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Dominance in Spatial Voting with Imprecise Ideals: A New Characterization of the Yolk.

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Abstract

We introduce a dominance relationship in spatial voting with Euclidean preferences, by treating voter ideal points as balls of radius δ. Values $\delta > 0$ model imprecision or ambiguity as to voter preferences, or caution on the part of a social planner. The winning coalitions may be any consistent monotonic collection of voter subsets. We characterize the minimum value of δ for which the δ-core, the set of undominated points, is nonempty. In the case of simple majority voting, the core is the yolk center and δ is the yolk radius. Thus the δ-core both generalizes and provides a new characterization of the yolk. We then study relationships between the δ-core and two other concepts: the ϵ-core and the finagle point. We prove that every finagle point must be within 2.32472 yolk radii of every yolk center, in all dimensions $m \geq 2$.

Key words : Spatial voting, dominance, core, yolk, finagle.

JEL Classification : *C62, C70, D71, D72.*

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1 Introduction

A classical problem in social choice theory is the existence of a core in voting games. The core is the set of undominated alternatives, those for which there is no winning coalition, that is to say a group of individuals which can enforce a decision, that wants to replace them by other alternatives. A non-empty core assures the stability and predictability of the collective choice. In this paper, we consider spatial voting, where the alternatives are the points in real m-dimensional space, and each voter has an ideal point in that space. In the standard spatial model individual preferences are Euclidean and winning coalitions are defined by majority rule. A voter with ideal point v has Euclidean preferences if s/he strictly prefers alternative x to alternative y when v is closer to x than to y. According to majority rule for n voters, the winning coalitions are all subsets of more than $n/2$ voters.

For all dimensions $m \geq 2$, the standard spatial model rarely admits of a nonempty core. For n odd, Plott [23] and Davis *et al.*[8] prove that in majority rule, the core is nonempty if and only if all the median hyperplanes have a common point of intersection. It follows that the set of configurations of ideal points for which the core is not empty has measure zero. For n even, results by Rubenstein [25], Banks [2], Banks *et al.*[3], and Saari [26] imply the same measure zero phenomenon for all dimensions $m \geq 3$. For n even and dimension $m = 2$, the set of configurations with nonempty core has strictly positive measure[1]. Nonetheless, the probability of a nonempty core is less than $\frac{2\sqrt{\pi n}}{e^{\frac{1}{2}(n-\frac{1}{3n})}}$ [34] which converges rapidly to zero as n increases.

There are many results about the existence or non-existence of the core of more general spatial voting games (for an interesting review of literature, see Miller [20]). One line of research considers preferences more general than Euclidean such as convex quadratic or smooth convex [2, 27, 11]. Another line of research considers more

[1] Place $n-1$ ideal points at the vertices of a regular $(n-1)$-polygon, and place the nth ideal point at the polygon center. Then for any small perturbation of the points, the nth point remains undominated.

general forms of the set of winning coalitions. Supermajority voting is a generalization of majority voting, in which an incumbent alternative can only be defeated by a vote of at least a fraction $\alpha \geq \frac{1}{2}$ of the voters. Greenberg [12] (among many others) analyzes the more general quota voting, where each voter v has positive weight w_v and a set S of voters is winning iff the sum of their weights $\sum_{v \in S} w_v$ is at least the quota requirement q. Saari [26] proves that in quota voting the core is the intersection of convex hulls of the ideal points of every minimal winning coalition. Even more generally, for arbitrary monotonic sets of winning coalitions, Nakamura [21] shows that the core of a voting game is non-empty if and only if the size of the space m is less or equal to the Nakamura number, which is the minimum number of winning coalitions with an empty intersection.

Beginning with Plott's classic paper [23], various elegant conditions necessary or sufficient for a nonempty core have been found. However, in high dimensions, all of the *necessary-and-sufficient* conditions for a nonempty core that have been obtained require a vast enumeration. This is because, as Bartholdi *et al.* [4] have shown, it is co(NP)-complete to show that a point is in the core, even in the standard Euclidean spatial model. That is, one can quickly demonstrate that a point is not in the core, by exhibiting a median hyperplane that does not pass through it. But unless $NP = co(NP)$, which is unlikely, it is not possible to quickly demonstrate that a point is undominated. This complexity property justifies the computational difficulty (in high dimension) of checking the conditions of Saari's and Nakamura's described above.

Many solution concepts for the spatial model in $m \geq 2$ dimensions have been proposed in order to provide predictive and/or prescriptive guidance when the core is empty. We have the minmax or Simpson-Kramer point, the point against which the maximum possible coalition is minimal [15]; the strong point or Copeland winner, the point dominated by the least m-dimensional volume of points [22], the uncovered set, those points not defeated in either one or two steps by another point [19]; the yolk, a smallest ball that intersects all median hyperplanes [17, 10]; the finagle point, a point from which the least distance must be traveled to dominate any other point [35]. Other solution concepts include the epsilon-core [30, 9, 33, 31, 6], the heart [28], and the soul [1].

All of these solution concepts coincide with the core when the core is nonempty. As a rule, solution concepts are clearly motivated by a plausible rationale. The minmax and strong points are those points that are most like a core point, in terms of voter opposition and dominating alternatives, respectively. The finagle point and epsilon core are those most like a core point, in terms of ambiguity of the alternative's location, and voter resistance to change, respectively. The uncovered set is the set of possible outcomes of one-step lookahead strategic sequential majority voting. The yolk is an interesting exception to this rule. McKelvey invented the yolk to get a handle on the uncovered set, and proved that the former contains the latter, if the radius is inflated by a factor of four [17].

In this paper we propose a new solution concept for spatial voting with Euclidean preferences and arbitrary monotonic winning coalitions, motivated by imprecision or ambiguity of voter ideal point locations. We treat ideal points as balls of radius $\delta \geq 0$ and derive a δ-dominance relationship from the viewpoint of a prudent mechanism designer or social planner. We characterize the minimal value of δ for which there is an undominated point, the δ-core, using a mathematical approach like in [26] and [16].

We then focus on the basic case of majority rule. We show that in this case the δ-core is the yolk center and the minimal δ is the yolk radius. Consequently, this paper provides a clear rationale for the yolk center as a solution concept: it is the point most like a core point, in terms of ambiguity of the voter ideal point locations. This reads much like the rationale for the finagle point. The rationales differ only in that the ideal point locations rather than the alternative location are imprecise. To conclude, we prove that, despite their conceptual difference, the distance between the finagle point and the yolk center (i.e., δ-core) is always less than 2.325 yolk radii, in any dimension $m \geq 2$. This improves the best claimed bound in the literature of 2.5.

The paper is organized as follows. Section 2 formally defines the spatial voting game and associated basic terminology. Section 3 motivates and defines the δ-dominance relationship and δ-core. It also compares them with the ϵ-core and its implicit dominance relationship. Section 4 characterizes the threshold value of δ for non-emptiness of the δ-core, and shows its correspondence to the yolk in the case of majority rule. This is the cornerstone of the paper. Section 5 relates the yolk to the finagle point and concludes. All proofs are deferred to the Appendix.

2 Definitions and Notation

A spatial voting model V in m dimensions may be written $V = (N, W, \{q^i\}_{i \in N})$ where $N = \{1, ..., n\}$ is a finite set of individuals (voters), $W \subset 2^N$ is the set of winning coalitions (that is, the set of groups of individuals that can enforce a decision), and $q^i \in \mathbb{R}^m$ is the ideal point of individual i. The set of winning coalitions must satisfy these conditions:

1) $\phi \notin W; N \in W$ (nonvacuous and nonempty)
2) $S \in W \Rightarrow W \setminus S \notin W$ (consistent)
3) $\forall S, T \in 2^N : S \subset T, S \in W \Rightarrow T \in W$ (monotonic).

As stated previously, V employs *quota voting* if there exist positive weights w_i : $i \in N$ and a quota q such that $S \in W \Leftrightarrow \sum_{i \in S} w_i \geq q$. Supermajority voting with $\frac{1}{2} < \alpha \leq 1$ is the special case $w_i = 1\ \forall i$, $q = \lceil \alpha n \rceil$. Majority rule is the special case $w_i = 1\ \forall i$, $q = \lceil \frac{n+1}{2} \rceil$.

The set of alternatives is $\mathbb{R}^m$. Denote by $d(x, y)$ the Euclidean distance between points x and y in $\mathbb{R}^m$. For any set of points $\mathcal{F} \subset \mathbb{R}^m$, let $Conv(\mathcal{F})$ denote the convex hull of $\mathcal{F}$. To simplify notation, for a set of voters $S \subset N$, we also let $Conv(S)$ denote the convex hull of their ideal points, which otherwise would entail the cumbersome notation $Conv(\cup_{i\in S}q^i)$ because ideal points need not be unique.

According to Euclidean preferences, voter $i \in N$ is indifferent between alternatives x and y when $d(q^i, x) = d(q^i, y)$, and strictly prefers x to y when $d(q^i, x) < d(q^i, y)$. Alternative y is *dominated by alternative x via coalition S*, denoted $y \prec_S x$, if S is a winning coalition and all individuals in S strictly prefer x to y:

1) $S \in W$
2) $\forall i \in S, d(q^i, x) < d(q^i, y)$

The point $y \in \mathbb{R}^m$ is *dominated by x*, denoted $y \prec x$, if $y \prec_S x$ for some (winning) coalition S. The point y is *undominated* if it is not dominated by any x. The core of a spatial voting game, denoted $C(V)$, is the set of alternatives that are undominated.

For $x \in \mathbb{R}^m$ and $\tau \geq 0$ let $B(x, \tau) = \{z : d(x, z) \leq \tau\}$ be the (closed) ball around x of radius τ. The *finagle radius* $f(x)$ of alternative x is the minimum value such that for each $y \in \mathbb{R}^m$, there exists $z \in B(x, f(x))$ that is not dominated by y. This means that whatever alternative y is proposed to defeat x, x can avoid defeat by "finagling" to a nearby point z that y does not dominate. The rationale is that an incumbent x takes advantage of ambiguity in its location to avoid defeat by a challenger y. A finagle point is a point x with minimal finagle radius $f(x)$. Thus, a finagle point is one that requires the least ambiguity to be undominated.

A hyperplane $H = \{x \in \mathbb{R}^m : c{\cdot}x = c_0\}$ is *median* if both of the closed halfspaces it defines contain the ideal points of at least $\frac{n}{2}$ voters. That is, H is a median hyperplane if $|\{i \in N : c \cdot q^i \leq c_0\}| \geq \frac{n}{2}$ and $|\{i \in N : c \cdot q^i \geq c_0\}| \geq \frac{n}{2}$. The *yolk* is a smallest ball that intersects all median hyperplanes. Equivalently, following [18], define the *yolk radius* $r(x)$ of alternative x as the minimum value such that $B(x, r(x)) \cap H \neq \phi$ for all median hyperplanes H. Then the yolk is a ball with center $c = \arg\min r(x)$ and radius $r(c)$.

To define the ϵ-core, one posits a change in the voters' behavior: the Euclidean preferences are replaced by a new preference relation which we call the ϵ-preference.

Formally, voter i strictly ϵ-prefers an alternative x to an alternative y if $d(x,q^i) < d(y,q^i) - \epsilon$ and i is ϵ-indifferent between the alternatives if $|d(x,q^i) - d(y,q^i)| \leq \epsilon$. Alternative x is ϵ-dominated if there are an alternative y and a winning coalition S such that every member of S, ϵ-prefers y to x. The ϵ-core of a spatial voting game is the set of alternatives which are not ϵ-dominated.

3 The δ-core as a new core concept

The idea of our proposed δ-domination solution concept is to treat ideal points as balls of radius δ. One motivation for doing so comes from the perspective of a social planner. The ideal point is an intrinsic characteristic of the individual; it completely defines an individual's preferences. Thus, it is probably impossible for an external operator, as a social planner, to have enough information to determine perfectly the location of this ideal point. Instead, we suppose that from number of indicators, the social planner can locate each ideal point with a margin of error that is a positive real number δ. This assumption induces an imperfect perception of the ideal point of the voter which is seen as a ball of radius δ. We call $B(q^i,\delta)$ the *imprecise ideals area* of voter i.

Another motivation for treating ideal points in this way is to model the ambiguity or imprecision of preferences as reported by individuals. For example, political pollsters routinely have to handle inconsistent responses from an individual within a single survey. Thus, even if the ideal points are chosen by the individuals themselves, as Serra [29] argues, they are often uncertain about their own placement in the space.

For any spatial voting game $V = (N, W, \{q^i\}_{i\in N})$ and any real positive number δ, the pair (V,δ) is the game associated with V, in which the voters have imperfect ideal points: the ideal point of the voter i can be any point in $B(q^i,\delta)$. The convex hull of the imprecise ideals areas of the voters belonging to a coalition S is denoted $Conv(S,\delta)$. We can introduce a new domination relation called the δ-domination, which is a direct generalization of the classical one.

We say that $y \in \mathbb{R}^m$ is δ-dominated by $x \in \mathbb{R}^m$ via a coalition S, denoted $y \prec_{S,\delta} x$ if:

1) $S \in W$
2) $\forall i \in S$, $\forall z \in B(q^i, \delta)$, $d(x, z) < d(y, z)$

This definition has a natural interpretation. A prudent social planner, before replacing an alternative x by y, wants to be sure that whatever the location of the voters in their imprecise ideals areas, x is dominated by y in the usual sense of the term. In other words, the social planner makes changes to the status quo cautiously. Likewise, a prudent social choice mechanism designer, knowing that voters flip-flop on some issues, allows policy change from x to y only if the voters in the winning coalition would consistently prefer y to x. We say that $y \in \mathbb{R}^m$ is $\delta-$dominated by $x \in \mathbb{R}^m$, denoted $y \prec_\delta x$, if there exists $S \in W$ such that $y \prec_{S,\delta} x$. The δ-core of a spatial voting game, denoted $C(V, \delta)$, is the subset of $\mathbb{R}^m$ of δ-undominated alternatives.

3.1 Individual preferences

Figure 1-(a) illustrates the usual case for $m = 2$ dimensions in which the individual preferences of a voter i are entirely determined by his/her single ideal point q^i. Voter i is indifferent between a given alternative x and another alternative y such that $d(y, q^i) = d(x, q^i)$. These points y form the circle of center q^i and radius $d(x, q^i)$. Any alternative l such that $d(l, q^i) < d(x, q^i)$, is strictly preferred to x, while x is strictly preferred to any alternative z such that $d(z, q^i) > d(x, q^i)$.

Figure 1-(b) shows the case that leads to the ϵ-core. The individual is indifferent between two alternatives x and y belonging to the dark area since $|d(y, q^i) - d(x, q^i)| \leq \epsilon$. S/he strictly prefers l to x since $d(x, q^i) - d(l, q^i) > \epsilon$. Finally, s/he prefers x to any alternative z outside the dark area since $\epsilon < d(z, q^i) - d(x, q^i)$.

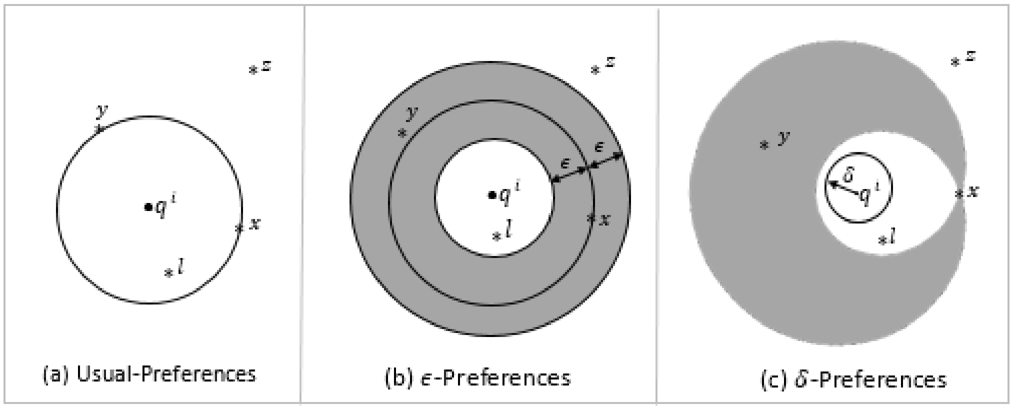

Figure 1

Figure 1-(c) corresponds to our approach. The hypothetical social planner, unable to choosing the ideal points perfectly, is faced with another version of the individual preferences that we will illustrate. In Figure 2(a), the disc with center q^i and radius δ represents the imprecise ideals area of the voter i, the set of the possible ideal points of voter i. If z^1 is the ideal point of voter i then s/he prefers any point l in the open disc of center z^1 and radius $d(x, z^1)$ to x. In the same way, if z^2 (Figure 2(b)) is the ideal point of the voter i, s/he prefers any point l in the open disc (with dashed border) of center z^2 and radius $d(x, z^2)$ to x. Now, if we know that the ideal point of voter i is an element of the set $\{z^1, z^2\}$ then, with respect to our approach, any point l belonging to the intersection of these two discs is better than x for individual i. On the dark area of Figure 2(b), voter i is indifferent between x and y since for the point z^1, we have $d(x, z^1) < d(y, z^1)$ but for the point z^2 we have $d(y, z^2) < d(x, z^2)$.

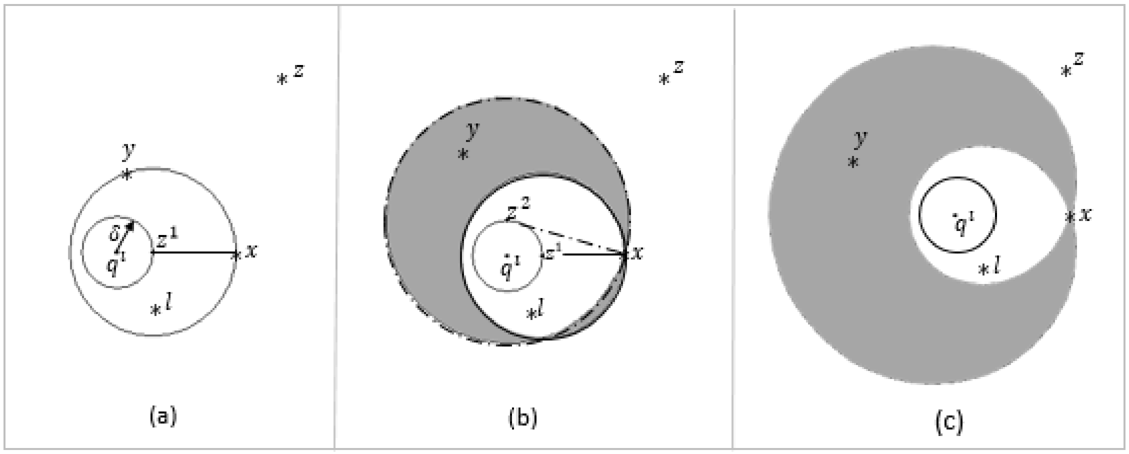

Figure 2

Extending this reasoning to all the points of the imprecise ideals area, we obtain Figure 2(c), which corresponds to Figure 1(c). Individual i prefers x to any alternative z located on the outer white area and prefers any alternative l located on the inner white area to x. Individual i is indifferent between the point x and any point located on the dark area.

The dark area represents all the points for which i is indifferent compared to x.

3.2 Collective preferences

Given a coalition $S \subseteq N$, we want to characterize the set of alternatives that are dominated via S and those that are not dominated via S. To do this, we will give a result that works with the δ-preferences. Clearly, $\delta = 0$ corresponds to the usual case, that is why this result remains valid for classical preferences.

Proposition 1 *Let (V, δ) be a spatial voting game with δ the radius of the imprecise ideals area. An alternative $z \in \mathbb{R}^m$ is not δ-dominated via $S \subseteq N$ if and only if $z \in Conv(S, \delta)$.*

Proof. All the proofs are in appendix. ■

Note that, the set of δ-Pareto-optimal alternatives for a coalition S refers to alternatives that are not δ-dominated via S. For $\delta = 0$, we have $Conv(S, \delta) = Conv(S)$ and this proposition means that the set of Pareto-optimal alternatives for the members of a coalition S is the convex hull of the ideal points of the voters belonging to S. Figure 3(a), illustrates the situation with two ideal points. The convex hull corresponds to the straight line segment defined by these two points. From a collective point of view, the two voters are indifferent between two alternatives x and y belonging to the segment whereas any alternative z outside the segment is dominated.

When $\delta > 0$, the set of δ-Pareto-optimal alternatives for a coalition S is the convex hull of the imprecise ideals areas on S. $Conv(S, \delta)$ is geometrically deduced from $Conv\,(S)$ by the enlargement of the boundaries over a distance δ. It corresponds to Figure 3(b).

Figure 3(c) corresponds to the ϵ-preference. The construction is less obvious and comes from Figure 4.

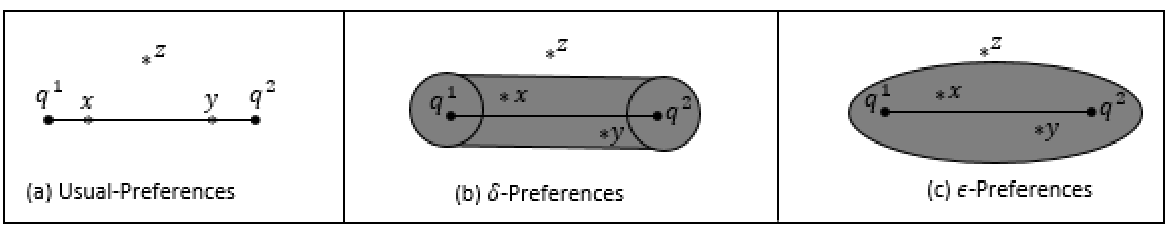

Figure 3

Consider Figure 4(a) with two ideal points: in the dark area (excluded borders), any point x is such that $d(x, q^i) < d(y, q^i) - \epsilon$, $i = 1, 2$ and then y is dominated by x. In Figure 4(b), we have $d(x, q^i) = d(y, q^i) - \epsilon$, $i = 1, 2$ and y is undominated. We can conclude that the set of undominated alternatives is bounded by the set of points such that $d(x, q^i) = d(y, q^i) - \epsilon$, $i = 1, 2$. Since x belongs to the segment $[q^1, q^2]$, we have $d(x, q^1) + d(x, q^2) = d(q^1, q^2)$, which is a constant, say D. Thus $d(y, q^1) - \epsilon + d(y, q^2) - \epsilon = D$ and $d(y, q^1) + d(y, q^2) = D + 2\epsilon$ which means that the undominated area is an ellipse with foci q^1 and q^2. We obtain Figure 4(c) where any x in the grey area is undominated via the coalition $\{1, 2\}$.

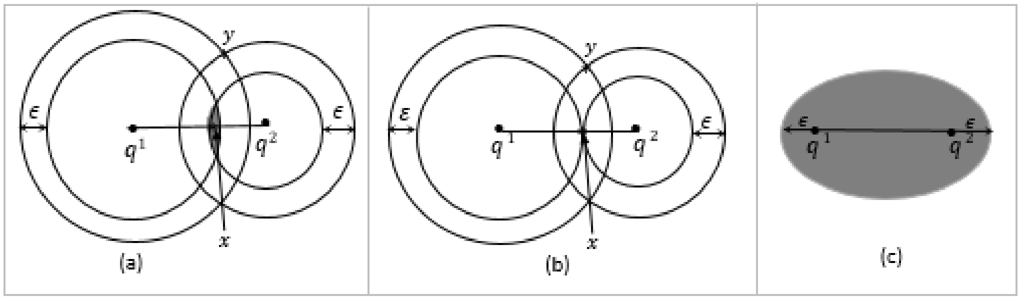

Figure 4

Given a coalition S, it is not easy to characterize the set of alternatives that are not ϵ-dominated via S. For instance, with five voters in a coalition, we obtain the following graph (by computer programming).

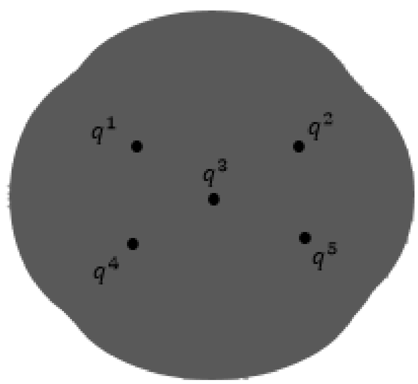

Figure 5

This figure is the intersection of several ellipses, which explains why the boundaries are not perfectly circular. We know that the classical case corresponds to the convex hull of the points, it is clear that there is no a simple transformation which connects the two sets.

4 Properties of the δ-core

In this section we derive a characterization of the δ-core and use it to specify the threshold value for which the δ-core is non-empty.

The next result is a direct consequence of Proposition 1

Corollary 1 *For every spatial voting game $V \equiv (N, W, \{q^i\}_{i \in N}, \delta)$, we have:*
$C(V, \delta) = \underset{S \in W}{\cap} Conv(S, \delta)$.

Corollary 1 can be seen as a generalization of Proposition 3 in Saari [26] for the particular case of quota games in the classical model where $\delta = 0$. Simple examples are presented in the graphs below: $N = \{1, 2, 3, 4, 5\}$, $W = \{S \subseteq N : |S| \geq 3\}$ and $(q^i)_i$ are represented on the graph.

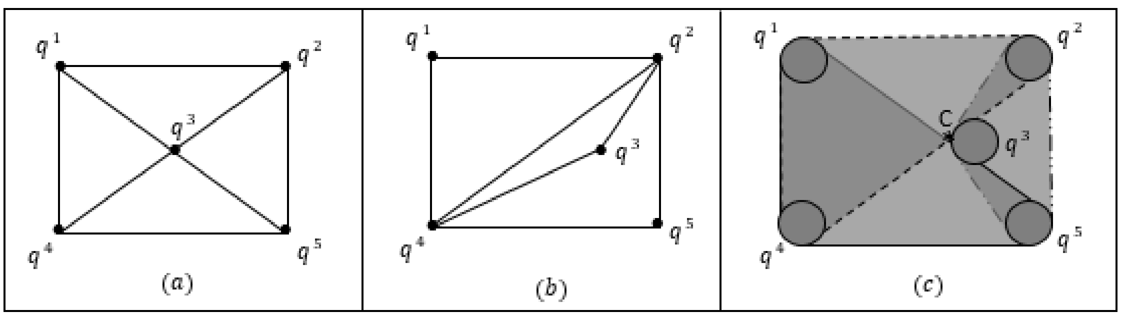

Figure 6

In Figure 6(a), $C(V) = \{q^3\}$ since every convex hull of a winning coalition (triangle formed by three ideal points) contains q^3 which is the only point belonging to the two triangles $q^1q^2q^3$ and $q^3q^4q^5$. In Figure 6(b), the core is empty since the three triangles $q^1q^2q^4$, $q^2q^3q^5$ and $q^3q^4q^5$ have an empty intersection. Visually, one can see what happens at the minimum value of δ such that the δ-core is non-empty. As δ increases, we obtain the Figure 6(c), for which the δ-core becomes the singleton $\{C\}$.

4.1 Non-emptiness of the δ-core

In this section we are looking for the minimum value of δ that ensures a non-empty δ-core. The essential idea turns out to be one employed previously by the authors [16] to demonstrate the non-uniqueness of the yolk in $m \geq 3$ dimensions. The yolk may be characterized as a smallest ball that intersects the convex hull of every subset of at least half of the ideal points. Obviously the non-minimal subsets are irrelevant. The relevant subsets are precisely the minimal winning coalitions under majority rule. Motivated by this characterization, we define the generalized yolk (g-yolk) as follows.

Definition 1 *A generalized yolk, or g-yolk, is a smallest ball that meets the convex hull of every minimal winning coalition.*

The non-emptiness of the δ-core is then characterized as follows:

Theorem 1 *Let (V, δ) be a spatial voting game with imprecise ideals areas of radius δ.*

Then the δ-core is nonempty, $C(V, \delta) \neq \phi$, if and only if $\delta \geq r$, where r is the radius of a g-yolk.

Note that the core is non-empty thanks to the degree of imprecision in the choice of the ideal points. In other words, the lack of information implies more stability in the collective choice. From a geometrical point of view, it is intuitive: without information on the preferences, the radius of the ideal points tends to infinity and then the intersection of the convex hulls cited before is of course non-empty. The particularity of our result is to propose an optimal degree of imprecision which guarantees the smallest size of the non-empty core.

Corollary 2 *Let (V, δ) be a spatial voting game for which the g-yolk is unique and has radius r. Then for $\delta = r$ the δ-core is the singleton point $\{C\}$, the center of the g-yolk (see Figure 6-(c)). If there are multiple g-yolks (with common radius r) then for $\delta = r$ the δ-core is the set of centers of the g-yolks.*

Focusing now on majority rule, we restate Corollary 2 as our promised rationale for the yolk.

Corollary 3 *In the standard spatial model with Euclidean preferences and majority rule, the set of yolk centers is the minimal nonempty δ-core and the yolk radius equals the threshold value of δ.*

5 Link with the Finagle Point

Given the similarity between the definitions of the Finagle point and the δ-core, and the equivalence between the δ-core and the yolk in the standard spatial model, we probe for relationships between the finagle point and the yolk center. Wuffle *et al.* [35, Lemma 1] prove that for all x, $f(x) \leq r(x)$. That is, the finagle radius at x can not exceed the x-yolk radius. They also claim to be able to show that the finagle point is always within 2.5 yolk radii of the center of the yolk. We have not found a formal proof of this claim, nor have we ascertained whether this claim is with regard to two dimensions or arbitrary dimension. Regardless, we obtain a tighter bound for any dimension. Our bound depends on the following theorem, which gives a lower bound of the ratio of the finagle radius to that of the yolk.

Theorem 2 *Let F be a finagle point with finagle radius $t > 0$ and let $C \neq F$ be a yolk center with yolk radius $r > 0$. Let $\tau = t/r$ (τ is known to be ≤ 1). Then either*

$$||F - C|| \leq t + r$$

or

$$\tau^2 \geq \frac{(||F - C|| - r)^3}{||F - C||}.$$

The result holds in all dimensions ≥ 2.

Corollary 4 *Under the conditions of the theorem, $||F - C|| < 2.32471796 \times r$, that is, the finagle point must be closer than $2.32471796 \times r$ to the yolk center.*

It is known that (Grofman et al. [13], Koehler [14], Tovey [32]) under certain conditions the radius of the yolk is small. In this case, Corollary 2 implies that the center of the yolk and the finagle point are very close. This is also what Bräuninger [6] points out.

6 Conclusion

Several final observations can be made. Firstly, we remark the paradoxical situation: the uncertainty on the choice of the ideal points permits a collective choice. In other words, too much information can be counterproductive in a social choice point of view. Secondly, our δ-dominance relation elevates the yolk center as a meaningful solution concept in its own right, rather than as being a way to bound the uncovered set. It also generalizes the yolk to Euclidean preferences and arbitrary monotonic winning coalitions, as has already been done by Saari [26] for supermajority and quota voting. Thirdly, to further enhance the theoretical properties of the yolk, it would be nice to have a dynamical justification of the yolk, a proof that the natural forces of majority voting tend to drive the group decision toward it. A dynamic justification would complement the essentially normative nature of our δ-dominance justification. Many other well-known solution concepts possess a dynamical justification. For example, Kramer [15] showed that repeated proposals by competing vote-maximizing parties will produce sequences converging to the minmax (Simpson-Kramer) set. Miller [19] showed that for many agenda rules, the outcome of strategic voting will be in the uncovered set. Tovey [31] showed that if $\epsilon > 2r$ any sequence of ϵ-voting from arbitrary starting point x will reach the ϵ-core in at most $\frac{\lfloor \, ||x-c||-r \rfloor}{\epsilon-2r}$ votes. Ferejohn *et al.* [10] gave a partial result for the yolk: if proposals are made at random with majority voting, then the process never terminates, but the incumbent proposal will be in the yolk with frequency more than $\frac{1}{2}$ or even $\frac{2}{3}$ for some parameter values. It seems likely to us that the addition of some "stickiness" from $\delta > 0$, incorporated into Ferejohn *et al.*'s analysis, would yield an upper bound on the expected number of votes until reaching and remaining in the yolk.

Appendix : Proofs

Throughout this section, $(N, W, \{q^i\}_{i\in N}, \delta) \equiv (V, \delta)$ is a $m-$dimensional spatial voting game where $B(q^i, \delta)$ is the imprecise ideals area of the voter $i \in N$.

Proof of Proposition 1

Suppose y is δ-dominated by some point z with respect to a coalition $T \in W$. Let $\mathcal{H} = \{x : ||x - z|| = ||x - y||\}$ be the hyperplane orthogonal to and bisecting segment $[y, z]$, Figure 7-(a). Let $\mathcal{H}^z$ (respectively $\mathcal{H}^y$) be the open halfspace defined by $\mathcal{H}$ that contains z (respectively y). By definition of δ-dominance, $\forall i \in T$, $B(q^i, \delta) \subset \mathcal{H}^z$. Hence the set $\mathcal{S} \equiv \bigcup_{i\in T} B(q^i, \delta) \subset \mathcal{H}^z$. Since $\mathcal{H}^z$ is convex, the convex hull $Conv(T, \delta) \subset \mathcal{H}^z$ as well. Then $y \in \mathcal{H}^y \Rightarrow y \notin \mathcal{H}^z \Rightarrow y \notin Conv(T, \delta)$.

Conversely, suppose $y \notin Conv(T, \delta)$. The set $\mathcal{S}$ is a finite union of compact sets and therefore is compact. It is known from Caratheodory's Theorem (see e.g. Theorem 11.1.8.6 in Berger [5]) that in $\mathbb{R}^m$ the convex hull of any compact set is compact. Hence the convex hull $Conv(T, \delta)$ is compact. By Rockafellar [24, Corollary 11.4.2 p. 99], there exists a hyperplane $\Pi = \{x : \pi \cdot x = \pi_0\}$ strictly separating $\{y\}$ from $Conv(T, \delta)$, such that $\pi \cdot y < \pi_0$ and $\pi \cdot x > \pi_0$, $\forall x \in Conv(T, \delta)$, Figure 7-(b). Let p be the projection of y onto Π, so $\pi \cdot p = \pi_0$ and $p = y + \alpha\pi$ for some scalar $\alpha > 0$.

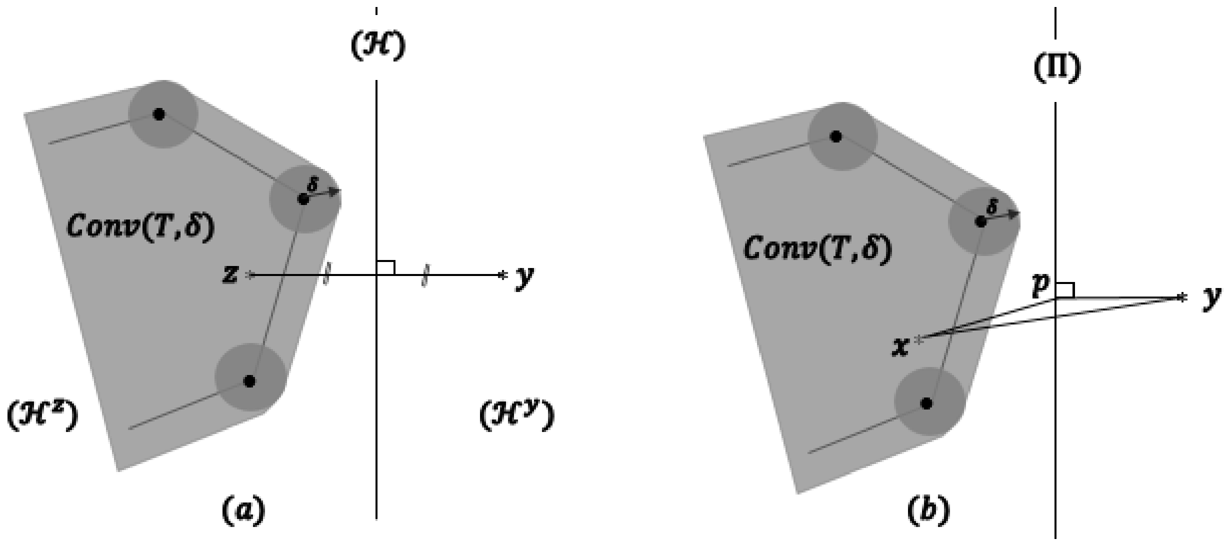

Figure 7

Geometrically one can see that p is closer than y to every point in $Conv(T, \delta)$ because $p \in \Pi$ and y is on the other side of Π from $Conv(T, \delta)$. Algebraically, for all $x \in Conv(T, \delta)$,

$$\begin{aligned}
||p - x||^2 &= (p - x) \cdot (y + \alpha\pi - x) \\
&= \alpha\pi \cdot p - \alpha\pi \cdot x + p \cdot y + ||x||^2 - x \cdot p - x \cdot y \\
&< \alpha\pi_0 - \alpha\pi_0 + p \cdot y + ||x||^2 - x \cdot p - x \cdot y \\
&< \alpha\pi \cdot x - \alpha\pi \cdot y + p \cdot y + ||x||^2 - x \cdot p - x \cdot y \\
&= (x - y) \cdot (x + \alpha\pi - p) = ||y - x||^2.
\end{aligned}$$

Therefore, p δ-dominates y with respect to coalition T, which completes the proof.■

Proof of Corollary 1

The proof of the Corollary follows immediately from Proposition 1 and the definition of δ-core. ■

Proof of Corollary 2

Consider the function ρ defined from $\mathbb{R}^m$ to $\mathbb{R}$ by: $\forall z \in \mathbb{R}^m$, $\rho(z) = \max\limits_{S \in W} d\left(z, Conv(S)\right)$. Finding a minimum for the function ρ may be restricted on the convex hull of $\{q^i\}_{i \in N}$ that is $Conv\left(N\right)$. Indeed, if $z \notin Conv\left(N\right)$, then there is a point l such that for all $S \in W$, $d\left(l, Conv(S)\right) < d\left(z, Conv(S)\right)$. However for all $S \in W$, $d\left(l, Conv(S)\right) < d\left(z, Conv(S)\right)$ implies that $\max\limits_{S \in W} d\left(l, Conv(S)\right) < \max\limits_{S \in W} d\left(z, Conv(S)\right)$, i.e. $\rho(l) < \rho(z)$.

Considering that ρ is continuous (as maximum of continuous functions) on the convex $Conv\left(N\right)$, it follows that ρ reaches its minimum at a point z^*, we denote $\delta^* = \rho(z^*) = \min\limits_{Z}\left(\max\limits_{S} d\left(z, Conv(S)\right)\right)$, it is clear that δ^* coincides with r the radius of a g-yolk, by definition.

$$\begin{aligned} \text{Otherwise, } \delta^* = \rho(z^*) = \max_{S \in W} d\left(z^*, Conv(S)\right) \quad &\Longleftrightarrow \quad \forall S \in W,\ d\left(z^*, Conv(S)\right) \leq \delta^* \\ &\Longleftrightarrow \quad \forall S \in W,\ z^* \in Conv(S, \delta^*) \\ &\Longleftrightarrow \quad z^* \in \bigcap_{S \in W} Conv(S, \delta^*) = C(V, \delta^*) \end{aligned}$$

We get the proof that $C(V, \delta^*) \neq \varnothing$, furthermore, if $\delta \geq \delta^*$ then $C(V, \delta^*) \subseteq C\left(V, \delta\right)$ i.e. $C\left(V, \delta\right) \neq \varnothing$.

Now, for a $\delta \geq 0$, assume that $C\left(V, \delta\right) \neq \varnothing$, then there exists $x \in C\left(V, \delta\right) = \bigcap\limits_{S \in W} Conv(S, \delta)$, for all $S \in W$, $d\left(x, Conv(S)\right) \leq \delta$. It follows that $\rho\left(x\right) = \max\limits_{S \in W} d\left(x, Conv(S)\right) \leq \delta$. We know that $\delta^* \leq \rho\left(x\right)$, then $\delta^* \leq \delta$ i.e. $C\left(V, \delta\right) \neq \varnothing \Rightarrow \delta \geq \delta^*$. ■

Proof of Corollary 3

According to the proof of Corollary 2, $z \in C\left(V, \delta^*\right) = C\left(V, r\right)$, where r is the radius of a g-yolk if and only if for all $S \in W$, $B\left(z, r\right) \cap Conv(S) \neq \varnothing$. Taking into account that r is minimal, it is clear that if the g-yolk of center c and radius r is unique, then the $\delta - core$ is the singleton c. ■

Proof of Theorem 2

Let $f = ||F - C||$. If $f \leq r + t$ the first alternative of the Theorem holds and nothing needs to be proved. Henceforth $f > r + t$ and the yolk and finagle ball are disjoint.

We implicitly use the following reasoning. If A is in the yolk, P is not in the yolk, and $P \prec A$, then the open halfspace containing P defined by the hyperplane H that orthogonally bisects segment $[P, A]$ contains a strict majority of the ideal points. Then the unique median hyperplane H' parallel to H is farther from A than H is. By definition of the yolk, H' intersects the yolk. By convexity of the yolk and since A is in the yolk, H intersects the yolk. See Figure 8(a).

Without loss of generality translate, scale, and rotate so that $C = 0 \in \mathbb{R}^m$, $r = 1$, and $F = (0, f, 0, \ldots, 0) \in \mathbb{R}^m$,, as depicted in Figure 8-(b). Thus $\tau = t$. Let $A = (0, r, 0, \ldots, 0)$. For F to finagle A there must be a point P on the ball $B(F, t)$ that defeats A. According to the Figure 8-(a), the extreme case is obtained when P belongs to the border of the finagle ball and L is tangent to the yolk. Without loss of generality rotate so that $P = (p_1, p_2, 0, \ldots, 0)$ and $p_1 \leq 0$. We have assumed $f > r+t$ so the yolk $B(0, r)$ and the finagle ball $B(F, t)$ are disjoint. The point P cannot be $(0, f - t, 0, \ldots, 0)$ because that would require the existence of a median hyperplane orthogonal to the line $(0, \theta, 0, \ldots, 0)$ and intersecting the line at a point where $\theta > r$, which would not intersect the yolk, a contradiction. Therefore $p_1 < 0$. From here on everything takes place in the two dimensional plane defined by the three non-collinear points A, F, P. We will suppress all the zeroes for dimensions higher than two.

Let B be the point where the median line L is tangent to the yolk. $[B, C]$ segment is perpendicular to L. L is perpendicular to $[P, A]$. Hence $[B, C]$ is parallel to $[P, A]$. Hence $\theta \equiv \angle BCA = \angle PAF$. Let $2x = ||P - A||$. So x is the distance from A to the intersection of L and segment $[P, A]$. Let D be the point on $[B, C]$ intersected by the line through A that is parallel to L. Then A, B, D are three corners of a rectangle whose side length between D and B equals x. Considering triangle ACD,

$$1 = x + \cos\theta. \tag{1}$$

(Remember that segments $[A, C]$ and $[B, C]$ both have length $r = 1$.)

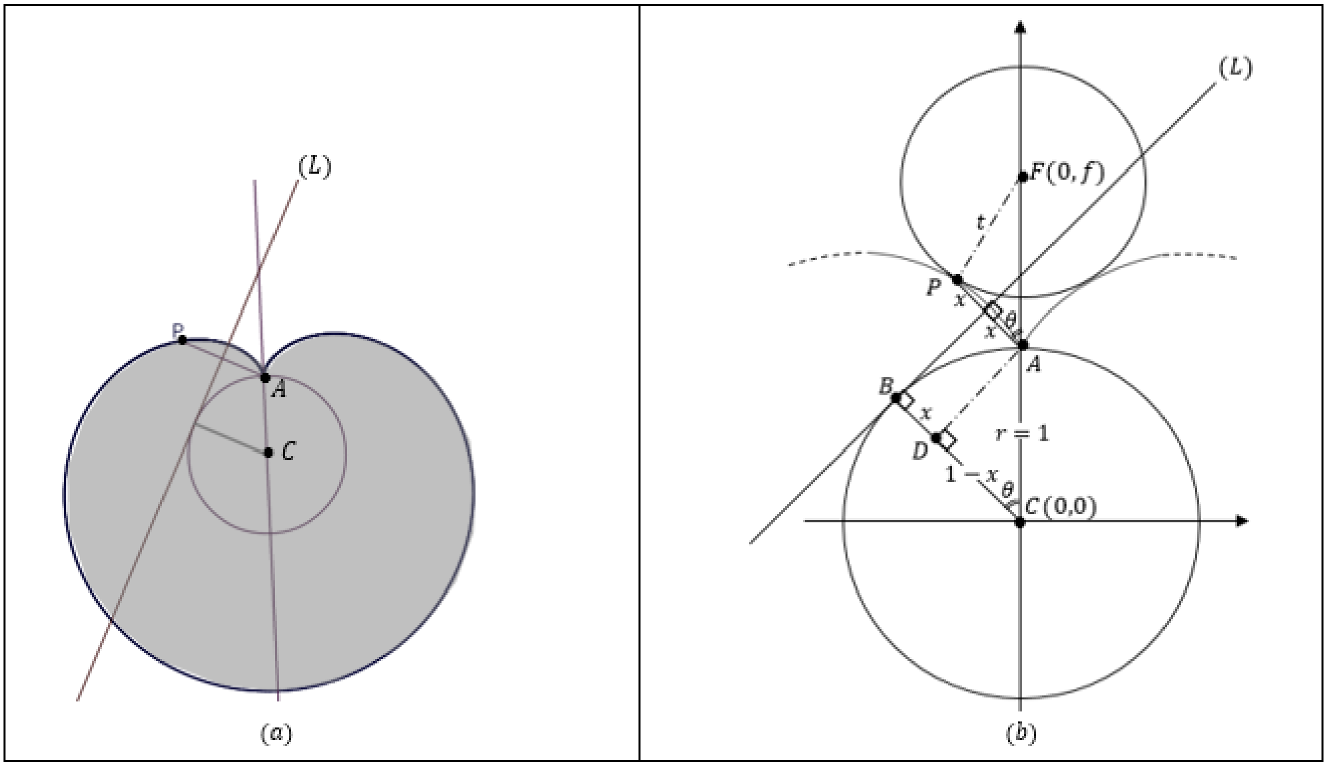

Figure 8

Apply the law of cosines to triangle FAP. $||F-P||^2 = t^2 = ||F-A||^2 + (2x)^2 - 2||F-A||(2x)\cos\theta = (f-1)^2 + 4x^2 - 4(f-1)x\cos\theta$. Applying equation (1) gives our key equation:

$$t^2 = (f-1)^2 + 4x^2 + 4(f-1)x^2 - 4(f-1)x = 4fx^2 - 4(f-1)x + (f-1)^2. \quad (2)$$

For this to be physically possible x must have a real value. The discriminant of the quadratic equation (2) in x must be ≥ 0. So $16(f-1)^2 \geq 16f((f-1)^2 - t^2)$. This simplifies to $t^2 \geq (f-1)^3/f$ to conclude the proof.■

Proof of Corollary 4

In the first case of Theorem 2, $||F-C|| \leq r + t \leq 2r$ and the Corollary holds. Otherwise, since $t \leq 1$ it must always be true that $(f-1)^3 \leq f$. The upper bound is such that $(f-1)^3 = f$, i.e., $(f-1)^3 - f = 0$. If we set $e = f - 1$, then $f = e + 1$ and $(f-1)^3 - f = 0$ can be written as $e^3 - e - 1 = 0$.

We will find the exact solution to $e^3 - e - 1 = 0$. This is a so-called *reduced cubic* because it has no quadratic term. We solve it with Scipione del Ferro's method, published by Cardano [7]. Let $e = y - z$ and constrain y, z to also satisfy $zy = -\frac{1}{3}$, where the numerator of the right hand side derives from the coefficient on the linear term. Then $0 = e^3 - e - 1 = y^3 - z^3 + (y - z) - e - 1 = y^3 - z^3 - 1 = y^3 + \frac{1}{27y^3} - 1$. Let $w = y^3$. Then $0 = w^2 - w + \frac{1}{27}$.

From this quadratic we take the pertinent root $w = \frac{1+\sqrt{23/27}}{2}$, whence

$$y = \sqrt[3]{\frac{1 + \sqrt{23/27}}{2}}$$

and then

$$e = y - z = y + \frac{1}{3y} = \sqrt[3]{\frac{1 + \sqrt{23/27}}{2}} + \frac{1}{3\sqrt[3]{\frac{1+\sqrt{23/27}}{2}}} < 1.32471796.$$

It follows that $f = e + 1 < 2.32471796$. Hence 2.32471796 is inconsistent with x having a real value. Hence f must be strictly less. Undoing the scaling by r completes the proof.■

Note that in cases where the finagle radius t is strictly less than the yolk radius r, this result forces the finagle point to be even closer to the yolk.

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