# Risk-Sharing in the Small and in the Large

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## 1 Introduction

Consider an exchange economy with a single consumption good, symmetric information, and no aggregate uncertainty. It is well understood that in such economy risk-averse and (subjective) expected-utility-maximizing agents will choose to introduce individual uncertainty in the final allocation if and only if they have different beliefs. That is, betting can occur in equilibrium only if agents disagree on the probabilities of some events. At the same time, it is also well understood that agents may not have precise beliefs, i.e., violate subjective expected utility, when some events relevant for the economy are more ambiguous than others (Ellsberg, 1961). The result on the connection between disagreement of beliefs and betting has been extended to take into account such imprecision. For instance, Billot, Chateauneuf, Gilboa, and Tallon (2000) show that risk-averse agents whose preferences satisfy the maxmin expected utility model of Gilboa and Schmeidler (1989) will bet if and only if they do not *share a prior*; i.e., if the sets of priors that they employ (in the maxmin representation) do not intersect. This result has been significantly extended by Rigotti, Shannon, and Strzalecki (2008, henceforth RSS), who showed that exactly the same is true for any collection of agents whose preferences satisfy convexity (which embodies both risk aversion and uncertainty aversion in the sense of Gilboa and Schmeidler (1989)) plus some mild structural conditions.

On the other hand, convexity has been questioned recently, both from a theoretical and from an experimental point of view; see e.g. Epstein (1999), Baillon, L'Haridon, and Placido (2011), L'Haridon and Placido (2010) among others. In this paper, we investigate the extent to which convexity can be dispensed with, while preserving the connection between "no betting" and "sharing priors" found by RSS. We do so by leveraging the definition of ambiguity aversion introduced by Ghirardato and Marinacci (2002, henceforth GM), as well as results from non-smooth calculus and optimization. In particular, we identify conditions that accommodate substantial departures from convexity of preferences, but still allow one to draw global conclusions from local analysis. Furthermore, we allow for a broad class of preferences: the only assumptions maintained throughout are strong monotonicity and local Lipschitz continuity of the preference functional $I$, in addition to standard assumptions on Bernoulli utility $u$.

To elaborate, the equivalence result in RSS has two main components: (1) the existence of a shared prior implies that every Pareto-efficient allocation involves full insurance; i.e., no agent is betting; (2) the existence of a Pareto-efficient allocation involving full insurance implies the existence of a shared prior; i.e., the absence of a shared prior implies the occurrence of betting in any Pareto-efficient allocation (hence in equilibrium).

We show that ambiguity aversion in the sense of GM is sufficient to obtain (1): with GM-ambiguity-averse preferences, the existence of a shared prior still implies that there will be no betting at any Pareto optimum (Proposition 7). The key is how to define the "priors" for agent $i$. It turns out that the appropriate notion is the "core" of the preference functional $I_i$, introduced by GM.[1] This is the collection of all probabilities $P$ for which $\int a\,dP \geq I_i(a)$ for all real functions $a$. It is non-empty if and only if the agent is GM-ambiguity-averse: see Theorem 12 in GM and Proposition 8 in Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2011). We also exhibit an economy in which the core of one agent's preference functional is empty, so a fortiori there cannot be a shared prior; in this economy there is a Pareto-efficient allocation that exhibits betting (Example 7).

Next, we show that (2) also generalizes, and that such generalization does not even require GM-ambiguity aversion. Specifically, if there exists a Pareto-efficient full-insurance alloca-

[1] Marinacci and Pesce (2013) consider a subclass of the preferences we analyze and study the impact of changes in GM-ambiguity aversion on efficient and equilibrium allocations. Though they do not focus on risk-sharing, they independently derive a version of our Proposition 7. Also Billot et al (2002) have a version of Proposition 7 for CEU preferences. Furthermore, they have a risk-sharing result which does not require convexity of preferences but holds for economies with a continuum of agents of each "type."

tion, then agents share a prior (Proposition 6). However, in this case the "shared prior" need not be an element of the core of each preference functional $I_i$. Rather, it is a common element of the Clarke differential of each $I_i$ at the proposed allocation, which Ghirardato and Siniscalchi (2012) interpret as the set of locally relevant priors. For this result, we build on a non-convex version of the Second Welfare Theorem due to Bonnisseau and Cornet (1988).

Thus, in the non-convex case, there is a gap between (1) and (2). Indeed we exhibit an Edgeworth-box economy (Example 3) in which both agents are GM-ambiguity-averse, there is a full-insurance allocation, there is a unique "prior" that belongs to the Clarke differentials of both agents' preference functionals at that allocation, and yet this prior belongs to the core for only one of the agents.

The main result of this paper closes this gap. As the above example indicates, GM-ambiguity aversion must be strengthened. We identify a class of preferences for which the core always contains the (normalized) Clarke differential of the representing functional $I_i$ at certainty. This class contains all GM-ambiguity-averse preferences that are convex or smooth, but also includes preferences that are neither smooth nor convex (Example 2). Theorem 9 then shows that, for this class of preferences, the existence of an interior, full-insurance, Pareto-efficient allocation is equivalent to the existence of a shared prior in the core of the agents' preference functionals. Moreover, each full-insurance Pareto-efficient allocation is a competitive equilibrium with transfers.

Finally, we provide additional examples that illustrate the results. In particular, we exhibit a non-convex, but GM-ambiguity-averse preference that accommodates the rankings in Machina (2009)'s reflection example, and satisfies the assumptions of Theorem 9 (Example 5).

# 2 Setup

## 2.1 Decision-theoretic framework

We follow RSS; see also Billot et al. (2000). We consider an Arrow-Debreu economy under uncertainty with finitely many states $S$, a single good that can be consumed in non-negative quantity, and $N$ consumers. This section and the next focus on the preferences of an individual consumer; to simplify notation, we do not use consumer indices. These will be introduced in Section 4, which deals with efficient and equilibrium allocations in the Arrow-Debreu economy under consideration.

Behavior is described by a preference relation $\succcurlyeq$ over bundles (contingent consumption plans) $f \in \mathbb{R}_+^S$. Preferences are represented by the pair $(I, u)$, where $u : \mathbb{R}_+ \to \mathbb{R}$ and, letting $\mathbb{U} = u(\mathbb{R}_+)$, $I : \mathbb{U}^S \to \mathbb{R}$; that is, for all $f, g \in \mathbb{R}_+^S$, $f \succcurlyeq g$ iff $I(u \circ f) \geq I(u \circ g)$, where for every $h = (h_1, \ldots, h_S) \in \mathbb{R}_+^S$, $u \circ h$ denotes the vector $\big(u(h_1), \ldots, u(h_S)\big) \in \mathbb{U}^S$. Assume throughout that every $I$ is normalized[2] ($I(1_S\gamma) = \gamma$ for every $\gamma \in \mathbb{U}$), locally Lipschitz and strongly monotonic (that is, $f \geq g$ and $f \neq g$ imply $f \succ g$), and $u$ is continuously differentiable, strictly concave, and strictly increasing.

To simplify notation, if $Q$ is any measure (not necessarily a probability measure) on $S$, then for every $a \in \mathbb{R}^S$, $Q(a) = \sum_s Q(s)a_s$. [Of course here a measure is characterized by a vector in $\mathbb{R}^S$, and sometimes we will treat $Q$ as such.]

## 2.2 Clarke differential

Given an open subset $B$ of $\mathbb{R}^S$, the **Clarke derivative** of a locally Lipschitz function $J : B \to \mathbb{R}$ at $b \in B$ in the direction $a \in \mathbb{R}^S$ is defined by

$$J^\circ(b; a) \equiv \limsup_{t \downarrow 0, c \to b} \frac{J(c + ta) - J(c)}{t}. \tag{1}$$

The **Clarke differential** of $J$ at $b \in B$ is then

$$\partial J(b) = \{Q \in \mathbb{R}^S : \forall a \in \mathbb{R}^S, Q(a) \leq J^\circ(b; a)\}. \tag{2}$$

If $J$ is monotonic, every element $Q$ of its Clarke differential at any given point is non-negative (Rockafellar, 1980, Theorem 6, Corollary 3).

The function $J$ is **nice** at $b \in B$ if $0_S \notin \partial J(b)$, where $0_S = (0, \ldots, 0) \in \mathbb{R}^S$: this notion is discussed and axiomatized in Ghirardato and Siniscalchi (2012). In particular, if $J$ is monotonic and concave, or if it is translation-invariant, it is nice everywhere in the interior of its domain.

[2]For virtually all preference models in the literature, either $I$ is normalized, or else an equivalent, normalized representation can be readily obtained.

The function $J$ is **regular** at $b \in B$ if its directional derivative

$$J'(b;a) = \lim_{t \downarrow 0} \frac{J(b+ta) - J(b)}{t} \tag{3}$$

is well-defined for all $a \in \mathbb{R}^S$, and coincides with $J^\circ(b;a)$: see Clarke (1983, Def. 2.3.4). If $J$ is continuously differentiable at $b$, then it is regular there (Clarke, 1983, Corollary to Proposition 2.2.1, and Proposition 2.3.6 (a)).

For the following two definitions, recall that preferences $\succcurlyeq$ are represented by $(I, u)$. First, the **normalized Clarke differential** of $I$ at $h \in \mathbb{U}^S$ is

$$C(h) = \left\{ \frac{Q}{Q(S)} : Q \in \partial I(u \circ h), Q \neq 0_S \right\}. \tag{4}$$

As noted in the Introduction, Ghirardato and Siniscalchi (2012) provide characterizations of the (normalized) Clarke differential; they also provide a behavioral condition that ensures that the functional representing preferences is locally Lipschitz (this condition is automatically satisfied by many popular decision models).

Second, define $I^u : \mathbb{R}^S_+ \to \mathbb{R}$ by $I^u(f) = I(u \circ f)$ for all $f \in \mathbb{R}^S_+$; thus, $f \succcurlyeq g$ iff $I^u(f) \geq I^u(g)$.[3]

**Remark 2.1** For every $i \in N$, the Clarke differential at $f \in \mathbb{R}^S_{++}$ of $I^u$ is

$$\partial I^u(f) = \left\{ Q^u \in \mathbb{R}^S : \forall h \in \mathbb{R}^S,\ Q^u(h) = \sum_s Q(s) u'(f(s)) h(s) \text{ for some } Q \in \partial I(u \circ f) \right\}.$$

# 3 The core, local beliefs, and normalized Clarke differentials

The **core** of the preference functional $I$ is defined as follows (Ghirardato and Marinacci, 2002):

$$\text{Core } I = \{P \in \Delta(S) : \forall a \in (\mathbb{U})^S,\ I(a) \leq P(a)\}. \tag{5}$$

A preference is **GM-ambiguity-averse** if its core is non-empty. As discussed in GM, this happens if there is a SEU preference (with the same Bernoulli utility $u$ used in the representation of $\succcurlyeq$) which acts as an "ambiguity-neutral" model for the preference $\succcurlyeq$.

[3]In RSS, preferences are assumed to be represented by some functional $J : \mathbb{R}^S_+ \to \mathbb{R}$: $f \succcurlyeq g$ iff $J(f) \geq J(g)$. The functional $I^u$ just defined corresponds to their $J$.

It is useful to relate the core to certain other sets of measures. The first plays a key role in RSS's result. For every bundle $f \in \mathbb{R}^S_+$, let

$$\pi(f) = \{P \in \Delta(S) : \forall g \in \mathbb{R}^S_+,\ I(u \circ g) \geq I(u \circ f) \text{ implies } P(g) \geq P(f)\}. \tag{6}$$

That is, $\pi(f)$ is the set of prices (normalized to lie in the unit simplex) such that any bundle that is weakly preferred to $f$ is not less expensive than $f$. This is the usual notion of "quasi-optimality." Alternatively, we can interpret each $P \in \pi(f)$ as representing a risk-neutral SEU preference whose better-than set at $f$ contains the better-than set of $\succcurlyeq$ at $f$.

RSS interpret it as a definition of *local beliefs*. They also introduce a condition, 'Translation invariance at certainty,' that ensures that $\pi(1_S x) = \pi(1_S)$ for all $x > 0$.

The second set of measures of interest is

$$\bar{\pi}(f) = \{P \in \Delta(S) : \forall g \in \mathbb{R}^S_+,\ I(u \circ g) \geq I(u \circ f) \text{ implies } P(u \circ g) \geq P(u \circ f)\}. \tag{7}$$

Each element $P$ of this set can be interpreted as representing an SEU preference (with the same Bernoulli utility $u$ used in the representation of $\succcurlyeq$), whose better-than set at $f$ contains the better-than set of $\succcurlyeq$ at $f$. Notice that, if one assumes that $u$ is linear, as in RSS (so that $I$ also reflects risk attitudes), then $\pi = \bar{\pi}$. Thus, $\pi(f)$ and $\bar{\pi}(f)$ differ only in that the measures in $\pi(f)$ are effectively risk-adjusted probabilities, whereas the measures in $\bar{\pi}(f)$ are not.[4]

Finally, the core is also related to the normalized differential of $I$ at certainty, i.e., $C(1_S x)$. The following result sheds some light on the relations between these sets[5]

**Proposition 1**

1. *for every* $x > 0$*,* Core $I \subseteq \bar{\pi}(1_S x) \subseteq \pi(1_S x)$*;*

2. *for every* $x > 0$*, if* $I$ *is nice at* $1_S u(x)$*, then* $\pi(1_S x) \subseteq C(1_S x)$*;*

3. Core $I = \bigcap_{x>0} \bar{\pi}(1_S x)$*;*

[4]Remark A.2 in the Appendix implies that, for any certain consumption bundle $f = 1_S x$, $\bar{\pi}(1_S x)$ is also the (normalized) Greenberg-Pierskalla differential of $I$ at $u \circ f$ (Greenberg and Pierskalla, 1973; Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio, 2008).

[5]In all parts of this result, the case $x = 0$ is excluded. This is because $\pi(1_S 0) = \bar{\pi}(1_S 0) = \Delta(S)$, and furthermore $\partial I(1_S \gamma)$ is not defined for $\gamma = 0$.

4. *for every* $x > 0$, Core $I \subseteq \partial I(1_S u(x))$; *hence,* Core $I \subseteq C(1_S x)$.

We illustrate some of these definitions and results in the following

**Example 1** Let $S = \{s_1, s_2\}$ and consider the risk-neutral preferences represented by

$$I(h) = \max\left(\left[\frac{1}{2}\sqrt{h_1} + \frac{1}{2}\sqrt{h_2}\right]^2, \epsilon + \min_{p\in[0.3,0.7]}[p h_1 + (1-p)h_2]\right)$$

for some small $\epsilon > 0$. Three indifference curves are depicted in Figure 1 (thick lines). The

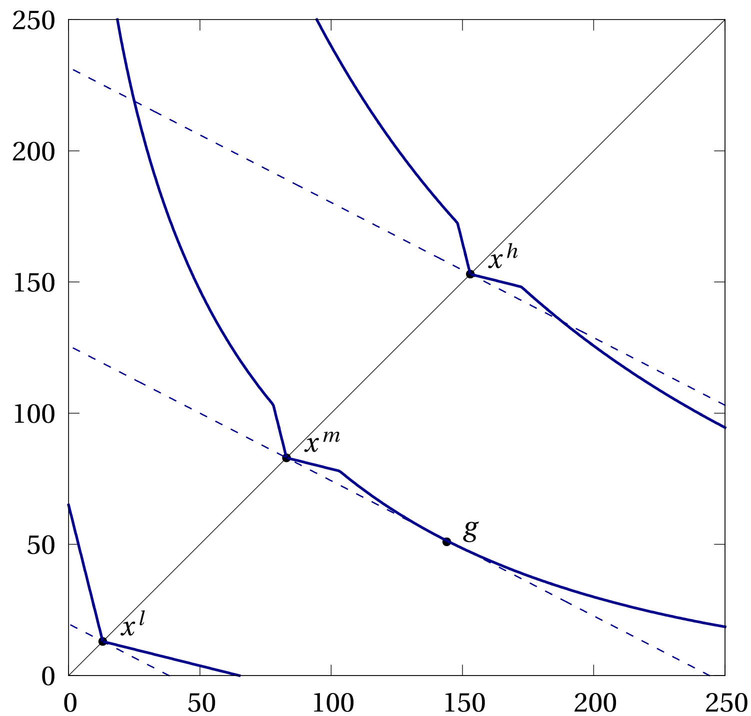


Figure 1: Relationship between the core, Clarke differential, and local beliefs

indifference curves have two features of interest. First, there is a small inward "dent" at certainty; in a neighborhood of the 45° line, this preference coincides with the risk-neutral MEU preference with priors $C = \{P \in \Delta(S) : 0.3 \le P(s_1) \le 0.7\}$. Second, away from the certainty line, indifference curves "flatten out" as they move farther away from the origin; thus, sufficiently far away from the origin, preferences are close to being risk-neutral EU with a uniform probability distribution on $S$, except in a neighborhood of the certainty line.[6]

[6]For bundles $g$ with high values of one of the coordinate and low values of the other, the preference again coincides with MEU (not shown in Figure 1). This is immaterial for the present purposes.

Since $u$ is the identity, the core of this preference is the set of probabilities that support the indifference curves of $I$ at *every* $1_S x$. There is a single such probability, namely the uniform distribution. For any other probability $P$ there is a sufficiently high prize $x > 0$ such that the line with slope determined by $P$ and going through $1_S x$ intersects the indifference curve of $I$ going through $1_S x$.

Next, consider the local belief sets $\pi(1_S x)$. Since $u$ is the identity, $\pi(1_S x)$ is the collection of probabilities that induce supporting lines at $1_S x$. Clearly, the core of $I$—the uniform distribution—is included in each $\pi(1_S x)$. However, the sets $\pi(1_S x)$ contain additional points. Furthermore, these sets are not constant: they shrink as $x$ increases. For instance, the dashed lines in Figure 1 are the level curves of probability distribution $P$ that supports the indifference curve at $1_S x^m$, and hence belongs to $\pi_1(1_S x^m)$. Furthermore, since $P$ is also tangent to the same indifference curve at the bundle $g \neq 1_S x^m$, any probability distribution that puts more weight on the vertical coordinate (and hence induces flatter level curves) cannot belong to $\pi(1_S x^m)$. However, by inspecting the level curves of $P$ going through $1_S x^\ell$ and $1_S x^h$ respectively it is apparent that (i) there are probabilities $P' \in \pi(1_S x^l)$ that induce flatter level curves than $P$, and (ii) $P$ itself does *not* belong to $\pi(1_S x^h)$.

Finally, since $I$ behaves like a MEU preference with priors $C$ around certainty, we have $C(1_S x) = \partial I(1_S u(x)) = C$ for all $x > 0$. In particular, this functional is nice at certainty, so part 2 of Proposition 1 applies. Note however that, for this preference, Core $I \subsetneq \pi(1_S x) \subsetneq C(1_S x)$ for every $x > 0$. □

Example 1 shows that the core and the Clarke differential at certainty may differ. However, there is a sufficient and, under additional assumptions, necessary condition that ensures consistency between local behavior at certainty and global behavior. Consider the following definition.[7]

**Definition 1** The functional $I$ is **differentially quasiconcave at** $b \in (\mathrm{int}(\mathbb{U}))^S$ if

$$\forall a \in \mathbb{U}^S, \qquad I(a) \geq I(b) \quad \Longrightarrow \quad \forall Q \in \partial I(b), \quad Q(a-b) \geq 0. \tag{8}$$

The functional $I$ satisfies **differential quasiconcavity at certainty (DQC)** if it is differentially

[7] $\mathrm{int}(\mathbb{U})$ denotes the interior of $\mathbb{U}$.

quasiconcave at $1_S\gamma$ for all $\gamma \in \text{int}(\mathbb{U})$.

The intuition for this definition is sharpest in case $I$ is continuously differentiable at a point $b = u \circ g$, in which case the Clarke differential equals the gradient of $I$ at $b$. In this case, $I$ is differentially quasiconcave at $b$ if, whenever a bundle $f$ (having utility profile $a$) is weakly preferred to $g$, then moving from $g$ in the direction of $f$ by a small (infinitesimal) amount is also beneficial. Proposition 3.1 in Penot and Quang (1997) implies that a continuous and strictly monotonic function is quasiconcave *if and only if* it satisfies Eq. (8) everywhere on its domain. The key observation is that condition DQC requires that Eq. (8) hold *only at certainty*. This allows for violations of quasiconcavity elsewhere on its domain, as the following example illustrates. We shall also demonstrate in Section 5 that such violations can accommodate interesting patterns of behavior.

**Example 2** Let $S = \{s_1, s_2\}$ and consider the risk-neutral VEU preferences defined by

$$I(h) = \frac{1}{2}(h_1 + h_2) - \max\left(\log\left(1 + \frac{1}{4}(h_1 - h_2)^2\right), \left|\frac{1}{2}\theta(h_1 - h_2)\right|\right).$$

At each point $1_S x$, the upper-contour sets of this preference are contained in the upper-contour sets of the risk-neutral MEU preference characterized by the priors $C = \{P \in \Delta(S) : \frac{1}{2}(1+\theta) \geq P(\{s_1\}) \geq \frac{1}{2}(1-\theta)\}$; denote the functional representation of this MEU preference by $J$. For $h$ sufficiently close to the 45° line, and for $h$ sufficiently far from it, $I(h) = J(h)$; for bundles $h$ at an intermediate distance from the 45° line, the indifference curves of $I$ are bent inward, so $I(h) \leq J(h)$. See Figure 2.

This preference is thus neither convex nor smooth. Its core is $C$, which is also its Clarke differential at any point on the 45° line. Condition DQC holds; to see this, note that, if $I(h) \geq I(x)$, then also $J(h) \geq J(x)$; since $J$ is concave, Proposition 3 below implies that it satisfies condition DQC, so $Q(1_S x; h - x) \geq 0$ for every $Q \in \partial J(1_S x) = \partial I(1_S x)$. □

On the other hand, the preferences in Example 1 do not satisfy condition DQC. For instance, consider the point $g$: since it lies on the indifference curve going through $1_S x^m$, it is indifferent to it, but if $Q \in \partial I(1_S x^m)$ is the probability that assigns weight 1 to the vertical coordinate, clearly $Q(g - 1_S x^m) < 0$.

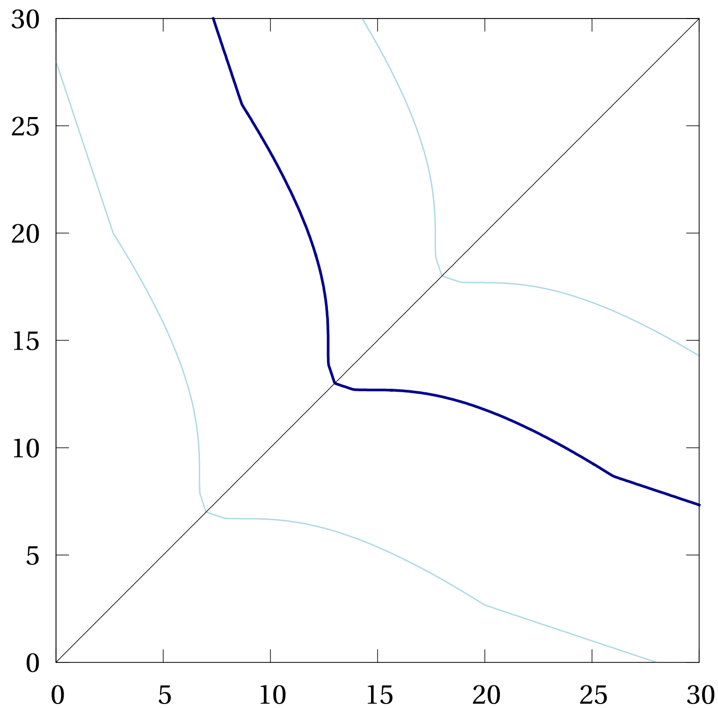


Figure 2: Indifference curves of a non-smooth, non-convex preference ($\theta = \frac{1}{2}$)

The following result, and its corollary, show that condition DQC provides the required tight connection between the core of each $I$ and its normalized Clarke differentials at certainty.

**Proposition 2**

1. *If DQC holds, then* $\bigcap_{x>0} C(1_S x) \subseteq \mathrm{Core}\ I$ *(so, by Proposition 1,* $\bigcap_{x>0} C(1_S x) = \mathrm{Core}\ I$*);*

2. *For every* $x > 0$*, if* $C(1_S x) \subseteq \mathrm{Core}\ I$*, then* $I$ *is differentially quasiconcave at* $1_S u(x)$*.*

3. *If* $\bigcup_{x>0} C(1_S x) \subseteq \mathrm{Core}\ I$*, then DQC holds.*

Condition DQC always holds in two important cases, the first of which was already noted above.

**Proposition 3** *DQC holds if one of the conditions below is satisfied:*

1. $I$ *is quasiconcave, or*

2. $\mathrm{Core}\ I \neq \emptyset$ *and* $I$ *is regular at every* $1_S \gamma$*,* $\gamma > 0$*.*

Condition DQC holds for all preference models considered in RSS, because they all satisfy convexity. Furthermore, recall that if a function is continuously differentiable at a point, it is regular there; hence, condition DQC also applies to all smooth representations of GM-ambiguity-averse preferences. However, as Example 2 suggests, this condition allows for non-differentiabilities, in addition to non-convexities.

Finally, recall from Proposition 1 that, for any $x > 0$, $\pi(1_S x) \subseteq C(1_S x)$, provided $I$ is nice at certainty; however, Example 1 shows that in general $\pi(1_S x)$ and $C(1_S x)$ may differ. It turns out that, under DQC, RSS's local belief sets always contain the corresponding normalized Clarke differentials, so, under niceness, the two sets coincide. Indeed, under niceness $C(1_S x) \subseteq \bar{\pi}(1_S x)$ as well, so that then all the sets $\pi(1_S x)$, $\bar{\pi}(1_S x)$ and $C(1_S x)$ coincide.

**Proposition 4** *If DQC holds, then for every $x > 0$, $C(1_S x) \subseteq \bar{\pi}(1_S x)$, and therefore $C(1_S x) \subseteq \pi(1_S x)$. If in addition $I$ is nice at $1_S u(x)$, then $C(1_S x) = \bar{\pi}(1_S x) = \pi(1_S x)$.*

The preference in Example 2 is not quasiconcave, but it does satisfy DQC and niceness (because in a neighborhood of certainty it coincides with a MEU preference). Consistently with Proposition 4, the normalized Clarke differential and local belief sets coincide (and also happen to be constant across all $x > 0$).

# 4 Risk Sharing

## 4.1 Notation and preliminaries

An economy is a tuple $(N, (\succcurlyeq_i, \omega_i)_{i \in N})$, where $N$ is the collection of agents, and for every $i$, agent $i$ is characterized by preferences $\succcurlyeq_i$ over $\mathbb{R}^S_+$ and has an endowment $\omega_i \in \mathbb{R}^S_+$.

We fix throughout the set $N$ of consumers, and their preferences $(\succcurlyeq_i)_{i \in N}$; further, we assume that each $\succcurlyeq_i$ is represented by $(I_i, u_i)$ as in Section 2. We let the endowments vary, so long as the aggregate endowment is constant and strictly positive, as in RSS. Formally, we consider the collection $\bar{\mathscr{E}}$ of economies $(N, (\succcurlyeq_i, \omega_i)_{i \in N})$ in which $\sum_i \omega_i \equiv 1_S \bar{\omega}$ for some $\bar{\omega} > 0$.

An *allocation* is tuple $(f_1, \ldots, f_N)$ such that $f_i \in \mathbb{R}^S_+$ for every $i \in N$; as usual $f_i$ is the contingent-consumption bundle assigned to agent $i$. For any economy $(N, (\succcurlyeq_i, \omega_i)_{i \in N})$, the allocation $(f_1, \ldots, f_N)$ is *feasible* if $\sum_i f = \sum_i \omega_i$; it is a *full-insurance allocation* if, for every consumer

$i$, $f = 1_S x$ for some $x \in \mathbb{R}_+$; it is *Pareto-efficient* if it is feasible, and there is no other feasible allocation $(g_1, \ldots, g_N)$ such that $g \succcurlyeq f$ for all $i$, and $g_j \succ_j f_j$ for some $j$.

It is useful to state the main result of Rigotti et al. (2008) for convex preferences.[8]

**Theorem 5 (cf. Rigotti et al. (2008), Proposition 9)** *Fix an economy $(N, (\succcurlyeq_i, \omega_i)_{i \in N}) \in \bar{\mathscr{E}}$. In addition to the maintained assumptions in Sec. 2, suppose that every $\succcurlyeq_i$ is strictly convex,[9] and that $\pi_i(1_S x) = \pi_i(1_S)$ for every $x > 0$. Then the following are equivalent:*

*(i) There exists an interior, full-insurance Pareto-efficient allocation;*

*(ii) Any Pareto-efficient allocation is a full-insurance allocation;*

*(iii) Every feasible, full-insurance allocation is Pareto-efficient;*

*(iv) $\bigcap_i \pi_i(1_S) \neq \emptyset$.*

The implications (ii) ⇒ (iii) ⇒ (i) hold for general preferences.[10] However, the implication (i) ⇒ (iv) uses the standard Second Welfare Theorem, which requires convexity. The argument for (iv) ⇒ (ii) also invokes convexity.

## 4.2 Necessary and sufficient conditions for efficiency

We now state two results that are reminiscent of the implications (i) ⇒ (iv) and (iv) ⇒ (ii) of Theorem 5, and are economically interesting in their own right. As argued in the Introduction, we then point out that there is a "gap" between these results, and show how to close it.

The first result generalizes the standard result that smooth indifference curves must be tangent at any interior Pareto-efficient allocation. With convex preferences, the common slope at the point of tangency determines a supporting price vector; as we discuss momentarily, a "local price vector" is also identified in the non-convex, non-smooth case, though the

[8] Strictly speaking, the assumptions in Theorem 5 are slightly stronger than those in RSS's Proposition 9. Specifically, we maintain the assumption that each $I_i$ is locally Lipschitz; RSS only assume continuity. We retain all our assumptions to streamline the exposition. Also note that all the parametric representations considered in RSS are concave, and hence locally Lipschitz.

[9] That is, for $f, g \in \mathbb{R}^S_+$ with $f \neq g$, $f \succcurlyeq_i g$ implies $\alpha f + (1-\alpha) g \succ_i g$ for all $\alpha \in (0,1)$.

[10] For (ii) ⇒ (iii), the key step is in Remark A.4, which follows from standard results. See also the proof of Proposition 7.

sense in which it "supports" the allocation is more delicate (see below). Thus, the following result can also be viewed as a local version of the Second Welfare Theorem.

**Proposition 6** *Fix an economy* $(N,(\succcurlyeq_i,\omega_i)_{i\in N})\in\bar{\mathscr{E}}$. *Let* $(f_i)_{i\in N}$ *be an allocation such that each functional* $I_i$ *is nice at* $u_i\circ f_i$. *If* $(f_i)_{i\in N}$ *is Pareto-efficient, then there exists a price vector* $p\in\mathbb{R}^S_+\setminus\{0\}$ *and, for each* $i\in N$, *scalars* $\lambda_i>0$ *and vectors* $Q_i^u\in\partial I_i^u(f_i)$ *such that* $p=\lambda_i Q_i^u$ *for every* $i$. *In particular, if* $(f_i)_{i\in N}$ *is a full-insurance allocation, then for each* $i\in N$ *there are scalars* $\mu_i>0$ *and vectors* $Q_i\in\partial I_i(u_i\circ f_i)$ *such that* $p=\mu_i Q_i$; *therefore,* $\bigcap_{i\in N}C_i(f_i)\neq\emptyset$.

The key step in the proof of the first claim is provided by Bonnisseau and Cornet (1988), who show that, under the stated assumptions, there is a vector $p$ such that $-p$ lies in the intersection of the Clarke normal cones of the upper contour set of $I_i^u$ at the bundle $f_i$ (see the Appendix for a precise statement and a definition of the required terms). If preferences are convex, this set coincides with the normal cone of the upper contour set of $I_i^u$ at $f_i$ in the sense of convex analysis. (Indeed Clarke's notion of normal cone is meant as a generalization of the normal cone of convex analysis.) This suggests interpreting $p$ as a "local price vector."

The second claim states that, if the Pareto-efficient allocation $(f_i)_{i\in N}$ is a *full-insurance* allocation, then the normalized Clarke differentials of the functionals $I_i$ themselves have non-empty intersection. For arbitrary Pareto optimal allocations, this conclusion only applies to the (normalized) Clarke differentials of the composite functional $I_i^u$.

Thus, if a full-insurance allocation is Pareto-efficient, then (up to rescaling) the Clarke differentials of the agents' functionals $I_i$ at that allocation have non-empty intersection. One may then wonder if the converse is also true: is it the case that, if the normalized Clarke differentials at some full-insurance allocation intersect, that allocation is Pareto-efficient? The next example shows that the answer is negative.

**Example 3** Interpret Figure 1 as an Edgeworth box: agent 1 has preferences inducing the solid indifference curves, whereas agent 2 has risk-neutral preferences inducing the dashed lines as indifference curves; of course, for agent 2, utility increases in the south-western direction. (Both consumers could be made strictly risk-averse without changing the analysis.)

Notice that the allocation $\left(1_S x^h, 1_S(\bar{\omega}-x^h)\right)$ provides full insurance. The normalized Clarke differential of $I_1$ at $1_S x^h$ contains that of $I_2$, which coincides with the probability $P$ representing 2's preferences. However, this allocation is not Pareto-efficient.

Also recall that, in this example, the core of $I_1$ is the uniform probability; hence, Core $I_1 \cap$ Core $I_2 = \emptyset$. Finally, note that the allocation $(g, 1_S\bar{\omega} - g)$ is Pareto-efficient, but does not provide full insurance. □

The intuition behind this example is as follows. Clarke differentials provide information about the local behavior of preferences (again, see Ghirardato and Siniscalchi, 2012, for a precise characterization). If the normalized Clarke differentials have non-empty intersection at an allocation, then *locally* there are no mutually beneficial trades. However, the notion of Pareto-efficiency involves more than just local comparisons: there may be Pareto-superior allocations sufficiently far from the given one. This is indeed the case for the allocation $\left(1_S x^h, 1_S(\bar{\omega} - x^h)\right)$. Thus, the example suggests that, in order to obtain a converse to Proposition 6, one needs to refer to a set of priors that conveys *global* information about preferences.

The second result we present shows that the core of the functional $I_i$ provides the required global information.

**Proposition 7** *If $\bigcap_i$ Core $I_i \neq \emptyset$, then, in any economy $(N, (\succcurlyeq_i, \omega_i)_{i \in N}) \in \bar{\mathcal{E}}$, a feasible allocation is Pareto-efficient if and only if it provides full insurance. Moreover, such an allocation is a competitive equilibrium allocation (with transfers).*

Thus, if the cores intersect, not only are full-insurance allocations efficient: they are the *only* efficient allocations. This generalizes the standard result that, if agents share a common prior, then an interior allocation is Pareto-efficient if and only if it provides full insurance.

Observe that the equivalence between Pareto-efficiency and full insurance obtains for all economies with the given agents and no aggregate uncertainty. This is because the assumption that the cores intersect involves only the agents' preferences, and is independendent of their endowments.

Proposition 7 also strengthens the conclusion of Proposition 6. If the cores intersect, then any probability $P \in \bigcap_i$ Core $I_i$ yields a price vector that supports any feasible, full-insurance allocation as a competitive equilibrium in the usual sense. By way of contrast, the vector $p$

identified in Proposition 6 is a supporting price only in the local sense discussed above.

Further, observe that, in Example 3, the set of Pareto-efficient allocations does not coincide with the set of full-insurance allocations. Proposition 7 states that a non-empty intersection of the cores is sufficient for these sets to coincide. Is this condition also necessary? The following example shows that it is not.

**Example 4** Let $S = \{s_1, s_2\}$. Assume that consumer 2 has EU preferences, with a prior $P$ that assigns probability 0.4 to state $s_1$ (on the horizontal axis) and power utility $u(x) = x^{0.2}$. Consumer 1 has preferences represented by

$$I_1(h) = \max\left(\frac{1}{2}h_1 + \frac{1}{2}h_2, \delta + \min_{p\in[0,1]}[p h_1 + (1-p)h_2]\right).$$

Thus, consumer 1's preferences are risk-neutral EU, with a uniform prior, except within $\delta$ of the certainty line. The value of $\delta$ is chosen so that, given the curvature of 2's utility function, there is no tangency anywhere except at certainty. Then, in this economy, a feasible allocation

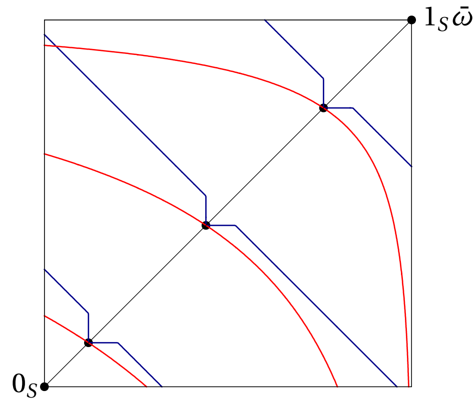


Figure 3: Relationship between the core, Clarke differential, and local beliefs

is Pareto-efficient if and only if it provides full insurance. Both preferences are GM-ambiguity-averse. In particular, as in Example 1, the core of $I_1$ consists solely of the uniform measure, whereas the core of 2's EU preference functional is $\{P\}$. Thus, the intersection of the cores is empty, even though the sets of Pareto-efficient and full-insurance allocations coincide.

Note also that the normalized Clarke differentials are constant at certainty: they equal $\Delta(S)$ for consumer 1, and $\{P\}$ for consumer 2. Consequently, at any $x > 0$, the core and the normalized Clarke differential of $I_1$ at $1_S x$ differ. $\square$

To sum up, the condition that $\bigcap_i C_i(1_S x_i) \neq \emptyset$ is necessary for the full-insurance allocation $(1_S x_1, \ldots, 1_S x_N)$ to be Pareto-efficient (Proposition 6), but it is not sufficient (Example 3). On the other hand, the condition that $\bigcap_i \text{Core } I_i \neq \emptyset$ is sufficient (Proposition 7) but not necessary (Example 4). This points to a gap between Propositions 6 and 7. We now propose conditions aimed at closing this gap.

### 4.3 Closing the gap

We begin with a preliminary result. Note that the gap between Propositions 6 and 7 is ultimately due to the fact that, for every agent $i$ and every $x > 0$, Core $I_i \subseteq C_i(1_S x)$, but the inclusion may be strict (as in Examples 3 and 4). However, the results in Section 3 provide a way to close the gap between Core $I_i$ and $C_i(1_S x)$ at some suitable constant bundle $1_S x$. In particular, under Condition DQC, part (1) of Proposition 2 immediately implies that this is the case if, for every agent $i$, there is some $x_i^* > 0$ such that the set $C_i(1_S x_i^*)$ is *minimal*, in the sense that it is contained in all other sets $C_i(1_S x)$, $x > 0$. Intuitively, $x_i^*$ is a certain consumption bundle where the functional $I_i$ is "least kinked." Notice that this assumption can be characterized in terms of preferences via Theorem 6 in Ghirardato and Siniscalchi (2012).

Then, we can consider an economy in $\bar{\mathscr{E}}$ in which the aggregate endowment is $1_S \sum_i x_i^*$, so that the allocation $(1_S x_1^*, \ldots, 1_S x_N^*)$ is feasible. In this economy, a non-empty intersection of the normalized Clarke differentials at this allocation—equivalently, a non-empty intersection of the cores—is both necessary (Proposition 6) and sufficient (Proposition 7) for the allocation to be Pareto-efficient. Furthermore, under this condition, in *every* economy in $\bar{\mathscr{E}}$, every feasible, full-insurance allocation is Pareto-efficient, and indeed there are no other Pareto-efficient allocations. This leads to the following result.

**Theorem 8** *Assume that, for every $i \in N$, DQC holds, and there is $x_i^* > 0$ such that $C_i(1_S x_i^*) \subseteq C_i(1_S x)$ for all $x > 0$, and $I_i$ is nice at $1_S u_i(x_i^*)$. Then the following are equivalent:*

*(i) The allocation $(1_S x_1^*, \ldots, 1_S x_N^*)$ is Pareto-efficient in some economy $\mathscr{E}^* \in \bar{\mathscr{E}}$;*

*(ii) For every $\mathscr{E} \in \bar{\mathscr{E}}$, every Pareto-efficient allocation is a full-insurance allocation;*

*(iii) For every $\mathscr{E} \in \bar{\mathscr{E}}$, every feasible, full-insurance allocation is Pareto-efficient;*

*(iv) $\bigcap_i \text{Core}\, I_i \neq \emptyset$.*

*Furthermore, under the above equivalent conditions, for every $\mathscr{E} \in \bar{\mathscr{E}}$, every interior, feasible full-insurance allocation is a competitive equilibrium with transfers.*

Observe that, under the assumptions in Theorem 8, every preference $\succcurlyeq_i$ is GM-ambiguity-averse. The reason is that, as noted above, DQC implies that Core $I_i = C_i(1_S x_i^*)$ at the certain consumption $x_i^*$ where the normalized differential is minimal; under niceness, $C_i(1_S x_i^*) \neq \emptyset$, so Core $I_i \neq \emptyset$.

**Proof:** For every $\mathscr{E} \in \bar{\mathscr{E}}$, if every Pareto-efficient allocation is a full-insurance allocation, then every feasible, full-insurance allocation is Pareto-efficient. Hence, (ii) $\Rightarrow$ (iii). Clearly (iii) $\Rightarrow$ (i): consider the economy $\mathscr{E}^*$ in which $\omega_i = 1_S x_i^*$ for each $i$; then $\sum_i \omega_i = 1_S \sum_i x_i^*$, a constant bundle which by assumption is strictly positive, so $\mathscr{E}^* \in \bar{\mathscr{E}}$, and of course $(1_S x_1^*, \dots, 1_S x_N^*)$ is feasible in $\mathscr{E}^*$ and provides full insurance; under (iii), it is Pareto-efficient. Finally, if $\bigcap_i$ Core $I_i \neq \emptyset$, then for every $\mathscr{E} \in \bar{\mathscr{E}}$, Proposition 7 implies that every Pareto-efficient allocation must be a full-insurance allocation; thus, (iv) $\Rightarrow$ (ii).

To complete the argument, we show that (i) $\Rightarrow$ (iv). Since under (i) the allocation $(1_S x_1^*, \dots, 1_S x_N^*)$ is Pareto-efficient and provides full insurance in some economy $\mathscr{E} \in \bar{\mathscr{E}}$, and each $I_i$ is nice at $1_S u_i(x_i^*)$, by Proposition 6 $\bigcap_i C_i(1_S x_i^*) \neq \emptyset$. Consider an agent $i$. Since $C_i(1_S x_i^*) \subseteq C_i(1_S x)$ for all $x > 0$, $C_i(1_S x_i^*) = \bigcap_{x>0} C_i(1_S x)$; since DQC holds, by Proposition 2 part (i), Core $I_i = C_i(1_S x_i^*)$. Since this is true for all agents $i \in N$, $\bigcap_i$ Core $I_i = \bigcap_i C_i(1_S x_i^*) \neq \emptyset$. ■

Observe that, differently from Theorem 5 and RSS's original result, conditions (ii) and (iii) in Theorem 8 apply to all the economies in $\bar{\mathscr{E}}$, regardless of their (riskless) aggregate endowment, rather than to one economy with a given, riskless aggregate endowment. Notice however that Theorem 5 and RSS's original result could also be stated by quantifying over economies in conditions (ii) and (iii). The reason is that condition (iv) only concerns preferences, and the implications (iv) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (i) apply to any economy without aggregate uncertainty.

We now provide a condition under which the aggregate endowment can be fixed, so as to obtain a result that is more directly comparable to Theorem 5 and RSS's Proposition 9.

Consider the following definition, which can be viewed as the natural counterpart of RSS's

"Translation Invariance at Certainty" in our setting.

**Definition 2** Let $\succcurlyeq$ be represented by $(I, u)$. Then $\succcurlyeq$ satisfies **condition IDC** (Invariant normalized Differentials at Certainty) if $C(1_S x) = C(1_S)$ for all $x > 0$.

Again, condition IDC can be readily characterized in terms of preferences via Theorem 6 in Ghirardato and Siniscalchi (2012). Clearly, under this assumption, every certain consumption bundle is "minimal," so the following result is obtained along the lines of Theorem 8.

**Theorem 9** *Fix an economy $(N, (\succcurlyeq_i, \omega_i)_{i\in N} \in \bar{\mathscr{E}}$. Assume that, for every $i \in N$, $I_i$ is nice at $1_S u_i(x)$ for every $x > 0$, and that conditions DQC and IDC hold. Then the following are equivalent:*

*(i) There exists an interior, full-insurance Pareto-efficient allocation;*

*(ii) Any Pareto-efficient allocation is a full-insurance allocation;*

*(iii) Every feasible, full-insurance allocation is Pareto-efficient;*

*(iv)* $\bigcap_i \mathrm{Core} I_i = \bigcap_i \bar{\pi}(1_S) = \bigcap_i \pi_i(1_S) = \bigcap_i C_i(1_S) \neq \emptyset$.

*Furthermore, under the above equivalent conditions, every interior, feasible full-insurance allocation is a competitive equilibrium with transfers.*

**Proof:** The implications (ii) ⇒ (iii) and (iv) ⇒ (ii) are as in Theorem 8. The implication (iii) ⇒ (i) is trivial. For (i) ⇒ (iv), by IDC and DQC Proposition 2 implies that Core $I_i = C_i(1_S x) = C_i(1_S)$ for all $i$ and all $x > 0$. If $(1_S x_1, \ldots, 1_S x_n)$ is an interior, full-insurance Pareto-efficient allocation, since each $I_i$ is nice at $1_S u_i(x_i)$, Proposition 6 implies that $\bigcap_i C_i(1_S x_i) \neq \emptyset$, so equivalently $\bigcap_i$ Core $I_i \neq \emptyset$; the other equalities follow from Proposition 4. ■

To relate our result to RSS's (Theorem 5), recall that, under the assumptions of Theorem 9, $\pi_i(1_S x) = C_i(1_S x)$ for all $x > 0$ by Proposition 4. Hence, condition IDC actually coincides with RSS's assumption of "Translation Invariance at Certainty." Per Proposition 3, DQC is implied by RSS's convexity assumption. Thus, if preferences are convex, the only difference between Theorems 9 and 5 is the assumption of niceness at certainty (see also footnote 8); if furthermore preferences have a concave representation, this assumption holds, so for such preferences Theorem 5 follows directly from Theorem 9.

We briefly compare Theorems 8 and 9. As noted above, the latter is meant to provide a more direct counterpart to RSS's original result. On the other hand, the former puts the emphasis on the role of aversion to ambiguity in risk sharing. In particular, risk sharing does not require that preferences satisfy condition IDC (or RSS's "Translation Invariance at Certainty."), if one is willing to consider the class $\bar{\mathscr{E}}$ of economies with fixed preferences but varying riskless aggregate endowment.

# 5 More Examples

In this section we provide three additional examples that illustrate our results. Example 5 shows that our Theorem 9 can accommodate behaviorally interesting preferences that are not covered by prior results on risk-sharing. Example 6 considers the special case of invariant biseparable preferences (Ghirardato, Maccheroni, and Marinacci, 2004). Finally, Example 7 emphasizes the need for assumptions such as IDC and GM-ambiguity aversion, even if preferences are convex.

**Example 5 (Smooth VEU preferences)** A convenient class of preferences that satisfies all conditions of Theorem 9, but is not necessarily covered by RSS's result, is the family of VEU preferences that are both smooth (hence, regular) and GM-ambiguity-averse, but not necessarily convex. These preferences admit a representation $(I, u)$ with[11]

$$I(a) = P(a) + A\big(P(\zeta_0 a), \ldots, P(\zeta_{J-1} a)\big),$$

where $P \in \Delta(S)$, $0 \leq J \leq |S|$, each $\zeta_j \in \mathbb{R}^S$ (an *adjustment factor*) satisfies $P(\zeta_j) = 0$, and $A : \mathbb{R}^J \to \mathbb{R}$ (the *adjustment function*) is continuously differentiable and satisfies $A(\phi) = A(-\phi)$ for all $\phi \in \mathbb{R}^J$, and $A \leq 0$. To ensure strict monotonicity, additionally assume that $P(\{s\}) > 0$ for all $s$ and, for all $a \in \mathbb{U}^S$ and $s \in S$, $1 + \sum_{0 \leq j < J} \frac{\partial A}{\partial \phi_j}(P(\zeta_0 a), \ldots, P(\zeta_{J-1} a))\zeta_j(s) > 0$. Note that $I$ is translation-invariant, so IDC holds; it is GM-ambiguity-averse, so Core $I \neq \emptyset$; and it is regular at certainty, so by Proposition 3 it satisfies DQC. Furthermore, by Propositions 1 and 2, and straightforward calculations (in the Appendix),

$$\text{Core } I = \pi(1_S) = \bar{\pi}(1_S) = C(1_S) = \{P\}.$$

[11] If $a, b : \mathbb{U}^S \to \mathbb{R}$, "$ab$" denotes the function that assigns the value $a(s)b(s)$ to each state $s$.

Smooth, GM-AA VEU preferences can provide a tractable model of behavior that can be deemed averse to ambiguity, even though it is inconsistent with convexity of preferences. To illustrate, we show that they can accommodate the modal preferences in the "reflection example" of Machina (2009) (see also Baillon et al., 2011). Let $S = \{s_1, s_2, s_3, s_4\}$ and assume that the events $\{s_1, s_2\}$ and $\{s_3, s_4\}$ are unambiguous and equally likely, but no further information is provided as to the relative likelihood of $s_1$ vs. $s_2$ and $s_3$ vs. $s_4$. Furthermore, the draw of $s_1$ vs. $s_2$ and $s_3$ vs. $s_4$ are perceived as being independent. Consider the bets in Table 1.

| | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|
| $f^1$ | \$4,000 | \$8,000 | \$4,000 | \$0 |
| $f^2$ | \$4,000 | \$4,000 | \$8,000 | \$0 |
| $f^3$ | \$0 | \$8,000 | \$4,000 | \$4,000 |
| $f^4$ | \$0 | \$4,000 | \$8,000 | \$4,000 |

Table 1: Machina's reflection example. Reasonable preferences: $f^1 \prec f^2$ and $f^3 \succ f^4$

Machina (2009) argues on the basis of symmetry considerations that the preference ranking $f^1 \prec f^2$ and $f^3 \succ f^4$ is plausible and consistent with aversion to ambiguity; L'Haridon and Placido (2010) verify that these rankings do occur in an experimental setting. However, Baillon et al. (2011) show that preference models that satisfy convexity cannot accommodate this behavior, while respecting natural probabilistic formulations of the noted symmetry and independence assumptions. We now demonstrate that, by way of contrast, smooth, GM-AA VEU preferences can do so. A similar example is provided in Siniscalchi (2009), but the VEU preferences described there are not smooth and violate DQC.

Assume a uniform baseline prior $P$ and two adjustment factors $\zeta_0, \zeta_1 \in \mathbb{R}^S$:

$$\zeta_0 = [1, -1, 0, 0] \qquad \text{and} \qquad \zeta_1 = [0, 0, 1, -1].$$

The adjustment function takes the form

$$A(\phi) = A(\phi_0, \phi_1) = -\frac{1}{2}\theta \sum_{j=0,1} \log\left(1 + \frac{\phi_j^2}{\theta}\right)$$

where $\theta \in (0, 4)$; note that $\lim_{\theta \to 0} A(\phi) = 0$, so the limiting case $\theta = 0$ corresponds to EU. We verify in Appendix A.5 that this specification of the parameters $P, A, \zeta_0, \zeta_1$ yields a strictly mono-

| Act | $P(\zeta_0 u \circ f^k)$ | $P(\zeta_1 u \circ f^k)$ | Adjustment (omitting $\frac{1}{2}\theta$) |
|---|---|---|---|
| $f^1$ | $\alpha - 1$ | $\alpha$ | $-\log(1+\theta^{-1}(\alpha-1)^2) - \log(1+\theta^{-1}\alpha^2)$ |
| $f^2$ | $0$ | $1$ | $-\log(1+\theta^{-1})$ |
| $f^3$ | $-1$ | $0$ | $-\log(1+\theta^{-1})$ |
| $f^4$ | $-\alpha$ | $1-\alpha$ | $-\log(1+\theta^{-1}\alpha^2) - \log(1+\theta^{-1}(1-\alpha)^2)$ |

Table 2: Adjustments

tonic preference, and that higher values of $\theta$ correspond to greater ambiguity aversion in the sense of GM. Finally, let $u(0) = 0$, $u(8,000) = 4$, and $u(4,000) = 4\alpha$, for some $\alpha \in (0,1)$.

All four acts $f^1, \ldots, f^4$ have the same expected baseline utility: $P(u \circ f^k) = 2\alpha + 1$ for $k = 1, \ldots, 4$. Hence, their ranking is entirely determined by the adjustment terms $A(P(\zeta_0 u \circ f^k), P(\zeta_i u \circ f^k))$. These are displayed in Table 2.

In order to generate the preferences $f^1 \prec f^2$, we need to ensure that $A(P(\zeta_0 u \circ f^1), P(\zeta_1 u \circ f^1)) < A(P(\zeta_0 u \circ f^2), P(\zeta_1 u \circ f^2))$. Notice that, since $(\alpha - 1)^2 = (1-\alpha)^2$, this will also ensure that $A(P(\zeta_0 u \circ f^3), P(\zeta_1 u \circ f^3)) > A(P(\zeta_0 u \circ f^4), P(\zeta_1 u \circ f^4))$ and therefore $f^3 \succ f^4$, as the adjustments for $f^1$ and $f^2$ are the same as the adjustments for $f^4$ and $f^3$ respectively. Thus, we require

$$-\log(1+\theta^{-1}(\alpha-1)^2) - \log(1+\theta^{-1}\alpha^2) < -\log(1+\theta^{-1})$$

which, as shown in Appendix A.5, holds iff $0 < \theta < \frac{\alpha(1-\alpha)}{2}$. $\square$

**Example 6 (Invariant Biseparable preferences)** A preference is *invariant biseparable* (Ghirardato et al., 2004) if its representation $(I, u)$ is such that $I$ is positively homogeneous and translation-invariant on its domain. We now show that MEU preferences are the only invariant biseparable preferences for which the conditions of Theorems 8 or 9 hold. Recall that, similarly, the only invariant biseparable preferences to which the results in RSS apply —i.e., the convex invariant biseparable preferences— are MEU preferences. Thus, both RSS's main risk-sharing result and our Theorems 8 and 9 only add to the risk-sharing result in Billot et al. (2000) insofar as they apply to preferences that do not satisfy either positive homogeneity or translation invariance.

Recall from Ghirardato et al. (2004) that, for an invariant biseparable preference represented by $(I,u)$, the functional $I$ admits a unique extension to all of $\mathbb{R}^S$, and the Clarke differential $\partial I(0_S)$ consists of probability measures, and coincides with the Clarke differential at any point on the certainty line. Hence, $I$ is nice at $1_S u(x)$ for every $x>0$, and condition IDC holds.

Let $C=\partial I(0_S)=\partial I(1_S u(x))\subseteq\Delta(S)$ for any $x\geq 0$. By Proposition 2, condition DQC holds *if and only if* $C=\bigcap_{x>0}C_i(1_S x)=\bigcup_{x>0}C_i(1_S x)=\text{Core } I$. But by Proposition 16 in Ghirardato et al. (2004), $C=\text{Core } I$ if and only if $I$ is concave.

Hence, an invariant biseparable preference satisfies condition DQC if and only if it is MEU. Equivalently, for an invariant biseparable preference, there is no gap between Propositions 6 and 7 if and only if preferences are in fact MEU. □

**Example 7** Consider the preferences represented by the utility function $u(x)=\sqrt{x}$ and the differentiable, quasiconcave, but not concave functional

$$I(a)=\frac{1}{2}a_2+\sqrt{4+\frac{1}{4}a_2^2+2a_1}-2.$$

Figure 4 shows indifference curves for this preference, as well as supporting lines at certainty. (The functional $I$ itself has linear level curves, so the supporting lines can also be interpreted as level curves for $I$.)

Since each supporting line corresponds to the gradient of $I^u$ at some point on the certainty line, the figure shows that the gradients are all different, and all non-zero; indeed, it may be verified that the slope of the indifference curve of $I^u$ at $1_S x$ is $-\frac{2}{u(x)+2}$. Hence (cf. Remark 2.1), $I$ is nice at certainty. Furthermore, since $I$ is quasiconcave, it satisfies DQC by Proposition 3, and therefore by Proposition 4, $\bar{\pi}(1_S x)=\pi(1_S x)=C(1_S x)$ for all $x>0$.

Since the differentials at certainty are all different, this preference does not satisfy IDC. Furthermore, while it is quasiconcave, it is not GM-ambiguity-averse because Core $I=\emptyset$. To see this, note that, by Proposition 1 the core must be contained in the sets $\pi(1_S x)$ for all $x>0$, but as noted above these sets are all singleton and different. Finally, since the normalize Clarke differentials at certainty are all singleton and different, there is no set $C(1_S x^*)$ that is contained in all $C(1_S x)$, $x>0$ (i.e., a point where $I$ is "least kinked").

From a decision-theoretic point of view, this shows that convexity ("Uncertainty Aversion")

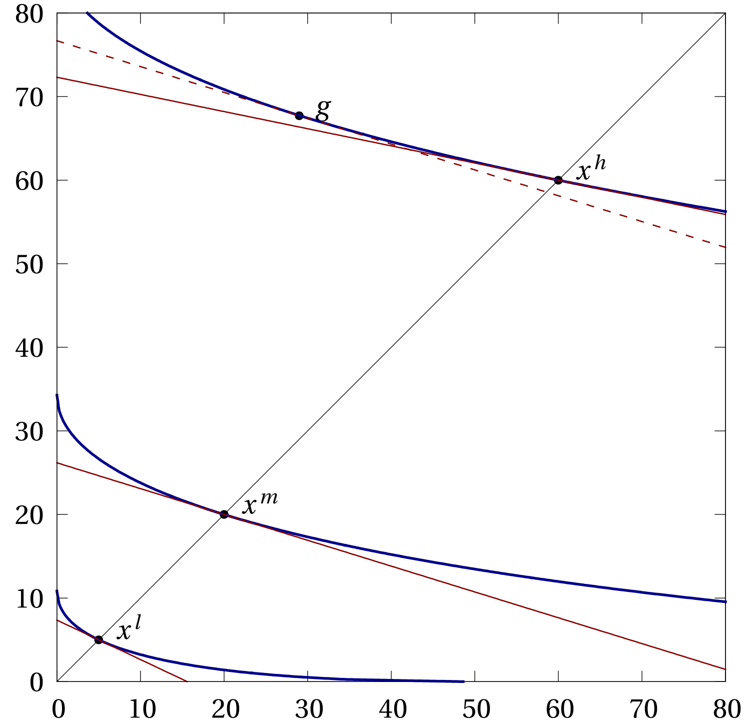


Figure 4: A convex preference with empty core.

of preferences does not imply GM-ambiguity aversion,[12] IDC,[13] or even the existence of a certain bundle where $I$ is least kinked.

As regards risk sharing, this example shows that condition IDC (respectively, the existence of a full-insurance allocation where the functionals $I_i$ are least kinked) is required in Theorem 9 (respectively, Theorem 8). To see this, interpret Figure 4 as an Edgeworth box; consumer 1 has the preferences described above, and consumer 2 has risk-neutral EU preferences, with beliefs corresponding to the slope of the supporting line going through $1_S x^m$. The dashed line in Figure 4 has the same slope, and so it represents an indifference curve for consumer 2. It follows that the allocation $(g, 1_S\bar{\omega} - g)$ is Pareto-efficient; note however that it does not provide full insurance. On the other hand, the interior allocation $(1_S x^m, 1_S(\bar{\omega} - x^m))$ is Pareto-

[12]Another example of a preference which is convex but not GM-ambiguity-averse can be found in Cerreia-Vioglio et al. (2011).

[13]Rigotti et al. (2008) provide an example of a convex preference that does not satisfy "Translation invariance at certainty."

efficient and provides full insurance. Hence, items (i) and (ii) in Theorem 9 are not equivalent. (As in Example 3, one can modify 2's preferences so as to make consumer 2 strictly risk-averse without changing the conclusions.)

# 6 Extensions

Strzalecki and Werner (2011) consider risk sharing in economies with aggregate uncertainty and convex preferences. While we leave a full investigation of such environments to future work, we can state a partial generalization of Proposition 7 to the case in which the aggregate endowment is non-constant, but "unambiguous" in a suitable sense.

We need two definitions. The first is a weakening of the notion of "unambiguous act" studied in Cerreia-Vioglio et al. (2011); the second adapts Strzalecki and Werner (2011)'s notion of "conditional beliefs" to the class of preferences we consider.

**Definition 3** Consider a preference $\succcurlyeq$ on $\mathbb{R}^S_+$ represented by $(I,u)$. An act $f : S \to \mathbb{R}_+$ is **core-unambiguous** for $(I,u)$ if $P(u \circ f) = I(u \circ f)$ for all $P \in \text{Core } I$. A partition $\mathscr{G}$ of $S$ is core-unambiguous for $(I,u)$ if every $\mathscr{G}$-measurable[14] bundle $f$ is core-unambiguous for $(I,u)$.

In Cerreia-Vioglio et al. (2011) an act $f$ is *unambiguous* if $P(u \circ f) = I(u \circ f)$ for all priors $P$ in the set $C$ that represents the largest independent subrelation of $\succcurlyeq$ in the sense of Bewley (2002).[15] It turns out that Core $I \subseteq C$, so Definition 3 is less demanding.[16]

**Definition 4 (cf. Strzalecki and Werner, 2011, Definition 3)** Consider a preference $\succcurlyeq$ on $\mathbb{R}^S_+$ represented by $(I,u)$, and a partition $\mathscr{G}$ of $S$. The $\mathscr{G}$**-conditional core** of $I$, written $\text{Core}_{\mathscr{G}}\, I$, is the collection of all probabilities $Q \in \Delta(S)$ such that

[14]A bundle $f$ is $\mathscr{G}$-measurable if $f(s) = f(s')$ for all $s, s' \in G$ and $G \in \mathscr{G}$.

[15]Ghirardato and Siniscalchi (2012) show that $C$ is the union of all sets $C(a)$, for all $a \in \big(\text{int}(\mathbb{U})\big)^S$.

[16]Consider a risk-neutral CEU preference with $S = \{s_1, s_2, s_3\}$ and capacity $\nu$ given by $\nu(\{s_1\}) = \frac{1}{3}$, $\nu(\{s_2\}) = \nu(\{s_3\}) = 0$, and $\nu(E) = \frac{2}{3}$ for every 2-element set $E$. The core of this preference consists solely of the uniform probability $P_u$, so e.g. for $a = (3,2,1)$ (obvious notation), $P_u(a) = 2 = I(a) = 3 \cdot \nu(\{s_1\}) + 2 \cdot [\nu(\{s_1, s_2\}) - \nu(\{s_1\})] + 3 \cdot [1 - \nu(\{s_1, s_2\})]$. Thus, $a$ is core-unambiguous. However, it is not unambiguous in the sense of Cerreia-Vioglio et al. (2011): by results in Ghirardato et al. (2004), the set $C$ contains, for example, the measure $P = (\frac{1}{3}, 0, \frac{2}{3})$ (obvious notation) for which $P(a) = \frac{5}{3}$.

(i) $Q(G) > 0$ for all $G \in \mathscr{G}$; and

(ii) there exists $P \in \text{Core } I$ with $P(G) > 0$ and $P(\cdot|G) = Q(\cdot|G)$ for all $G \in \mathscr{G}$.[17]

Loosely speaking, $\text{Core}_{\mathscr{G}}\ I$ is the set of all probabilities that "match" the probabilities conditional upon each event $G \in \mathscr{G}$ induced by some $P \in \text{Core } I$. If every $P$ in the core assigns positive probability to the elements of $\mathscr{G}$, then $\text{Core}_{\mathscr{G}}\ I$ is a larger set than $\text{Core } I$. Note also that, if $\mathscr{G} = \{S\}$, then $\text{Core}_{\mathscr{G}}\ I = \text{Core } I$. For further interpretation, see Strzalecki and Werner (2011).

We can now state the promised partial generalization of Proposition 7. Let $\mathscr{E}$ be the partition induced by the aggregate endowment $\omega \equiv \sum_i \omega_i \in \mathbb{R}^S_+$: that is, the coarsest partition $\mathscr{G}$ such that $\omega$ is $\mathscr{G}$-measurable. If the aggregate endowment is constant, then $\mathscr{E} = \{S\}$ and a bundle if $\mathscr{E}$-measurable if and only if it provides full insurance.

**Proposition 10** *If, for every $i \in N$, $\mathscr{E}$ is core-unambiguous for $(I_i, u_i)$, and $\bigcap_i \text{Core}_{\mathscr{G}}\ I_i \neq \emptyset$, then every Pareto-efficient allocation is $\mathscr{E}$-measurable.*

The other assertions in Proposition 7 do not generalize under the assumption that $\bigcap_i \text{Core}_{\mathscr{G}}\ I_i \neq \emptyset$. Consider a two-state, two-agent economy with aggregate uncertainty: then $\mathscr{E}$ is the discrete partition, so every bundle is $\mathscr{E}$-measurable, and furthermore the $\mathscr{E}$-conditional cores are degenerate and always intersect no matter what the preferences. However (except for degenerate cases) not every feasible allocation is Pareto-efficient; moreover, one can easily construct examples of Pareto-efficient allocations that are not competitive equilibria. Whether one can obtain positive results under stronger assumptions is left to future research.

# A Appendix: Proofs

**Note**: a preference $\succcurlyeq$ is *strictly monotonic* if, for all $f, g \in \mathbb{R}^S_+$, $f(s) > g(s)$ for all $s$ implies $f \succ g$. Hence, a strongly monotonic preference is also strictly monotonic, but the converse is not true. Many of our results apply to strictly, as well as strongly monotonic preferences; for this reason, we take care to invoke "strong monotonicity" only when it is necessary, and invoke "strict monotoncity" otherwise.

[17] $P(\cdot|G)$ and $Q(\cdot|G)$ denote conditional probabilities given $G$.

## A.1 Preliminaries

**Proof of Remark 2.1:** The map $F:\mathbb{R}^S_+ \to \mathbb{U}^S$ defined by $F(f)=(u(f_1),\dots,u(f_S))$ is strictly differentiable (pp. 30-31 Clarke, 1983) and, furthermore, it maps every neighborhood of $f$ to a neighborhood of $F(f)$.[18] Hence, since $I^u = I\circ F$, by Theorem 2.3.10 in Clarke $\partial I^u(f) = \partial I(u\circ f)\circ D_s F(f)$; that is, more explicitly, every $Q^u \in \partial I^u(f)$ is defined by

$$\forall h \in \mathbb{R}^S, \qquad Q^u(h) = \sum_s Q(s)u'(f_s)h_s$$

for some $Q\in\partial I(u\circ f)$. ■

The following geometric notions will be useful. For every bundle $f\in\mathbb{R}^S_+$, let

$$U(f)=\{g\in\mathbb{R}^S_+ : g \succcurlyeq f\},$$

the upper countour set of the preference $\succcurlyeq$ at $f$. For every set $C\subset\mathbb{R}^S_+$ and bundle $f\in\mathbb{R}^S_+$, let

$$d_C(f)=\inf\{\|f-g\| : g\in C\}$$

The *Clarke tangent cone* to $C$ at some $f\in C$ is

$$T_C(f)=\{v\in\mathbb{R}^S : (d_C)^0(f;v)=0\},$$

i.e. the set of directions $v$ for which the Clarke derivative of the distance function (which is Lipschitz and convex) is zero. The following characterization (Clarke, 1983, Theorem 2.4.5) is useful:

$$T_C(f)=\{v\in\mathbb{R}^S : \forall(f^k,t^k)\subset C\times\mathbb{R}_{++} \text{ s.t. } f^k\to f,\ t^k\downarrow 0,\ \exists(v^k)\subset\mathbb{R}^S \text{ s.t. } v^k\to v,\ f^k+t^kv^k\in C\ \forall k\}.$$

Finally, define the *Clarke normal cone* to $C$ at $f$ by polarity:

$$N_C(f)=\{Q\in ba(S)=\mathbb{R}^S : Q(v)\le 0\,\forall v\in T_C(f)\}.$$

[18] To see this, fix a strictly positive bundle $f$ and consider the set $\{g\in\mathbb{R}^S_+ : f_s-\epsilon<g_s<f_s+\epsilon \forall s\in S\}$, which is open. The image of this set via $F$ is $\{v\in\mathbb{U}^S : u(f_s-\epsilon)<v_s<u(f_s+\epsilon)\forall s\in S\}$, because $u$ is continuous and strictly increasing. This set is also open.

Specializing to our environment, we have

$$T(f) \equiv T_{U(f)}(f) = \big\{ v \in \mathbb{R}^S : \forall (f^k, t^k) \subset \mathbb{R}^S_+ \times \mathbb{R}_{++} \text{ s.t. } f^k \succcurlyeq f \; \forall k,\; f^k \to f,\; t^k \downarrow 0, \\ \exists (v^k) \subset \mathbb{R}^S \text{ s.t. } v^k \to v,\; f^k + t^k v^k \succcurlyeq f \; \forall k \big\}.$$

and it is convenient to define

$$N(f) \equiv N_{U(f)}(f) = \{ Q \in \mathbb{R}^S : Q(v) \le 0 \; \forall v \in T(f) \}.$$

Loosely speaking, $T(f)$ is the set of directions $v$ with the property that any sequence of bundles preferred to $f$ and converging to it can be perturbed in the direction $v$ without leaving the upper contour set of $f$. More informally, moving from bundles near $f$ in the direction $v$ by a small amount leads to an act that is at least as good as $f$. Then, if $Q$ is in the normal cone, $-Q$ is a price vector that assigns non-negative value to such changes.

The following two results pertain to the Clarke normal cone. Note that the first does not require any particular assumption on the functional $I$.

**Remark A.1** For every bundle $f \in \mathbb{R}^S_{++}$, $-\pi(f) \subseteq N(f)$.

**Proof:** Fix $P \in \pi(f)$. Consider $v \in T(f)$, the constant sequence $f^k \equiv f$, and an arbitrary sequence $(t^k) \downarrow 0$. Since $v \in T(f)$, there exists a sequence $(v^k) \to v$ such that, for every $k$, $f^k + t^k v^k \succcurlyeq f$, i.e., $I(u \circ [f + t^k v^k]) \ge I(u \circ f)$. Since $P \in \pi(f)$, $P(f + t^k v^k) \ge P(f)$, and therefore $P(v^k) \ge 0$ for every $k$. By continuity, $P(v) \ge 0$. Therefore, $-P \in N(f)$. ■

**Lemma 11** *For any agent $i$ and bundle $f \in \mathbb{R}^S_+$, if $I$ is nice at $u \circ f$, then $N(f) \subseteq \bigcup_{\lambda \ge 0} \lambda\big(-\partial I^u(f)\big)$. In particular, for any $x > 0$, if $R \in N(1_S x) \setminus \{0_S\}$, then there is $\mu > 0$ and $Q \in \partial I(1_S u(x))$ such that $R = -\mu Q$.*

**Proof:** Let $J^u = -I^u$, and note that $U(f) = \{ g \in \mathbb{R}^S_+ : J^u(g) \le J^u(f) \}$. By Proposition 2.3.1 in Clarke (1983), $\partial J^u(f) = -\partial I^u(f)$. Recall that every $Q^u \in \partial I^u(1_S x)$ maps $a \in \mathbb{R}^S$ to $\sum_s Q(s) u'(f_s) a_s = u'(x) Q(a)$ for some $Q \in \partial I(1_S u(x))$; since $u$ is strictly increasing, and $Q$ is non-negative because $I$ is monotonic, it follows that $Q^u = 0_S$ only if $Q = 0$; but $I$ is nice at $u \circ f$ by assumption, so $0_S \notin \partial I^u(f)$, i.e., $I^u$ and hence $J^u$ are nice at $f$. Then, by Corollary 1 to Theorem 2.4.7 in Clarke (1983), $N(f) \subset \bigcup_{\lambda \ge 0} \lambda \partial J^u(f) = \bigcup_{\lambda \ge 0} \lambda\big(-\partial I^u(f)\big)$, as claimed.

If $f = 1_S x$ for some $x > 0$, then every $Q^u \in \partial I^u(1_S x)$ maps $a \in \mathbb{R}^S$ to $\sum_s Q(s)u'(x)a_s = u'(x)Q(a_s)$ for some $Q \in \partial I(1_S u(x))$, where $u'(x) > 0$ by assumption. By the preceding claim, every $R \in N(1_S x)$ not equal to $0_S$ can be written as $R = -\lambda Q^u$ for some $\lambda > 0$ and $Q^u \in \partial I^u(1_S x)$, and hence also as $R = -\lambda u'(x)Q$ for some $Q \in \partial I(1_S u(x))$. The second claim follows by taking $\mu = \lambda u'(x)$. ■

Conclude this section with a remark that restates the definition of $\bar{\pi}(f)$ for $f = 1_S x$.

**Remark A.2** For every $x > 0$, $\bar{\pi}(1_S x) = \{P \in \Delta(S) : \forall g \in \mathbb{R}^S_+,\ u(x) \geq P(u \circ g)$ implies $u(x) \geq I(u \circ g)\}$.

**Proof:** Denote the set on the rhs of the Remark by $\hat{\pi}(1_S x)$. Suppose that $P \in \bar{\pi}(1_S x)$. We show that, for every $g \in \mathbb{R}^S_+$, $I(u \circ g) > u(x)$ implies $P(u \circ g) > u(x)$, so $P \in \hat{\pi}(1_S x)$. Fix $g$ and suppose $I(u \circ g) > u(x)$. Since $P \in \bar{\pi}(1_S x)$, $P(u \circ g) \geq u(x)$. By contradiction, suppose $P(u \circ g) = u(x)$. Let $y > 0$ be such that $I(u \circ g) > u(y) > u(x)$, which exists because $u(\cdot)$ is continuous and strictly increasing. Then a fortiori $I(u \circ g) \geq u(y)$, but $P(u \circ g) = u(x) < u(y)$, which contradicts the assumption that $P \in \bar{\pi}(1_S x)$. Hence, $P(u \circ g) > u(x)$, as claimed.

Conversely, suppose that $P \in \hat{\pi}(1_S x)$. We show that, for every $g \in \mathbb{R}^S_+$, $u(x) > P(u \circ g)$ implies $u(x) > I(u \circ g)$, so $P \in \bar{\pi}(1_S x)$. Fix $g$ and suppose that $u(x) > P(u \circ g)$. Since $P \in \hat{\pi}(1_S x)$, $u(x) \geq I(u \circ g)$. By contradiction, suppose $u(x) = I(u \circ g)$. Choose $y > 0$ such that $u(x) > u(y) > P(u \circ g)$. Then a fortiori $u(y) \geq P(u \circ g)$, but $u(y) < u(x) = I(u \circ g)$, which contradicts the assumption that $P \in \hat{\pi}(1_S x)$. Hence, $u(x) > I(u \circ g)$, as required. ■

## A.2 Proof of the results in Section 3

**Proof of Proposition 1:** (1): fix $x > 0$. Suppose $P \in \text{Core } I$ and consider a bundle $g \in \mathbb{R}^S_+$. Assume that $P(1_S u(x)) \geq P(u \circ g)$. Since $I$ is normalized and $P$ is in the core of $I$, $I(1_S u(x)) = u(x) = P(1_S u(x)) \geq P(u \circ g) \geq I(u \circ g)$. Since this holds for all $g \in \mathbb{R}^S_+$, $P \in \bar{\pi}(1_S x)$. Hence, $\text{Core } I \subseteq \bar{\pi}(1_S x)$.

For the second inclusion, note that $P \in \bar{\pi}(1_S x)$ iff, for all $g \in \mathbb{R}^S$, $I(u \circ g) > u(x)$ implies

$P(u \circ g) > u(x)$. Now fix $P \in \bar{\pi}(1_S x)$ and consider a bundle $g \in \mathbb{R}^S_+$. Assume that $I(u \circ g) \geq u(x)$. Since $\mathbb{U}$ does not include its upper bound, there is $\bar{\epsilon} > 0$ such that, for all $\epsilon \in (0, \bar{\epsilon})$, $u \circ g + 1_S \epsilon \in \mathbb{U}^S$ (i.e., there is $g_\epsilon \in \mathbb{R}^S_+$ with $u \circ g_\epsilon = u \circ g + 1_S \epsilon$). Then, for any such $\epsilon$, by strict monotonicity, $I(u \circ g + 1_S \epsilon) > I(u \circ g) \geq u(x)$, and so $P(u \circ g) + \epsilon = P(u \circ g + 1_S \epsilon) > u(x)$ because $P \in \bar{\pi}(1_S x)$. Since this holds for all $\epsilon \in (0, \bar{\epsilon})$, $P(u \circ g) \geq u(x)$; since $u$ is (strictly) concave, $u(P(g)) \geq P(u \circ g) \geq u(x)$; since $u$ is strictly increasing, $P(g) \geq x$; and since $g \in \mathbb{R}^S_+$ was arbitrary, $P \in \pi(1_S x)$.

(2): fix $x > 0$ and consider $P \in \pi(1_S x)$. By Remark A.1, $-P \in N(1_S x)$. By Lemma 11, if $I$ is nice at $1_S u(x)$, then $N(1_S x) \setminus \{0_S\} \subseteq \bigcup_{\mu > 0} (-\partial I(1_S u(x)))$. Therefore, there are $\mu > 0$ and $Q \in \partial I(1_S u(x))$ such that $-P = \mu(-Q)$, i.e., $P = \mu Q$. Furthermore, $1 = P(S) = \mu Q(S)$, so $\mu = Q(S)^{-1}$ and $P = \frac{Q}{Q(S)} \in C(1_S x)$, as required.

(3): By the first inclusion in part (1), Core $I \subseteq \bigcap_{x>0} \bar{\pi}(1_S x)$. Conversely, suppose $P \in \bigcap_{x>0} \bar{\pi}(1_S x)$. We claim that $P(\{s\}) > 0$ for all $s \in S$. By contradiction, suppose that $P(\{s\}) = 0$ for some $s \in S$. Then, for every $x > 0$, $P(1_S u(x)) = u(x) > u(0) = P(u \circ 1_{\{s\}})$, and therefore, since $P \in \bar{\pi}(1_S x)$, $I(1_S u(x)) \geq I(u \circ 1_{\{s\}})$. Since this holds for all $x > 0$, by continuity of $u$ and $I$, $I(1_S u(0)) \geq I(u \circ 1_{\{s\}})$, i.e., $0 \succcurlyeq 1_{\{s\}}$, which contradicts strong monotonicity.

Now fix $g \in \mathbb{R}^S_+$ and let $x \in \mathbb{R}_+$ be such that $u(x) = P(u \circ g)$. If $x = 0$, then the preceding claim implies that $g = 0$, and so $P(1_S u(0)) = u(0) = I(u \circ g)$. Otherwise, since by assumption $P \in \bar{\pi}(1_S x)$, $P(1_S u(x)) = u(x) = P(u \circ g)$ implies $I(1_S u(x)) \geq I(u \circ g)$; but $I(1_S u(x)) = u(x) = P(1_S u(x)) = P(u \circ g)$, so indeed $P(u \circ g) \geq I(u \circ g)$.

(4): Fix $P \in \text{Core } I$. $x > 0$, and $a \in \mathbb{R}^S$. Let $\gamma = u(x)$. We calculate:

$$\begin{aligned}
I^\circ(1_S\gamma; a) = &\limsup_{c \to 1_S\gamma, t \downarrow 0} \frac{I(c + ta) - I(c)}{t} = \\
= &\limsup_{d \to 1_S\gamma, t \downarrow 0} \frac{I(d) - I(d - ta)}{t} \geq \\
\geq &\limsup_{t \downarrow 0} \frac{I(1_S\gamma) - I(1_S\gamma - ta)}{t} \geq \\
\geq &\limsup_{t \downarrow 0} \frac{\gamma - P(1_S\gamma - ta)}{t} = \\
= &\limsup_{t \downarrow 0} \frac{\gamma - \gamma + tP(a)}{t} = \\
= &P(a).
\end{aligned}$$

The second equality follows because, if $c \to 1_S\gamma$ and $t \downarrow 0$, then $d \equiv c + ta \to 1_S\gamma$; conversely, if $d \to 1_S\gamma$ and $t \downarrow 0$, then $c \equiv d - ta \to 1_S\gamma$. The first inequality follows by considering the constant sequence $d \equiv 1_S\gamma$. The second inequality follows from normalization and the fact that $P \in \text{Core } I$: $I(1_S\gamma - ta) \leq P(1_S\gamma - ta)$, so $-I(1_S\gamma - ta) \geq -P(1_S\gamma - ta)$.

Hence, for every $a \in \mathbb{R}^S$, $\max_{Q \in \partial I(1_S\gamma)} Q(a) = I^\circ(1_S\gamma; a) \geq \max_{P \in \text{Core } I} P(a)$, so by standard results (e.g., Clarke, 1983, Prop. 2.1.4 (b)), $\text{Core } I \subseteq \partial I(1_S\gamma)$. Furthermore, by definition, $Q \in \text{Core } I$ implies $Q(S) = 1$, so $Q = \frac{Q}{Q(S)} \in C(1_S x)$.

This completes the proof of Proposition 1. ■

**Proof of Proposition 2** (1): Assume that DQC holds, and let $P \in \bigcap_{x>0} C(1_S x)$. Fix $a \in \mathbb{U}^S$ and let $\gamma = I(a)$. By monotonicity of $I$, $\min_s a_s \leq \gamma \leq \max_s a_s$. Since $a \in \mathbb{U}^S$, by continuity of $u$ there is $x \geq 0$ such that $\gamma = u(x)$.

If $x = 0$, then, since $u(x) = u(0) = \min \mathbb{U}$, $a \geq 1_S u(0)$, and therefore $P(a) \geq P(1_S u(0)) = u(0) = u(x) = \gamma = I(a)$. Now consider the case $x > 0$, so $\gamma > \min \mathbb{U}$. By definition, since $P \in C(1_S x)$, there is $Q \in \partial I(1_S\gamma)$ such that $Q(S) > 0$ and $P = \frac{Q}{Q(S)}$. Since $\mathbb{U} = u(\mathbb{R}_+)$ does not contain its upper bound and is connected because $u$ is continuous, $\gamma \in \text{int}(\mathbb{U})$. Hence, by DQC, $I(a) = \gamma$ implies $Q(a - 1_S\gamma) \geq 0$, i.e., $Q(a) \geq Q(1_S\gamma)$; hence also $P(a) \geq P(1_S\gamma)$. Therefore, $P$ satisfies $P(a) \geq P(1_S\gamma) = \gamma = I(a)$. Since this holds for all $a \in \mathbb{U}^S$, $P \in \text{Core } I$. Thus, $\bigcap_{x>0} C(1_S x) \subseteq \text{Core } I$.

(2): Assume that $C(1_S x) \subseteq \text{Core } I$ for some $x > 0$, let $\gamma = u(x)$, and fix $a \in \mathbb{U}^S$. Suppose that $I(a) \geq \gamma$: then, for every $P \in \text{Core } I$, $P(a) \geq I(a) \geq \gamma = P(1_S \gamma)$, i.e, $P(a - 1_S \gamma) \geq 0$. Since, by assumption, $C(1_S x) \subseteq \text{Core } I$, if $Q \in \partial I(1_S \gamma)$, so that $\frac{Q}{Q(S)} \in C(1_S x)$, one has $Q(a - 1_S \gamma) \geq 0$. Therefore, $I^\ell(1_S \gamma; a - 1_S \gamma) = \min_{Q \in \partial I(1_S \gamma)} Q(a - 1_S \gamma) \geq 0$, i.e., $I$ is differentially quasiconcave at $1_S \gamma$.

(3): since $\gamma \in \text{int}(\mathbb{U})$ iff $u^{-1}(\gamma) > 0$, the result is immediate from (2). ■

To prove results involving the condition in DQC, it is convenient to define the **Clarke lower derivative** of $I$ (cf. Ghirardato et al., 2004, pp. 150 and 157) as

$$I^\ell(b; a) = \liminf_{t \downarrow 0, c \to b} \frac{I(c + ta) - I(c)}{t};$$

It is readily verified that $I^\ell(b; a) = -I^\circ(b; -a)$ and, therefore, $I^\ell(b; a) = \min_{Q \in \partial I(b)} Q(a)$ for all interior $b \in \mathbb{U}^S$ and all $a \in \mathbb{R}^S$. Then, the condition in DQC can equivalently be restated as follows:

$$\forall \gamma \in \text{int}(\mathbb{U}),\ a \in \mathbb{U}^S, \qquad I(a) \geq \gamma \quad \Longrightarrow \quad I^\ell(1_S \gamma; a - 1_S \gamma) \geq 0. \tag{9}$$

**Proof of Proposition 3:** For both results, we use the equivalent characterization in Eq. (9). As noted in the text, part (1) follows from a result in Penot and Quang (1997); however, since their assumptions are formulated somewhat differently from ours, invoking their result requires some work. We provide a direct proof.

(1) Fix $\gamma \in \text{int}(\mathbb{U})$ and $a \in \mathbb{U}^S$ such that $I(a) \geq \gamma$. Also fix $\epsilon > 0$ such that $a + 1_S \epsilon \in \mathbb{U}^S$ (this must exist, because $\mathbb{U} = u(\mathbb{R}_+)$ does not contain its supremum). By strict monotonicity, $I(a + 1_S \epsilon) > \gamma$. Consider sequences $(c^k) \subset \mathbb{U}^S$ and $(t^k) \subset \mathbb{R}_{++}$ such that $c^k \to 1_S \gamma$ and $t^k \downarrow 0$. Note that

$$t^k[(a + 1_S \epsilon) - 1_S \gamma] + c^k = t^k[(a + 1_S \epsilon) - 1_S \gamma + c^k] + (1 - t^k) c^k$$

and, since $c^k \to 1_S \gamma$, eventually $(a + 1_S \epsilon) - 1_S \gamma + c^k \in \mathbb{U}^S$; furthermore, by continuity $I(a + 1_S \epsilon - 1_S \gamma + c^k) \to I(a + 1_S \epsilon)$ and $I(c^k) \to I(1_S \gamma) = \gamma$. Therefore, for $k$ sufficiently large, $I(a + 1_S \epsilon - 1_S \gamma + c^k) > I(c^k)$. Then, by quasiconcavity, for all such $k$,

$$I(t^k[(a + 1_S \epsilon) - 1_S \gamma] + c^k) = I(t^k[(a + 1_S \epsilon) - 1_S \gamma + c^k] + (1 - t^k) c^k) \geq I(c^k).$$

It follows that

$$I^{\ell}(1_S\gamma;(a+1_S\epsilon)-1_S\gamma)=\liminf_{c\to 1_S\gamma,t\downarrow 0}\frac{I(t[(a+1_S\epsilon)-1_S\gamma]+c)-I(c)}{t}\geq 0.$$

Finally, by continuity of $I^{\ell}(1_S\gamma;\cdot)$, $I^{\ell}(1_S\gamma;a-1_S\gamma)\geq 0$ as well.

(2): if $I$ is regular, $I^{\ell}(1_S\gamma;a-1_Sx)=-I^{\circ}(1_S\gamma;1_S\gamma-a)=-I'(1_S\gamma;1_S\gamma-a)$; furthermore, if $I(a)\geq I(1_S\gamma)=\gamma$, by GM-ambiguity aversion and normalization, for any $P\in\text{Core } I$,

$$\begin{aligned}-I^{\ell}(1_S\gamma;a-1_S\gamma)=I'(1_S\gamma;1_S\gamma-a)=&\\ &=\lim_{t\downarrow 0}\frac{I(1_Sx+t[1_Sx-a])-I(1_Sx)}{t}=\\ &=\lim_{t\downarrow 0}\frac{I(1_Sx+t[1_Sx-a])-x}{t}\leq\\ &\leq\lim_{t\downarrow 0}\frac{P(1_Sx+t[1_Sx-a])-x}{t}=\\ &=\lim_{t\downarrow 0}\frac{x+tx-tP(a)-x}{t}=x-P(a)\leq I(a)-P(a)\leq 0,\end{aligned}$$

so DQC holds. ■

**Proof of Proposition 4:** Fix $x>0$ and $P\in C(1_Sx)$. We first claim that $P(\{s\})>0$ for all $s\in S$. By assumption, there is $Q\in\partial I(1_Su(x))$ such that $Q(S)>0$ and $P=\frac{Q}{Q(S)}$. By strong monotonicity, $I(1_Su(x)+1_{\{s\}})>u(x)$; by continuity, there exists $\epsilon\in(0,u(x))$ such that $1_Su(x)+1_{\{s\}}-\epsilon 1_{S\setminus\{s\}}\in\mathbb{U}^S$ and

$$I(1_Su(x)+1_{\{s\}}-\epsilon 1_{S\setminus\{s\}})>u(x).$$

Therefore, since $I$ is differentially quasiconcave at $1_Su(x)$ by DQC,

$$Q(1_Su(x)+1_{\{s\}}-\epsilon 1_{S\setminus\{s\}}-1_Su(x))\geq 0\quad\Leftrightarrow\quad Q(\{s\})\geq\epsilon Q(S\setminus\{s\})\quad\Leftrightarrow\quad P(\{s\})\geq\epsilon P(S\setminus\{s\}).$$

If $P(\{s\})=0$, the last inequality reduces to $0\geq\epsilon$, a contradiction. Thus, $P(\{s\})>0$.

We now show that, for any $g\in\mathbb{R}^S_+$, $u(x)\geq P(u\circ g)$ implies $u(x)\geq I(u\circ g)$; thus, $P\in\bar{\pi}(1_Sx)$. We show that the contrapositive holds. Suppose that $I(u\circ g)>u(x)$; notice that we cannot have $g(s)=0$ for all $s$, because by assumption $x>0$ and so $u(x)>u(0)=I(1_Su(0))$ by strict monotonicity of $u$ and normalization. Hence, for every $\alpha\in(0,1)$, $g(s)\geq\alpha g(s)$ in every state $s$, and there is at least one state $s^*$ such that $g(s^*)>\alpha g(s^*)$. Furthermore, by continuity there is

$\alpha^* \in (0,1)$ such that $I(u \circ (\alpha^* g)) > u(x)$. By DQC, $I$ is differentially quasiconcave at $1_S u(x)$, so $Q(u \circ (\alpha^* g) - 1_S u(x)) \geq 0$, and so $P(u \circ (\alpha^* g)) \geq u(x)$. Finally, since there is at least one state $s^*$ with $g(s^*) > \alpha^* g(s^*)$, and we showed above that $P(\{s^*\}) > 0$, $P(u \circ g) > P(u \circ (\alpha^* g)) \geq u(x)$, as required.

Hence, $C(1_S x) \subseteq \bar{\pi}(1_S x)$, and indeed by Proposition 1 part 1, $C(1_S x) \subseteq \bar{\pi}(1_S x) \subseteq \pi(1_S x)$. If in addition $I$ is nice at $1_S u(x)$, part 2 of Proposition 1 implies that $\pi(1_S x) \subseteq C(1_S x)$, so $C(1_S x) = \bar{\pi}(1_S x) = \pi(1_S x)$. ■

**Note:** the above argument shows that it is enough to assume quasiconcavity at $1_S u(x)$ in order to obtain the noted inclusions.

### A.3 Proof of the results in Section 4

The key step in the proof of Proposition 6 is contained in the following result.

**Lemma 12** *If $(f_i)_{i \in N}$ is a Pareto-efficient allocation, then there exists a price vector $p \in \mathbb{R}^S_+ \setminus \{0\}$ such that $-p \in N_i(f_i)$ for all $i \in N$.*

**Proof:** Apply Prop. 2.1 (a) and (e) and Theorem 2.1 in Bonnisseau and Cornet (1988) to get $-p \in \bigcap_{i \in N} N_i(f_i)$. We only need to show that $p$ is non-negative. By monotonicity, $\mathbb{R}^S_+ \subset T_i(f_i)$: to see this, note that, if $v \in \mathbb{R}^S_+$, then for any sequence $(f^k, t^k)$ such that $f^k \succcurlyeq_i f_i$, $f^k \to f_i$, and $t \downarrow 0$, the constant sequence $v^k = v$ satisfies $f^k + t^k v^k \geq f^k \succcurlyeq_i f_i$ for all $k$.

Now consider $v \in \mathbb{R}^S_+$ s.t. $v_s = 0$ iff $p_s \geq 0$, and $v_s = 1$ otherwise. If $p_s < 0$ for some $s$, then $p \cdot v < 0$, i.e. $-p \cdot v > 0$, which contradicts the fact that $v \in T_i(f_i)$ and $-p \in N_i(f_i)$ for all $i$. Thus, $p \geq 0$. ■

**Proof of Proposition 6:** For the first implication, Lemma 12 yields $p \in \mathbb{R}^S_+ \setminus \{0_S\}$ such that $-p \in N_i(f_i)$ for all $i$; by Lemma 11, $-p \in \bigcup_{\lambda > 0} \lambda\big(-\partial I_i^u(f)\big)$ for all $i \in N$, and the claim follows. The second claim follows from the second part of Lemma 11. Finally, at a full-insurance allocation $(1_S x_1, \ldots, 1_S x_N)$, $p = \mu_i Q_i$ for every $i$, where $\mu_i > 0$ and $Q_i \in \partial I_i(1_S u_i(x_i))$; then $Q_i(S) = \frac{\sum_s p_s}{\mu_i}$, and therefore $\frac{Q_i}{Q_i(S)} = \frac{\mu_i^{-1} p}{\mu_i^{-1} \sum_s p_s} = \frac{p}{\sum_s p_s} \equiv P$; hence, $P \in \bigcap_i C_i(1_S x)$. ■

**Remark A.3** If $-I_i$ is regular, then by Theorem 2.3.10 and Corollary 1 to Theorem 2.4.7 in Clarke (1983) $-I_i^u$ is also regular, and $N_i(f_i) = \bigcup_{\lambda \geq 0} \lambda\big(-\partial I_i^u(f_i)\big)$.

The next Remark follows from standard arguments; we include the proof for completeness. Observe that the argument relies on continuity and strong monotonicity.

**Remark A.4** If a feasible allocation $(f_1, \ldots, f_N)$ is not Pareto-efficient, then it is Pareto-dominated by a Pareto-efficient allocation.

**Proof:** By assumption, there exists a feasible allocation $(g_1, \ldots, g_N)$ that Pareto-dominates $(f_1, \ldots, f_N)$. Assume wlog that $g_1 \succ_1 f_1$. Consider the following problem: maximize $I_1(u_1 \circ h_1)$ subject to $(h_1, \ldots, h_N)$ being feasible and $h_i \succcurlyeq_i g_i$ for all $i = 2, \ldots, N$. Notice that the allocation $(g_1, \ldots, g_N)$ satisfies these constraints. By standard arguments (e.g. Mas-Colell, Whinston, and Green, 1995, §16.F), since preferences are continuous and strongly monotonic, a solution $(h_1^*, \ldots, h_N^*)$ to this problem exists and is Pareto-efficient. Furthermore, for every $i > 1$, $h_i^* \succcurlyeq_i g_i \succcurlyeq_i f_i$, and $h_1^* \succcurlyeq_i g_1 \succ_1 f_1$; that is, $(h_1^*, \ldots, h_N^*)$ is a Pareto-efficient allocation that Pareto-dominates $(f_1, \ldots, f_N)$. ■

**Proof of Proposition 7:** Let $P \in \bigcap \text{Core } I_i$. Since each $I_i$ is strongly monotonic, $P(\{s\}) > 0$ for every $s \in S$: to see this, note that, if $f \in \mathbb{R}_+^S$ is such that $f(s) = 1$ and $f(s') = 0$ for all $s' \neq s$, then $P(\{s\})u_i(1) + [1 - P(\{s\})]u_i(0) = P(u_i \circ f) \geq I_i(u_i \circ f) > u_i(0)$, which implies $P(\{s\}) > 0$.

Now suppose $(f_1, \ldots, f_N)$ is a Pareto-efficient allocation that is not full-insurance, i.e., wlog $f_1$ is not constant. Let $x_i = P(f_i)$ for every $i$; since $f_1, \ldots, f_N$ is feasible, $\sum_i x_i = \sum_i P(f_i) = P(\sum_i f_i) = P(\sum_i \omega_i) = P(1_S \bar{\omega}) = \bar{\omega}$, i.e., $(1_S x_1, \ldots, 1_S x_N)$ is feasible as well. Since every $u_i$ is strictly concave,

$$P(u_i(x_i)) = u_i(x_i) = u_i\big(P(f_i)\big) \geq P(u_i \circ f_i),$$

and the inequality is strict for agent 1 and any other agent for whom $f_i$ is not constant. If $x_i = 0$, then, since $P$ is strictly positive, $f_i = 0_S$, and so trivially $x_i \succcurlyeq_i f_i$. In particular, this implies that $x_1 > 0$, because $f_1$ is non-constant. Now consider the case $x_i > 0$. Since $P \in \text{Core } I_i$, by part 3 of Proposition 1, also $P \in \bar{\pi}_i(1_S x_i)$, so the above inequality implies $x_i \succcurlyeq_i f_i$ for all $i$. Furthermore, for $i = 1$, since $x_1 > 0$ and $u_1(x_1) > P(u_1 \circ f_1)$, by continuity of $u_1$ there is $\epsilon \in (0, x_1)$ such that

$u_1(x_1-\epsilon) > P(u_1 \circ f_1)$ as well. Then, again by part 3 of Proposition 1, $x_1-\epsilon_1 \succcurlyeq_1 f_1$. Since preferences are strictly monotonic, $x_1 \succ_1 f_1$. This contradicts the assumption that $(f_1,\dots,f_N)$ was Pareto-efficient.

Conversely, consider a feasible, full-insurance allocation $(1_S y_1,\dots,1_S y_N)$, and suppose that it is not Pareto-efficient. Then, by Remark A.4, it is Pareto-dominated by a Pareto-efficient allocation; by the result just proved, since $\bigcap_i \text{Core } I_i \neq \emptyset$, this allocation must be a full-insurance allocation, say $(1_S x_1,\dots,1_S x_N)$. Since preferences are strictly monotonic, this implies that $x_i \geq y_i$ for all $i$, and the inequality is strict for at least one $i$. But then $\sum_i x_i > \sum_i y_i = \bar{\omega}$, i.e., $(1_S x_1,\dots,1_S x_N)$ is not feasible: contradiction.

Finally, let $(1_S x_1,\dots,1_S x_N)$ be a full-insurance, hence Pareto-efficient allocation, and suppose that $g \succ_i x_i$ for some $g \in \mathbb{R}^S_+$ and $i \in N$. Fix $P \in \bigcap_i \text{Core } I_i$. If $P(1_S x_i) = x_i \geq P(g)$, then by (strict) concavity and strict monotonicity of $u_i$, $u_i(x_i) \geq u_i(P(g)) \geq P(u_i \circ g)$. If $x_i = 0$, then, as above, $g = 0_S$ because $P$ is strictly positive; hence, $1_S x_i = g \sim_i g$, which contradicts the assumption that $g \succ_i x_i$. If instead $x_i > 0$, then, since $P \in \bar{\pi}_i(1_S x_i)$, $x_i \succcurlyeq_i g$, which again contradicts the assumed strict preference. Hence, for all $g$, $g \succ_i x_i$ implies $P(g) > P(1_S x_i) = x_i$; equivalently, $P(g) \leq x_i$ implies $x_i \succcurlyeq_i g$. We can then let $t = P(1_S x_i) - P(\omega_i) = x_i - P(\omega_i)$: we get $\sum_i t = \sum_i x_i - \sum_i P(\omega_i) = \bar{\omega} - P(\sum_i \omega_i) = \bar{\omega} - P(1_S\bar{\omega}) = 0$. Hence $t_1,\dots,t_N$ define feasible transfers. Since preferences are strictly monotonic (hence local non-satiated), consumers will exhaust their budget $P(\omega_i)+t_i = x_i$, and the argument just given shows that they will demand $1_S x_1,\dots,1_S x_N$. ■

## A.4 Proof of Proposition 10

We need a generalization of the first inclusion in part 3 of Proposition 1:

**Lemma 13** *If $\mathscr{G}$ is core-unambiguous for $(I,u)$, then* Core $I \subseteq \bar{\pi}(g)$ *for every $\mathscr{G}$-measurable bundle $g \in \mathbb{R}^S_+ \setminus \{0_S\}$.*

**Proof:** Fix a $\mathscr{G}$-measurable $g \in \mathbb{R}^S_+ \setminus \{0_S\}$. Suppose $P \in \text{Core } I$ and consider a bundle $f \in \mathbb{R}^S_+$. Assume that $P(u \circ g) \geq P(u \circ f)$. Since $P$ is in the core of $I$ and $g$ is $\mathscr{G}$-measurable, hence core-unambiguous, $I(u \circ g) = P(u \circ g) \geq P(u \circ f) \geq I(u \circ f)$. Since this holds for all $f \in \mathbb{R}^S_+$,

$P \in \bar{\pi}(g)$. Hence, Core $I \subseteq \bar{\pi}(g)$. ■

**Proof of Proposition 10:** Let $Q \in \bigcap_i \text{Core}_{\mathscr{E}}\ I_i$ and, for every $i \in N$, let $P_i \in \text{Core}\ I_i$ be the probability that satisfies Condition (ii) in Definition 4. Since each $I_i$ is strongly monotonic, $P_i(\{s\}) > 0$ for every $s \in S$: to see this, note that, if $f \in \mathbb{R}^S_+$ is such that $f(s) = 1$ and $f(s') = 0$ for all $s' \neq s$, then $P_i(\{s\})u_i(1) + [1 - P_i(\{s\})]u_i(0) = P_i(u_i \circ f) \geq I_i(u_i \circ f) > u_i(0)$, which implies $P_i(\{s\}) > 0$. Therefore, by Condition (ii) in Definition 4, $Q(\{s\}) > 0$ for every $s \in S$ as well.

By contradiction, suppose $(f_1, \ldots, f_N)$ is a Pareto-efficient allocation but some bundle $f_i$, say wlog $f_1$, is not $\mathscr{E}$-measurable. We construct a new allocation $(g_1, \ldots, g_N)$ that is $\mathscr{E}$-measurable and Pareto-dominates it. For every $i \in N$, every $G \in \mathscr{E}$, and every $s \in G$, let

$$g_i(s) \equiv \sum_{s' \in G} Q(\{s'\}|G) f_i(s') = \sum_{s' \in G} P_i(\{s'\}|G) f_i(s'), \tag{10}$$

where the equality follows from the choice of $P$ and Condition (ii) in Definition 4. That is, $g_i(s)$ is the conditional expectation of $f_i$ given $G$, where $s \in G$.

First, verify feasibility: for every $G \in \mathscr{E}$ and $s \in G$,

$$\sum_i g_i(s) = \sum_i \sum_{s' \in G} Q(\{s'\}|G) f_i(s') = \sum_{s' \in G} Q(\{s'\}|G) \sum_i f_i(s') = \sum_{s' \in G} \omega(s') = \omega(s).$$

The next-to-last equality follows from the assumption that $(f_1, \ldots, f_N)$ is feasible. The last equality follows from the assumption that $\mathscr{E}$ is the partition induced by $\omega$, so that, if $s \in G$, then $\omega(s') = \omega(s)$ for all $s' \in G$.

Turn to Pareto-dominance. For every $G \in \mathscr{E}$, fix $s_G \in G$. For every $i \in N$, since $u_i$ is strictly concave,

$$P_i(u_i \circ f_i) = \sum_{G \in \mathscr{E}} P_i(G) \sum_{s \in G} P_i(\{s\}|G) u_i(f_i(s)) \leq \sum_{G \in \mathscr{E}} P_i(G) u_i\left(\sum_{s \in G} P_i(\{s\}|G) f_i(s)\right) = \sum_{G \in \mathscr{E}} P_i(G) u_i(g_i(s_G)) = P_i(u_i \circ g_i).$$

The inequality follows from Jensen's inequality, and it is strict for agent 1 and any other agent for whom $f_i$ is not $\mathscr{E}$-measurable (i.e., for which $f_i$ is not constant on every $G \in \mathscr{E}$). The penultimate equality follows from the fact that $\sum_{s \in G} P_i(\{s\}|G) f_i(s) = \sum_{s \in G} Q(\{s\}|G) f_i(s) = g_i(s_G)$.

If $g_i = 0_S$, then, since $P_i$ is strictly positive, $f_i = 0_S$ as well, and so trivially $g_i \succcurlyeq_i f_i$. Furthermore, recall that by assumption there is $G \in \mathscr{E}$ such that $f_1$ is not constant on $G$; then, $g_1(s_G) > 0$.

Now consider $i \in N$ such that $g_i \in \mathbb{R}^+_S \setminus \{0_S\}$ (including $i = 1$). Since $P_i \in \text{Core } I_i$, by Lemma 13, also $P_i \in \bar{\pi}_i(g_i)$, because by construction $g_i$ is $\mathscr{E}$-measurable and $\mathscr{E}$ is core-unambiguous for $(I_i, u_i)$. Thus, $P_i(u_i \circ g_i) \geq P_i(u_i \circ f_i)$ implies $g_i \succcurlyeq_i f_i$.

Conclude that $g_i \succcurlyeq_i f_i$ for all $i \in N$. Furthermore, for $i = 1$, since $g_1(s_G) > 0$ for some $G \in \mathscr{E}$, and $P_1(u_1 \circ g_1) > P_1(u_1 \circ f_1)$, by continuity of $u_1$ and the fact that $P_i(G) > 0$ there is $\epsilon \in (0, g_1(s_G))$ such that $P_1(u_1 \circ (g_1 - 1_G \epsilon)) > P_1(u_1 \circ f_1)$ as well. Then, again by Lemma 13, $g_1 - 1_G \epsilon \succcurlyeq_1 f_1$. Since preferences are strictly monotonic, $g_1 \succ_1 f_1$. This contradicts the assumption that $(f_1, \ldots, f_N)$ was Pareto-efficient. ■

## A.5 Calculations for Example 5

Observe first of all that, for all $\phi \in \mathbb{R}^n$,

$$\nabla I(a) \equiv \left( \frac{\partial I(a)}{\partial a(s)} \right)_{s \in S} = \left( P(\{s\}) \left[ 1 + \sum_{0 \leq j < J} \frac{\partial A(P(\zeta_0 a), \ldots, P(\zeta_{n-1} a))}{\partial \phi_j} \zeta_j(s) \right] \right)_{s \in S}. \tag{11}$$

Thus, the condition in the text ensuring that preferences are strongly monotonic is simply the requirement that all partial derivatives be strictly positive almost everywhere on $\mathbb{U}^S$.

Next, we show that $\nabla A(0_J) = 0_J$. Fix $0 \leq j < J$. Since $A$ is continuously differentiable at $0_J$, satisfies $A(0_J) = 0$ and is symmetric about $0_J$,

$$\nabla A(0_J) \cdot 1_j = \lim_{t \downarrow 0} \frac{A(0_J + t 1_j) - A(0_j)}{t} = \lim_{t \downarrow 0} \frac{A(t 1_j)}{t} = \lim_{t \downarrow 0} \frac{A(t(-1_j))}{t} = \lim_{t \downarrow 0} \frac{A(0_J + t(-1_j)) - A(0_J)}{t} = \nabla A(0_J) \cdot (-1_j),$$

which clearly requires that $\nabla A \cdot 1_j = \frac{\partial A(0_J)}{\partial \phi_j} = 0,$ as claimed. Since $P(\zeta_j 1_S x) = x P(\zeta_j) = 0$, it follows that $\nabla I(1_S x) = P$ for all $x > 0$.

Next, we verify that the specification of adjustment factors and function in Example 5, together with a uniform baseline prior, ensures strict monotonicity. We use Eq. (11): first, note that

$$\frac{\partial A}{\partial \phi_j} = -\frac{1}{2} \theta \cdot \frac{2\theta^{-1} \phi_j}{1 + \theta^{-1} \phi_j^2} = -\frac{\phi_j}{1 + \theta^{-1} \phi_j^2}.$$

Hence,

$$\left| \frac{\partial A}{\partial \phi_j} \right| = \frac{|\phi_j|}{1 + \theta^{-1} \phi_j^2} = \frac{|\phi_j|}{1 + \theta^{-1} |\phi_j|^2}.$$

Letting $t = |\phi_j|$, this is less then one iff $t < 1+\theta^{-1}t^2$, i.e. iff $t^2-\theta t+\theta > 0$. We study the function $t \mapsto t^2-\theta t+\theta$ for $t \geq 0$. If $t = 0$, the function takes the value $\theta$, so we need $\theta > 0$. The derivative of this function at any $t > 0$ (which is also the right derivative at 0) is $2t-\theta$, which shows that this function is strictly convex and has a minimum at $t = \frac{1}{2}\theta$, where it is equal to $\frac{1}{4}\theta^2-\frac{1}{2}\theta^2+\theta$. This is strictly positive iff $-\frac{1}{4}\theta+1>0$, i.e. iff $\theta<4$, as claimed.

Now consider states $s = s_1, s_2$. Only $\zeta_0$ has non-zero values, and $\zeta_0(s) \in \{1,-1\}$. Therefore, if $\theta \in (0,4)$,

$$1-\frac{\phi_0}{1+\theta^{-1}\phi_0^2}\zeta_0(s)-\frac{\phi_1}{1+\theta^{-1}\phi_1^2}\zeta_1(s) \geq 1-\left|\frac{\phi_0}{1+\theta^{-1}\phi_0^2}\right|>0.$$

Similarly, in states $s = s_3, s_4$, $\zeta_0(s)=0$ and $\zeta_1(s) \in \{1,-1\}$, so

$$1-\frac{\phi_0}{1+\theta^{-1}\phi_0^2}\zeta_0(s)-\frac{\phi_1}{1+\theta^{-1}\phi_1^2}\zeta_1(s) \geq 1-\left|\frac{\phi_1}{1+\theta^{-1}\phi_1^2}\right|>0.$$

so $I$ is strictly increasing.

We now show that, if $\theta$ increases, the resulting preference is more GM-ambiguity-averse. By the characterization result in Siniscalchi (2009), it suffices to show that $A(\phi)$ is decreasing in $\theta$ for every $\phi$. Differentiating $A(\phi)$ with respect to $\theta$,

$$\frac{\partial A(\phi)}{\partial \theta} = -\frac{1}{2}\sum_j \log(1+\theta^{-1}\phi_j^2)-\frac{1}{2}\theta\sum_j \frac{1}{1+\theta^{-1}\phi_j^2}(-\theta^{-2}\phi_j^2);$$

it suffices to show that, for every $j$ and $\phi_j$, $\log(1+\theta^{-1}\phi_j^2) > \frac{\theta^{-1}\phi_j^2}{1+\theta^{-1}\phi_j^2}$. Let $t \equiv \theta^{-1}\phi_j^2$, so we need to show that $\log(1+t) > \frac{t}{1+t}$. Both functions equal zero at $t=0$. For $t>0$, the derivatives of the lhs and rhs are $\frac{1}{1+t}$ and $\frac{1\cdot(1+t)-t(1)}{(1+t)^2} = \frac{1}{(1+t)^2}$ respectively. Since $(1+t)^2 > 1+t$ for $t>0$, $\frac{1}{1+t} < \frac{1}{(1+t)^2}$, and therefore, for all $t>0$, $\log(1+t) = \int_0^t \frac{1}{1+s}ds > \int_0^t \frac{1}{(1+s)^2}ds = \frac{t}{1+t}$, as claimed.

Finally, we derive the condition on $\theta$ for the desired rankings to hold:

$$\begin{aligned}
& -\log(1+\theta^{-1}(\alpha-1)^2)-\log(1+\theta^{-1}\alpha^2) < -\log(1+\theta^{-1}) \\
\Leftrightarrow \quad & (1+\theta^{-1}(1-\alpha)^2)(1+\theta^{-1}\alpha^2) > 1+\theta^{-1} \\
\Leftrightarrow \quad & 1+\theta^{-1}(1-\alpha)^2+\theta^{-1}\alpha^2+\theta^{-2}(1-\alpha)^2\alpha^2 > 1+\theta^{-1} \\
\Leftrightarrow \quad & (1-\alpha)^2+\alpha^2+\theta^{-1}(1-\alpha)^2\alpha^2 > 1 \\
\Leftrightarrow \quad & \theta^{-1} > \frac{1-\alpha^2-(1-\alpha^2)}{\alpha^2(1-\alpha)^2} = \frac{1-\alpha^2-1-\alpha^2+2\alpha}{\alpha^2(1-\alpha)^2} = \frac{2\alpha(1-\alpha)}{\alpha^2(1-\alpha)^2} = \frac{2}{\alpha(1-\alpha)} \\
\Leftrightarrow \quad & \theta < \frac{\alpha(1-\alpha)}{2}.
\end{aligned}$$

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