that $\bar{\epsilon} = \epsilon^*$ that is to say $C^{\epsilon^*}(V) = A$. We have just to verify that A belongs to ϵ-$Conv(S)$, for any minimal winning coalition S. By construction, this is true for S_1, S_2, S_3 and S_4. For $\epsilon = 0.14 < \bar{\epsilon}$, we have $d(Q_3, H) + d(Q_6, H) - d(Q_3, Q_6) - 2\epsilon = (9 - 8.35) + \sqrt{8.35^2 + 2^2} - \sqrt{9^2 + 2^2} - 0.28 \approx -0,41 \leq 0$. According to Lemma 1, we can conclude that $H \in \epsilon$-$Conv(\{3, 6\})$. We know that $Q_3 \in \epsilon$-$Conv(\{3, 6\})$ and ϵ-$Conv(\{3, 6\})$ is convex, it follows that $[H, Q_3] \subseteq \epsilon$-$Conv(\{3, 6\}) \subseteq \epsilon$-$Conv(\{1, 2, 3, 6\})$. According to step 2, $A \in [H, Q_3]$, this induces $A \in \epsilon$-$Conv(\{1, 2, 3, 6\})$. We can deduce by symmetry that $A \in \epsilon$-$Conv(\{3, 4, 5, 7\})$. Other cases naturally deduce. For example : $A \in \epsilon$-$Conv(\{1, 4\}) \subseteq \epsilon$-$Conv(S)$ for $S = \{1, 3, 4, 5\}, \{1, 2, 3, 5\}, \{1, 2, 3, 5\}$. Similarly, $A \in \epsilon$-$Conv(\{3, 6\}) \subseteq \epsilon$-$Conv(S)$ for $S = \{1, 2, 4, 6\}, \{1, 2, 5, 6\}$, etc. ■

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