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Holding Period under Risk Aversion

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Abstract

This paper deals with real estate portfolio optimization when investors are risk averse. In this framework, we examine an important decision making problem, namely the determination of the optimal time to sell a diversified real estate portfolio. The optimization problem corresponds to the maximization of a concave utility function defined on both the free cash flows and the terminal value of the portfolio. We determine several types of optimal times to sell and analyze their properties. We extend previous results, established for the quasi linear utility case, where investors are risk neutral. We consider four cases. In the first one, the investor knows the probability distribution of the real estate index. In the second one, the investor is perfectly informed about the real estate market dynamics. In the third case, the investor uses an intertemporal optimization approach which looks like an American option problem. Finally, the buy-and-hold strategy is considered. For these four cases, we analyze in particular how the solutions depend on the market volatility and we compare them with those of the quasi linear case. We show that the introduction of risk aversion allows to better account for the real estate market volatility. We also introduce the notion of compensating variation to better measure the impacts of both the risk aversion and the volatility.

Key Words Real estate portfolio, Optimal holding period, Risk aversion, Real estate market volatility.

JEL Classification C61, G11, R21

1. Introduction

Holding period is defined as the expected period of time during which an investment is or should be kept. The holding period is an important topic in finance and has been the subject of numerous theoretical and empirical research studies (see Atkins and Dyl, 1997; In, Kim and Gençayc, 2011 and Lim and Kim, 2011). The holding period in real estate portfolio management is a topic that has only recently drawn the attention of both investors and academics. Traditionally, real estate investment had been a rather passive process, with investors adopting a buy-and-hold strategy for real estate, an asset class capable of generating relatively stable recurring cash flow derived from rental agreements. The strategy was to hold real estate for many years, a valid strategy given the large transaction costs. However, the sophistication of the real estate industry and to some extent the general perception that real estate cycles tend to be shorter have led to more attention to the notion of holding period and especially of ex-ante holding period.[2]

Calculations of the optimal holding period are nearly always empirical and the holding period is assumed to depend upon many factors, including market conditions, regulation, transactions costs and tax, types of property, lease length, and investment style. Hendershott and Ling (1984), Gau and Wang (1994) or Fisher and Young (2000) show that, for the US, the holding durations depend mainly on tax laws. Brown and Geurts (2005) show the average holding period is around 5 years through a sample of small residential investments over the period 1970-1990 in San Diego. They conclude that investors sell their assets earlier when values rise faster than rents. For the UK market, Rowley, Gibson and Ward (1996) prove the existence of ex ante expectations about holding periods, related to depreciation or obsolescence factors. Collett, Lizieri and Ward (2003) show ex post holding periods are higher than those usually claimed by investors using a commercial real estate database of properties in the UK. Their empirical analysis shows that the median holding period is about seven years. They also suggest a link between price volatility and holding period but they fail to highlight a proxy for measuring the relationship. For residential real estate, Cheng et al. (2010) demonstrate that higher illiquidity and transaction costs lead to longer holding periods, while higher return volatility implies shorter holding periods. These latter results are consistent with previous papers on financial assets. These kind of empirical studies does not allow concluding about the relation between real estate asset volatility and optimal holding period.[3]

Many attempts have sought to develop models to determine optimal holding period for real estate portfolio.[4] Baroni et al. (2007) determine the optimal holding period ex-ante (e.g. for closed funds, when the initial investment is realized). They model terminal values as diffusion processes, and derive a closed formula for the optimal holding period. This model has been

further developed by Amédée-Manesme et al. (2015) who incorporate lease structure effect in order to better account for the specificities of real estate. They show how the volatility can influence the optimal time to sell in the context of rational risk neutral investors. Barthélémy and Prigent (2009) also compute an optimal ex-ante time to sell using American option approach. Nevertheless, they assume that the investor is risk-neutral. Due to this latter hypothesis, the volatility of the real estate asset either has no impact on the optimal holding period or its role is only implicit. Regarding the volatility, Rehring (2012) examines the U.K. real estate market and shows that the conditional standard deviation of commercial real estate returns depends on the investment maturity as it is the case for usual stocks in particular on the long-term horizons. The transaction costs and marketing period are also discussed.

In this paper, our aim is to better emphasize the impact of the volatility when the investor is risk averse. For this purpose, we consider a risk-averse investor that maximizes its expected utility at maturity over a given time period. In this approach, time horizon and risk aversion are the key parameters. We thus introduce expected utility (EU) theory as suggested by Arrow (1965) to model decisions under uncertainty for risk averse investors. This way, we account for preferences of individual investors who seek to maximize their preference over possible events according to their corresponding probabilities. We concentrate here on the Hyperbolic Absolute Risk Aversion (HARA)[5] utility functions class and particularly on the sub-class Constant Relative Risk Aversion (CRRA) utility function. Our results show that the relative risk aversion plays a key role to evaluate the monetary loss from not having access to the "best" horizon. This feature has to be related to previous works about the influence of risk aversion such as Kallberg and Ziemba (1983). We also examine the robustness of our results with respect to the utility specification.

The model is built up on previous work. First, we determine the optimal holding period when it has to be chosen at initial date, extending previous results of Baroni *et al.* (2007). The investor is assumed to know probability distribution of real estate asset. We illustrate what are the impacts of the risk aversion, the real asset value and the volatility on the selling strategies. We determine this latter one when the investor is perfectly informed about the growth rate dynamics but must choose his strategy only at initial time. However, usually such a solution is not time consistent since the same determination of optimal time to sell at a future date leads to a different solution. Second, we study the best ideal case where the investor knows exactly the price dynamics, as soon as a new period starts. In that case, he can immediately choose the best time to sell the asset. This approach provides the upper bound of the present value of the portfolio as a function of holding period policy. Indeed, the present value is maximized using perfect foresight. We use this special framework as a benchmark. Finally, we determine the optimal holding period according to the American option approach. In this context, at each time during a given management period, the investor compares the present expected utility of portfolio value with the maximal expected utility he could have if he would keep the asset. We show that the investor must sell as soon as the present utility is higher than its expectation.

We also introduce the notion of compensating variation to evaluate the monetary loss of not

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[5] The specification of utility functions is a tough problem because different utility functions have different behavioural implications. In this line, the work of de Palma and Prigent (2009) can be consulted.

having the "best" portfolio (here not having chosen the best optimal time to sell). Compensating variation is the adjustment that returns the consumer to the original utility after an economic change has occurred. As shown in de Palma and Prigent (2008, 2009), the compensating variation allows to measure the adequacy of a given portfolio to investor's utility.

Our work contributes to the academic literature in optimal holding period in real estate. Prior studies have mainly been empirical and did not propose models to explain investment horizon. Here, we derive a closed-form formula to determine the ex-ante optimal holding period of a real estate asset. It takes account of the risk aversion of the real estate investors. Indeed, real estate investors are usually considered to be risk-neutral while seeking high yield returns (see Fugazza et al., 2007). In sum, this paper contributes by considering the risk aversion of investors, by proposing a model that allows computing optimal holding period and finally by providing solutions whose properties may explain most of the previous empirical results. In addition, practitioners may find here an interesting approach to better model their ex-ante optimal holding period.

The structure of the paper is laid out as follows. Section 2 presents the continuous-time framework and the optimal time to sell we get in the neutral risk investor case. Results for the optimal holding period when the date must be chosen at initial time is developed in section 3 for quadratic utility function and CRRA utility functions. Section 4 gives a theoretical framework for other portfolio strategies, as the perfectly informed investor, the American option solution and the buy-and-hold strategy. All these strategies are compared in Section 5 using compensating variations. Section 6 concludes.

2. Continuous-time model and risk neutral investor

In this section, the time of sale is pre-set, committed irrevocably at time 0, based on the expected dynamics of the portfolio value and its cash flow. The real estate portfolio value is defined as the sum of the discounted free cash flows (FCF) and the discounted terminal value (the selling price). We denote k as the discount rate of the free cash flows, and V_T the terminal value. We assume that the free cash flow grows at a constant rate g [6].

2.1. Continuous-time model

Following Baroni *et al.* (2007), we suppose that the price dynamics follows a geometric Brownian motion:

$$\frac{dP_t}{P_t} = \mu dt + \sigma dW_t, \tag{1}$$

where W_t is a standard Brownian motion.

This equation assumes that the real estate return can be modelled as a simple diffusion process where parameters μ and σ are respectively equal to the trend and to the volatility.

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[6] This assumption allows explicit solutions for the probability distributions of the optimal times to sell and of the optimal portfolio values. The introduction of stochastic rates would lead to only simulated solutions.

Then the future real estate index value at time t, discounted at time 0, can be expressed as:

$$P_t = P_0 \exp\left[\left(\mu - k - \tfrac{1}{2}\sigma^2\right)t + \sigma W_t\right] \text{ with } E\left[\frac{P_t}{P_0}\right] = \exp\left([\mu - k]t\right). \quad (2)$$

Denote by FCF_0 the initial value of the free cash flow. The continuous-time version of the sum of the discounted free cash flows FCF_s is equal to:

$$C_t = \int_0^t FCF_s e^{-ks} ds = \int_0^t FCF_0 e^{-[k-g]s} ds = \frac{FCF_0}{k-g}\left(1 - e^{-[k-g]t}\right), \quad (3)$$

Introduce the real estate portfolio value process V, which is the sum of the discounted free cash flows and the future real estate index value at time t, discounted at time 0:

$$V_t = C_t + P_t = c\left(1 - e^{-at}\right) + P_0\, e^{\left[\left(\mu - k - \tfrac{1}{2}\sigma^2\right)t + \sigma W_t\right]}. \quad (4)$$

We determine the portfolio value $V_{\overline{T}}$ for a given maturity $\overline{T}$. This assumption on the time horizon allows accounting of selling constraints before a limit date. The higher $\overline{T}$, the less stringent this limit. Additionally, this hypothesis allows the study of buy-and-hold strategies (see section 6).

2.2. Computation with the linear utility function:

The optimization problem is:

$$Max_{t \in [0,\overline{T}]} E\left[V_t\right]. \quad (5)$$

Since the expectation of V_t is equal to:

$$E\left[V_t\right] = \frac{FCF_0}{k-g}\left(1 - e^{-[k-g]t}\right) + P_0 e^{[\mu-k]t}, \quad (6)$$

Then, the optimal holding period is determined as follows (see Barthélémy and Prigent, 2009).

- *1: The initial price P_0 is smaller than $\frac{FCF_0}{k-\mu} e^{-(\mu-g)\overline{T}}$.*

Then, the optimal time to sell T^* corresponds to the maturity $\overline{T}$. Since *the Price Earning Ratio* (PER)[7] is too small ($< \frac{e^{-(\mu-g)\overline{T}}}{k-\mu}$), the sell is not relevant before maturity.

- *2: The initial price P_0 lies between the two values $\frac{FCF_0}{k-\mu} e^{-(\mu-g)\overline{T}}$ and $\frac{FCF_0}{k-\mu}$.*

The optimal time to sell T^* is solution of the following equation:

$$\frac{\partial E\left[V_t\right]}{\partial t} = 0, \quad (7)$$

which solves to (see Baroni et al., 2007):

$$T^* = \frac{1}{\mu - g} \ln\left(\frac{FCF_0}{P_0} \times \frac{1}{k-\mu}\right). \quad (8)$$

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[7] The PER $\frac{P_0}{FCF_0}$ is defined as the initial price of a property or group of properties divided by the free cash flows produced by this or these properties. It measures the current price relative to its cash flows production. The PER is often known in real estate as the price multiple.

In particular, note that T^* is a decreasing function of the initial price P_0 and of the difference between the index return μ and the growth rate g of the free cash flows. This latter property was empirically observed by Brown and Geurts (2005). It means that investors sell property sooner when values rise faster than rent.

- *3: The initial price P_0 is higher than $\frac{FCF_0}{k-\mu}$.*

The optimal time to sell T^* corresponds to the initial time 0. Since the PER, $\frac{P_0}{FCF_0}$, is sufficiently large ($> \frac{1}{k-\mu}$), there is no reason to keep the asset P.

We note that the discounted expected value V_t of the portfolio is concave. Knowing the optimal time to sell T^* which is deterministic, the probability distribution of the discounted portfolio value V_{T^*} can be determined. The value V_{T^*} is equal to:

$$V_{T^*} = \frac{FCF_0}{(k-g)}\left[1-e^{-(k-g)T^*}\right] + P_0\exp\left[\left(\mu-k-1/2\sigma^2\right)T^* + \sigma W_{T^*}\right].$$

Denote $A = \frac{FCF_0}{(k-g)}\left[1-e^{-(k-g)T^*}\right]$ the cumulative discounted free cash flow value at T^*. Since, from (13), the optimal time to sell satisfies:

$$T^* = \frac{1}{\mu-g}\ln\left(\frac{FCF_0}{P_0(k-\mu)}\right),$$

then, we deduce:

$$A = \frac{FCF_0}{(k-g)}\left(1-\left[\frac{FCF_0}{P_0(k-\mu)}\right]^{\left(-\frac{k-g}{\mu-g}\right)}\right),$$

and the cdf $F_{V_{T^*}}$ of V_{T^*} is given by:

$$F_{V_{T^*}}(v) = \begin{cases} 0, \text{if } v \leq A \\ N\left[\frac{1}{\sigma\sqrt{T^*}}\left(\ln\left(\frac{v-A}{P_0}\right) - \left(\left(\mu-k-1/2\sigma^2\right)T^*\right)\right)\right], \text{if } v > A \end{cases} \qquad (9)$$

where N denotes the cdf of the standard Gaussian distribution.

3. Optimal time to sell T^*, chosen at time 0 with risk aversion

In what follows, we determine the optimal solution at time 0, for a given maturity $\bar{T}$ and for an investor maximizing expected utility. We have to analyze the expected utility of the future real estate index value at time t, discounted at time 0: if the price return μ is higher than the discount rate k, then, the optimal solution for the linear utility case is simply equal to the maturity $\bar{T}$. Thus we consider the case $\mu < k$. Consequently, not selling the asset implies higher cumulated cash but a smaller discounted expected terminal value $P_0 e^{(\mu-k)t}$. Hence, the investor has to choose between more (discounted) flows and less expected discounted index

value. We also focus on the sub case $g < \mu$, which corresponds to empirical data.

We investigate two main numerical cases. These cases are those of Baroni et al. (2007):

- **Case 1** corresponds to an early selling, due in particular to weak expected return of the real estate asset. We set :

$$\mu = 4.4\%, \sigma = 5\%, g = 3\%, k = 8.4\%, P_0 = 100, FCF_0 = 100 / 22 .$$

- **Case 2** corresponds to a late selling, due in particular to higher expected return of the real estate asset. We set :

$$\mu = 6\%, \sigma = 5\%, g = 2\%, k = 9.5\%, P_0 = 100, FCF_0 = 100 / 15 .$$

To model risk aversion, we introduce a standard decision criterion based on a separable utility function which is additive with respect to current time. It is defined by:

$$\int_0^t e^{-\rho s} u(C_s) ds + E\left[U(P_t)\right] e^{-\rho t}$$

where u and U are utility functions respectively defined on the cash flows and the current market value. The assumption is that the free cash flows can be used to consume along the time period. Such kind of inter temporal utility function allows getting the time consistency. The term $e^{-\rho t} u(C_t)$ can be interpreted as the utility of the consumption at date t viewed at initial time 0. For the choice of the utility function, we first analyse some of the specificities of the quadratic function and then, we concentrate to the CRRA function[8].

3.1. Computation with the quadratic utility function

The expected utility of the portfolio value V_t at time t in the case of the quadratic utility function is given by the sum of two terms:

- The first one corresponds to the utility defined on the cash flows:

$$E\left[\int_0^t e^{-\rho s} u(C_s) ds\right] = \int_0^t e^{-\rho s}\left(C_s - \frac{\lambda_c}{2} C_s^2\right) ds$$

$$= FCF_0\left[\frac{1 - e^{-(\rho+k-g)t}}{(\rho+k-g)}\right] - \frac{\lambda_c}{2} FCF_0^2\left[\frac{1 - e^{-(\rho+2k-2g)t}}{(\rho+2k-2g)}\right] \quad (10)$$

- The second one is the utility provided by the market value

$$E\left[e^{-\rho t} U(P_t)\right] = e^{-\rho t}\left[E(P_t) - \frac{\lambda_P}{2} E(P_t^2)\right] = P_t^2 = P_0^2\, e^{\left[2\left(\mu-k-\frac{1}{2}\sigma^2\right)t+2\sigma W_t\right]} \quad (11)$$

Knowing that

$$E\left[e^{\left(A-\frac{1}{2}B^2\right)t+BW_t}\right] = e^{At} , \text{ where } W_t \,\square\, N\left(0, \sqrt{t}\right),$$
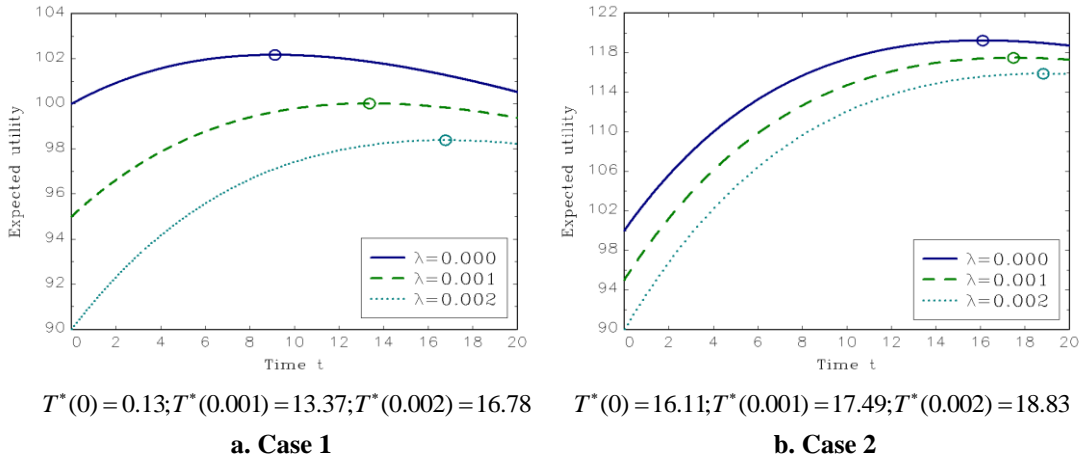
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we deduce that Relation (11) is equivalent to

$$E\left[e^{-\rho t}U\left(P_t\right)\right]=P_0\,e^{(\mu-k-\rho)t}-\frac{\lambda_P}{2}e^{-\rho t}\left[P_0^2\,e^{2\left(\mu-k+\frac{1}{2}\sigma^2\right)t}\right] \tag{12}$$

Then the discounted expected value of the portfolio at time t described, adding (10) and (12) is

$$E\left[V_t\right]=FCF_0\left[\frac{1-e^{-(\rho+k-g)t}}{(\rho+k-g)}\right]-\frac{\lambda_c}{2}FCF_0^{\ 2}\left[\frac{1-e^{-(\rho+2k-2g)t}}{(\rho+2k-2g)}\right]+P_0\,e^{(\mu-k-\rho)t}-\frac{\lambda_P}{2}e^{-\rho t}\left[P_0^2\,e^{2\left(\mu-k+\frac{1}{2}\sigma^2\right)t}\right]$$

For instance, Figure 1 illustrates the impact of λ on the portfolio utility function according to the selling time. We consider the two previous numerical cases 1 and 2. Notice that $\lambda=0$ corresponds to the risk neutral case presented in section 2.1. In both cases the optimal time to sell is increasing with the level of the risk aversion λ .[9]

$T^*(0)=0.13; T^*(0.001)=13.37; T^*(0.002)=16.78$ $T^*(0)=16.11; T^*(0.001)=17.49; T^*(0.002)=18.83$

a. Case 1   **b. Case 2**

Fig 1. Quadratic utility, optimal time T^* with respect to risk aversion λ

3.2. Computation with the CRRA utility function

The "Constant relative risk aversion" (CRRA) function is of the type $u(x)=x^{1-c}/(1-c)$ if $c\neq1$, $\ln[x]$ if $c=1$. This type of function exhibits decreasing absolute risk-aversion (DARA). Three main cases can be distinguished for the CRRA case:

- If $0<c<1$ the individual has a small relative risk aversion. If his wealth becomes null, his utility is equal to 0. Thus, it is lower bounded. On the contrary, if his wealth becomes high, he is never satiated.

- For the special case $c=1$ the utility converges to -∞ when the wealth converges to 0 but the individual is still never satiated when his wealth increases.

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[9] If risk aversion raises, the expected utility can be no longer monotone with respect to the selling time. The quadratic utility function is not clearly defined for all the values of λ . If λ becomes high enough the utility function is then a decreasing function for values higher than $1/\lambda$.

- Finally, for $c > 1$ the utility converges to $-\infty$ when the wealth converges to 0 more quickly than for the previous logarithm case but now the individual can be satiated (his utility is upper bounded). In that latter case, such individual searches first of all for limiting the downside risk while accepting to give up potential high returns.

The expected utility is given by:

$$\int_0^t e^{-\rho s} u(C_s) ds + e^{-\rho t} E[U(P_t)]$$

with

$$\int_0^t e^{-\rho s} u(C_s) ds \quad = \quad \int_0^t e^{-\rho s} \frac{C_s^{(1-\gamma_c)}}{(1-\gamma_c)} ds$$

$$= \quad \frac{FCF_0^{(1-\gamma_c)}}{(1-\gamma_c)} \left[\frac{1 - e^{-[\rho - (1-\gamma_c)(g-k)]t}}{[\rho - (1-\gamma_c)(g-k)]} \right]$$

and

$$e^{-\rho t} U(P_t) = e^{-\rho t} \frac{\left(P_0 e^{\left[\left(\mu-k-\frac{1}{2}\sigma^2\right)t + \sigma W_t\right]} \right)^{(1-\gamma_P)}}{(1-\gamma_P)} = \frac{P_0^{(1-\gamma_P)}}{(1-\gamma_P)} e^{-\rho t + \left[\left(\mu-k-\frac{1}{2}\sigma^2\right)t + \sigma W_t\right](1-\gamma_P)}$$

Finally, we get:

$$E\left[\int_0^t e^{-\rho s} u(C_s) ds + e^{-\rho t} U(P_t) \right] \quad = \quad \frac{FCF_0^{(1-\gamma_c)}}{(1-\gamma_c)} \left[\frac{1 - e^{-[\rho - (1-\gamma_c)(g-k)]t}}{[\rho - (1-\gamma_c)(g-k)]} \right]$$

$$+ \frac{P_0^{(1-\gamma_P)}}{(1-\gamma_P)} e^{\left[-\rho + (1-\gamma_P)(\mu-k) - \frac{1}{2}\sigma^2 \gamma_P (1-\gamma_P)\right]t}$$

and the first derivative with respect to time t is equal to:

$$\frac{FCF_0^{(1-\gamma_c)}}{(1-\gamma_c)} \left[e^{-[\rho - (1-\gamma_c)(g-k)]t} \right]$$

$$+ \frac{P_0^{(1-\gamma_P)}}{(1-\gamma_P)} \left[-\rho + (1-\gamma_P)(\mu-k) - \frac{1}{2}\sigma^2 \gamma_P (1-\gamma_P) \right] e^{\left[-\rho + (1-\gamma_P)(\mu-k) - \frac{1}{2}\sigma^2 \gamma_P (1-\gamma_P)\right]t}$$

Previous formula allows getting explicit relations between the optimal selling time T^* and various parameters such as the relative risk aversions γ_c and γ_P, the volatility σ and the initial real estate asset value P_0.

4. Solution analysis for the CRRA utility

4.1. Conditions on existence of a non degenerated solution

First, we note that, under specific assumptions about both real estate and utility parameters, the sign of the previous derivative with respect to time is constant, independently from the initial cash flow and real asset values.

We have two cases:

A) If $(1-\gamma_c)>0$ and $(1-\gamma_P)\left[\rho+(1-\gamma_P)(k-\mu+\frac{1}{2}\sigma^2\gamma_P)\right]<0$ then the derivative is non negative. Thus, the global utility is an increasing function of the time. In such a case, it is never optimal to sell before maturity. The second assumption implies that $\gamma_P>1$. It means also that the coefficient ρ must be relatively high, which corresponds to a high degree of impatience.

B) If $(1-\gamma_c)<0$ and $(1-\gamma_P)\left[\rho+(1-\gamma_P)(k-\mu+\frac{1}{2}\sigma^2\gamma_P)\right]>0$, then the derivative is non positive. Thus, the global utility is a decreasing function of the time. In such a case, it is never optimal to wait for selling. The first assumption is not too realistic since here the cash flows are deterministic.

In what follows, we analyze the other more realistic cases since they depend on initial cash flow and real asset values. We assume in particular:

$$(1-\gamma_P)\left[\rho+(1-\gamma_P)(k-\mu+\frac{1}{2}\sigma^2\gamma_P)\right]>0.$$

Sub case B.1. The real estate and risk aversion parameters are such that:

$$(1-\gamma_P)(\mu-k)-(1-\gamma_c)(g-k)-\frac{1}{2}\sigma^2\gamma_P(1-\gamma_P)<0$$

This case happens for instance when the relative risk aversion γ_P is high. The optimality corresponds here to a minimum as underlined by Figure 2. On the whole range $[0,\ \overline{T}\ ]$, the utility is always increasing. Then, it is optimal to wait for selling: $T^*=\overline{T}$.
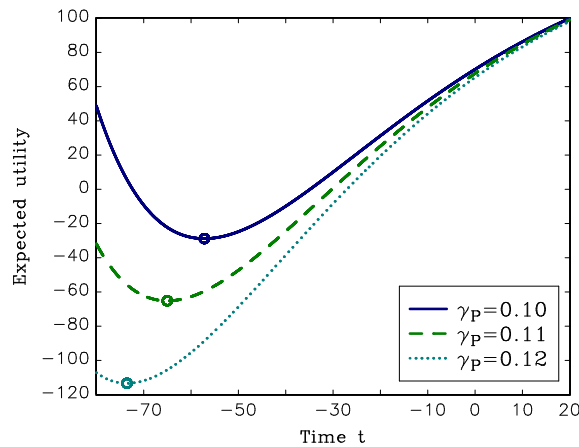
Fig 2. Condition for the existence of the analytical solution for T^*

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Sub case B.2. The real estate and risk aversion parameters are such that:

$$(1-\gamma_P)(\mu-k)-(1-\gamma_c)(g-k)-\tfrac{1}{2}\sigma^2\gamma_P(1-\gamma_P)>0$$

This happens for example when $0<\gamma=\gamma_C=\gamma_P<1$ and the volatility has usual values. Then, we have to distinguish three main cases for the initial value P_0 of the real asset. For this purpose, denote respectively by $\underline{P}$ and $\overline{P}$ the following terms:

$$\underline{P}=FCF_0^{\frac{(1-\gamma_c)}{(1-\gamma_P)}}\left(\frac{(1-\gamma_P)}{(1-\gamma_C)}\frac{\exp\left[-\overline{T}\left[(1-\gamma_P)(\mu-k)-(1-\gamma_c)(g-k)-\tfrac{1}{2}\sigma^2\gamma_P(1-\gamma_P)\right]\right]}{\left[\rho-(1-\gamma_P)(\mu-k)+\tfrac{1}{2}\sigma^2\gamma_P(1-\gamma_P)\right]}\right)^{\frac{1}{(1-\gamma_P)}}$$

And

$$\overline{P}=\left(\frac{FCF_0^{(1-\gamma_c)}}{\left[\rho-(1-\gamma_P)(\mu-k)+\tfrac{1}{2}\sigma^2\gamma_P(1-\gamma_P)\right]}\frac{(1-\gamma_P)}{(1-\gamma_C)}\right)^{\frac{1}{1-\gamma_P}}$$

By dividing $\underline{P}$ and $\overline{P}$ by FCF_0, we get the corresponding limits on the PER coefficient, $\underline{PER}$ and $\overline{PER}$. The range between $\overline{PER}$ and $\underline{PER}$ is an increasing function of the spread between μ and g as illustrated by Figure 3. Note that the dashed line indicates a PER of 22, which corresponds to the one of the numerical case 1.
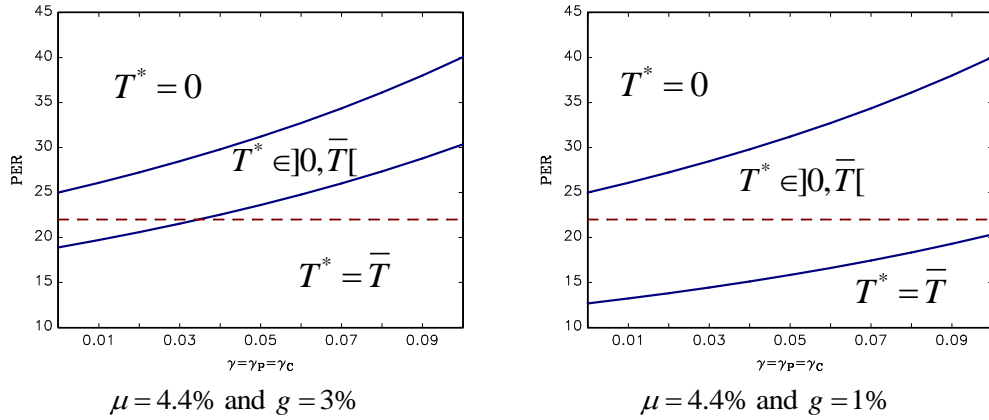
Fig 3. PER limits as a function of $\mu-g$

i) The initial value P_0 is smaller than $\underline{P}$

In that case, the optimal time to sell corresponds to the maturity. Indeed, the PER P_0/FCF_0 is too small, which implies that the sell is not relevant before maturity.

ii)   The initial value lies between $\underline{P}$ and $\overline{P}$.

The optimal solution $T*$ corresponding to a null first derivative is given by:

$$T^* = \frac{1}{\left(1-\gamma_P\right)(\mu-k)-(1-\gamma_c)(g-k)-\frac{1}{2}\sigma^2\gamma_P\left(1-\gamma_P\right)} \times$$

$$Log\left[\frac{FCF_0^{(1-\gamma_c)}}{P_0^{(1-\gamma_P)}}\frac{(1-\gamma_P)}{(1-\gamma_C)}\frac{1}{\left[\rho-(1-\gamma_P)(\mu-k)+\frac{1}{2}\sigma^2\gamma_P\left(1-\gamma_P\right)\right]}\right]$$

(17)

For the special case $\left[\gamma = \gamma_C = \gamma_P\right]$, we get:

$$T^* = \frac{1}{\left(1-\gamma\right)\left(\left(\mu-g\right)-\frac{1}{2}\sigma^2\gamma\right)} \times$$

$$Log\left[\left(\frac{FCF_0}{P_0}\right)^{(1-\gamma)}\frac{1}{\left[\rho+(1-\gamma)\left[(k-\mu)+\frac{1}{2}\sigma^2\gamma\right]\right]}\right]$$

(18)

Note that, for $\gamma = 0$ and $\rho = 0$, we recover the result corresponding to the quasi linear utility (no risk aversion):

$$T^* = \frac{1}{\left(\mu-g\right)} \times Log\left[\left(\frac{FCF_0}{P_0}\right)\frac{1}{\left[k-\mu\right]}\right]$$

In what follows, we begin by examining the shape of the utility function according to the relative risk aversion γ. Then, we study the impact of the volatility.

iii)   The initial value P_0 is higher than $\overline{P}$

In that case, the optimal time to sell corresponds to the initial time 0. Since the PER P_0 / FCF_0 is sufficiently large, there is no reason to wait for selling.

Remark: For $0 < \gamma_C, \gamma_P < 1$, both lower and upper bounds $\underline{P}$ and $\overline{P}$ are decreasing functions with respect to parameter ρ. This is also true for the optimal solution given in Relation (17).

In what follows, concerning the sensitivities, we present the results for the case 1 only as we obtain the same qualitative effects for the second numerical, case 2.

4.2. Sensitivities to low risk aversion and to volatility with $\gamma_C = \gamma_P$

The optimal time to sell is an increasing function of the risk aversion. Higher the risk aversion, higher the rents weight in the portfolio. Hence, it leads to wait more before selling the portfolio because the rents may balance longer the loss in capital. This is shown on Figure 4 representing the expected utility for the first numerical example (case 1) and small values of γ. Let us notice that, with $\gamma = \gamma_C = \gamma_P = 0$, we get the optimal time to sell obtained in

Barthélémy and Prigent (2009), $T^* = 9.13$ (upper curve in both Figures 4a and 4b). With $\gamma = \gamma_C = \gamma_P = 0.001$, we get $T^* = 9.43$, and with $\gamma = 0.002$, $T^* = 9.73$. The effects on T^* are more important for a higher level of risk aversion (see Figure 4b). Indeed the optimal time to sell can be equal to the maturity itself (see the curve corresponding to $\gamma = \gamma_C = \gamma_P = 0.004$, for which $T^* = \overline{T} = 20$).
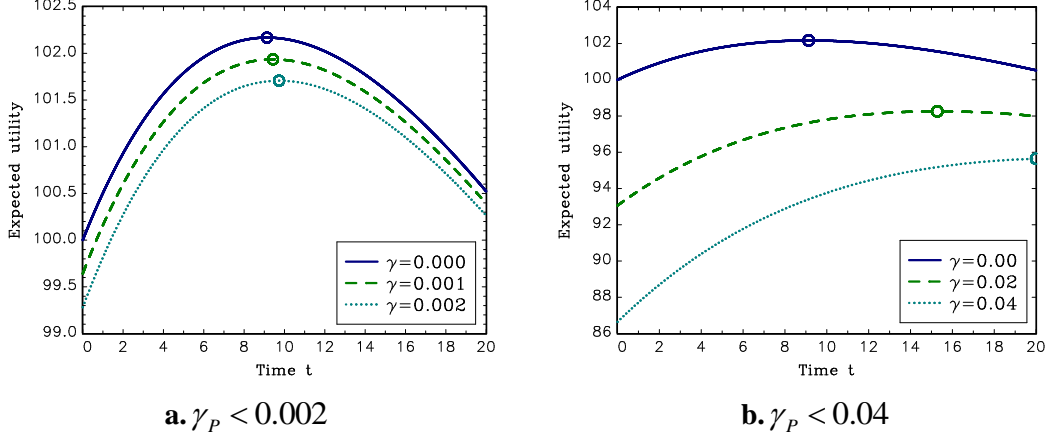
a. $\gamma_P < 0.002$ **b.** $\gamma_P < 0.04$

Fig 4. Expected utility as a function of the risk aversion, first numerical example

Small values of $\gamma = \gamma_C = \gamma_P$.

The volatility has a negative impact on the optimal time to sell. When the volatility is increasing, the optimal time to sell is decreasing ($dT^*/d\sigma < 0$). But, for the given parameters values of cases 1 or 2, the impact is negligible as presented in Table 1 and in Table 2.

| σ	$T^*(\gamma = 0)$	$T^*(\gamma = 0.001)$	$T^*(\gamma = 0.002)$
0.05	9.131	9.431	9.732
0.15	9.131	9.420	9.710
0.25	9.131	9.398	9.660

Table 1. Optimal time to sell as function of σ - first numerical example

σ	$T^*(\gamma = 0)$	$T^*(\gamma = 0.01)$	$T^*(\gamma = 0.02)$
0.05	16.109	17.206	18.328
0.15	16.109	17.177	18.274
0.30	16.109	17.079	18.094

Table 2. Optimal time to sell as function of σ - second numerical example

4.3. Sensitivities to low risk aversion and to volatility with $\gamma_C = 0$ and $0 \le \gamma_P < 1$

As in subsection 4.1, we can analyze the range between $\overline{PER}$ and $\underline{PER}$ (see Figure 3). Figure 5a shows that the spread $\overline{PER} - \underline{PER}$ is increasing with γ_P as well as the two bounds, $\underline{PER}$ and $\overline{PER}$. Moreover, with a PER of 22 (the one of the numerical case 1), values of γ_P up to around 0.04 lead to an interior solution for T^*, as underlined by the dashed line. This is illustrated in a other way on Figure 5b where values of γ_P equal to 0.015 or 0.030 lead to a $T^* < 20$, while the curve corresponding to $\gamma_P = 0.045$ imply that $T^* = 20$. Finally, As in the previous section with $\gamma_C = \gamma_P$, the increase of the risk parameter γ_P leads to a higher optimal time to sell (see Figure 5.b).
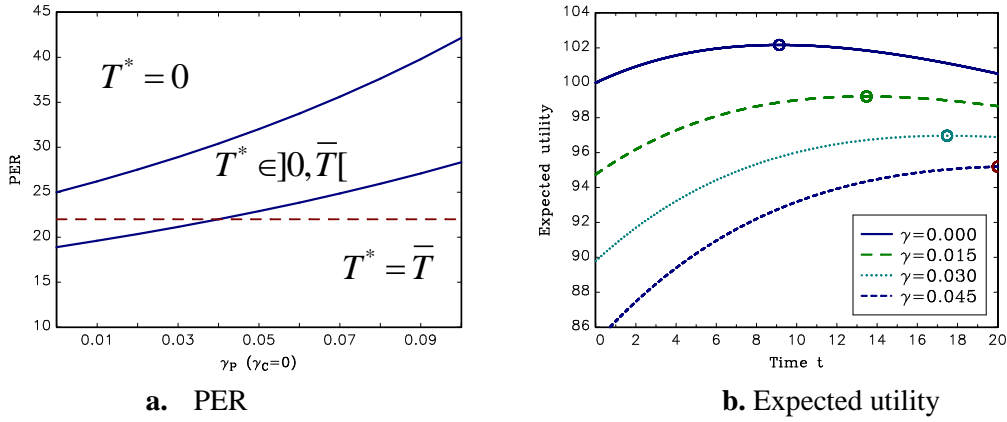
a. PER   **b.** Expected utility

Fig 5. PER limits and expected utility for the numerical example 1 - w.r.t. the risk aversion, first numerical example $\gamma_C = 0$

Table 3 exhibits the same relation between T^* and σ.

| σ	$T^*(\gamma_P = 0)$	$T^*(\gamma_P = 0.01)$	$T^*(\gamma_P = 0.02)$	$T^*(\gamma_P = 0.03)$
0.05	9.13	12.06	14.84	17.48
0.15	9.13	11.97	14.70	17.32
0.25	9.13	11.79	14.41	16.99

Table 3. Optimal time to sell as function of σ - first numerical example, $\gamma_C = 0$

Remark: If we consider much higher volatilities (as for example for equity markets during the recent financial crisis), we note that the optimal time to sell can be decreasing with respect to the relative risk aversion but takes high values for a relative risk aversion lying between 0 and 1, as shown in next subsection (see Figure 6a).

4.4. Sensitivities to volatility and to high risk aversion to the real estate asset with $\gamma_C = 0$ and $\gamma_P > 1$

In what follows, we assume that the relative risk aversion to the real asset can be moderate or high. It means that we set $\gamma_P > 1$. In such a case, the optimal time to sell as a function of the risk aversion is decreasing from a given aversion level, as shown in Figure 6b, contrary to the case $\gamma_P < 1$, illustrated in Figure 6a.

a. $\gamma_P < 1$ **b.** $\gamma_P > 1$
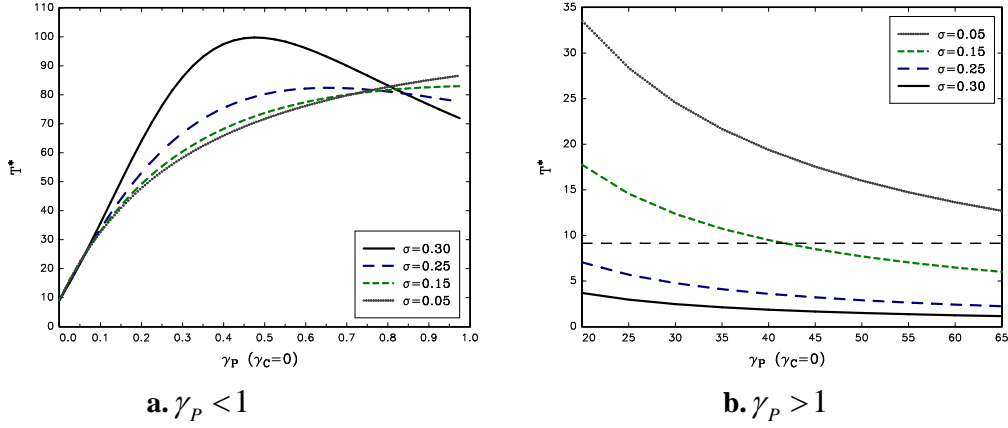
Fig 6. T^* as a function of volatility, first numerical example, $\gamma_C = 0$

Additionally, if the volatility takes high values, the optimal time to sell can be smaller than for the risk-neutral case (see Figure 6b, where this latter value is equal to 9.13 and corresponds to the straight line).

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5. Other optimal times to sell for a risk averse investor

5.1. Perfectly informed investor T^{**}

In this section, the investor is supposed to have a perfect foresight about the entire future price path. Trajectories are random (the investor does not choose the realized path) but, at time 0, the whole path is known. Therefore, the investor can maximize with respect to this trajectory. Thus, the optimal solution is deterministic conditionally to this information. Nevertheless, the path is unknown just before time 0. Consequently, the optimal time to sell is a random variable. This 'ideal' framework is not realistic but provides an upward benchmark. Note that, since the investor is rational, his utility function is increasing. Therefore, since the path is known, the maximization of the utility of his portfolio value is equivalent to the maximization of a linear utility. This means that we recover previous solution provided in Barthélémy and Prigent (2009), which does not depend on risk aversion. In what follows, we recall the distributions of the optimal holding period T^{**} and of the optimal value $V_{T^{**}}$ and provide the explicit formula by means of a mild approximation of the aforementioned paper. Introduce the function G defined by:

$$G(m,y,t) = 1 - \frac{1}{2} Erfc\left(\frac{y}{\sqrt{2t}} - m\frac{\sqrt{t}}{\sqrt{2}}\right) - \frac{1}{2}e^{2my}Erfc\left(\frac{y}{\sqrt{2t}} + m\frac{\sqrt{t}}{\sqrt{2}}\right),$$

where the function $Erfc$ is given by:

$$Erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}du.$$

Denote also

$$A(v) = \frac{FCF_0}{v} + \mu - k - 1/2\sigma^2, \text{and } B(v) = \ln\left(\frac{v}{P_0}\right).$$

Then, the approximated cdf of $V_{T^{**}}$ is given explicitly by:

$$P[V_{T^{**}} \leq v] = \begin{cases} 0, \text{ for } v < P_0, \\ G\left(\frac{A(v)}{\sigma}, \frac{B(v)}{\sigma}, \bar{T}\right) \text{for } v > P_0 \end{cases}. \tag{15}$$

The probability that the real estate portfolio value is higher than P_0 is equal to 1. Thus, whatever the path, the investor receives at least P_0. Indeed, if all the future discounted portfolio values are lower than the initial price, he knows he has to sell at time 0 and then receives exactly P_0. Summing up, time T^{**} does not depend on risk aversion but its probability distribution does.

5.2. American optimal selling time T^{***}

In this third case, we allow that the investor may choose the optimal time to sell, according to market fluctuations and information from past observations. In this case, he faces an "American" option problem. Recall that the investor preferences are modelled by means of utility function. At any time t before selling, he compares the utility of the present value P_t with the maximum of the future utility value he expects given the available information at time t (mathematically speaking he computes the maximum expected utility of his portfolio on all $\mathsf{J}_{t,\bar{T}}$-measurable stopping times τ). It means that he decides to sell at time t only if the utility of his portfolio value at this time is higher than the maximal expected utility that he can expect to reach if he does not sell at this time t. Thus, he has to compare $U(P_t)$ with $\sup_{\tau \in \mathsf{J}_{t,\bar{T}}} E\left[\int_t^\tau e^{-\rho(s-t)} u(C_s)ds + e^{-\rho(\tau-t)} U(P_\tau) \Big| \mathsf{J}_t\right]$. Intuitively, the optimal time T^{***} must be the first time at which the utility $U(P_t)$ is "sufficiently" high. At this price level, the future free cash flows (received in case of no sell) will not be high enough to balance an index value lower than the price P_t at time t (the expected index value decreases with time as the discounted trend $\mu - k$ is negative). The optimal time T^{***} corresponds exactly to the first time at which the asset price P_t is higher than a deterministic level (see Appendix B). This result generalizes the case considered in Barthélémy and Prigent (2009) where the investor has a linear utility. In that case, he sells directly the asset if the price P_0 is higher than $\frac{FCF_0}{k-\mu}$. Then, since the return of the discounted free cash flows is equal to $e^{-(k-g)t}$, the price P_t has to be compared with the value $\frac{FCF_0}{k-\mu} e^{-(k-g)t}$. Here, we provide an extension of this result when the individual has risk aversion.

5.2.1. The American option problem

Denote by $\mathsf{V}(x,t)$ the following value function:

$$\mathsf{V}(x,t) = \sup_{\tau \in \mathsf{J}_{t,\bar{T}}} E\left[\int_t^\tau e^{-\rho(s-t)} u(C_s)ds + e^{-\rho(\tau-t)} U(P_\tau) \Big| P_t = x\right]$$

Note that we always have $\mathsf{V}(x,t) \geq U(x)$, since $\tau = t \in \mathsf{J}_{t,\bar{T}}$ and, in that case, $\mathsf{V}(x,t) = U(x)$.

As usual for American options[10], two "regions" have to be considered:

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[10] See Elliott and Kopp (1999, p. 193).

- *The continuity region:*

$$C = \left\{ (x,t) \in R^+ \times [0,\overline{T}] \,\middle|\, V(x,t) > U(x) \right\}$$

- *The stopping region:*

$$S = \left\{ (x,t) \in R^+ \times [0,\overline{T}] \,\middle|\, V(x,t) = U(x) \right\}$$

The first optimal stopping time T_t^{***} after time t is given by

$$T_t^{***} = \inf \left\{ \theta \in [t,\overline{T}] \,\middle|\, V(P_\theta,\theta) = \int_t^\theta e^{-\rho(s-t)} u(C_s) ds + e^{-\rho(\theta-t)} U(P_\theta) \right\}.$$

Then:

$$T_t^{***} = \inf \left\{ \theta \in [t,\overline{T}] \,\middle|\, (P_\theta,\theta) \notin C \right\}.$$

5.2.2. Computation of the value function V

To determine T_t^{***}, we have to calculate $V(x,t)$. We have to compute:

$$\sup_{\tau \in J_{t,\overline{T}}} E\left[\int_t^\tau e^{-\rho(s-t)} u(C_s) ds + e^{-\rho(\tau-t)} U\left(P_t \exp\left[\left(\mu - k - 1/2\sigma^2 \right)(\tau - t) + \sigma(W_\tau - W_t) \right] \right) \,\middle|\, P_t = x \right]$$

In particular, we have to search for the value τ_t^* for which the maximum

$$\sup_{\tau \in J_{t,\overline{T}}} E\left[\int_t^\tau e^{-\rho(s-t)} u(C_s) ds + e^{-\rho(\tau-t)} U\left(x \exp\left[\left(\mu - k - 1/2\sigma^2 \right)(\tau - t) + \sigma(W_\tau - W_t) \right] \right) \right]$$

is achieved.

This problem is the dynamic version of the determination of T^* presented in Section 4. Introduce the function $f_{t,x}$ defined by:

$$f_{t,x,z}(\theta) = \int_t^{t+\theta} e^{-\rho(s-t)} u(C_s) ds + e^{-\rho(\tau-t)} U\left(x \exp\left[\left(\mu - k - 1/2\sigma^2 \right)\theta + \sigma\sqrt{\theta} z \right] \right)$$

This function is strictly increasing with respect to x.

According to the distribution of the random variable z (which here is the standard Gaussian distribution), we have to solve:

$$\sup_{\theta \in J_{t,\overline{T}}} E\left[f_{t,x,z}(\theta) \right].$$

Case 1. The optimal solution is equal to the maturity $\overline{T}$.

We have:

$$V(x,t) = E\left[\int_t^{\overline{T}} e^{-\rho(s-t)} u(C_s) ds + e^{-\rho(\overline{T}-t)} U\left(x \exp\left[\left(\mu - k - 1/2\sigma^2 \right)(\overline{T} - t) + \sigma(W_{\overline{T}} - W_t) \right] \right) \right]$$

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Using Jensen inequality, we deduce in that case that $V(x,t) > U(x)$, for all three standard utility functions (since they are concave and strictly increasing).

Case 2: The optimal solution lies strictly between t and $\overline{T}$.

The optimal time τ_t^* is equal to $(t + \theta^*)$, where θ^* is the solution of the following equation:

$$\frac{\partial E\left[f_{t,x,z}\right]}{\partial \theta}(\theta) = 0.$$

$$V(x,t) = E\left[\int_t^{\tau_t^*} e^{-\rho(s-t)} u(C_s) ds + e^{-\rho(\tau_t^*-t)} U\left(x \exp\left[\left(\mu - k - 1/2\sigma^2\right)\left(\tau_t^* - t\right) + \sigma(W_{\tau_t^*} - W_t)\right]\right)\right]$$

Case 3: The optimal time τ_t^* corresponds to the present time t, and

$$V(x,t) = U(x).$$

Consequently, from the three previous cases, we deduce the value of $V(x,t)$.

Finally, the American optimal time T^{***} is determined by:

$$T^{***} = \inf\left\{t \in [0,\overline{T}] \mid V(P_t,t) = U(P_t)\right\}.$$

Therefore, we can check that $V(P_t,t) = U(P_t)$ if and only if:

$$\frac{\partial E\left[f_{t,P_t,z}\right]}{\partial \theta}(\theta) < 0, \forall \theta \in \left[0,\overline{T} - t\right].$$

Thus, we have:

$$T^{***} = \inf\left\{t \in [0,\overline{T}] \mid P_t \geq l\left(t, FCF_0, k, g, \mu, \rho, RA_C, RA_P\right)\right\},$$

where RA_C and RA_P denote respectively the parameters characterizing the risk aversions to the free cash flows and the real estate asset and where $l\left(t, FCF_0, k, g, \mu, \rho, RA_C, RA_P\right)$ is determined from optimality condition of 0 being optimal for the first problem (see Section 4).

5.2.3. The American option problem for the CRRA case

For the CRRA case, we can give explicit conditions on the current value P_t of the real estate asset to determine the American optimal time T^{***}. In Appendix, we prove that $V(x,t)$ has indeed an explicit value and that the price P_t has to be compared with the value:

$$\left(\frac{FCF_0^{(1-\gamma_c)}}{\left[\rho - (1-\gamma_P)(\mu-k) + \frac{1}{2}\sigma^2\gamma_P(1-\gamma_P) \right] \frac{(1-\gamma_P)}{(1-\gamma_C)}} \right)^{\frac{1}{1-\gamma_P}} e^{-(k-g)\frac{(1-\gamma_C)}{1-\gamma_P}t}$$

Previous formula allows getting explicit relations between the optimal selling time T^{***} and various parameters such as the relative risk aversions γ_C and γ_P, the volatility σ and the initial cash flow value FCF_0.

6. Compensating variations of the three optimal strategies and the buy-and hold one

The ratio of expected utilities characterizes the investor's choice behaviour but it is only a qualitative criterion since utilities are defined up to affine transformations. In what follows, we use instead a quantitative index of investor's satisfaction based on the standard economic concept of compensating variation. The compensating variation (CV) is a measure of utility change. It is the amount of money required to reach the initial utility when a change occurs in prices or in the market. CV can thus be used to find the effect of changes on the net welfare (of an agent or of a portfolio). As illustrated in de Palma and Prigent (2008, 2009), the notion of CV is very useful to evaluate the monetary loss of not having the "best" portfolio. The utility loss from not having access to a "better" portfolio is provided by the compensating variation measure. If an investor with risk aversion γ and initial investment V_0 faces a choice between two (random) horizons $T^{(1)}$ and $T^{(2)}$, he has to compare the two expected utilities $E[U_\gamma(V_{T^{(i)}}); V_0]$. Assume that horizon $T^{(2)}$ provides higher utility than maturity $T^{(1)}$. If the investor selects maturity $T^{(1)}$ instead of $T^{(2)}$, he will get the same expected utility provided that he invests an initial amount $V_0 \geq V_0$ such that:

$$E\left[U_\gamma(V_{T^{(1)}}); V_0 \right] = E\left[U_\gamma(V_{T^{(2)}}); V_0 \right]$$

Therefore, this investor requires (theoretically) a monetary compensation that can be evaluated by means of the ratio V_0 / V_0. This amount is in line with the certainty equivalent concept in expected utility analysis. It can be viewed as an implicit initial investment necessary to keep the same level of expected utility.

Recall also that, at any time t of the management period $\left[0, \overline{T} \right]$, the "portfolio" value is given by:

$$V_t = C_t + P_t,$$

with

$$C_t = \frac{FCF_0}{k-g}\left(1 - e^{-[k-g]t} \right) \text{ and } P_t = P_0 \exp\left[\left(\mu - k - 1/2\sigma^2 \right)t + \sigma W_t \right]$$

We introduce the returns $R_{T^{(1)}} = V_{T^{(1)}} / V_0$ and $R_{T^{(2)}} = V_{T^{(2)}} / V_0$.

6.1. The compensating variation for the quadratic case

Suppose that the investor's utility U is of quadratic. Function U is equal to:

$$U(v) = v - \lambda \frac{v^2}{2}, \text{ with } \lambda > 0.$$

If we fix the level of risk aversion λ, then relation $E[U_\lambda(V_{T^{(1)}}); V_0] = E[U_\lambda(V_{T^{(2)}}); V_0]$ is equivalent to:

$$V_0 E\left[R_{T^{(1)}}\right] - \frac{\lambda}{2} V_0^2 E\left[R_{T^{(1)}}^2\right] = V_0 E\left[R_{T^{(2)}}\right] - \frac{\lambda}{2} V_0^2 E\left[R_{T^{(2)}}^2\right].$$

The previous relation provides the expression of the compensating variation for the quadratic case, through the resolution of the following polynomial equation:

$$\frac{\lambda V_0}{2} x^2 E\left[R_{T^{(1)}}^2\right] - x E\left[R_{T^{(1)}}\right] + E\left[R_{T^{(2)}}\right] - \frac{\lambda V_0}{2} E\left[R_{T^{(2)}}^2\right] = 0,$$

where x denotes the possible values of the compensating variation V_0 / V_0. Set:

$$\Delta = \left(E\left[R_{T^{(1)}}\right]\right)^2 - 2\lambda V_0 E\left[R_{T^{(1)}}^2\right]\left(E\left[R_{T^{(2)}}\right] - \frac{\lambda V_0}{2} E\left[R_{T^{(2)}}^2\right]\right).$$

Then, we deduce:

$$\frac{V_0}{V_0} = \frac{E\left[R_{T^{(1)}}\right] + \sqrt{\Delta}}{\lambda V_0 E\left[R_{T^{(1)}}^2\right]}.$$

Since the relative risk aversion is increasing for the quadratic case, it is not surprising that the compensating variation depends on the wealth level V_0.

6.2. The compensating variation for the CRRA case

Suppose that the investor's utility u and U are of CRRA type. Function u and U are respectively equal to:

$$u(FCF) = \frac{FCF^{1-\gamma_c}}{1-\gamma_c}, \text{ with } \gamma_c > 0,$$

$$U(P) = \frac{P^{1-\gamma_P}}{1-\gamma_P}, \text{ with } \gamma_P > 0.$$

We have:

$$\int_0^t e^{-\rho s} u(FCF_s) ds + e^{-\rho t} U(P_t) = \frac{FCF_0^{(1-\gamma_c)}}{(1-\gamma_c)}\left[\frac{1 - e^{-[\rho-(1-\gamma_c)g]t}}{[\rho-(1-\gamma_c)g]}\right] + \frac{P_0^{(1-\gamma_P)}}{(1-\gamma_P)} e^{\left[-\rho+(1-\gamma_P)\mu-\frac{1}{2}\sigma^2\gamma_P(1-\gamma_P)\right]t}$$

Then, relation

$$\int_0^T e^{-\rho s} u(FCF_s, FCF_0) ds + e^{-\rho t} U(P_t, P_0) = \int_0^{T^*} e^{-\rho s} u(FCF_s, FCF_0) ds + e^{-\rho t} U(P_t, P_0)$$

is equivalent to

$$\frac{FCF_0^{(1-\gamma_c)}}{(1-\gamma_c)}\left[\frac{1-e^{-[\rho-(1-\gamma_c)g]T}}{[\rho-(1-\gamma_c)g]}\right]+\frac{P_0^{(1-\gamma_P)}}{(1-\gamma_P)}e^{\left[-\rho+(1-\gamma_P)\mu-\frac{1}{2}\sigma^2\gamma_P(1-\gamma_P)\right]T}$$

$$=$$

$$\frac{FCF_0^{(1-\gamma_c)}}{(1-\gamma_c)}\left[\frac{1-e^{-[\rho-(1-\gamma_c)g]T^*}}{[\rho-(1-\gamma_c)g]}\right]+\frac{P_0^{(1-\gamma_P)}}{(1-\gamma_P)}e^{\left[-\rho+(1-\gamma_P)\mu-\frac{1}{2}\sigma^2\gamma_P(1-\gamma_P)\right]T^*},$$

which yields to a linear relation between FCF_0 and P_0.

In that case, several additional conditions can be imposed to determine the compensating variations. For example:

1) We can adjust the initial free cash flow FCF_0 and fix the initial index price: $\overline{P}_0 = P_0$.

2) We can adjust proportionally both the initial free cash flow FCF_0 and the initial index price. In that case, we have:

$$V_0 = FCF_0 + P_0 \text{ with } \frac{\overline{V}_0}{V_0} = \frac{\overline{FCF}_0}{FCF_0} = \frac{\overline{P}_0}{P_0}$$

Then, we deduce that the compensating variation $\dfrac{\overline{V}_0}{V_0}$ when the investor selects the maturity T instead of the optimal one T^* is given by:

$$\frac{\left(\overline{V}_0/V_0\right)^{(1-\gamma_c)}FCF_0^{(1-\gamma_c)}}{(1-\gamma_c)}\left[\frac{1-e^{-[\rho-(1-\gamma_c)g]T}}{[\rho-(1-\gamma_c)g]}\right]+\frac{\left(\overline{V}_0/V_0\right)^{(1-\gamma_c)}P_0^{(1-\gamma_P)}}{(1-\gamma_P)}e^{\left[-\rho+(1-\gamma_P)\mu-\frac{1}{2}\sigma^2\gamma_P(1-\gamma_P)\right]T}$$

$$=$$

$$\frac{FCF_0^{(1-\gamma_c)}}{(1-\gamma_c)}\left[\frac{1-e^{-[\rho-(1-\gamma_c)g]T^*}}{[\rho-(1-\gamma_c)g]}\right]+\frac{P_0^{(1-\gamma_P)}}{(1-\gamma_P)}e^{\left[-\rho+(1-\gamma_P)\mu-\frac{1}{2}\sigma^2\gamma_P(1-\gamma_P)\right]T^*}$$

For the special case where $\gamma = \gamma_C = \gamma_P$, the following relation provides the expression of the compensating variation for the CRRA case:

$$\left(\overline{V}_0/V_0\right)=\left(\frac{FCF_0^{(1-\gamma)}\left[\dfrac{1-e^{-[\rho-(1-\gamma)g]T^*}}{[\rho-(1-\gamma)g]}\right]+P_0^{(1-\gamma)}e^{\left[-\rho+(1-\gamma)\mu-\frac{1}{2}\sigma^2\gamma(1-\gamma)\right]T^*}}{FCF_0^{(1-\gamma)}\left[\dfrac{1-e^{-[\rho-(1-\gamma)g]T}}{[\rho-(1-\gamma)g]}\right]+P_0^{(1-\gamma)}e^{\left[-\rho+(1-\gamma)\mu-\frac{1}{2}\sigma^2\gamma(1-\gamma)\right]T}}\right)^{\frac{1}{(1-\gamma)}} \qquad (20)$$

In what follows, we numerically illustrate the CV for the CRRA case, according to the values of the parameter γ_P in the cases where $\gamma_C = \gamma_P$ (see eq. 20) or when $\gamma_C = 0$. Moreover, we

consider the parameter values of the numerical cases 1 and 2.

6.2.1 - The relative risk aversions are smaller than 1

The compensating variation increases with the risk aversion as illustrated on Figure 7, where Figure 7a refers to the numerical case 1 and Figure 7b to the numerical case 2. The corresponding T^* are respectively 9.13, 14.20 and 19.32 (in 7a) and 16.11, 18.07 and 19.97 (in 7b). The first case (7a) raises an additional comment: although compensating variation increases with the risk aversion for short holding period, the contrary may occur for longer holding period, even if the impact is not that significant.


a. Case 1   **b. Case 2**
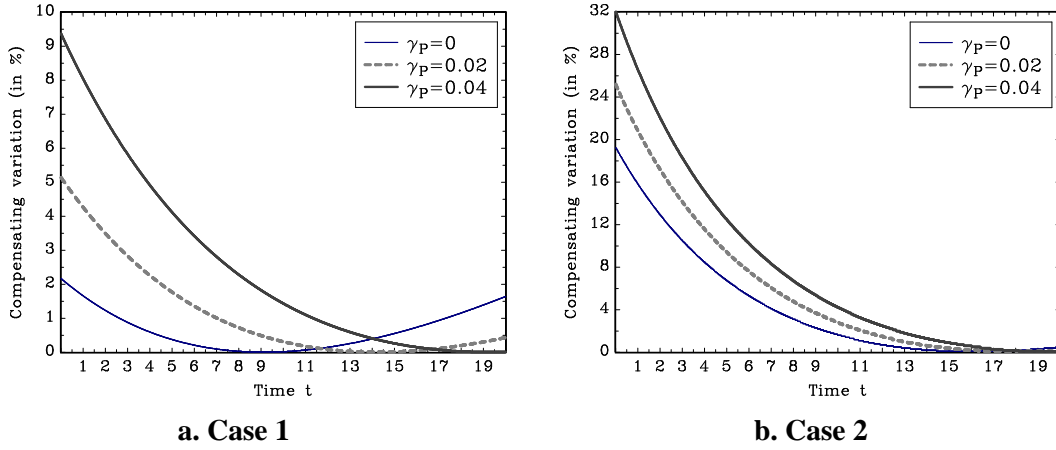
Fig. 7. Compensating variation for very small RRA ($\gamma_C = 0$)

6.2.2 - The relative risk aversion to the real asset γ_P is higher than 1

In such a case, we have to set $\gamma_C = 0$ to get non degenerated solutions (since the free cash flows are deterministic).

The effects of the RRA on T^* appear with standard error around 30%. Figure 8 illustrates the compensating variations for moderate relative risk aversions for the first numerical case. The optimal time to sell $T^*(\gamma_P)$ are $T^*(5) = 19.10$, $T^*(6) = 16.00$ and $T^*(7) = 13.80$. The compensating variations can be very high in that case. Note that the results for the second numerical case are quite similar to those of the first numerical case one presented here.
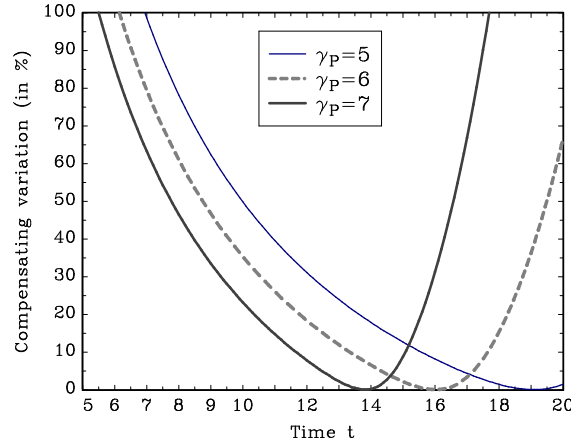
Fig. 8. Compensating variation with $\sigma = 30\%$ for medium RRA γ_P ($\gamma_C = 0$)

The higher the RRA, the smaller T^* and the higher the compensating variations. We obtain the same effects when increasing the standard deviation for a given RRA. This is illustrated on figure 9, where the RRA is set to 7.
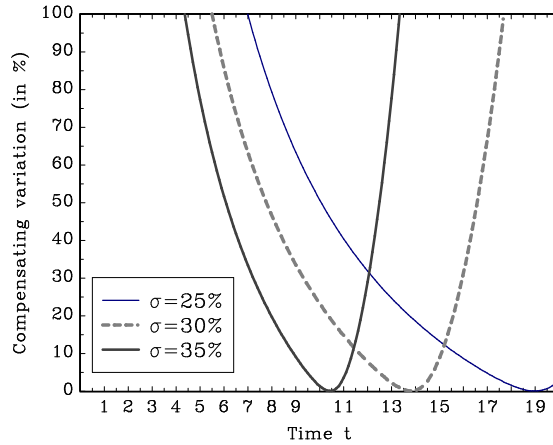
Fig. 9. Compensating variation for medium RRA $\gamma_P = 7$ ($\gamma_C = 0$)

7. Conclusion

This paper emphasizes the impact of the real estate market volatility on optimal holding period. For this purpose, the investor is assumed to be risk-averse, which is an usual assumption when dealing with portfolio optimization but not in the standard real estate literature. The investor is also assumed to consume his free cash flows along the time period. We investigate several kinds of optimal times to sell, illustrating their sensitivities to real estate parameters and risk aversion level. Note in particular that, in the CRRA case, we provide a quite explicit solution of American optimal time to sell, extending previous results which correspond only to the no risk aversion case. Our findings show that, for usual parameter values for the real estate markets, the optimal times to sell are increasing with respect to weak risk aversions while, for high risk aversion levels, it is the converse. Finally, we evaluate the monetary loss of not choosing the "best" optimal time to sell (the so-called compensating variations). We show that this loss can be severe which emphasizes that the

optimality of the holding period is crucial when dealing with real estate investment. In practice, the compensating variation allows investors to measure the cost of not waiting the optimal time or of holding the asset too long.

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Appendix: The American case T* for the CRRA case**

First, to compute:

$$\int_t^\tau e^{-\rho(s-t)} u(C_s)\,ds + e^{-\rho(\tau-t)} U(P_\tau),$$

note that we have:

$$\int_t^\tau e^{-\rho(s-t)} u(C_s)\,ds = \int_t^\tau e^{-\rho(s-t)} \frac{C_s^{(1-\gamma_c)}}{(1-\gamma_c)}\,ds = \frac{FCF_0^{(1-\gamma_c)}}{(1-\gamma_c)} \int_t^\tau e^{-\rho(s-t)} e^{(1-\gamma_c)(g-k)s}\,ds$$

$$= \frac{FCF_0^{(1-\gamma_c)}}{(1-\gamma_c)} e^{\rho t} \left[\frac{e^{-[\rho-(1-\gamma_c)(g-k)]t} - e^{-[\rho-(1-\gamma_c)(g-k)]\tau}}{[\rho-(1-\gamma_c)(g-k)]} \right]$$

and

$$e^{-\rho(\tau-t)} U(P_\tau) = e^{-\rho(\tau-t)} \frac{\left(P_t e^{\left[\left(\mu-k-\frac{1}{2}\sigma^2\right)(\tau-t)+\sigma(W_\tau-W_t)\right]} \right)^{(1-\gamma_P)}}{\left(1-\gamma_P\right)}$$

$$= \frac{P_t^{(1-\gamma_P)}}{\left(1-\gamma_P\right)} e^{-\rho(\tau-t)} e^{\left[\left(\mu-k-\frac{1}{2}\sigma^2\right)(\tau-t)+\sigma(W_\tau-W_t)\right](1-\gamma_P)}$$

Therefore, we have:

$$E\left[e^{-\rho(\tau-t)} U(P_\tau) \middle| P_t = x \right] = \frac{x^{(1-\gamma_P)}}{\left(1-\gamma_P\right)} E\left[e^{\left[-\rho+\left(\mu-k-\frac{1}{2}\gamma_P\sigma^2\right)(1-\gamma_P)\right](\tau-t)} \right]$$

Finally, we get:

$$E\left[\int_t^\tau e^{-\rho(s-t)} u(C_s)\,ds + e^{-\rho(\tau-t)} U(P_\tau) \middle| P_t = x \right] =$$

$$\frac{FCF_0^{(1-\gamma_c)}}{(1-\gamma_c)} e^{\rho t} \left[\frac{e^{-[\rho-(1-\gamma_c)(g-k)]t} - e^{-[\rho-(1-\gamma_c)(g-k)]\tau}}{[\rho-(1-\gamma_c)(g-k)]} \right] + \frac{x^{(1-\gamma_P)}}{\left(1-\gamma_P\right)} E\left[e^{\left[-\rho+\left(\mu-k-\frac{1}{2}\gamma_P\sigma^2\right)(1-\gamma_P)\right](\tau-t)} \right]$$

As in Section 4, we introduce respectively $\underline{P_t}$ and $\overline{P}_t$ defined by:

$$FCF_0^{\frac{(1-\gamma_c)}{(1-\gamma_P)}} \left(\frac{(1-\gamma_P)}{(1-\gamma_C)} \frac{\exp\left[-\overline{T}\left((1-\gamma_P)(\mu-k)-(1-\gamma_c)(g-k)-\frac{1}{2}\sigma^2\gamma_P\left(1-\gamma_P\right)\right)\right]}{\left[\rho-(1-\gamma_P)(\mu-k)+\frac{1}{2}\sigma^2\gamma_P\left(1-\gamma_P\right)\right] e^{-\left[(\mu-k)-\frac{1}{2}\sigma^2\gamma_P\right](1-\gamma_P)t}} \right)^{\frac{1}{(1-\gamma_P)}}$$

and

$$\left(\frac{FCF_0^{(1-\gamma_c)} e^{(g-k)(1-\gamma_C)t}}{\left[\rho-(1-\gamma_P)(\mu-k)+\frac{1}{2}\sigma^2\gamma_P\left(1-\gamma_P\right)\right]} \frac{(1-\gamma_P)}{(1-\gamma_C)} \right)^{\frac{1}{1-\gamma_P}}$$

Consequently, we have three cases:

Case 1. The real estate asset value x is smaller than $\underline{P_t}$.

Then the optimal time corresponds to the maturity $\overline{T}$ and the value function $V(x,t)$ is given by:

$$V(x,t) = \frac{FCF_0^{(1-\gamma_c)}}{(1-\gamma_c)} e^{\rho t} \left[\frac{e^{-[\rho-(1-\gamma_c)(g-k)]t} - e^{-[\rho-(1-\gamma_c)(g-k)]\overline{T}}}{[\rho-(1-\gamma_c)(g-k)]} \right] + \frac{x^{(1-\gamma_P)}}{(1-\gamma_P)} E\left[e^{\left[-\rho+\left(\mu-k-\frac{1}{2}\gamma_P\sigma^2\right)(1-\gamma_P)\right](\overline{T}-t)} \right]$$

This is a linear function with respect to $\dfrac{x^{(1-\gamma_P)}}{(1-\gamma_P)}$.

Case 2. The asset value lies between the two values $\underline{P_t}$ and $\overline{P_t}$.

The optimal solution τ_t^* corresponding to a null first derivative is given by:

$$\tau_t^*(x) = \frac{1}{(1-\gamma_P)(\mu-k)-(1-\gamma_c)(g-k)-\frac{1}{2}\sigma^2\gamma_P(1-\gamma_P)} \times$$

$$Log\left[\frac{FCF_0^{(1-\gamma_c)}}{x^{(1-\gamma_P)}} \frac{(1-\gamma_P)}{(1-\gamma_C)} \frac{e^{\left[(\mu-k)-\frac{1}{2}\sigma^2\gamma_P\right](1-\gamma_P)t}}{\left[\rho-(1-\gamma_P)(\mu-k)+\frac{1}{2}\sigma^2\gamma_P(1-\gamma_P)\right]} \right]$$

And we have:

$$V(x,t) = \frac{FCF_0^{(1-\gamma_c)}}{(1-\gamma_c)} e^{\rho t} \left[\frac{e^{-[\rho-(1-\gamma_c)(g-k)]t} - e^{-[\rho-(1-\gamma_c)(g-k)]\tau_t^*}}{[\rho-(1-\gamma_c)(g-k)]} \right] + \frac{x^{(1-\gamma_P)}}{(1-\gamma_P)} e^{\left[-\rho+\left(\mu-k-\frac{1}{2}\gamma_P\sigma^2\right)(1-\gamma_P)\right](\tau_t^*-t)}$$

This is a power function with respect to $\dfrac{x^{(1-\gamma_P)}}{(1-\gamma_P)}$.

Case 3. The asset value is higher than $\overline{P_t}$.

Then, the optimal time τ_t^* is equal to the present time t itself. In that case, we get:

$$V(x,t) = U(x) = \frac{x^{(1-\gamma_P)}}{(1-\gamma_P)}$$

Consequently, the American optimal time is determined by:

$$T^{***} = \inf\left\{ t \in [0,\overline{T}] \,\middle|\, V(P_t,t) = \frac{P_t^{(1-\gamma_P)}}{(1-\gamma_P)} \right\}.$$

Therefore, we can check that $V(P_t,t) = P_t$ if and only if $P_t \geq \underline{P_t}$. Thus, we have:

$$T^{***} = \inf \left\{ t \in [0,\overline{T}] \,\middle|\, P_t \geq \left(\frac{FCF_0^{(1-\gamma_c)}}{\left[\rho - (1-\gamma_P)(\mu-k) + \frac{1}{2}\sigma^2 \gamma_P (1-\gamma_P) \right] (1-\gamma_C)} \frac{(1-\gamma_P)}{(1-\gamma_C)} \right)^{\frac{1}{1-\gamma_P}} e^{-(k-g)\frac{(1-\gamma_C)}{1-\gamma_P} t} \right\}.$$

For $\rho = \gamma_C = \gamma_P = 0$ (linear case), we recover the lower bound $\underline{P_t} = \frac{FCF_0}{(k-\mu)} e^{-(k-g)t}$.

In what follows, we denote by A the term

$$\left(\frac{FCF_0^{(1-\gamma_c)}}{\left[\rho - (1-\gamma_P)(\mu-k) + \frac{1}{2}\sigma^2 \gamma_P (1-\gamma_P) \right] (1-\gamma_C)} \frac{(1-\gamma_P)}{(1-\gamma_C)} \right)^{\frac{1}{1-\gamma_P}}$$

Then, we have:

$$T^{***} = \inf \left\{ t \in [0,\overline{T}] \,\middle|\, P_t \geq A e^{-(k-g)\frac{(1-\gamma_C)}{(1-\gamma_P)} t} \right\}$$

Using standard results about the first time $T_y^{(m)}$ at which a Brownian motion with drift $W^{(m)}$ reaches a given level y, we can derive the pdf and cdf of T^{***}. Indeed, the condition $P_t \geq \frac{FCF_0}{(k-\mu)} e^{-(k-g)t}$ is equivalent to

$$P_0 \exp\left[\left(\mu - k - 1/2\sigma^2 \right) t + \sigma W_t \right] \geq A e^{-(k-g)\frac{(1-\gamma_C)}{(1-\gamma_P)} t},$$

$$\left[\mu - k + (k-g)\frac{(1-\gamma_C)}{(1-\gamma_P)} - 1/2\sigma^2 \right] t + \sigma W_t \geq \ln\left[\frac{A}{P_0} \right],$$

$$\left(\frac{(\mu-k)}{\sigma} + \frac{(k-g)}{\sigma} \frac{(1-\gamma_C)}{(1-\gamma_P)} - 1/2\sigma \right) t + W_t \geq \frac{1}{\sigma} \ln\left[\frac{A}{P_0} \right].$$

Setting $m = \left(\frac{(\mu-k)}{\sigma} + \frac{(k-g)}{\sigma} \frac{(1-\gamma_C)}{(1-\gamma_P)} - 1/2\sigma \right)$ and $y = \frac{1}{\sigma} \ln\left[\frac{A}{P_0} \right]$, the cdf of the random T_y is given by:

- For the case $\frac{A}{P_0} \leq 1$, we have:

$$T^{***} = 0.$$

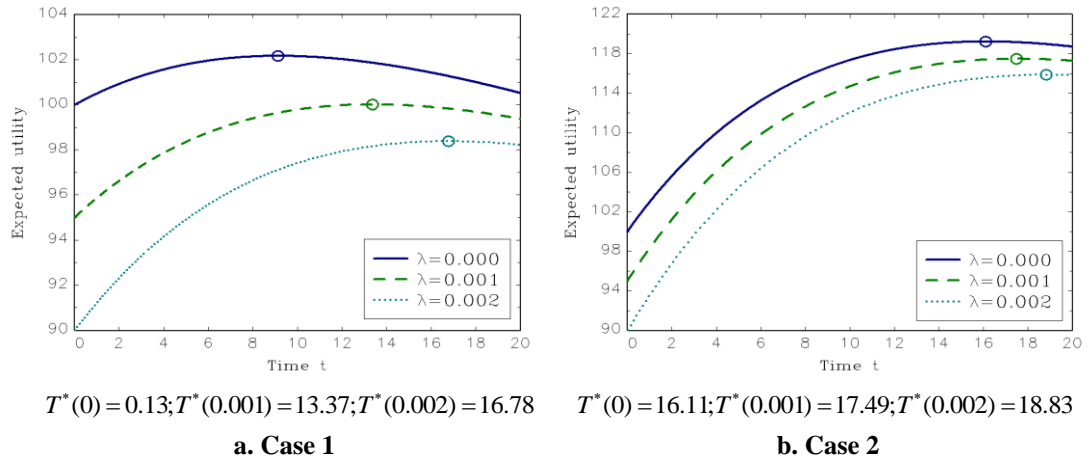- For the case $\frac{A}{P_0} > 1$, we have:

$$P[T_y \leq t] = 1 - G(m, y, t).$$

Thus, since $P[T^{***} \leq t] = P[T_y^{(m)} \leq t]$ for any $t < \overline{T}$, we deduce

$$P\left[T^{***} \leq t\right] = \frac{1}{2} Erfc\left(\frac{y}{\sqrt{2t}} - m\frac{\sqrt{t}}{\sqrt{2}}\right) + \frac{1}{2} e^{2my} Erfc\left(\frac{y}{\sqrt{2t}} + m\frac{\sqrt{t}}{\sqrt{2}}\right),$$

and

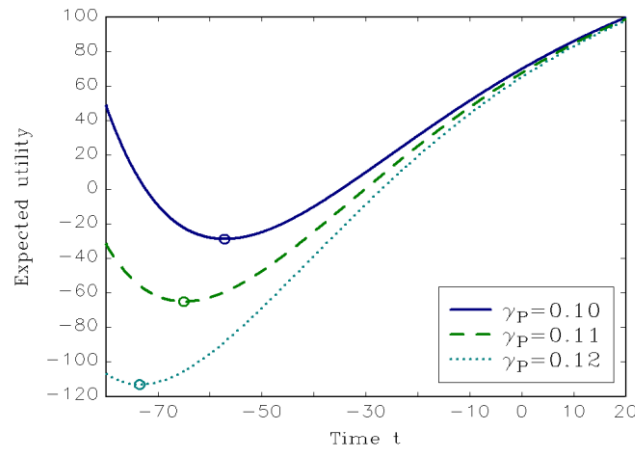$$P\left[T^{***} = \overline{T}\right] = P\left[T_y^{(m)} = \overline{T}\right] + P\left[T_y^{(m)} > \overline{T}\right] = 0 + P\left[T_y^{(m)} > \overline{T}\right] =$$

$$= 1 - \frac{1}{2} Erfc\left(\frac{y}{\sqrt{2\overline{T}}} - m\frac{\sqrt{\overline{T}}}{\sqrt{2}}\right) - \frac{1}{2} e^{2my} Erfc\left(\frac{y}{\sqrt{2\overline{T}}} + m\frac{\sqrt{\overline{T}}}{\sqrt{2}}\right).$$

$$T^*(0)=0.13; T^*(0.001)=13.37; T^*(0.002)=16.78 \qquad T^*(0)=16.11; T^*(0.001)=17.49; T^*(0.002)=18.83$$

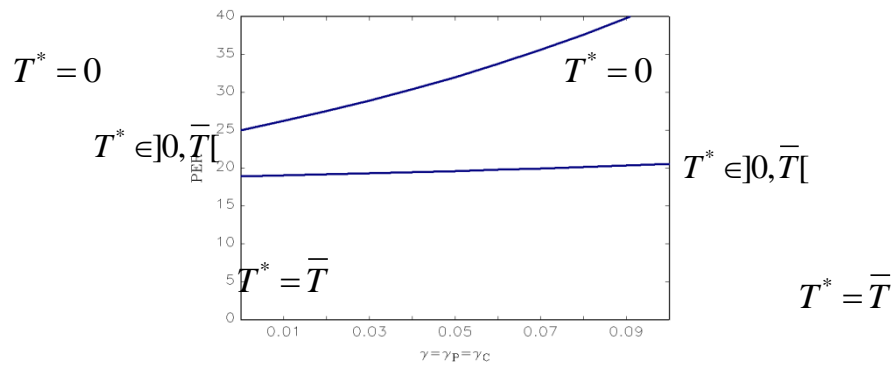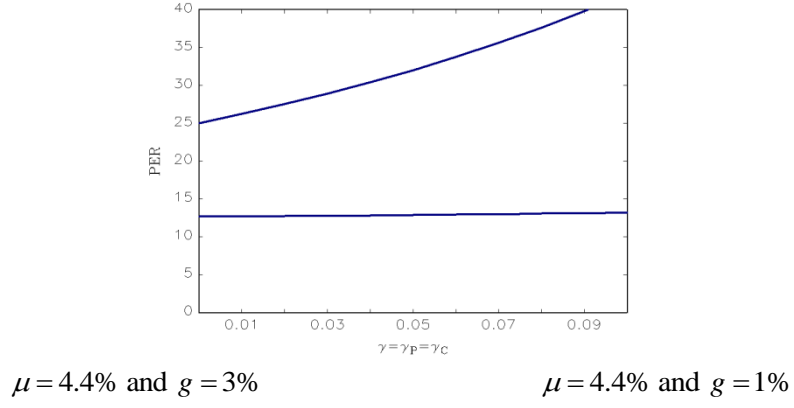**a. Case 1**   **b. Case 2**

Fig 1. Quadratic utility, optimal time T^* with respect to risk aversion λ

Fig 2. Case 1 - condition for the existence of the analytical for T^*

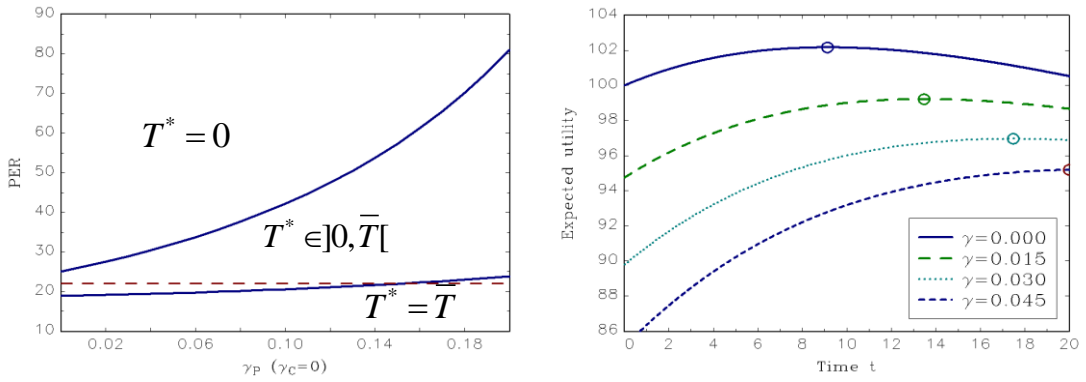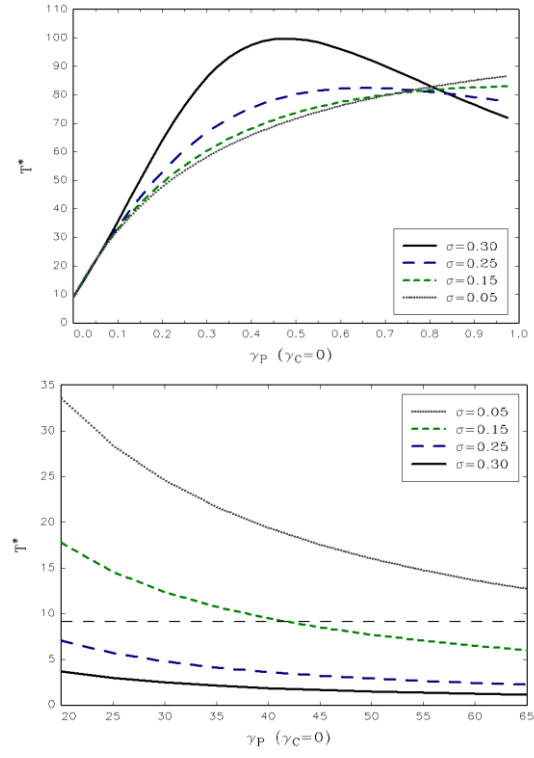$\mu = 4.4\%$ and $g = 3\%$ $\mu = 4.4\%$ and $g = 1\%$

Fig 3. PER limits for the case 1 as a function of $\mu - g$
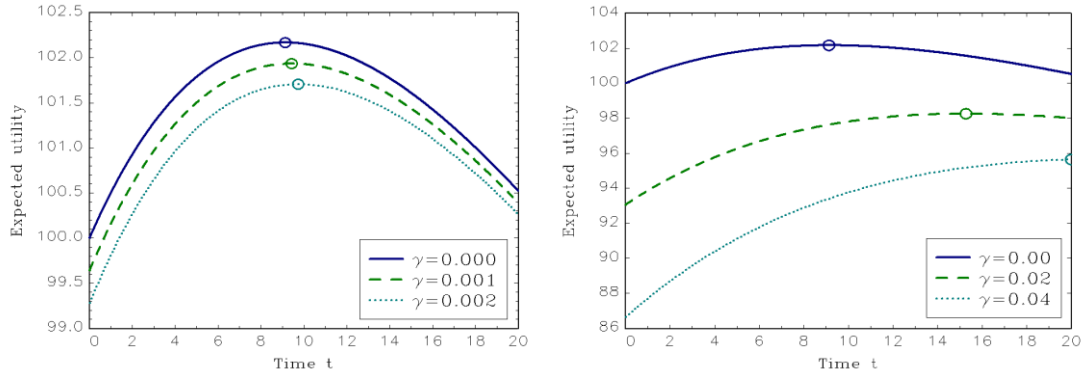
Fig 4. Expected utility as a function of the risk aversion, first numerical example
Small values of γ

Fig 5. PER limits and expected utility for the numerical example 1 - w.r.t. the risk aversion,
first numerical example $\gamma_C = 0$

Fig 6. T^* as a function of volatility, first numerical example, $\gamma_C = 0$