Collateral constraints and self-fulfilling macroeconomic fluctuations*

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Preliminary

Abstract

We show how realistic occasionally binding collateral constraints cause macroeconomic fluctuations in a representative-agent model. Collateral constraints imply that the effect of choices on the price of collateral feeds back into the set of feasible choices, thus giving rise to multiple equilibria. We characterize how the possibility of multiple equilibria depends on aggregate wealth: for low levels of wealth the economy is vulnerable to changes of consumer confidence (sunspots) which cause non-optimal fluctuations of the price of collateral and consumption. We point out the sources of equilibrium inefficiency and discuss which policies improve welfare.

Keywords: Collateral constraints, Multiple equilibria, Consumer confidence, Sunspots, Endogenous gridpoint method.
JEL: E21, D91, C63.

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1 Introduction

The median working-age household in the U.S. owns a home and borrows against its collateral value.[1] During the 2007-2009 recession the value of this housing collateral fell sharply by 29.5% (see, for example, Table 3 in Glover, Heathcote, Krueger and Ríos-Rull, 2012) and consumer spending fell by 5.4%, more than twice as much as in the average postwar recession. Across countries, consumer spending seems to have fallen more in countries with a more indebted household sector: for example, consumption fell by 7.7% in the UK compared with 2.9% in Germany (see Ohanian, 2011, OECD Factbook, 2010, and Mian and Sufi, 2011, Mian et al., 2011, for evidence across counties within the US).[2]

In this paper we show that an occasionally binding collateral constraint may cause endogenous fluctuations of consumption and the price of housing collateral. Qualitatively consistent with the facts above, these fluctuations depend on the aggregate wealth of the economy. The reason is that the collateral constraint gives rise to multiple equilibria since the effect of agents' choices on the price of housing collateral feeds back into the set of feasible choices. Equilibrium multiplicity occurs for levels of aggregate wealth at which the collateral constraint may or may not bind, depending on whether agents coordinate on an equilibrium with high or low demand. Since agents form expectations about which equilibrium will occur, changes of consumer confidence (sunspots) about the probability of a high or low demand equilibrium cause fluctuations in consumption and the price of housing collateral.

The paper makes a positive and normative contribution to the large literature on financial frictions and macroeconomic fluctuations surveyed by Quadrini (2011). We provide a parsimonious explanation for observed sizeable fluctuations in consumption and the price of housing collateral which has been difficult to achieve in previous research (see, for example, Sánchez-Marcos and Ríos-Rull, 2009). Our explanation is not based on bubbles (see Kocherlakota, 2009, and his references) which cannot arise in our model with a finite horizon.[3] Instead, it is essential for the multiplicity in our model that housing collateral has intrinsic value.

We find that perturbing equilibrium aggregate wealth in the competitive equi-

[1] See the data of the Survey of Consumer Finances (SCF) in the 2000s for households with a head between ages 24 and 65. In the SCF 2007, for example, 68% of these households own their primary residence and 57% borrow against housing collateral. For the whole sample the percentages are 69% and 49%, respectively.

[2] See also the empirical evidence in Gourinchas and Obstfeld (2012) on the positive association between credit market booms and financial crises for developed and developing countries.

[3] As in Kocherlakota (2009), one can construct multiple steady state equilibria in the infinite-horizon version of our economy, assuming parameter values for which no state-dependent equilibrium multiplicity occurs in the economy with a finite horizon.

librium of the economy with occasionally binding collateral constraints improves efficiency. This is because the competitive equilibrium is not first best due to the collateral constraint. Thus, perturbations to aggregate wealth can improve welfare as they change the price of the collateral good or change the extrinsic uncertainty due to sunspots. The inefficiency due to equilibrium multiplicity is new in our model while the "pecuniary externality," that equilibrium price changes matter for welfare, is similar to Lorenzoni (2008)'s analysis of inefficient credit booms in a model without multiple equilibria and without a collateral channel. Compared to Lorenzoni (2008), the "pecuniary externality" in our model operates through the collateral channel and not through the revenue from asset sales. In this respect, the "pecuniary externality" is closest to Bianchi (2011) who characterizes the constrained efficient allocation in a model without equilibrium multiplicity but with collateral.[4] While the collateral in Bianchi (2011) is determined differently by the stochastic endowment in tradable and non-tradable goods rather than by an asset housing, as in our model, the results for the "pecuniary externality" are similar: whether aggregate wealth is larger or smaller than required for constrained efficiency generally depends on the binding pattern of the collateral constraint, i.e. whether the constraint is currently binding or possibly binding in the future.

Concerning equilibrium multiplicity, we find that the structure of our model imposes plausible restrictions on the occurrence of multiple equilibria and sunspots: whether equilibrium multiplicity and changes in consumer confidence can cause fluctuations of consumption and house prices depends on the state variable aggregate wealth, as suggested by the empirical evidence above. This is different from He, Wright and Zhu (2012), and the examples of equilibrium multiplicity with collateral provided by Stein (1995) and Tirole (2006), ch.14. We thus need to carefully model how multiple equilibria enter in the recursive formulation of our intertemporal model.

Closely related to our research are the papers by Perri and Quadrini (2011) and Heathcote and Perri (2011). Compared with Perri and Quadrini (2011), collateral constraints in our model are not restricting the financing opportunities of firms (see also the seminal paper by Kiyotaki and Moore, 1997, or Mendoza, 2010) but the spending of consumers. More importantly, whereas the possible values of collateral in Perri and Quadrini (2011) are exogenous, in our model these values are endogenously determined.

As in Heathcote and Perri (2011) we find that extrinsic uncertainty due to equilibrium multiplicity is more likely to arise in economies with low aggregate wealth.[5] Whereas the analysis of the dynamics in Heathcote and Perri (2011)

[4] See also Jeanne and Korinek (2011).

[5] The effect of leverage on fluctuations has also been analyzed in the literature on leverage cycles. In this literature ex-ante heterogeneity of agents allows to endogenously determine the

is local around the stable steady state, as in the classic sunspot literature surveyed in Benhabib and Farmer (1999), in our model extrinsic uncertainty, due to random coordination on one of the multiple equilibria, directly enters in the recursive formulation of the maximization problem as in Cole and Kehoe (2000).

The multiplicity which occurs in our model is related to the equilibrium multiplicity in Cole and Kehoe (2000)'s model of debt roll-over with limited commitment. As in our model, the current set of feasible choices depends on agents' expectations about the equilibrium on which agents coordinate in the future: a coordination problem across periods. Since the current value of collateral determines the set of feasible current choices in our model, there is also a coordination problem within a period due to the simultaneous choices of all agents, as in models of bank runs (Diamond and Dybvig, 1983) or currency crises (Obstfeld, 1996, and references therein).[6]

An important difference to the analysis of government debt by Cole and Kehoe (2000) is that our model is concerned with private-sector debt that is secured by collateral. Borrowing opportunities are thus endogenously determined by the price of collateral in our model whereas in Cole and Kehoe (2000) they are determined by an outside option whose value crucially depends on exogenous default costs. Another difference is that agents' utility is strictly concave in consumption which is key in our model without production to obtain the plausible prediction that equilibrium multiplicity and thus the importance of changes in consumer confidence depend on aggregate wealth.

The rest of the paper is organized as follows. In Section 2 we present the model, recursive formulation and the novel solution method which handles efficiently that equilibrium multiplicity depends on the endogenous state variable aggregate wealth. In Section 3 we characterize equilibrium multiplicity before we analyze the role of confidence in Section 4 and the welfare properties of the economy in Section 5. We conclude in Section 6.

2 Model

A representative consumer makes choices in periods $T-2$, $T-1$ and final period T. The consumer derives utility $U(c_t, h_t)$ from housing h_t and consumption c_t and receives labor income y_t which may differ across periods due to a deterministic trend. The choices in each period are consumption c_t and the endogenous asset positions in the next period: housing h_{t+1} and the financial risk-free asset a_{t+1}. The financial asset earns return r, taken as given in the small-open

amount of leverage. See, for example, Geanakoplos (2009) and the references therein.

[6]For possible equilibrium multiplicity due to interest rate feedbacks on banks' loan supply see Caballero and Krishnamurthy (2001).

economy, and p_t denotes the relative price of housing. Housing may be interpreted as land which is in fixed supply and does not depreciate.[7] In Appendix A we show that we can replace the small open economy with an economy which consists of two types of agents: bankers who price the financial asset, as in Cole and Kehoe (2000), and consumers who are the "marginal" owner occupiers and thus price housing.

For clarity, only coordination on multiple equilibria is uncertain in our model and consumers have rational expectations about this extrinsic uncertainty which does not result from stochastic changes in economic fundamentals. Asset markets are incomplete so that consumers cannot insure the aggregate extrinsic uncertainty. They face a collateral constraint which allows them to borrow against the liquidation value of the housing collateral $\mu p_t h_{t+1}$, $\mu \in [0, 1]$. This constraint exists because a lender can at most seize the land –wasting fraction $1 - \mu$ in the process of appropriating it– but no other resources if the consumer defaults. As in Kiyotaki and Moore (1997) there are only one-period debt contracts since consumers can repudiate and renegotiate when portfolio choices are made in the next period. Hence, lenders have to ensure that the value of their loan never exceeds the liquidation value of the housing collateral. Note the timing assumption implicit in the constraint that lenders receive payment before the portfolio choices and coordination among consumers determine the new price p_{t+1} in the next period.[8]

2.1 Recursive consumer problem

We denote as Π_t the state variable which determines the equilibrium on which agents have coordinated in the *current* period in the presence of multiple equilibria. The belief B_t determines the probabilities which the representative agent attaches to multiple equilibria in the *next* period. We thus summarize the aggregate state variables as $s_t = (A_t, H_t, \Pi_t, B_t)$ where the aggregate housing stock

[7] Davis and Heathcote (2007) show that the price of land accounts for most of house price fluctuations at low and business cycle frequencies.

[8] The timing assumption, frequently made in the literature on collateral constraints, may be motivated with a moral-hazard problem in which the creditor can observe during the period of loan origination whether the borrower decides to cheat and default in the next period (Jeanne and Korinek, 2011). Under this "time to cheat" assumption, the creditor can appropriate the collateral good and resell it at current prices until the end of the period if the borrower cheats.

An alternative interpretation of the constraint is existing financial regulation which allows consumers to borrow against housing collateral up to a regulated loan-to-value ratio μ. Information about supervisory loan-to-value limits in the U.S. is available at http://www.fdic.gov/regulations/laws/rules/2000-8700.html (land) H_t is normalized to unity for all t. The recursive consumer problem is

$$
\begin{aligned}
& v_t\,(a_t, h_t; s_t) \\
&= \max_{a_{t+1},h_{t+1}} [U(\underbrace{(1+r)a_t + p_t h_t + y_t - a_{t+1} - p_t h_{t+1}}_{c_t}, h_t) \\
&+ \beta E_t v_{t+1}(a_{t+1}, h_{t+1}; s_{t+1})]
\end{aligned} \tag{1}
$$

subject to the constraints

$$
\begin{aligned}
a_{t+1} + p_t h_{t+1} + c_t &= (1+r)a_t + p_t h_t + y_t \\
(1+r)a_{t+1} + \mu p_t h_{t+1} &\geq 0 \\
h_{t+1} &\geq \underline{h} \\
y_t &= (1+g)^t y \\
A_{t+1} &= F_t(s_t) \\
H_{t+1} &= 1, \text{for all } t.
\end{aligned}
$$

Besides the standard budget constraint and the collateral constraint discussed previously, housing is restricted to have a minimum size $\underline{h} \geq 0$. We assume that $\underline{h} < 1 = H_t$ so that this constraint will be slack in equilibrium, and that labor income grows at a deterministic rate g. The last two constraints impose that the consumer rationally predicts the evolution of the aggregate state variables and thus the occurrence of multiple equilibria, which depend on aggregate financial assets A_{t+1} as we will see below. Because of uncertain coordination on multiple equilibria for some A_{t+1}, the expectation operator E_t enters in the recursive formulation. It is conditional on information available at time t and thus the state s_t which determines $A_{t+1} = F_t(s_t)$ so that the notation $E_t v_{t+1}(a_{t+1}, h_{t+1}; s_{t+1})$ is a convenient shorthand for $E\,[v_{t+1}(a_{t+1}, h_{t+1}; s_{t+1})|s_t]$.

For later reference, the envelope conditions of the consumer problem are

$$
\frac{\partial v_t}{\partial a_t} = (1+r)\frac{\partial U(c_t, h_t)}{\partial c_t} \tag{2}
$$

and

$$
\frac{\partial v_t}{\partial h_t} = \frac{\partial U(c_t, h_t)}{\partial h_t} + p_t \frac{\partial U(c_t, h_t)}{\partial c_t}. \tag{3}
$$

For the model solution, we will parametrize utility with the commonly used CRRA utility function

$$
U(c_t, h_t) = \frac{\psi(c_t, h_t)^{1-\sigma} - 1}{1-\sigma}
$$

with consumption basket $\psi(c_t, h_t) = c_t^{\theta} h_t^{1-\theta}$. Thus,

$$
U(c_t, h_t) = \frac{c_t^{\theta(1-\sigma)} h_t^{(1-\theta)(1-\sigma)} - 1}{1-\sigma}
$$

and

$$\frac{\partial U(c_t, h_t)}{\partial c_t} = \theta {c_t}^{\theta(1-\sigma)-1} h_t^{(1-\theta)(1-\sigma)}, \tag{4}$$

$$\frac{\partial U(c_t, h_t)}{\partial h_t} = (1-\theta)\, {c_t}^{\theta(1-\sigma)} h_t^{(1-\theta)(1-\sigma)-1}. \tag{5}$$

2.2 Equilibrium

An equilibrium consists of value functions v_t, policies a_{t+1} and h_{t+1}, price p_t and an equation of motion for aggregate financial assets A_{t+1} such that for given r in periods $t = T-2, T-1, T$:

- v_t is the value function of the representative consumer with maximizing choices a_{t+1} and h_{t+1},
- the price p_t clears the housing market,
- individual choices are consistent with aggregates: $a_{t+1} = A_{t+1}$ and $h_{t+1} = H_{t+1} = 1$,
- probabilities for equilibria in the successor period are consistent with the law of motion $A_{t+1} = F_t(s_t)$ mapping into A_{t+1} for these probabilities at which multiple equilibria exist.

The only non-standard element of this equilibrium definition is the last statement. It makes explicit that the set of probabilities attached to the multiple equilibria in the successor period is restricted by the structure of the model. The reason is that the aggregate law of motion depends on the probabilities attached to equilibria in the successor period. Rational expectations require that only those probabilities belong to the set of equilibrium probabilities for which the aggregate law of motion $A_{t+1} = F_t(s_t)$ maps into values A_{t+1} at which multiple equilibria exist. We will elaborate on this further when we discuss the role of confidence in Section 4.

2.2.1 Equilibrium conditions

Assigning the multiplier κ_t to the collateral constraint and recalling that $s_t = (A_t, H_t, \Pi_t, B_t)$ with $H_t = 1 > \underline{h}$ for $t = T-2, T-1, T$, the first-order conditions, which characterize equilibrium choices of problem (1), are

$$\frac{\partial U(c_t(s_t), 1)}{\partial c_t} = \beta(1+r)E_t\left(\frac{\partial U(c_{t+1}(s_{t+1}), 1)}{\partial c_{t+1}}\right) + \kappa_t(1+r) \tag{6}$$

and

$$\frac{\partial U(c_t(s_t),1)}{\partial c_t}p_t = \beta E_t\left(\frac{\partial U(c_{t+1}(s_{t+1}),1)}{\partial h_{t+1}} + p_{t+1}\frac{\partial U(c_{t+1}(s_{t+1}),1)}{\partial c_{t+1}}\right) + \kappa_t \mu p_t, \quad (7)$$

where we have used the envelope conditions (2) and (3).

Condition (6) is the standard Euler equation for financial assets, evaluated at the equilibrium where $a_{t+1} = A_{t+1}$ and $h_{t+1} = H_{t+1} = 1$ and augmented by the term $\kappa_t(1+r)$. This is the additional marginal gain of accumulating financial assets with a binding collateral constraint ($\kappa_t > 0$). Equilibrium condition (7) equates the marginal cost of purchasing an additional unit of housing with the marginal benefit of more utility from housing, of the resale value of the house in marginal utility terms and of the relaxation of the collateral constraint when it is binding ($\kappa_t > 0$). This makes explicit the triple role of housing as consumption good, asset and collateral.

2.3 Constructive results

For general utility functions there does not exist a closed-form solution of the model. Since the multiplicity of equilibria depends on aggregate wealth, the model is cumbersome to solve numerically (see Feng et al., 2011). For each aggregate state s_t, we need to find out how many equilibria exist in the next period given the law of motion for $A_{t+1} = F_t(s_t)$. This also requires to check for which probabilities, assigned to the multiple equilibria, the mapping $A_{t+1} = F_t(s_t)$ attains values of A_{t+1} at which the supposed number of equilibria indeed exists, a daunting task.

We develop a solution method which allows to handle this problem very efficiently. The algorithm builds on Hintermaier and Koeniger (2010) and uses the endogenous gridpoint method (EGM). By specifying an exogenous grid for the endogenous state variable in the *next* period A_{t+1}, the first-order conditions for the constrained and unconstrained equilibria, together with probabilities assigned to these equilibria, are used to determine the "endogenous" grid of the state variable in this period A_t implied by the equilibrium relationships. Compared with standard solution methods, the algorithm also avoids root-finding and forward maximization to determine the optimal state variable A_{t+1} for given current state variables.

Before we present the solution of the model in detail, it is useful to establish the following results which provide foundations for the solution algorithm.

Proposition 1 *For $\mu \in [0,1]$, a given $A_{t+1} < 0$ is attained by an unconstrained equilibrium of the economy if and only if it is attained also by a constrained equilibrium. The relative price of housing in such an unconstrained equilibrium* *is (weakly) larger than the price of housing in the constrained equilibrium. For $A_{t+1} \geq 0$, there exists an unconstrained equilibrium if and only if the price of housing is weakly positive.*

Proof. In a constrained equilibrium the economy borrows $A_{t+1} < 0$ where $a_{t+1} = A_{t+1}$ and $h_{t+1} = H_{t+1} = 1$. The collateral constraint implies that the price is given by

$$p_t = -\frac{(1+r)A_{t+1}}{\mu}.$$

Using equilibrium condition (6) to substitute out $\partial U(c_t(s_t), 1)/\partial c_t$ in condition (7), replacing p_t by the expression above and solving for κ_t, we get that the multiplier of the collateral constraint

$$\begin{aligned} \kappa_t &= \frac{\beta}{1+r-\mu} \left\{ \frac{-\mu}{(1+r)\, A_{t+1}} E_t \left(\frac{\partial U(c_{t+1}(s_{t+1}), 1)}{\partial h_{t+1}} \right) \right. \\ &\quad \left. + E_t \left(\frac{\partial U(c_{t+1}(s_{t+1}), 1)}{\partial c_{t+1}} \left(\frac{-\mu p_{t+1}(s_{t+1})}{(1+r)\, A_{t+1}} - (1+r) \right) \right) \right\} \\ &\geq 0 \end{aligned} \tag{8}$$

if and only if

$$\underbrace{\frac{E_t \left(\frac{\partial U(c_{t+1}(s_{t+1}),1)}{\partial h_{t+1}} + \frac{\partial U(c_{t+1}(s_{t+1}),1)}{\partial c_{t+1}} p_{t+1}(s_{t+1}) \right)}{(1+r) E_t \left(\frac{\partial U(c_{t+1}(s_{t+1}),1)}{\partial c_{t+1}} \right)}}_{\text{price in unconstrained equilibrium}} \geq \underbrace{-\frac{(1+r)A_{t+1}}{\mu}}_{\text{price in constrained equilibrium}}. \tag{9}$$

Note that the price in the constrained equilibrium is positive for $A_{t+1} < 0$ and recall that aggregate state variables are summarized by $s_t = (A_t, H_t, \Pi_t, B_t)$ with $H_t = 1$. Since $1 + r > \mu$ for $r > 0$, inequality (9) follows directly from rearranging (8), recalling that $A_{t+1} < 0$. To see that the left-hand side of inequality (9) equals the price in the unconstrained equilibrium, use equilibrium condition (6) to substitute out $\partial U(c_t(s_t), 1)/\partial c_t$ in condition (7), set $\kappa_t = 0$ and solve for p_t.

For $A_{t+1} \geq 0$, there exists an unconstrained equilibrium if and only if

$$\underbrace{\frac{E_t \left(\frac{\partial U(c_{t+1}(s_{t+1}),1)}{\partial h_{t+1}} + \frac{\partial U(c_{t+1}(s_{t+1}),1)}{\partial c_{t+1}} p_{t+1}(s_{t+1}) \right)}{(1+r) E_t \left(\frac{\partial U(c_{t+1}(s_{t+1}),1)}{\partial c_{t+1}} \right)}}_{\text{price in unconstrained equilibrium}} \geq 0. \tag{10}$$

This follows from (weakly) positive marginal utility and is straightforward to show by backward induction. Note that the price p_t depends on p_{t+1} and $p_{T+1} =$

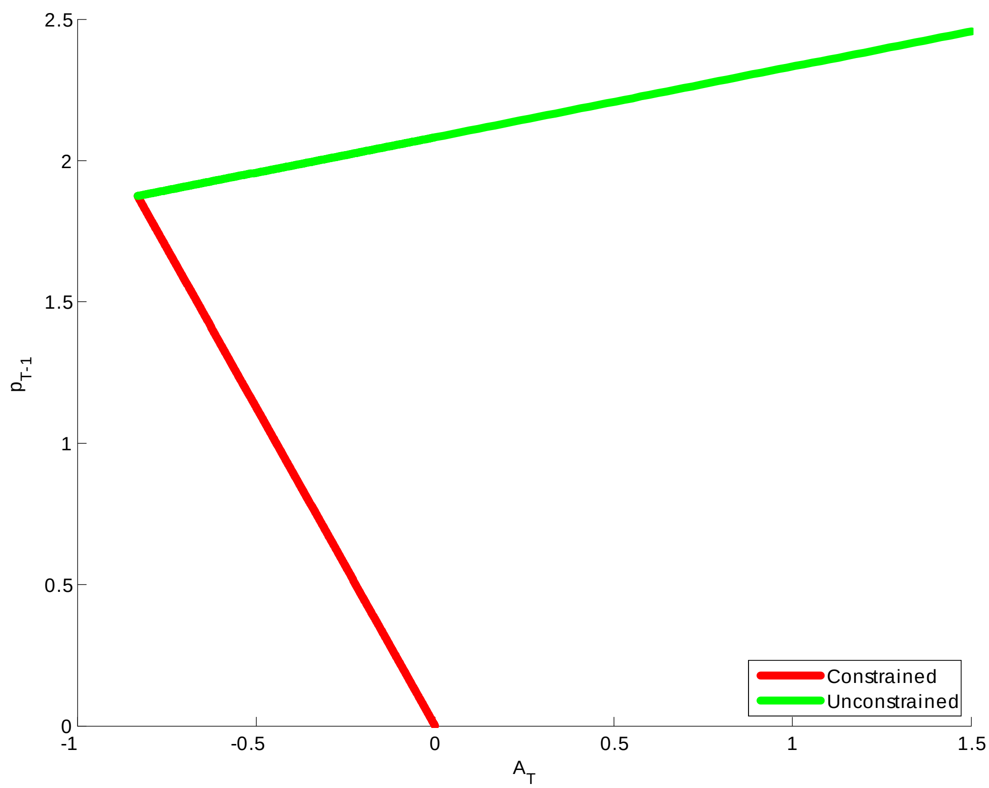

Figure 1: The relative price of housing p_{T-1} and financial wealth A_T in the constrained and unconstrained equilibrium. Notes: The unit is annual labor income. The parameter values are specified in Table 1, Section 3.

0 in the last period T since the representative consumer has no demand for housing after the terminal period.

■

Figure 1 illustrates Proposition 1 for period $T-1$. Every $A_T < 0$, which is attained in equilibrium, is attained by both unconstrained and constrained equilibria. This is important for the numerical solution of the problem in general, since we have to map back from any $A_{t+1} < 0$ into A_t using the equilibrium conditions for *both* the unconstrained and constrained equilibrium. For $A_{t+1} \geq 0$ instead we only need to use the equilibrium conditions for the unconstrained equilibrium. Equally important is the part of Proposition 1 which shows that the restriction $\kappa_t \geq 0$ is equivalent to the price being (weakly) larger in the unconstrained equilibrium than in the constrained equilibrium. This allows us to compute straightforwardly the lower bound for endogenous financial wealth A_{t+1}. In Figure 1 the lower bound is determined by the intersection of the unconstrained and constrained price schedule, which we formally state in the following corollary.

Corollary 1 *Aggregate financial wealth in the next period, which is attained in equilibrium, is bounded below by* $\underline{A}_{t+1}$. *The lower bound is implicitly determined by*

$$\frac{E_t\left(\frac{\partial U(c_{t+1}(\underline{A}_{t+1},1,\Pi_{t+1},B_{t+1}),1)}{\partial h_{t+1}} + \frac{\partial U(c_{t+1}(\underline{A}_{t+1},1,\Pi_{t+1},B_{t+1}),1)}{\partial c_{t+1}} p_{t+1}(s_{t+1})\right)}{(1+r)E_t\left(\frac{\partial U(c_{t+1}(\underline{A}_{t+1},1,\Pi_{t+1},B_{t+1}),1)}{\partial c_{t+1}}\right)} = -\frac{(1+r)\underline{A}_{t+1}}{\mu}$$

if $\underline{A}_{t+1} < 0$ *and*

$$\frac{E_t\left(\frac{\partial U(c_{t+1}(\underline{A}_{t+1},1,\Pi_{t+1},B_{t+1}),1)}{\partial h_{t+1}} + \frac{\partial U(c_{t+1}(\underline{A}_{t+1},1,\Pi_{t+1},B_{t+1}),1)}{\partial c_{t+1}} p_{t+1}(s_{t+1})\right)}{(1+r)E_t\left(\frac{\partial U(c_{t+1}(\underline{A}_{t+1},1,\Pi_{t+1},B_{t+1}),1)}{\partial c_{t+1}}\right)} = 0$$

if $\underline{A}_{t+1} \geq 0$.

Proof. This follows from Proposition 1. The lower bound $\underline{A}_{t+1}$ of aggregate wealth is given by the value of aggregate financial wealth in the next period at which (9) and (10) hold as equalities. ■

We can also provide a lower bound for financial wealth in the *current* period.

Lemma 1 *A constrained equilibrium with non-negative consumption exists for* $A_t \geq \underline{A}_t$. *The lower bound* $\underline{A}_t = -y_t/(1+r)$ *if at this lower bound there is no demand for housing so that* $p_t = 0$.

Proof. The lower bound is given by the smallest amount of financial assets A_t at which a constrained equilibrium with non-negative consumption exists. If agents at the bound have zero consumption and no demand for housing for given labor income y_t, $p_t = 0$ and the collateral constraint implies $A_{t+1} = -\mu p_t/(1+r) = 0$. Using $A_{t+1} = 0$, $H_{t+1} = H_t = 1$ and $c_t = 0$ in the aggregate resource constraint $A_{t+1} + p_t + c_t = (1+r)A_t + p_t + y_t$ then implies $\underline{A}_t = -y_t/(1+r)$. ■

2.4 Solution method

With these constructive results, we are able to present our method which solves the model with equilibrium multiplicity very efficiently. We start with an exogenous grid for the future endogenous state variable A_{t+1}, $G_{A_{t+1}} \equiv \{A_{t+1,1}, A_{t+1,2}, ..., A_{t+1,I}\}$. Given Proposition 1 and Corollary 1, we then use the equations that determine an unconstrained equilibrium to find the values $A_{t,i}$ which correspond to each of the gridpoints $A_{t+1,i} \geq \underline{A}_{t+1}$. Similarly, we use the equations that determine a constrained equilibrium to find the values $A_{t,i}$ which correspond to each of the gridpoints $0 > A_{t+1,i} \geq \underline{A}_{t+1}$. Clearly the bounds of the grid $A_{t+1,1}$ and $A_{t+1,I}$ need to be specified so that $A_{t+1,1} \leq \underline{A}_{t+1}$ and $A_{t+1,I} > 0$.

2.4.1 Unconstrained equilibrium

Since $\kappa_t = 0$ in the unconstrained equilibrium, the first-order equilibrium conditions (6) and (7) imply that the price in an unconstrained equilibrium is given by

$$\begin{aligned} p_{t,i} &= \frac{E_{t,i}\left(\frac{\partial U(c_{t+1}(s_{t+1,i}),1)}{\partial h_{t+1}} + \frac{\partial U(c_{t+1}(s_{t+1,i}),1)}{\partial c_{t+1}} p_{t+1}(s_{t+1,i})\right)}{(1+r)E_{t,i}\left(\frac{\partial U(c_{t+1}(s_{t+1,i}),1)}{\partial c_{t+1}}\right)} \\ &= \frac{E_{t,i}\left[(1-\theta)\, c_{t+1}(s_{t+1,i})^{\theta(1-\sigma)} + \theta c_{t+1}(s_{t+1,i})^{\theta(1-\sigma)-1} p_{t+1}(s_{t+1,i})\right]}{(1+r)E_{t,i}\left[\theta c_{t+1}(s_{t+1,i})^{\theta(1-\sigma)-1}\right]}, \end{aligned} \tag{11}$$

where for the second equality we substitute in the derivatives (4) and (5) for the CRRA utility function, evaluated at $h_{t+1} = H_{t+1} = 1$. The notation $s_{t+1,i} = (A_{t+1,i}, 1, \Pi_{t+1}, B_{t+1})$ makes explicit that the state variables depend on the gridpoint $A_{t+1,i}$. Importantly, the set of equilibrium beliefs B_t may change across $A_{t+1,i}$ so that the expectation operator $E_{t,i}$ is also indexed by the gridpoint $A_{t+1,i}$.

Noting that the first-order equilibrium condition for financial assets (6) is easily inverted for CRRA utility, we determine consumption in the unconstrained equilibrium by

$$c_{t,i} = \left\{\beta\,(1+r)\,E_{t,i}\left[c_{t+1}(s_{t+1,i})^{\theta(1-\sigma)-1}\right]\right\}^{\frac{1}{\theta(1-\sigma)-1}} \tag{12}$$

and use the equilibrium law of motion to compute the "endogenous" grid of current financial assets as

$$A_{t,i} = \frac{A_{t+1,i} + c_{t,i} - y_t}{1+r}. \tag{13}$$

2.4.2 Constrained equilibrium

In the constrained equilibrium, the price is given by

$$p_{t,i} = -\frac{1+r}{\mu} A_{t+1,i}, \text{ for } A_{t+1,i} \in [\underline{A}_{t+1}; 0],\ \underline{A}_{t+1} < 0. \tag{14}$$

The multiplier of the collateral constraint equals the expression in (8), which for CRRA utility is

$$\kappa_{t,i} = \frac{\beta}{1+r-\mu} \left\{ \frac{1}{p_{t,i}} E_{t,i} \left[(1-\theta) c_{t+1}(s_{t+1,i})^{\theta(1-\sigma)} \right] \right. \tag{15}$$
$$\left. + E_{t,i} \left[\theta c_{t+1}(s_{t+1,i})^{\theta(1-\sigma)-1} \left(\frac{p_{t+1}(s_{t+1,i})}{p_{t,i}} - (1+r) \right) \right] \right\}.$$

Inverting the equilibrium condition for financial assets for CRRA utility, we get

$$c_{t,i} = \left\{ \frac{1+r}{\theta} \left(\beta E_{t,i} \left[\theta c_{t+1}(s_{t+1,i})^{\theta(1-\sigma)-1} \right] + \kappa_{t,i} \right) \right\}^{\frac{1}{\theta(1-\sigma)-1}}. \tag{16}$$

The equilibrium law of motion (13) then determines the "endogenous" grid of the current state variable $A_{t,i}$ for the constrained equilibria.

For gridpoints $A_{t+1,i}$, $i = 1, ..., I$, we have thus solved for $p_{t,i}$, $c_{t,i}$, and $A_{t,i}$, using the conditions for unconstrained and constrained equilibria. We then use these solutions to construct equilibrium policies $A_{t+1}(s_t)$, $c_t(s_t)$ and the price $p_t(s_t)$, interpolating them for exogenous gridpoints of A_t which in general differ from $A_{t,i}$ and include the lower bound $\underline{A}_t$ specified in Lemma 1.

The efficiency gains of the solution method, which conditions on future and not on current financial wealth, are at least threefold: (i) the existence of unconstrained and constrained equilibria is easily characterized in terms of future financial wealth A_{t+1} which allows to handle multiplicity very efficiently computing the unconstrained and constrained equilibria separately; (ii) the binding patterns of the collateral constraint are known in terms of future financial wealth A_{t+1} so that the occasionally binding constraint is dealt with efficiently as in Hintermaier and Koeniger (2010); (iii) the closed form solutions of the equilibrium policies and price for each gridpoint $A_{t+1,i}$ avoid time-consuming root-finding procedures common in standard numerical solution methods based on forward maximization (Carroll, 2006).

We now apply the method to solve the model for periods $T-1$ and $T-2$. Since decisions in the terminal period T are trivial, multiplicity can arise first in period $T-1$ due to coordination failure resulting from simultaneous choices within a period. In the next Section 3 we characterize the multiplicity resulting from this coordination problem in detail, providing also analytic examples (see in particular appendices B and C). In the following Section 4 we show how beliefs in period $T-2$ about coordination on one of the multiple equilibria in period $T-1$ affect the equilibrium in period $T-2$, a coordination problem across periods.

3 Equilibrium multiplicity

We solve the model for three periods $T-2$, $T-1$, and T. The decisions in the terminal period are trivial. Since all is consumed in the last period, $p_T = 0$, $a_{T+1} = 0$ and $h_{T+1} = 0$, so that the economy's equilibrium law of motion implies

$$c_T = (1+r)A_T + y_T. \tag{17}$$

The first interesting period is thus period $T-1$. We now show that there exist multiple equilibria in this period for plausible values of the parameters and state variable A_{T-1}. Table 1 displays the parameter values for which we present the solution. We set the length of a period to 15 years so that the specific decisions of the terminal period receive less weight when the agents make decisions in the previous periods. We approximate the fraction of collateral value that is wasted, when the lender appropriates it, with 20%. This corresponds to a maximum loan-to-value ratio $\mu = 0.8$ which is in line with supervisory loan-to-value limits for land and real estate in the U.S. (see http://www.fdic.gov/regulations/laws/rules/2000-8700.html). As benchmark we keep the housing stock and labor income constant and show in Section 4 how expected income growth affects the solution. The preference parameters and interest rate are within the range of commonly used values. A small intertemporal elasticity of substitution is needed for changes in consumer confidence to matter in period $T-2$, which we discuss further in Section 4.

Figure 2 shows the solution for aggregate financial wealth in the interval $[-1.5; -0.5]$. If the representative agent who prices housing has financial debt between 108% and 121% of annual labor income, there exist multiple equilibria: two constrained equilibria (on the red part of the equilibrium locus) and one unconstrained equilibrium (on the green part of the equilibrium locus).[9] In the

[9]The equity carried into period T by the representative agent who prices housing is between 7.6% and 54% of annual earnings in the interval of financial wealth in which multiple equilibria occur.

	Parameters	
Discount factor (per annum)	β	0.95
Weight of c in consumption basket	θ	0.8
Intertemporal elasticity of substitution	$1/\sigma$	$1/20$
Interest rate (per annum)	r	0.04
Loan-to-value ratio	μ	0.80
Growth rate (per annum)	g	0
Labor income	y	1
Housing stock	H	1

Table 1: Parameter values for the model solution.

high-demand unconstrained equilibrium, the representative agent consumes more which is financed with more borrowing.[10] This is feasible since the higher demand drives up the relative price of collateral which in turn relaxes the collateral constraint. In the low-demand constrained equilibria the relative price of collateral is lower, tightening the collateral constraint and thus restricting demand to its lower level. Coordination failure due the simultaneous choices of agents within a period thus may generate fluctuations in consumption and the relative price of housing. These fluctuations do not result from changes in economic fundamentals and are sizeable: the changes across the different equilibria are between 2.5% and 4.8% in terms of consumption and between 51.6% and 79.7% in terms of the relative price of housing. Since the constrained equilibria are dominated by the unconstrained equilibrium in terms of welfare, there is scope for policy intervention which we analyze further in Section 5.

Equilibrium multiplicity occurs only for certain levels of financial debt because the utility function is concave in consumption c_t. We show in Appendix B that multiplicity still arises for the utility function

$$U(c_t, h_t) = -\frac{a}{2}(\bar{c} - c_t)^2 + h_t,$$

which is linear in housing and quadratic in consumption. As becomes explicit in Appendix B, concavity of the utility in consumption is important to generate

[10] Since there is no uncertainty in the terminal period T, we can solve explicitly for the lower bound of financial wealth $\underline{A}_T = -\alpha y_T$ with $\alpha \equiv \frac{\mu(1-\theta)}{(1+r)(\theta(1+r)+\mu(1-\theta))}$. Note that $0 \leq \alpha \leq 1$ for $\theta \in [0;1]$, $\mu \in [0;1]$, $r \geq 0$. From Corollary 1 it follows that the lower bound is the solution of

$$\underbrace{\frac{1-\theta}{\theta}\frac{(1+r)\underline{A}_T + y_T}{1+r}}_{\text{price in unconstr. equilibrium}} = \underbrace{-\frac{1+r}{\mu}\underline{A}_T}_{\text{price in constr. equilibrium}}.$$

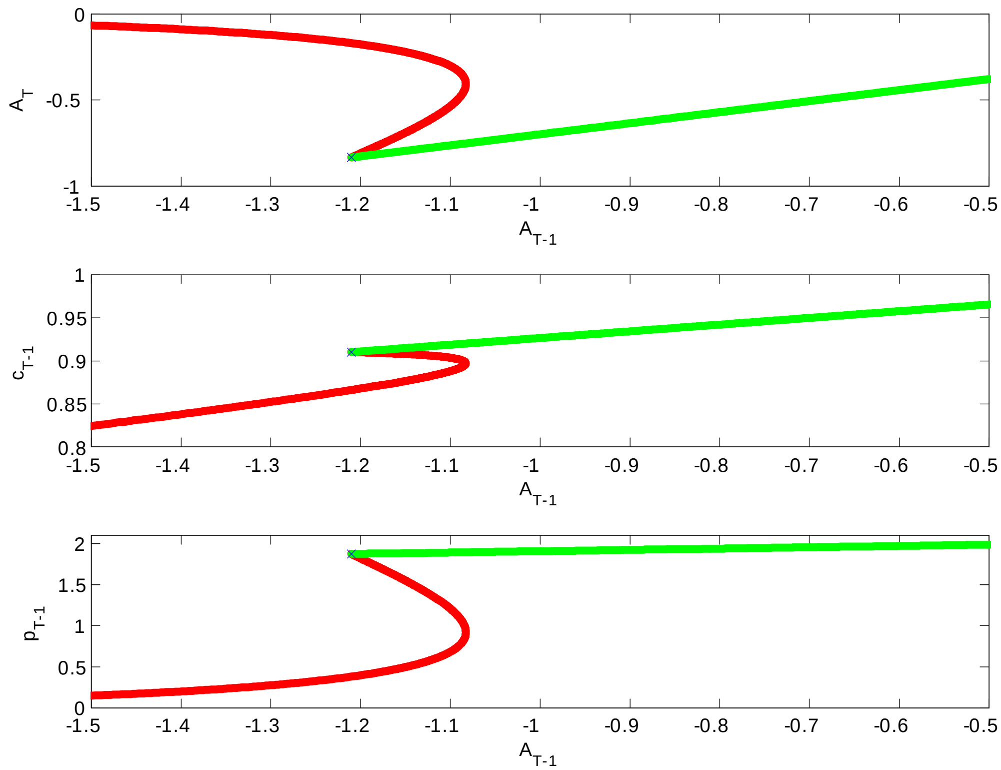

Figure 2: The model solution in period $T-1$ for aggregate financial wealth in the interval $[-1.5; -0.5]$. Notes: Constrained equilibrium in red color; unconstrained equilibrium in green color. The unit is annual labor income. The parameter values are specified in Table 1.

multiplicity in period $T-1$ that depends on aggregate wealth, non-linearity of the *marginal* utility is not.

In order to provide further intuition for the state-dependent equilibrium multiplicity, suppose that one wants to construct multiple equilibria for a given A_{T-1}. A necessary condition for equilibrium multiplicity is that there exist multiple constrained equilibria since the unconstrained equilibrium is unique (see appendices B and C). Graphically, we need for equilibrium multiplicity that the (red) locus for constrained equilibria in Figure 2 is backward bending. Equations (6) and (7) imply that, in any constrained equilibrium with $\kappa_{T-1} > 0$ and $0 \leq \mu < 1+r$, the return to housing is larger than the return to the bond

$$\frac{\frac{\partial U(c_T,1)}{\partial h_T} \Big/ \frac{\partial U(c_T,1)}{\partial c_T}}{p_{T-1}} + \frac{p_T}{p_{T-1}} > 1+r,$$

where $p_T = 0$ in the terminal period. The representative agent thus would like to invest more into housing but is prevented from doing so due to the collateral constraint. Since the collateral constraint depends on the relative price of the collateral good, multiple constrained equilibria may be supported by different prices of the collateral good. In these equilibria the implied return to housing has to be weakly larger than the return of the bond. Given that the return to housing is decreasing in the price of housing p_{T-1}, which depends on A_{T-1}, the existence of multiple constrained equilibria also depends on the state variable A_{T-1}. This aggregate state dependance of our intertemporal model with asset pricing is not present in Stein (1995). At the values of aggregate wealth for which multiple equilibria exist, however, the excess demand functions have a similar shape as in Stein (1995).[11]

The CRRA preferences in general do not allow for a closed-form solution if the economy is constrained. The unique unconstrained equilibrium instead can be characterized in closed form. For the special case of logarithmic utility, $\sigma = 1$ implies $U(c_t, h_t) = \theta \ln c_t + (1-\theta) \ln h_t$, we can characterize the behavior of the economy for both constrained and unconstrained equilibria. As shown in Appendix C, the law of motion is then quadratic in A_T for the constrained economy so that there exist two constrained equilibrium candidates for a given A_{T-1}. As illustrated in Figure 2, both solutions of the quadratic equation are indeed an equilibrium if the price in both equilibria is lower, and thus the return higher, than in the unconstrained equilibrium in which the return to housing

[11] Indeed, as in the classic examples of equilibrium multiplicity in microeconomics, the number of equilibria in period T-1 is odd unless excess demand does not exist for any positive relative price of the collateral good. This may occur if the demand for housing (and consumption) attains zero for a strictly positive p_{T-1} due to the collateral constraint. See, for example, the case of log utility with $\beta = 0.9993$, $r = 0.0007$ and $\mu = 0.95$ in which at most two equilibria exist in period T-1.

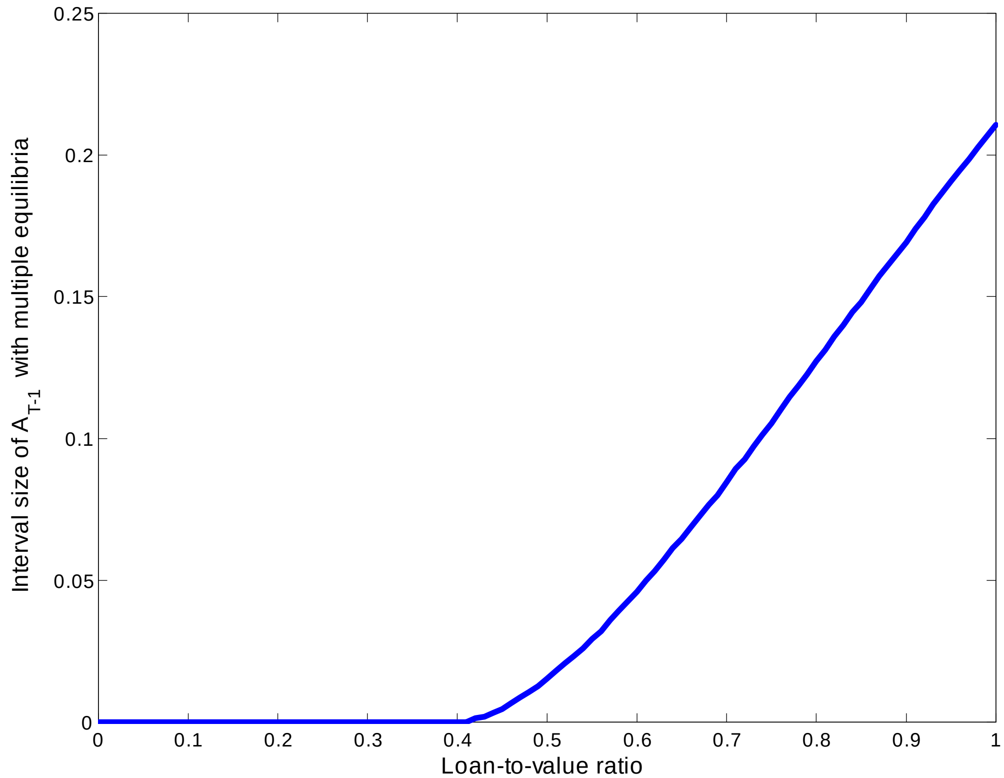

Figure 3: The loan-to-value ratio and equilibrium multiplicity in period $T-1$. Notes: The unit of A_{T-1} is annual labor income. Besides μ the parameter values are specified as in Table 1.

equals the return to financial assets. The same reasoning applies with more general preferences, $\sigma \neq 1$, for which no closed form solution is available.

Note that multiplicity can only occur for $\mu > 0$ since the price feedback from the collateral constraint is eliminated if $\mu = 0$. In the literature on collateral constraints, which abstracts from equilibrium multiplicity (see, for example, Bianchi, 2011, or Mendoza, 2010), values for μ are typically specified in the range between 0.1 and 0.4. Figure 3 shows that for plausible parameter values multiplicity arises in our model for values of μ larger than 0.4. We have argued above that a value $\mu = 0.8$ is realistic in our model, given the supervisory loan-to-value limits for land and real estate in the U.S.

Given that multiple equilibria exist for some values of financial debt, we proceed to show how this multiplicity affects the equilibrium in period $T-2$, when agents rationally form beliefs about the random coordination on these equilibria in period $T-1$. If the probability assigned to the high-demand equilibrium is higher, we call an economy more confident about its prospects.

4 The role of confidence

For a given state of the economy $s_{T-2} = (A_{T-2}, 1, \Pi_{T-2}, B_{T-2})$ agents rationally predict A_{T-1}, using the equilibrium law of motion $A_{T-1} = F_{T-2}(s_{T-2})$. If there exist multiple equilibria at A_{T-1}, beliefs on which of these equilibria the agents coordinate determine the equilibrium in period $T-2$. The beliefs are not arbitrary. They are restricted by the requirement that the equilibrium law of motion $A_{T-1} = F_{T-2}(s_{T-2})$ for a given belief indeed maps into a value of A_{T-1} at which multiple equilibria exist.

For a low enough intertemporal elasticity of substitution, there are equilibrium beliefs so that for some A_{T-2} the equilibrium law of motion maps into values of A_{T-1} where multiple equilibria exist. The intuition is that for very small A_{T-1} in the constrained economy (which may never be attained from any A_{T-2}), the representative agent with a low intertemporal elasticity of substitution is only willing to hold the housing stock $H = 1$ if the price of the collateral good is low: the agent is not willing to forego consumption in order to hold the housing stock at any higher price for which the return to housing is weakly larger than the return to financial assets. Multiple equilibria then occur at (larger) values of A_{T-1} where the agent has more resources. These A_{T-1} are attained with positive probability for some A_{T-2}.

In order to illustrate the local effect of the beliefs on the equilibrium in period $T-2$, consider a state of the economy s_{T-2} that maps into financial wealth A_{T-1} at which two equilibria exist. The equilibria are denoted by $(c_{T-1,1}, p_{T-1,1})$ and $(c_{T-1,2}, p_{T-1,2})$ where we use the notation $c_{T-1,i} \equiv c_{T-1}(A_{T-1}, 1, \Pi_{T-1,i}, B_{T-1})$ and $p_{T-1,i} \equiv p_{T-1}(A_{T-1}, 1, \Pi_{T-1,i}, B_{T-1})$, $i = 1, 2$. Note that the belief in period B_{T-1} is degenerate since there is a unique equilibrium in period T. Assigning confidence weights ω_i, $i = 1, 2$, to the two equilibria, we now characterize how the equilibrium in period $T-2$ responds locally to changes in beliefs B_{T-2} by varying the confidence weights for a given A_{T-2}, i.e., $dA_{T-2} = 0$. Since $\omega_1 + \omega_2 = 1$, $d\omega_1 = -d\omega_2$. Without loss of generality, we let ω_1 be the probability assigned to the high-demand equilibrium, which we call confidence.

4.1 Response of the unconstrained equilibrium to changes in confidence

The unconstrained equilibrium in $T-2$ is characterized by the three equations (11), (12) and (13) which, for the example of two possible equilibria in period

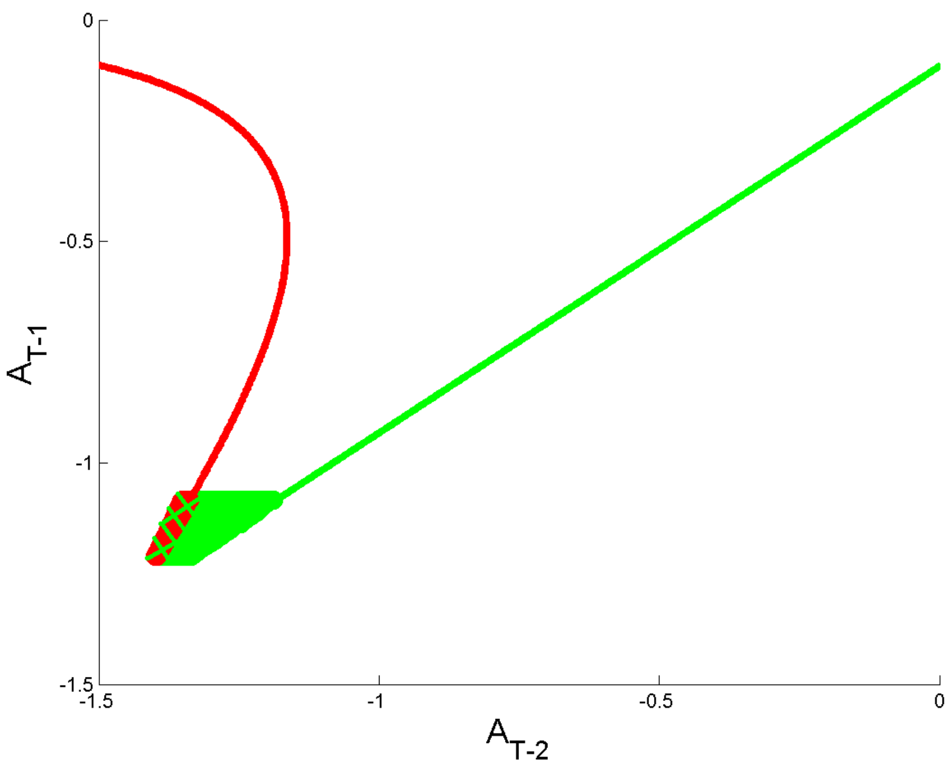

Figure 4: The equilibrium law of motion in period $T-2$. Notes: Constrained equilibrium in red color; unconstrained equilibrium in green color. In the hatched area both unconstrained and constrained equilibria exist. The unit is annual labor income. The parameter values are specified in Table 1, Section 3.

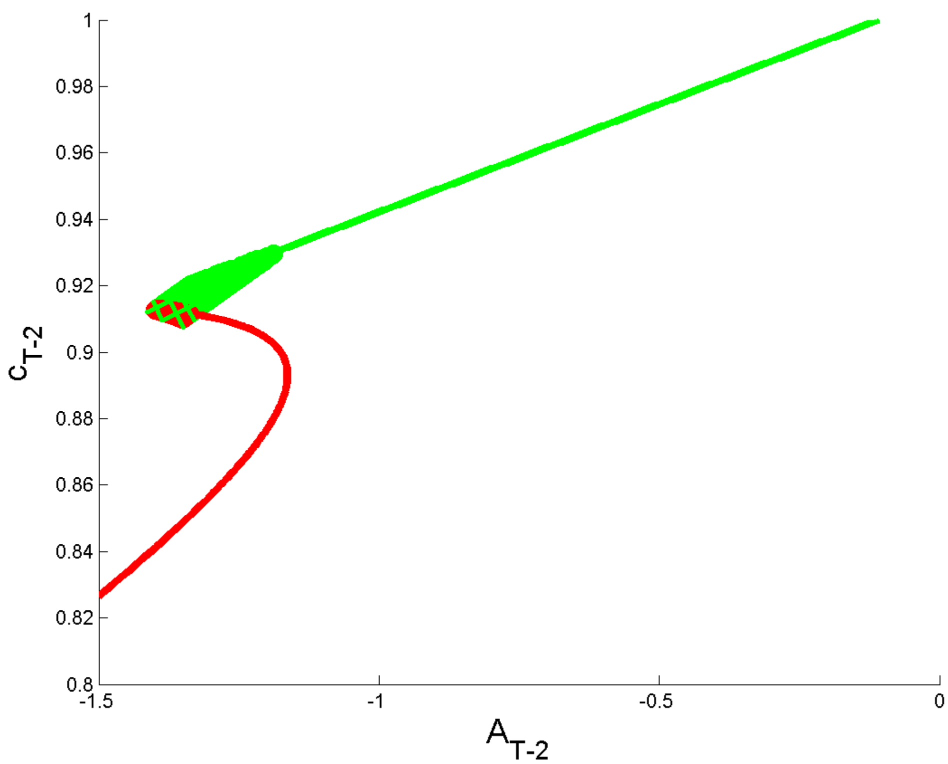

Figure 5: Equilibrium consumption in period $T-2$. Notes: Constrained equilibrium in red color; unconstrained equilibrium in green color. In the hatched area both unconstrained and constrained equilibria exist. The unit is annual labor income. The parameter values are specified in Table 1, Section 3.

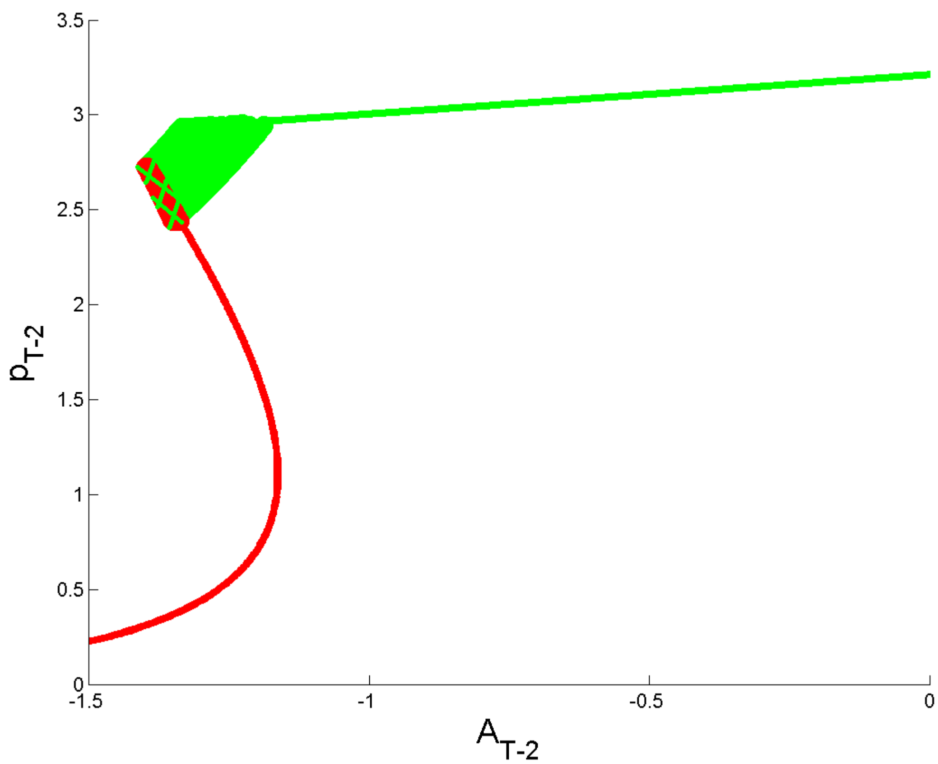

Figure 6: The equilibrium price of housing in period $T-2$. Notes: Constrained equilibrium in red color; unconstrained equilibrium in green color. In the hatched area both unconstrained and constrained equilibria exist. The unit is annual labor income. The parameter values are specified in Table 1, Section 3.

$T-1$, can be written as

$$\begin{aligned}
p_{T-2} &= \Big\{(1-\theta)\left[\omega_1 c_{T-1,1}^{\theta(1-\sigma)} + \omega_2 c_{T-1,2}^{\theta(1-\sigma)}\right] \\
&\quad +\theta\left[\omega_1 c_{T-1,1}^{\theta(1-\sigma)-1} p_{T-1,1} + \omega_2 c_{T-1,2}^{\theta(1-\sigma)-1} p_{T-1,2}\right]\Big\} / \\
&\quad \Big\{(1+r)\left[\omega_1 c_{T-1,1}^{\theta(1-\sigma)-1} + \omega_2 c_{T-1,2}^{\theta(1-\sigma)-1}\right]\Big\}, \\
c_{T-2} &= \Big\{\beta\,(1+r)\left[\omega_1 c_{T-1,1}^{\theta(1-\sigma)-1} + \omega_2 c_{T-1,2}^{\theta(1-\sigma)-1}\right]\Big\}^{\frac{1}{\theta(1-\sigma)-1}}, \\
A_{T-1} &= (1+r)A_{T-2} + y_{T-2} - c_{T-2}.
\end{aligned}$$

Totally differentiating, recalling $dA_{T-2}=0$ (and also $dy_{T-2}=0$), we get

$$\begin{aligned}
dp_{T-2} &= \alpha_1 d\omega_1 + \alpha_2 d\omega_2 + \alpha_3 dA_{T-1}, \\
dc_{T-2} &= \gamma_1 d\omega_1 + \gamma_2 d\omega_2 + \gamma_3 dA_{T-1}, \\
dA_{T-1} &= -dc_{T-2},
\end{aligned}$$

where the coefficients α_j and γ_j, $j=1,2,3$, contain the respective derivatives. For example α_3 contains derivatives of $c_{T-1,i}$ and $p_{T-1,i}$, $i=1,2$, with respect to A_{T-1}. Substituting dA_{T-1} in the second equation and rearranging,

$$dc_{T-2} = \frac{\gamma_1}{1+\gamma_3} d\omega_1 + \frac{\gamma_2}{1+\gamma_3} d\omega_2.$$

Since $d\omega_1 = -d\omega_2$, the response of consumption to an increase in confidence ω_1 is

$$dc_{T-2} = \frac{\gamma_1-\gamma_2}{1+\gamma_3} d\omega_1$$

which is the opposite of the response of financial wealth:

$$dA_{T-1} = \frac{\gamma_2-\gamma_1}{1+\gamma_3} d\omega_1.$$

The price response to an increase in confidence instead is given by

$$\begin{aligned}
dp_{T-2} &= \left(\alpha_1 - \frac{\alpha_3\gamma_1}{1+\gamma_3}\right) d\omega_1 + \left(\alpha_2 - \frac{\alpha_3\gamma_2}{1+\gamma_3}\right) d\omega_2 \\
&= \left(\alpha_1 - \alpha_2 - \frac{\alpha_3(\gamma_1-\gamma_2)}{1+\gamma_3}\right) d\omega_1.
\end{aligned}$$

4.2 Response of the constrained equilibrium to changes in confidence

The constrained equilibrium in $T-2$ is characterized by the equations (13), (14), (15) and (16). For the example of two possible equilibria in period $T-1$ we have

$$
\begin{aligned}
p_{T-2} &= -\frac{1+r}{\mu} A_{T-1}, \\
c_{T-2} &= \left\{ \beta\left(1+r\right)\left[\omega_1 c_{T-1,1}^{\theta(1-\sigma)-1} + \omega_2 c_{T-1,2}^{\theta(1-\sigma)-1}\right] + \frac{1+r}{\theta}\kappa_{T-2}(s_{T-1}; p_{T-2}, \omega_1, \omega_2)\right\}^{\frac{1}{\theta(1-\sigma)-1}}, \\
A_{T-1} &= (1+r)A_{T-2} + y_{T-2} - c_{T-2},
\end{aligned}
$$

where $\kappa_{T-2}(s_{T-1}; p_{T-2}, \omega_1, \omega_2)$ makes explicit that κ_{T-2} in (15) depends on the variables of interest.

Totally differentiating, recalling $dA_{T-2} = 0$ (and again $dy_{T-2} = 0$), we get

$$
\begin{aligned}
dp_{T-2} &= -\frac{1+r}{\mu} dA_{T-1}, \\
dc_{T-2} &= \gamma_4 d\omega_1 + \gamma_5 d\omega_2 + \gamma_6 dA_{T-1} + \gamma_7 dp_{T-2}, \\
dA_{T-1} &= -dc_{T-2},
\end{aligned}
$$

where the coefficients γ_j, $j = 4, 5, 6, 7$, also contain the respective derivatives of the multiplier κ_{T-2}. Note that the price only responds to changes in the confidence weights through changes of financial wealth $dA_{T-1} \neq 0$. Substituting dp_{T-2} and dA_{T-1} into the second equation and rearranging,

$$
dc_{T-2} = \frac{\gamma_4}{1+\gamma_6 - \frac{1+r}{\mu}\gamma_7} d\omega_1 + \frac{\gamma_5}{1+\gamma_6 - \frac{1+r}{\mu}\gamma_7} d\omega_2.
$$

Since $d\omega_1 = -d\omega_2$, the response of consumption to an increase in confidence ω_1 is

$$
dc_{T-2} = \frac{\gamma_4 - \gamma_5}{1+\gamma_6 - \frac{1+r}{\mu}\gamma_7} d\omega_1.
$$

Thus, if an increase in confidence increases consumption, the economy decreases financial wealth by the same amount both in the constrained and unconstrained economy. The relative price of housing increases proportionally with factor $(1+r)/\mu$ in the constrained economy. The price response in the unconstrained economy instead is also directly affected by the change of confidence and not only by changes in financial wealth.

Figures 4, 5 and 6 illustrate the effect of beliefs on the equilibrium in period $T-2$.[12] There is a set of equilibria at those A_{T-2} which map into A_{T-1} where multiple equilibria exist. In the hatched area both unconstrained and constrained equilibria exist. These equilibria are caused by coordination failure across periods which introduces the possibility of fluctuations in consumption and prices due to changes of confidence. Note that the figures also show the multiplicity, already present in period $T-1$, which resulted from the coordination failure within the period.

Inspecting more in detail the equilibrium sets, which are displayed in the figures, we find that a local increase in confidence increases consumption and the relative price of housing and reduces financial wealth. As the figures illustrate, changes in beliefs can change consumption by $1-2\%$ and the change of the relative price of housing is about ten times larger. These fluctuations in consumption and the price of housing are inefficient so that there is scope for welfare-improving policy intervention which we discuss further in Section 5.

4.3 Coordination through observable price and law of motion

One may wonder how agents coordinate on one of the many possible equilibria. As pointed out by Atkeson (2000), markets and prices coordinate agents by aggregating information. The observable price of the collateral good and the law of motion allow agents to perfectly coordinate for a given value of the state variable A_{T-2}. Figure 7 illustrates this point by plotting the price p_{T-2} against A_{T-2}, focussing on part of the state space in Figure 6 where beliefs affect equilibrium outcomes.[13]

Let us explain some important details that are visible in the figure. Firstly, for a given A_{T-1}, equation (14) implies that changes in beliefs do not affect p_{T-2} if the economy is constrained, but only the law of motion: different A_{T-2} map into a given A_{T-1}. The constrained equilibria attained by different beliefs, for a given A_{T-1}, are thus depicted by the horizontal red loci in Figure 7, with one locus for each A_{T-1}. Secondly, the unconstrained equilibria for different equilibrium beliefs are on the green upward sloping loci in Figure 7, again one locus for each A_{T-1}.

The main point of Figure 7 is that, for a given A_{T-1}, the unconstrained and constrained equilibria always imply a different price p_{T-2}, for any equilibrium belief. For a given A_{T-1}, the green locus intersects the red locus only exactly

[12] To produce the figures, we have drawn randomly from the simplex defined by the set of probabilities assigned to the (at most) three equilibria in period $T-1$.

[13] The points in the figure are slightly irregularly spaced for a given A_{T-1} since, as before, we have drawn randomly from the simplex defined by the set of probabilities assigned to the (at most) three equilibria in period $T-1$.

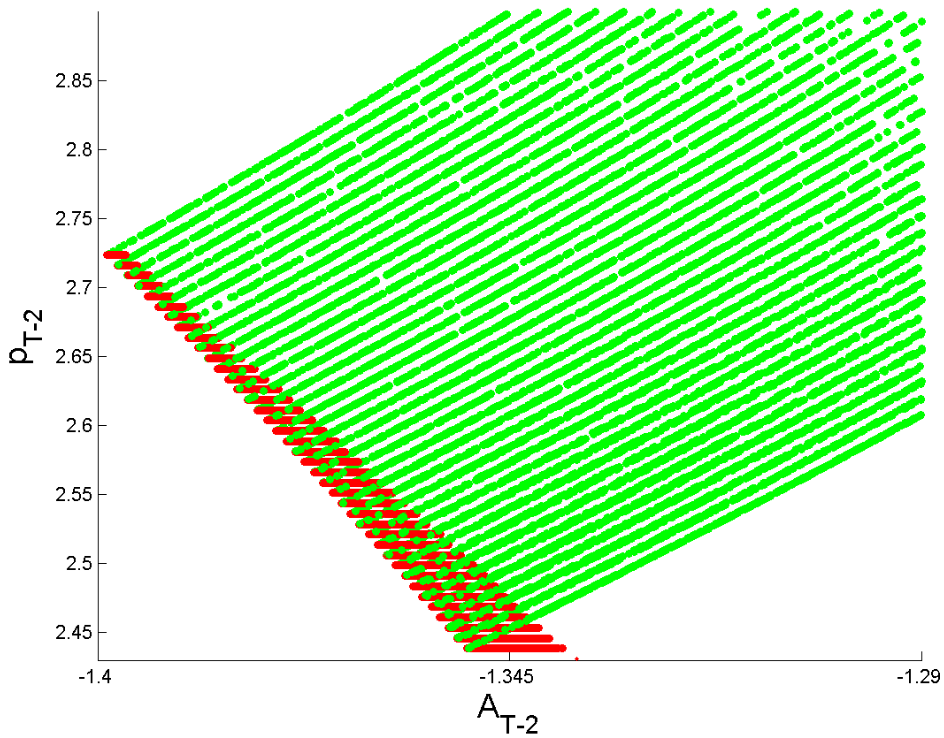

Figure 7: The equilibrium price of housing in $T-2$ in more detail. Notes: Constrained equilibrium in red color; unconstrained equilibrium in green color. The unit is annual labor income. The parameter values are specified in Table 1, Section 3.

at the boundary of the equilibrium set, where the constraint is slack so that $\kappa_{T-2} = 0$. Thus, for every state A_{T-2}, there is a unique mapping from p_{T-2} and A_{T-1}, determined by the aggregate law of motion, into the equilibrium belief. This enables agents to coordinate perfectly since they can infer exactly by which equilibrium belief the observable p_{T-2} and A_{T-1} have been generated.

4.4 Expected income growth

Before we discuss the sources of equilibrium inefficiency in our model, we perform an experiment which allows us to link our model to the recent debate on the drop and slow recovery of consumption in the last recession. DeNardi, French and Benson (2012) have argued that the drop and slow recovery of consumption may have been caused by a fall in expected income growth (see also

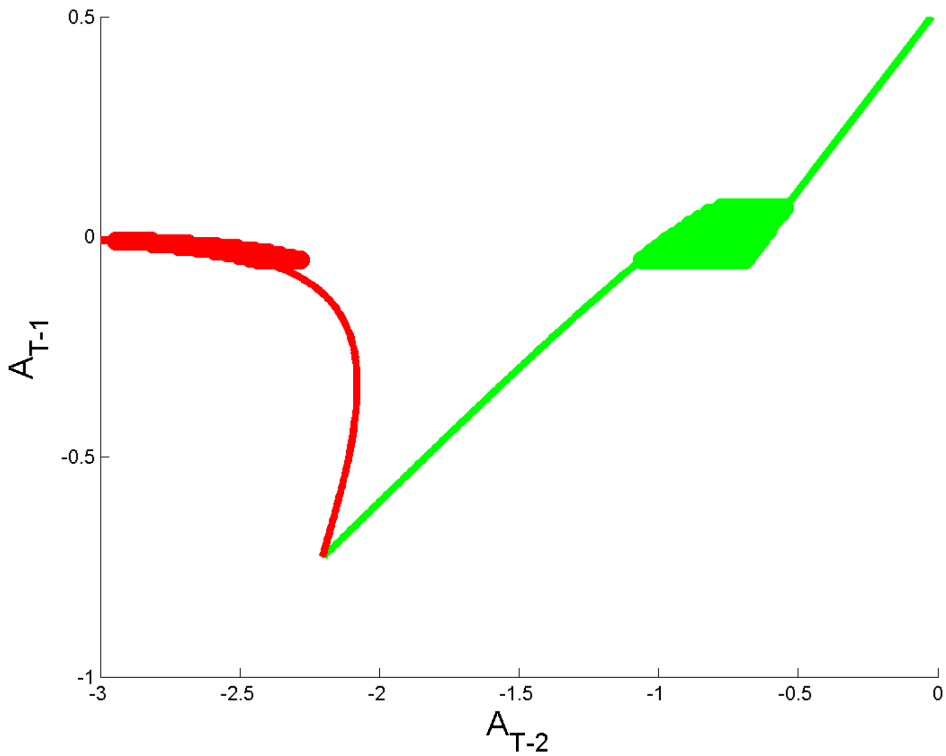

Figure 8: The equilibrium law of motion in period $T-2$ for an economy with falling income in period $T-1$. Notes: Constrained equilibrium in red color; unconstrained equilibrium in green color. The unit is annual labor income. The parameter values are as specified in Table 1, Section 3 but income falls at an annual rate of 1% in period $T-1$ and recovers to its intial level in period T.

Guerreri and Lorenzoni, 2011, for an alternative explanation based on precautionary savings). In order to gauge the effect of smaller expected income growth in our model, we assume that y_t falls by 1% per annum in period $T-1$ and recovers to its initial level in period T.

Figures 8, 9 and 10 display the solution for period $T-2$. Figure 9 shows that consumption indeed falls at most levels of financial wealth A_{T-2} since consumers anticipate the fall in income and save for a rainy day. Furthermore Figure 8 for the equilibrium law of motion illustrates that equilibrium multiplicity in period $T-1$ occurs at higher levels of financial wealth (due to the lower income in period $T-1$). This shifts the interval of financial wealth in which changes of beliefs affect the equilibrium. Changes in beliefs can cause sizeable fluctuations of consumption and the relative price of housing for levels of finan-

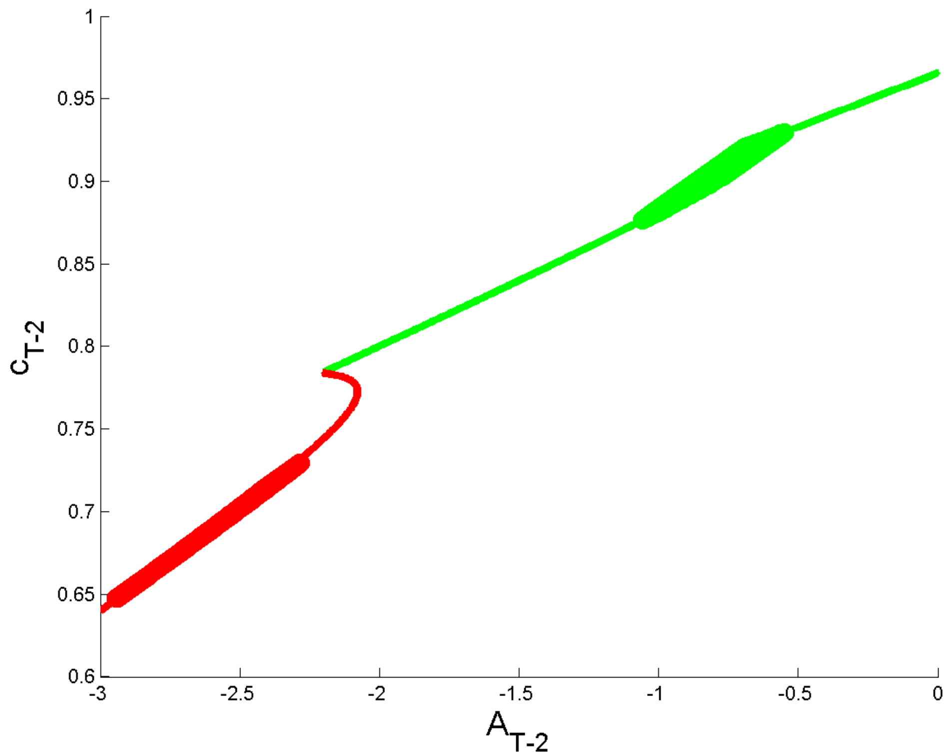

Figure 9: Consumption in period $T-2$ for an economy with falling income in period $T-1$. Notes: Constrained equilibrium in red color; unconstrained equilibrium in green color. The unit is annual labor income. The parameter values are as specified in Table 1, Section 3 but income falls at an annual rate of 1% in period $T-1$ and recovers to its intial level in period T.

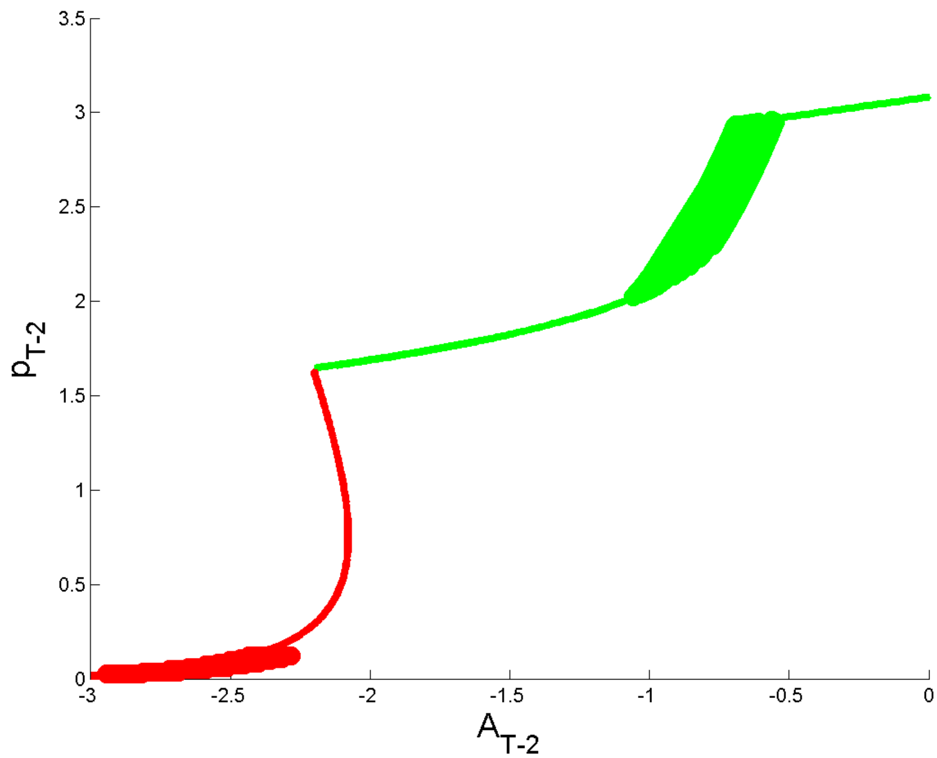

Figure 10: The relative price of housing in period $T-2$ for an economy with falling income in period $T-1$. Notes: Constrained equilibrium in red color; unconstrained equilibrium in green color. The unit is annual labor income. The parameter values are as specified in Table 1, Section 3 but income falls at an annual rate of 1% in period $T-1$ and recovers to its intial level in period T.

cial wealth at which the representative agent has equity larger than three times annual earnings. This suggests that some of the persistent drop in consumption and the price of collateral in the recent U.S. recession may be explained by more pessimistic beliefs about the equilibrium on which agents coordinate in the future, which adds to effect of anticipated lower income growth.

5 Equilibrium inefficiency

There are two sources of inefficiencies resulting from the collateral constraint in our model. Firstly, the representative agent does not take into account how the price of collateral determines the set of feasible consumption choices. Secondly, the representative agent does not internalize the effect of the accumulation decisions on aggregate financial wealth in the next period A_{t+1}. This decision determines whether coordination failure may arise due to equilibrium multiplicity.

In order to analyze equilibrium inefficiency formally and relate our model to the literature, in Appendix D we solve the problem of a social planner who internalizes the price effect (see Lorenzoni, 2008, Bianchi, 2011, and Jeanne and Korinek, 2011) to improve efficiency, maintaining the assumption of an occasionally binding collateral constraint. The prevalence of collateral constraints in real-world markets suggests that it is important to investigate how efficiency can be improved given that such a constraint is imposed on the economy.

Compared with the competitive equilibrium, the planner internalizes the effect of consumption and wealth accumulation on the price of the collateral good (see Appendix D for the derivations). If less current consumption (and more financial wealth A_{t+1}) reduces the current price of the collateral good, the collateral constraint tightens in period t. This is costly if the collateral constraint binds in the period t. The price effect lowers the marginal gain of accumulating financial assets at the collateral constraint. It implies that the social planner accumulates *less* wealth compared with agents in the competitive equilibrium.

There is an additional effect, however, if the collateral constraint binds in period $t + 1$. If less current consumption and more wealth accumulation A_{t+1} *increase* the price of the collateral good in the *next* period, this relaxes a binding collateral constraint in period $t + 1$. Ceteris paribus, this effect implies that the social planner accumulates *more* wealth than agents in the competitive equilibrium. Similar to Bianchi (2011), whether more or less wealth accumulation than in the competitive equilibrium improves efficiency depends on the binding pattern of the collateral constraint across periods.

We now discuss how coordination failure may be eliminated in the decentralized equilibrium. Suppose that there exist government bonds which can be collateralized since they are backed by the government's taxation authority (see Caballero and Krishnamurthy, 2006, and Kocherlakota, 2009). Then the planner can improve welfare by committing to provide the representative agent with government bonds which the agent can use as collateral to finance additional consumption and housing by borrowing from foreign investors. If the government can commit to this policy action, whenever agents coordinate on a Pareto-dominated constrained equilibrium, this equilibrium can be eliminated within a given period and thus also from an ex-ante perspective. Rational agents will then attach zero probability to such an equilibrium occurring and no government bonds need to be issued in equilibrium. If the government's commitment to use its tax authority to back the bonds is not credible, however, international investors will not accept them as collateral and such policy remedy is not available.

Given that multiple equilibria can only occur if $\mu > 0$, one may wonder whether regulation of home equity requirements could improve welfare by eliminating equilibrium multiplicity.[14] We find that a stricter equity requirement in general does not improve welfare in our economy. This is illustrated in Figure 11 which compares welfare of the benchmark economy with $\mu = 0.8$ to welfare of an economy with a tighter limit for the loan-to-value ratio $\mu = 0.7$. The figure shows that, at each level of wealth, the negative welfare effect of a smaller μ, and thus a larger equity requirement, dominates: the welfare loss due to the more restricted set of feasible consumption choices outweighs other effects, for example due to the change in the amount of uncertainty in the economy. Thus, our model with a collateral channel has different policy implications compared with Lorenzoni (2008) where capital requirements increase welfare since they reduce the pecuniary externality resulting from asset sales. We conjecture that there would be a bigger role for equity requirements in our model if we introduced costly default that occurs with positive probability in equilibrium.

6 Conclusion

We have provided a parsimonious model which identifies collateral constraints as cause for sizeable fluctuations of consumption and the relative price of housing. The constraints introduce coordination failure, both within and across periods. We have discussed the sources of equilibrium inefficiency and possible policy remedies. For our analysis, we have developed a new solution method

[14]We do not analyze housing policies since housing is really land in our model and thus in fixed supply. It would be interesting in future research to allow for housing construction in the model in order to broaden the welfare analysis.

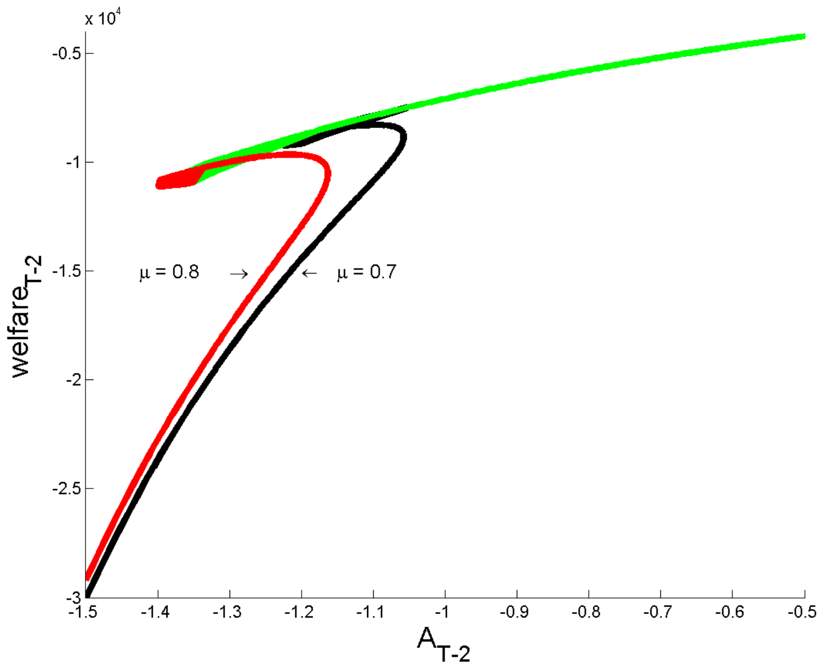

Figure 11: Welfare in period $T - 2$ for different loan-to-value ratios μ. Notes: Besides μ the parameter values are as specified in Table 1, Section 3 .

which handles equilibrium multiplicity and equilibrium beliefs about these multiple equilibria very efficiently.

It is important, albeit challenging, to investigate in future research whether our model can generate sizeable amplification of shocks to fundamentals if the structure is nested in a quantitative business cycle model. For quantitative applications it will be important to allow for more realistic debt contracts with longer maturity since the maturity of the representative debt contract will matter for the quantitative feedback from the price of collateral on consumption.

Appendix

A. Foundations for the implicit pricing assumptions in the model

We provide foundations for the structure of the economy presented in the main text. We show that we can replace the small open economy with an economy which consists of two types of agents: bankers who price the financial asset, as in Cole and Kehoe (2000), and consumers who are the "marginal" owner occupiers and thus price housing.

We assume that the utility of bankers is quasi-linear and separable in consumption and housing. Bankers are endowed with housing at the satiation level $\overline{h} > \underline{h}$ and receive an income flow $\overline{y}$ in every period. The assumptions about the endowment and preferences of bankers imply that the bankers price the financial asset but not housing if they are at an interior optimum. The rationale for these modeling assumptions is that financial wealth is concentrated among few households in US data. Due to the substantial wealth of these households, we assume that they are approximately risk neutral, are satiated in owner-occupied housing and thus are not marginal buyers of additional owner-occupied housing. More formally, the recursive problem of the bankers is given by

$$w_t\left(a_t, h_t; s_t\right) = \max_{a_{t+1}, h_{t+1}} [c_t - \frac{1}{2}\left(\overline{h} - h_t\right)^2 + \beta_b E_t v_{t+1}(a_{t+1}, h_{t+1}; s_{t+1})] \tag{18}$$

subject to the constraints

$$\begin{aligned} a_{t+1} + p_t h_{t+1} + c_t &= (1+r)a_t + p_t\overline{h} + \overline{y} \\ a_{t+1} &\geq \underline{a}, \\ h_t &\geq \underline{h}, \\ A_{t+1} &= F_t(s_t) \\ H_{t+1} &= 1 + \overline{h}, \text{ for all } t. \end{aligned}$$

We construct an equilibrium in which bankers provide the funds for the marginal buyers of housing. Hence, the borrowing constraint is slack and the equilibrium condition for housing

$$p_t = \overline{h} - h_{t+1} = 0$$

implies that bankers do not value housing beyond their endowment: $p_t = 0$ if $h_{t+1} = \overline{h}$. The remaining housing stock $H = 1$ is then priced by the rest of the population who are the marginal buyers of owner-occupied housing.

Since the borrowing constraint is not binding for the bankers in equilibrium, the equilibrium condition for financial assets implies that its price is determined by the discount factor of bankers

$$\beta_b = \frac{1}{1+r}.$$

If bankers are more patient than the rest of the population $\beta_b > \beta$, they price the financial asset and constrained marginal buyers of owner-occupied housing price the remaining housing stock $H = 1$.

B. Equilibrium multiplicity in period $T-1$ for quasi-linear utility

We first derive the mapping from A_T into A_{T-1}, c_{T-1} and p_{T-1} in closed form, using the equations in subsection 2.4. We then derive conditions for multiplicity.

For later reference, note that the quasi-linear utility function

$$U(c_t, h_t) = -\frac{a}{2}(\overline{c} - c_t)^2 + h_t$$

implies that for $h_t = 1$

$$\frac{\partial U(c_t, 1)}{\partial c_t} = a(\overline{c} - c_t) \text{ and } \frac{\partial U(c_t, 1)}{\partial h_t} = 1.$$

Unconstrained equilibrium

For given A_T and $\kappa_{T-1} = 0$ in the unconstrained equilibrium, the equilibrium first-order condition for financial assets (6) implies

$$a(\overline{c} - c_{T-1}) = \beta(1+r)E_{T-1}\left[a(\overline{c} - c_T)\right].$$

Using (17) and solving for c_{T-1}, we get

$$\begin{aligned} c_{T-1} &= \overline{c} - \frac{\beta}{a}(1+r)E_{T-1}\left[a(\overline{c} - (1+r)A_T - y_T)\right] \\ &= \left[1 - \beta(1+r)\right]\overline{c} + \beta(1+r)\left[(1+r)A_T + y_T\right], \end{aligned} \tag{19}$$

where $E_{T-1}A_T = A_T$. Since there is no income uncertainty in the last period, also $E_{T-1}y_T = y_T$. Income uncertainty could be added easily in this example because the preferences imply certainty equivalence.

The equilibrium first-order condition for housing (7) can be solved for the price p_{T-1}, using $p_T = 0$ and (19):

$$\begin{aligned} p_{T-1} &= \frac{\beta}{a(\bar{c} - c^u_{T-1})} \\ &= \frac{\beta / a}{\bar{c} - [1 - \beta(1+r)]\,\bar{c} - \beta(1+r)\,[(1+r)A_T + y_T]} \\ &= \frac{1}{a(1+r)} \frac{1}{\bar{c} - (1+r)A_T - y_T}. \end{aligned} \tag{20}$$

We then use the budget constraint to solve for A_{T-1} as a function of A_T:

$$\begin{aligned} A_{T-1} &= \frac{A_T + c_{T-1} - y_{T-1}}{1+r} \\ &= \frac{A_T + [1-\beta(1+r)]\,\bar{c} + \beta(1+r)\,[(1+r)A_T + y_T] - y_{T-1}}{1+r} \\ &= \frac{[1-\beta(1+r)]\,\bar{c} + [1+\beta(1+r)^2]\,A_T + \beta(1+r)y_T - y_{T-1}}{1+r}. \end{aligned} \tag{21}$$

Note that c_{T-1}, p_{T-1} and A_{T-1} positively depend on A_T for $0 < c_T < \bar{c}$. Given these equations, it is straightforward to retrieve the policies A_T and c_{T-1} as a function A_{T-1}:

$$A_T = \frac{(1+r)A_{T-1} + y_{T-1} - \beta(1+r)y_T - [1-\beta(1+r)]\,\bar{c}}{1+\beta(1+r)^2} \tag{22}$$

implies that

$$\begin{aligned} c_{T-1} &= [1-\beta(1+r)]\,\bar{c} + \beta(1+r)^2 A_T + \beta(1+r)y_T \\ &= \frac{1-\beta(1+r)}{1+\beta(1+r)^2}\bar{c} + \frac{\beta(1+r)}{1+\beta(1+r)^2}y_T \\ &\quad + \frac{\beta(1+r)^2}{1+\beta(1+r)^2}\,[(1+r)A_{T-1} + y_{T-1}]\,. \end{aligned}$$

Since A_T and c_{T-1} are linearly increasing in A_{T-1} for $0 < c_{T-1} < \bar{c}$, also

$$p_{T-1} = \frac{\beta}{a(\bar{c} - c_{T-1})}$$

is monotonically increasing in A_{T-1}.

Constrained equilibrium

If consumers are constrained in $T-1$,

$$p_{T-1} = -\frac{1+r}{\mu} A_T. \tag{23}$$

Consumption in the last period is given by

$$c_T = -\mu p_{T-1} + y_T. \tag{24}$$

Given that $p_T = 0$, it follows from equation (8) that the multiplier κ_{T-1} is determined by

$$\begin{aligned} &\kappa_{T-1}\left[1+r-\mu\right] \\ &= \beta \left[\frac{1}{p_{T-1}} - (1+r)a(\bar{c} - c_T) \right]. \end{aligned} \tag{25}$$

The equilibrium first-order condition for financial assets (6) with $\kappa_{T-1} > 0$ implies

$$\begin{aligned} a(\bar{c} - c_{T-1}) &= \beta(1+r)a(\bar{c} - c_T) \\ &+ \frac{\beta(1+r)}{1+r-\mu} \left[\frac{1}{p_{T-1}} - (1+r)a(\bar{c} - c_T) \right] \\ &= \frac{\beta(1+r)}{1+r-\mu} \left[\frac{1}{p_{T-1}} - \mu a(\bar{c} - c_T) \right]. \end{aligned}$$

Note that with linear utility in consumption, instead of quadratic utility, (6) would determine a unique κ_{T-1} which is independent of A_{T-1}. Hence, in this case equilibrium multiplicity does not depend on the state variable A_{T-1}.

Recalling (24), we solve the previous equation for c_{T-1}. Different from consumption in the unconstrained equilibrium, c_{T-1} in the constrained equilibrium is a non-linear function of A_T because of the price feedback from the collateral constraint:

$$\begin{aligned} c_{T-1} &= \bar{c} - \frac{\beta(1+r)}{a\left[1+r-\mu\right]} \left[\frac{1}{p_{T-1}} - \mu a(\bar{c} + \mu p_{T-1} - y_T) \right] \\ &= \bar{c} + \frac{\beta(1+r)}{a(1+r-\mu)} \left[\frac{\mu}{A_T(1+r)} + \mu a\left[\bar{c} - A_T(1+r) - y_T\right] \right] \end{aligned} \tag{26}$$

where the second equality uses (23).

We use the budget constraint to map A_T into A_{T-1}, for all admissible negative values of A_T which ensure that consumption $0 < c_{T-1} < \bar{c}$ (it is straightforward to derive the restrictions on A_T). Using the budget constraint with $h_T = H_T = 1$ and substituting out c_{T-1} given by (26), we get

$$A_{T-1} = \frac{\bar{c} + \frac{\beta(1+r)}{a(1+r-\mu)}\left[\frac{\mu}{A_T(1+r)} + \mu a\left[\bar{c} - A_T(1+r) - y_T\right]\right] + A_T - y_{T-1}}{1+r}. \qquad (27)$$

This equation implies a non-linear mapping from A_T to A_{T-1} if $1 + r > \mu > 0$, making it possible for multiple equilibria to arise. Note that for $\mu = 0$ the equation is linear in A_T as it simplifies to

$$A_{T-1} = \frac{A_T + \bar{c} - y_{T-1}}{1+r}.$$

For $1 + r > \mu > 0$, however, equation (27) is quadratic in A_T for a given A_{T-1} since rearranging (27) and multiplying by A_T results in

$$\begin{aligned} & \left(1 - \frac{\beta(1+r)^2\mu}{(1+r-\mu)}\right) A_T^2 \qquad (28) \\ & + \left(\bar{c} - (1+r)A_{T-1} - y_{T-1} + \frac{\beta(1+r)\mu}{(1+r-\mu)}(\bar{c} - y_T)\right) A_T + \frac{\beta\mu}{a(1+r-\mu)} \\ = \; & 0. \end{aligned}$$

Figure 12 displays the loci for the constrained (red) and unconstrained (green) equilibria which have a qualitatively similar shape as the loci for CRRA utility presented in the paper. Equilibrium multiplicity occurs for levels of financial wealth at which the demand for consumption c approaches the satiation point and thus housing becomes attractive. For the parameter values, which are chosen purely for illustrative purposes, this is the case for consumption values very close to $\bar{c}$.

The lower bound $\underline{A}_T$

Before we characterize multiplicity, we derive the lower bound $\underline{A}_T$ above which constrained and unconstrained equilibrium exist. By Corollary 1 the lower bound $\underline{A}_T$ is given by the equation

$$\underbrace{\frac{1}{a(1+r)}\frac{1}{\bar{c} - (1+r)\underline{A}_T - y_T}}_{\text{price in unconstr. equilibrium}} = \underbrace{-\frac{1+r}{\mu}\underline{A}_T}_{\text{price in constr. equilibrium}}.$$

with the solution

$$\underline{A}_T = \frac{\bar{c} - y_T - \sqrt{(\bar{c} - y_T)^2 + 4\frac{\mu}{a(1+r)}}}{2(1+r)},$$

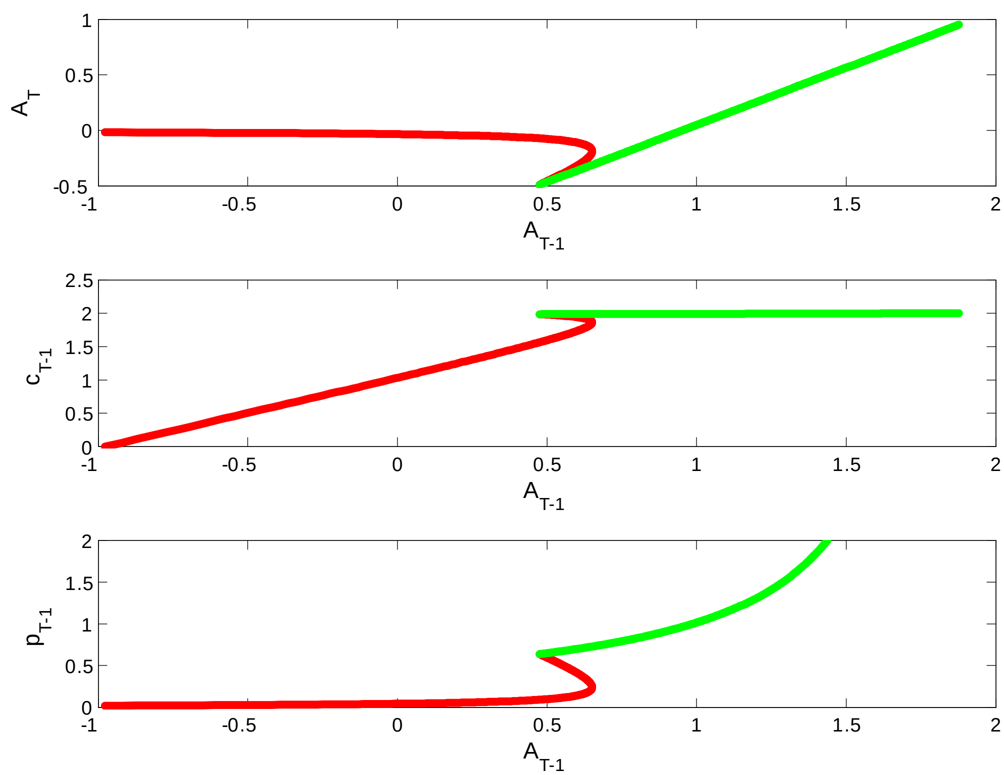

Figure 12: The model solution in period $T-1$ for quasi-linear utility. Notes: Constrained equilibrium in red color; unconstrained equilibrium in green color. The unit is period labor income. The parameter values are $\mu = 0.8$, $a = 1$, $\bar{c} = 2$, $\beta = 0.01$, $r = .04$, $y_t = y = 1$.

where $\bar{c} > y_T$ if the marginal utility of consumption in period T shall be positive for at least some $A_T > 0$. Note that, for the relevant root, $\underline{A}_T \leq 0$ if $\mu \geq 0$ so that a constrained equilibria may exist.

Equilibrium multiplicity

We use equation (28) which is quadratic in A_T for a given A_{T-1}. Thus, there may be two values of A_T which solve the equation for a given A_{T-1}. The solution to the equation is

$$\begin{aligned} A_T &= \left\{ -\left(\bar{c} - (1+r)A_{T-1} - y_{T-1} + \frac{\beta(1+r)\mu}{(1+r-\mu)}(\bar{c} - y_T) \right) \right. \\ &\quad \left. \pm \sqrt{\left(\bar{c} - (1+r)A_{T-1} - y_{T-1} + \frac{\beta(1+r)\mu}{(1+r-\mu)}(\bar{c} - y_T) \right)^2 - 4\left(1 - \frac{\beta(1+r)^2\mu}{(1+r-\mu)}\right)\frac{\beta\mu}{a(1+r-\mu)}} \right\} \\ &\quad /2\left(1 - \frac{\beta(1+r)^2\mu}{(1+r-\mu)}\right). \end{aligned}$$

Since the equilibrium law of motion implies $c_{T-1} = (1+r)\,A_{T-1} + y_{T-1} - A_T$ and $A_T < 0$ in the constrained equilibrium, it follows from $\bar{c} \geq c_{T-1}$ that $\bar{c} \geq (1+r)\,A_{T-1} + y_{T-1}$ and thus

$$\bar{c} - (1+r)A_{T-1} - y_{T-1} + \frac{\beta(1+r)\mu}{(1+r-\mu)}(\bar{c} - y_T) > 0,$$

where we maintain $\bar{c} > y_T$ and $\mu < 1+r$. Thus there only exist two constrained equilibria with $A_T < 0$, for given A_{T-1}, if

$$1 - \frac{\beta(1+r)^2\mu}{(1+r-\mu)} > 0 \text{ or } \beta < \frac{1+r-\mu}{(1+r)^2\mu}.$$

This parameter restriction is necessary but not sufficient since also $A_T > \underline{A}_T$ is required for equilibrium existence.

As illustrated by Figure 12, the unconstrained and constrained equilibrium map into the lower bound $\underline{A}_T$ from the same A_{T-1}. Thus, if there exist two constrained equilibria, there may also exist a third unconstrained equilibrium for some A_{T-1}. This is the case if the smallest A_{T-1} at which an unconstrained equilibrium exists is smaller than the largest A_{T-1} at which a constrained equilibrium exists. We now show under what conditions this is the case.

The positive monotonicity of $A_T(A_{T-1})$ in (22) implies that the smallest A_{T-1}, for which there exists an unconstrained equilibrium, maps into the lower bound $\underline{A}_T$. Using (21), this is the case for

$$A^u_{T-1}(\underline{A}_T) \equiv \frac{[1-\beta(1+r)]\,\bar{c} + [1+\beta(1+r)^2]\,\underline{A}_T + \beta(1+r)y_T - y_{T-1}}{1+r}$$

and an unconstrained equilibrium exists for $A_{T-1} \geq A^u_{T-1}(\underline{A}_T)$.

For constrained equilibria instead, the largest value of A_{T-1} is attained for the value of $A_T < 0$ which maximizes the right-hand side of (27). The vertex in Figure 12, where A_{T-1} as a function of A_T takes its maximal value, can be computed by differentiating the right-hand side with respect to A_T so that

$$\frac{\beta(1+r)}{a(1+r-\mu)}\left(-\frac{\mu}{A_T^2(1+r)} - \mu a(1+r)\right) + 1 = 0$$

which has the solution

$$A_T^v = -\sqrt{\frac{\frac{\beta\mu}{a(1+r-\mu)}}{1-\frac{\beta(1+r)^2\mu}{(1+r-\mu)}}},$$

where the necessary condition for multiplicity $1 - \beta(1+r)^2\mu/(1+r-\mu) > 0$ implies that the square root is well defined. The negative root is the relevant one since A_T^v is indeed a maximum if the second derivative

$$\frac{\beta(1+r)}{a(1+r-\mu)}\frac{2\mu}{A_T^3(1+r)} < 0.$$

This requires $A_T < 0$ and thus rules out the positive root. Moreover, $A_T^v < 0$ in a constrained equilibrium.

A condition for multiplicity is thus that $A^c_{T-1}(A^v_T)$, obtained by substituting A^v_T in equation (27), is larger than $A^u_{T-1}(\underline{A}_T)$.

An alternative condition for multiplicity is that the derivative of the right-hand side of (27), evaluated at the lower bound $\underline{A}_T$, has positive sign. That is

$$\begin{aligned}
& 1 - \frac{\beta\mu}{a(1+r-\mu)}\frac{1}{\underline{A}_T^2} - \frac{\beta(1+r)^2\mu}{(1+r-\mu)} \\
= \quad & 1 - \frac{\beta(1+r)^2\mu}{(1+r-\mu)} - \frac{\beta\mu}{a(1+r-\mu)}\frac{1}{\left(\frac{(\bar{c}-y_T)-\sqrt{(\bar{c}-y_T)^2+4\frac{\mu}{a(1+r)}}}{2(1+r)}\right)^2} \\
> \quad & 0,
\end{aligned}$$

where the necessary condition for multiplicity implies $1-\beta(1+r)^2\mu/(1+r-\mu) > 0$.

C. Equilibrium multiplicity in period $T-1$ for CRRA utility

We first derive the mapping from A_T into A_{T-1}, c_{T-1} and p_{T-1} in closed form, using the equations in subsection 2.4. We then derive conditions for multiplicity.

Unconstrained equilibrium

For given A_T and $\kappa_{T-1}=0$ in the unconstrained equilibrium, the equilibrium first-order condition for financial assets (6) implies

$$\theta c_{T-1}^{\theta(1-\sigma)-1} = \beta(1+r)\theta c_T^{\theta(1-\sigma)-1},$$

where the unique equilibrium $c_T = (1+r)A_T + y_T$ implies that the previous equation contains no expectation operator. Using (17) and solving for c_{T-1}, we get

$$c_{T-1} = (\beta(1+r))^{\frac{1}{\theta(1-\sigma)-1}} \left((1+r)A_T + y_T\right). \tag{29}$$

The equilibrium first-order condition for housing (7) can be solved for the price p_{T-1}, using $p_T = 0$ and (29):

$$\begin{aligned} p_{T-1} &= \frac{\beta\left(1-\theta\right)c_T^{\theta(1-\sigma)}}{\theta c_{T-1}^{\theta(1-\sigma)-1}} \\ &= \frac{\beta\left(1-\theta\right)c_T^{\theta(1-\sigma)}}{\beta(1+r)\theta c_T^{\theta(1-\sigma)-1}} \\ &= \frac{1-\theta}{\theta}\frac{c_T}{1+r} \\ &= \frac{1-\theta}{\theta}\frac{(1+r)A_T + y_T}{1+r}. \end{aligned} \tag{30}$$

We then use the budget constraint to solve for A_{T-1} as a function of A_T:

$$\begin{aligned} A_{T-1} &= \frac{A_T + c_{T-1} - y_{T-1}}{1+r} \\ &= \frac{A_T + (\beta(1+r))^{\frac{1}{\theta(1-\sigma)-1}}\left((1+r)A_T + y_T\right) - y_{T-1}}{1+r} \\ &= \frac{\left[1 + (1+r)\left(\beta(1+r)\right)^{\frac{1}{\theta(1-\sigma)-1}}\right]A_T + (\beta(1+r))^{\frac{1}{\theta(1-\sigma)-1}}y_T - y_{T-1}}{1+r}. \end{aligned} \tag{31}$$

Note that c_{T-1}, p_{T-1} and A_{T-1} positively depend on A_T. Given these equations, it is straightforward to retrieve the policies A_T and c_{T-1} as a function A_{T-1}:

$$A_T(A_{T-1}) = \frac{(1+r)A_{T-1} + y_{T-1} - (\beta(1+r))^{\frac{1}{\theta(1-\sigma)-1}}y_T}{1 + (1+r)\left(\beta(1+r)\right)^{\frac{1}{\theta(1-\sigma)-1}}} \tag{32}$$

implies that

$$c_{T-1} = (\beta(1+r))^{\frac{1}{\theta(1-\sigma)-1}} \left((1+r)A_T(A_{T-1}) + y_T\right)$$

and

$$p_{T-1} = \frac{1-\theta}{\theta}\frac{(1+r)A_T(A_{T-1}) + y_T}{1+r}$$

are monotonically increasing in A_{T-1}.

Constrained equilibrium

If consumers are constrained in $T-1$,

$$p_{T-1} = -\frac{1+r}{\mu}A_T. \tag{33}$$

As before, consumption in the last period is given by

$$c_T = (1+r)A_T + y_T. \tag{34}$$

Given that $p_T = 0$, it follows from equation (8) that the multiplier κ_{T-1} is determined by

$$\begin{aligned} &\kappa_{T-1}\left[1+r-\mu\right] \\ &= \beta\left[\frac{1}{p_{T-1}}\left(1-\theta\right)c_T^{\theta(1-\sigma)} - (1+r)\theta c_T^{\theta(1-\sigma)-1}\right] \\ &= \beta\left((1+r)A_T + y_T\right)^{\theta(1-\sigma)}\left[-\frac{\mu(1-\theta)}{(1+r)A_T} - \frac{(1+r)\theta}{(1+r)A_T + y_T}\right]. \end{aligned} \tag{35}$$

The equilibrium first-order condition for financial assets (6) with $\kappa_{T-1} > 0$ implies

$$\begin{aligned} \theta {c_{T-1}}^{\theta(1-\sigma)-1} &= \beta(1+r)c_T^{\theta(1-\sigma)-1} \\ &+ \frac{\beta(1+r)}{1+r-\mu}c_T^{\theta(1-\sigma)}\left[-\frac{\mu(1-\theta)}{(1+r)A_T} - \frac{(1+r)\theta}{c_T}\right] \\ &= \beta(1+r)((1+r)A_T + y_T)^{\theta(1-\sigma)-1} \\ &+ \frac{\beta(1+r)}{1+r-\mu}((1+r)A_T + y_T)^{\theta(1-\sigma)}\left[-\frac{\mu(1-\theta)}{(1+r)A_T} - \frac{(1+r)\theta}{((1+r)A_T + y_T)}\right]. \end{aligned}$$

Different from consumption in the unconstrained equilibrium, c_{T-1} in the constrained equilibrium is a non-linear function of A_T because of the binding collateral constraint. Note that the non-linearity would vanish if agents do not derive utility from housing, $\theta = 1$, or $\mu = 0$. In this case, equation (7) implies $p_{T-1} = 0$, and $A_T = 0$ in constrained equilibrium. In general consumption is highly non-linear in A_T so that we no longer have a closed form mapping from A_T to A_{T-1}. For log utility, $\sigma = 1$, however, the marginal utility of housing at the equilibrium $h = 1$ is constant $(1-\theta)((1+r)A_T + y_T)^{\theta(1-\sigma)} = (1-\theta)$ and we get

$$\begin{aligned}
\theta c_{T-1}^{-1} &= \beta(1+r)c_T^{-1} + \frac{\beta(1+r)}{1+r-\mu}\left[-\frac{\mu(1-\theta)}{(1+r)A_T} - \frac{(1+r)\theta}{c_T}\right] \\
&= \beta(1+r)((1+r)A_T + y_T)^{-1} \\
&+ \frac{\beta(1+r)}{1+r-\mu}\left[-\frac{\mu(1-\theta)}{(1+r)A_T} - \frac{(1+r)\theta}{((1+r)A_T + y_T)}\right] \\
&= \frac{\beta(1+r)^2 A_T - \frac{\beta(1+r)}{1+r-\mu}\mu(1-\theta)(1+r)A_T - \frac{\beta(1+r)}{1+r-\mu}\mu(1-\theta)y_T - \frac{\beta(1+r)}{1+r-\mu}(1+r)^2\theta A_T}{(1+r)^2 A_T^2 + (1+r)A_T y_T},
\end{aligned}$$

so that

$$c_{T-1} = \frac{\theta(1+r)^2 A_T^2 + \theta(1+r)A_T y_T}{\left(\beta\frac{(1+r)^2}{1+r-\mu}\left((1-\theta)(1+r) - \mu(2-\theta)\right)\right)A_T - \frac{\beta(1+r)}{1+r-\mu}\mu(1-\theta)y_T} \tag{36}$$

Using the budget constraint with $h_T = 1$ and substituting out c_{T-1} given by (36), we get

$$\begin{aligned}
&(1+r)A_{T-1} \\
= \ & A_T - y_{T-1} \\
&+ \frac{\theta(1+r)^2 A_T^2 + \theta(1+r)A_T y_T}{\left(\beta\frac{(1+r)^2}{1+r-\mu}\left((1-\theta)(1+r) - \mu(2-\theta)\right)\right)A_T - \frac{\beta(1+r)}{1+r-\mu}\mu(1-\theta)y_T}.
\end{aligned} \tag{37}$$

Note that for $\mu = 0$ the equation is linear in A_T as it simplifies to

$$(1+r)A_{T-1} = A_T - y_{T-1} + \frac{\theta(1+r)^2 A_T + \theta(1+r)y_T}{\beta(1+r)^2(1-\theta)}.$$

For $1 + r > \mu > 0$, however, equation (37) is quadratic in A_T for a given A_{T-1}. Rearranging (37) results in

$$mA_T^2 + nA_T + k = 0, \tag{38}$$

with

$$\begin{aligned}
m &= \beta \frac{(1+r)^2}{1+r-\mu}\left((1-\theta)(1+r)-\mu(2-\theta)\right)+\theta(1+r)^2, \\
n &= -\left(\left(\beta \frac{(1+r)^2}{1+r-\mu}\left((1-\theta)(1+r)-\mu(2-\theta)\right)\right)(1+r)A_{T-1}-\theta(1+r)y_T\right), \\
k &= \frac{\beta(1+r)}{1+r-\mu}\mu(1-\theta)y_T\left((1+r)A_{T-1}+y_{T-1}\right).
\end{aligned}$$

The lower bound $\underline{A}_T$

Before we characterize multiplicity, we derive the lower bound $\underline{A}_T$ above which constrained and unconstrained equilibrium exist. By Corollary 1 the lower bound $\underline{A}_T$ is given by the equation

$$\underbrace{\frac{1-\theta}{\theta}\frac{(1+r)\underline{A}_T+y_T}{1+r}}_{\text{price in unconstr. equilibrium}} = \underbrace{-\frac{1+r}{\mu}\underline{A}_T}_{\text{price in constr. equilibrium}}.$$

with the solution

$$\underline{A}_T = -\frac{\mu(1-\theta)}{(1+r)(\theta(1+r)+\mu(1-\theta))}y_T.$$

Equilibrium multiplicity

Equation (38), which is quadratic in A_T for a given A_{T-1}, implies that there may be two values of A_T which solve the equation for a given A_{T-1}. If both solutions are negative and larger than $\underline{A}_T$, there exist multiple constrained equilibria.

The unconstrained and constrained equilibrium map into the lower bound $\underline{A}_T$ from the same A_{T-1}. Thus, if there exist two constrained equilibria, there may also exist a third unconstrained equilibrium for some A_{T-1}. This is the case if the smallest A_{T-1} at which an unconstrained equilibrium exists is smaller than the largest A_{T-1} at which a constrained equilibrium exists.

The positive monotonicity of A_T as a function of A_{T-1} in (32) implies that the smallest A_{T-1}, for which there exists an unconstrained equilibrium, maps into the lower bound $\underline{A}_T$. Let us denote this value with A^-_{T-1}. An unconstrained equilibrium exists for $A_{T-1} \geq A^-_{T-1}$.

For constrained equilibria instead, the largest value of A_{T-1} is attained for the value of $A_T < 0$ which maximizes the right-hand side of (37). Let us denote this value with A^+_{T-1}. A condition for multiplicity of constrained and unconstrained equilibria is thus that $A^+_{T-1} > A^-_{T-1}$. An alternative condition is that the derivative of the right-hand side of (37), evaluated at the lower bound $\underline{A}_T$, has positive sign.

D. Social planner problem

The planner faces no coordination problem. Given $H_t = 1$, for all t, the recursive problem of the social planner is

$$V_t(A_t, 1) = \max_{A_{t+1}, C_t} [U(C_t, 1) + \beta V_{t+1}(A_{t+1}, 1)] \tag{39}$$

subject to the constraints

$$\begin{aligned} A_{t+1} + C_t &= (1+r)A_t + y_t \\ (1+r)A_{t+1} + \mu q_t(A_{t+1}, C_t) &\geq 0 \\ y_t &= (1+g)^t y \\ A_{t+1} &= F_t(s_t) \\ H_{t+1} &= 1, \text{ for all } t. \end{aligned}$$

There is no expectation operator in the recursive problem since the planner does not face a coordination problem. The notation $q_t(A_{t+1}, C_t)$ makes explicit how the price in the competitive equilibrium depends on the choices of the planner. Using the equilibrium condition (7) of the competitive equilibrium,

$$q_t(A_{t+1}, C_t) = \frac{\beta\left(\frac{\partial U(C_{t+1}(A_{t+1},1),1)}{\partial h_{t+1}} + \widetilde{p}_{t+1}(A_{t+1}, 1)\frac{\partial U(C_{t+1}(A_{t+1},1),1)}{\partial C_{t+1}}\right)}{\frac{\partial U(C_t,1)}{\partial C_t} - \mu\widetilde{\kappa}_t}$$

where equation (8) shows how the multiplier of the collateral constraint $\widetilde{\kappa}_t$ depends on A_{t+1}, and $\widetilde{p}_{t+1}$ denotes the price of the collateral good in $t+1$, given that allocations are chosen by the planner and are supported by the price of the collateral good obtained in competitive markets.

The first-order conditions

$$\frac{\partial U(C_t, 1)}{\partial C_t} + \mu\frac{\partial q_t(A_{t+1}, C_t)}{\partial C_t}\eta_t - \lambda_t = 0,$$

and

$$-\lambda_t + \beta(1+r)\lambda_{t+1} + \left(1 + r + \mu\frac{\partial q_t(A_{t+1}, C_t)}{\partial A_{t+1}}\right)\eta_t = 0$$

imply

$$\begin{aligned} \frac{\partial U(C_t, 1)}{\partial C_t} &= \beta(1+r)\frac{\partial U(C_{t+1}, 1)}{\partial C_{t+1}} \\ &+\eta_t\left(1 + r + \mu\left(\frac{\partial q_t(A_{t+1}, C_t)}{\partial A_{t+1}} - \frac{\partial q_t(A_{t+1}, C_t)}{\partial C_t}\right)\right) \\ &+\eta_{t+1}\beta(1+r)\mu\frac{\partial q_{t+1}(A_{t+2}, C_{t+1})}{\partial C_{t+1}} \end{aligned} \tag{40}$$

where η_t is the multiplier of the collateral constraint and λ_t is the multiplier of the resource constraint for period t in the social planner problem. Compared with the competitive equilibrium condition (6), the planner internalizes the effect of consumption and wealth accumulation on the price of the collateral good. If less consumption C_t (and more financial wealth A_{t+1} in period $t+1$) reduces the price q_t, the collateral constraint tightens in period t. This is costly if the collateral constraint binds in the period t so that $\eta_t > 0$. The price effect lowers the marginal gain of accumulating financial assets at the collateral constraint $(1+r)\,\eta_t$. It implies that the social planner accumulates *less* wealth compared with agents in the competitive equilibrium.

There is an additional effect, however, if the collateral constraint binds in period $t+1$ so that $\eta_{t+1} > 0$. If less consumption C_t and more wealth accumulation A_{t+1} *increase* the price q_{t+1} in the *next* period, this relaxes a binding collateral constraint in period $t+1$. Ceteris paribus, this effect implies that the social planner accumulates *more* wealth than agents in the competitive equilibrium.

Let us now consider how a planner may be able to decentralize the social optimum in which there is no coordination failure and the effect of wealth accumulation on the price of the collateral good is internalized. We derive how a tax/subsidy on wealth accumulation can make agents internalize the price effect if the planner's problem is concave.

Denote a state-dependent tax/subsidy for wealth accumulation with $\tau(A_t, 1)$. The tax/subsidy is rebated/financed lump-sum so that the government budget balances. For the equilibrium condition of the agent

$$\frac{\partial U(c_t(A_t,1),1)}{\partial c_t} = (1 + r - \tau(A_t,1))\left(\beta \frac{\partial U(c_{t+1}(A_{t+1},1),1)}{\partial c_{t+1}} + \kappa_t\right) \tag{41}$$

to implement the social optimum,

$$\tau(A_t,1) = \frac{-\eta_t \mu \left(\frac{\partial q_t(A_{t+1},C_t)}{\partial A_{t+1}} - \frac{\partial q_t(A_{t+1},C_t)}{\partial C_t}\right) - \eta_{t+1}\beta(1+r)\mu \frac{\partial q_{t+1}(A_{t+2},C_{t+1})}{\partial C_{t+1}}}{\beta \frac{\partial U(C_{t+1}(A_{t+1},1),1)}{\partial C_{t+1}} + \kappa_t},$$

with $A_{t+1} = F_t(A_t, 1)$. The solution for τ shows that whether the planner taxes capital income or subsidizes interest payments on loans depends on the binding patterns of the collateral constraint in periods t and $t+1$; and it depends on the strength of the general equilibrium effect of choices C_t and A_{t+1} on prices q_t and q_{t+1}. Clearly, the tax/subsidy is zero if the collateral constraint is not binding, $\eta_t = \eta_{t+1} = 0$, or the prices q_t and q_{t+1} are not affected by choices C_t and A_{t+1}. If $\eta_t > 0$ and $\eta_{t+1} = 0$, wealth accumulation in period t should be taxed if more demand in period t increases q_t and thus relaxes the collateral constraint.

Instead, wealth accumulation in period t should be subsidized if $\eta_t = 0$ and $\eta_{t+1} > 0$ and more demand in period $t+1$ relaxes the collateral constraint in that period.

References

[1] Atkeson, Andrew (2000): "Rethinking Multiple Equilibria in Macroeconomic Modeling: Comment," *NBER Macroeconomic Annual*, vol. 15, 162–171.

[2] Benhabib, Jess and Roger Farmer (1999): "Indeterminacy and Sunspots in Macroeconomics," ch.6, in: *Handbook of Macroeconomics*, Taylor, John B. and Michael Woodford (eds.), vol. 1A, 387–448.

[3] Bianchi, Javier (2011): "Overborrowing and Systemic Externalities in the Business Cycle," *American Economic Review*, vol. 101, 3400–3426.

[4] Caballero, Ricardo J. and Arvind Krishnamurthy (2001): "International and Domestic Collateral Constraints in a Model of Emerging Market Crisis," *Journal of Monetary Economics*, vol. 48, 513–548.

[5] Caballero, Ricardo J. and Arvind Krishnamurthy (2006): "Bubbles and Capital Flow Volatility: Causes and Risk Management," *Journal of Monetary Economics*, vol. 53, 35–53.

[6] Carroll, Christopher D. (2006): "The Method of Endogenous Gridpoints for Solving Dynamic Stochastic Optimization Problems," *Economics Letters*, vol. 91, 312–320.

[7] Cole, Harold L. and Timothy J. Kehoe (2000): "Self-Fulfilling Debt Crises," *Review of Economic Studies*, vol. 67, 91–116.

[8] Davis, Morris and Jonathan Heathcote (2007): "The Price and Quantity of Residential Land in the United States," *Journal of Monetary Economics*, vol. 54, 2595–2620.

[9] De Nardi, Mariacristina, Eric French and David Benson (2012): "Consumption and the Great Recession," *Economic Perspectives*, Federal Reserve Bank of Chicago, vol. 36, 1–16.

[10] Diamond, Douglas W. and Philip H. Dybvig (1983): "Bank Runs, Deposit Insurance, and Liquidity," *Journal of Political Economy*, vol. 91, 401–419.

[11] Feng, Zhigang, Jianjun Miao, Adrian Peralta-Alva and Manuel S. Santos (2011): "Numerical Simulation of Nonoptimal Dynamic Equilibrium Models," Federal Reserve Bank of St. Louis, Working Paper Series No. 2009-018B.

[12] Glover, Andrew, Jonathan Heathcote, Dirk Krueger and José-Víctor Ríos-Rull (2012): "Intergenerational Redistribution in the Great Recession," Manuscript, University of Pennsylvania.

[13] Gourinchas, Pierre-Olivier and Maurice Obstfeld (2012): "Stories of the Twentieth Century for the Twenty-First," *American Economic Journal: Macroeconomics*, vol. 4, 226–265.

[14] Guerrieri, Veronica and Guido Lorenzoni (2011): "Credit Crises, Precautionary Savings, and the Liquidity Trap," Manuscript, MIT.

[15] He, Chao, Randall Wright and Yu Zhu (2012): "Housing and Liquidity," Manuscript, University of Wisconsin – Madison.

[16] Heathcote, Jonathan and Fabrizio Perri (2011): "Wealth and Volatility," Manuscript, Federal Reserve Bank of Minneapolis.

[17] Hintermaier, Thomas and Winfried Koeniger (2010): "The Method of Endogenous Gridpoints with Occasionally Binding Constraints among Endogenous Variables," *Journal of Economic Dynamics & Control*, vol. 34, 2074–2088.

[18] Jeanne, Olivier and Anton Korinek (2011): "Managing Credit Booms and Busts: A Pigouvian Taxation Approach," Manuscript, University of Maryland.

[19] Kiyotaki, Nobuhiro and John Moore (1997): "Credit Cycles," *Journal of Political Economy*, vol. 105, 211–248.

[20] Kocherlakota, Narayana (2009): "Bursting Bubbles: Consequences and Cures," Manuscript, Federal Reserve Bank of Minneapolis.

[21] Lorenzoni, Guido (2008): "Inefficient Credit Booms," *Review of Economic Studies*, vol. 75, 809–833.

[22] Mendoza, Enrique G. (2010): "Sudden Stops, Financial Crises, and Leverage," *American Economic Review*, vol. 100, 1941–1966.

[23] Mian, Atif and Amir Sufi (2011): "Household Leverage and the Recession of 2007–09," *IMF Economic Review*, vol. 1, 74–117.

[24] Mian, Atif, Kamalesh Rao and Amir Sufi (2011): "Household Balance Sheets, Consumption, and the Economic Slump," Manuscript, University of Berkeley, California.

[25] Obstfeld, Maurice (1996): "Models of Currency Crises with Self-Fulfilling Features," *European Economic Review*, vol. 40, 1037–1047.

[26] OECD Factbook (2010): The Crisis and Beyond, DOI http://10.1787/18147364, http://10.1787/factbook-2010-en, OECD, Paris.

[27] Ohanian, Lee E. (2011): "Accounting for the Great Recession," Economic Policy Paper 11-1, Federal Reserve Bank of Minneapolis.

[28] Perri, Fabrizio and Vincenzo Quadrini (2011): "International Recessions," CEPR Discussion Paper No. 8483.

[29] Quadrini, Vincenzo (2011): "Financial Frictions in Macroeconomic Fluctuations," *Economic Quarterly*, Federal Reserve Bank of Richmond, vol. 97, 209–254.

[30] Sánchez-Marcos, Virginia and José-Víctor Ríos-Rull (2009): "Aggregate Shocks and House Prices Fluctuations," Manuscript, University of Minnesota.

[31] Stein, Jeremy C. (1995): "Prices and Trading Volume in the Housing Market: a Model with Down-Payment Effects," *Quarterly Journal of Economics*, vol. 110, 379–406.

[32] Tirole, Jean (2006): *The Theory of Corporate Finance*, Princeton University Press, Princeton and Oxford.