Competitive Fair Division under additive utilities

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Abstract

When utilities are additive we propose new arguments vindicating the familiar Competitive Equilibrium with Equal Incomes division rule, and criticizing its normative nemesis, the Egalitarian rule.

Resource Monotonicity says that more manna should not hurt anyone. *Responsive Shares* means that by raising my bid on one good I cannot end up with a smaller share of that good. *Independence of Lost Bids* says that changing my bid on a good I am not consuming before or after, does not affect the allocation at all. The latter property, and a mild invariance axiom allowing us to merge goods that the agents view as equivalent, characterize the Competitive rule.

1 Introduction

Additive utilities ? Modern economic analysis mostly dismisses additive utilities (equivalently, linear preferences) because in many contexts some degree of complementarity between goods is a fact, as when preferences aggregate housing, health, education, etc. But recent work on the theory and implementation of fair division rules gives a central role to additive utilities, for compelling practical reasons.

The goal is allocate divisible private goods, the common property of a set of beneficiaries, in a "closed economy": there is no outside market where the goods are available at a price, and monetary transfers between agents are not feasible. Think of distributing the family heirlooms between siblings: the emotional value attached to these objects bears little connection to their scrap value, and the siblings often want to avoid "distasteful direct monetary payments" ([22]); other examples include a divorcing couple splitting assets like pets, or the custody of children ([2]), managers dividing office space, students sharing overdemanded

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business school courses ([27], [4]), or substitutable workers allocating job shifts ([3]).

The empirical observation is that, in the context of the above examples, most people cannot form sophisticated preferences described by general utility functions. For the same reason, practical combinatorial auctions do not ask buyers to report a ranking of all subsets of objects, a complex task even with 6 objects, an outright impossible one with 10 or more ([1], [36], [8]). The websites like *Spliddit* and *Adjusted Winner*[1] applying theoretical concepts to solve concrete division problems, use a transparent *bidding system* to elicit individual preferences: you distribute 100 points over the different goods, and these weights define your additive utility. The proof of the pudding is in the eating: thousands of visitors use these sites every month, fully aware that their bid will be interpreted as their additive utility. ([12]).

Linear preferences are unpopular for another reason, of a purely technical nature: agents consume on the frontier of their budget set, which creates many technical challenges, illustrated in some of our proofs. The routine assumption that indifference curves do not cross the axis is meant precisely to avoid corner consumption. But in the fair division context, it is a good thing that many goods end up in the basket of a single agent: objects that cannot be physically split (pets, the grandfather's clock) can still be divided by randomization or time sharing, but we normally wish to minimize the number of such indirect splitting devices.

Two fair division rules and the punchline. Four decades ago, the literature on fair division under general Arrow Debreu preferences proposed two rules, the *Competitive Equilibrium with Equal Income* ([35]), here simply the *Competitive* rule, and the *Egalitarian Equivalent* rule[2] ([21]), here simply the *Egalitarian* rule. They are, respectively, the hero and the villain of this paper.

The high level principle behind the Competitive division rule is that the manna provides *equal opportunities* to all beneficiaries, in the sense of the No Envy test ([10]): I weakly prefer my share to yours because I can afford both. The Egalitarian rule equalizes *welfare* rather than opportunities, measuring welfare as the share of the entire manna equivalent to the actual share: in the bidding system above, this is the number of points you assigned to your share.

We discuss three natural axioms, each with a dual fairness and incentive interpretation, that the Competitive rule meets, but the Egalitarian rule fails, in the additive domain. And from the third axiom we derive a characterization of the Competitive rule. We submit these results as a strong normative vindication of the Competitive rule in the additive domain.

It is remarkable, and well known, that in our context the Competitive rule has an equivalent welfarist representation, as the maximizer of the Nash product of individual utilities.[3] Thus the Competitive allocations solve a convex opti-

[1] www.spliddit.org/
www.nyu.edu/projects/adjustedwinner/
[2] Aka the Adjusted Winner rule ([2]) in the case of two agents.
[3] See Section 3, or Chapter 7 in [19] for a textbook presentation.

mization program and, critical to our results, are characterized by a system of Karush–Kuhn–Tucker conditions (KKT for short) much easier to handle than competitive demands. But this welfarist representation of the Competitive rule holds for all homothetic preferences, and to that much bigger domain our results do not extend at all: in fact two of our axioms are not even defined outside the additive domain. See point 2 in the next Section.

Three axioms Start with the familiar *Resource Monotonicity* (RM) property: as we add new goods to the pot (or increase the quantity of some goods) the welfare of all beneficiaries should improve at least weakly. This is a compelling and popular solidarity property in the common property regime. Originally introduced for the fair division of private goods ([25]), it was applied to a broad range of resource allocation problems with production and/or indivisibilities (see the survey [30]). Its incentive aspect is that, if RM fails some agents have an incentive to sabotage the process by destroying some goods, or failing to discover them. The Competitive rule meets RM, the Egalitarian rule does not (Proposition 1).

The next two properties are new, and specific to the additive domain. *Responsive Shares* (RS), says that shares should respond monotonically to bids: if I raise my bid for a certain good, ceteris paribus,[4] my share of this good increases (weakly). Failure of RS invites disingenuous reporting of one's preferences: if I like good a today more than yesterday, by reporting this I may end up with a smaller share of a, or no share at all! The Competitive rule meets RS, the Egalitarian rule does not (Proposition 2).

To introduce the third axiom, *Independence of Lost Bids* (ILB), recall that in the additive domain we expect "corner solutions". To be precise: if n agents share p goods, at almost all utility profiles *every* efficient allocation of the goods has at least $(n-1)(p-1)$ null entries: most agents do not consume most goods (Lemma 1). In terms of the bid profiles (one bid per agent per good), most bids are *losing*.

Independence of Lost Bids means that when we revise a losing bid (a bid on a good that we do not consume), and the bid remains losing, nothing happens to anybody: the allocation selected by the rule does not change. It is an incentive property inasmuch as a change of report on a good that we do not end up consuming is "cheap": it is presumably harder to verify *ex post* my marginal utility for that good than for a good I am actually eating.

Proposition 3 says that the Competitive rule meets ILB. The Egalitarian rule fails it spectacularly: a fake *increase* of my lost bids, small enough that they remain losing, *always* benefits me strictly, and hurts every other agent.

Our main result, Theorem 1, characterizes the Competitive rule by the combination of standard efficiency and symmetry properties, Independence of Lost Bids, and a mild invariance property allowing us to merge goods that the agents view as equivalent.

[4] In our model we allow bids (vectors of rates of substitution between goods) with an arbitrary sum, and require rules to be invariant when the scale of the bids profile changes. If instead the sum of the bids is fixed (to 100 points), raising the bid on good a implies a proportional lowerig of bids on other goods.

Contents After reviewing the literature, we define the model and introduce the Competitive and Egalitarian rules in Section 3. Section 4 discusses our three properties RM, RS and ILB. The characterization result is the object of Section 5. Most proofs are in Section 6, and two open questions in Section 7.

2 Related literature

1. The fair division of private goods (*manna*) under heterogenous ordinal preferences is a benchmark problem in distributive justice. The earliest microeconomic model was proposed before 1950 by famous mathematicians as the "cake-division" problem: the cake is a measure space and utilities are non atomic *additive* measures ([29]), and a key concern is to achieve a fair division by means of a small number of cuts ([24]). But from the 1970s the literature insists on allowing general microeconomic preferences ([34] is an early survey), and finds that the Egalitarian rule is the most robust of the two: continuous and monotonic preferences is all we need to define the Egalitarian allocations and unique utility profile. By contrast the existence of a Competitive allocation is only guaranteed if preferences are also convex, and even then multiple allocations with different utility profiles are possible.

These difficulties disappear in the domain of homothetic preferences. There the Competitive allocations maximize the (Nash) product of individual utilities (and vice-versa: [9]), hence the Competitive utility profile is unique. Although this equivalence is more than 50 years old, interest in special homothetic domains arose only recently. In the linear domain Schummer shows that Strategyproofness is incompatible with Efficiency and a minimally fair treatment of the participants ([26]), a result recently strenghtened by Cho and Thomson ([6]). Megiddo and Vazirani ([18]) show that the competitive utility profile depends continuously upon the rates of substitution and the total endowment; it is also computed in time polynomial in the dimension $n + p$ of the problem ([13]). As for the type of monotonicity properties we promote in this paper, its only precursor appears is an unpublished note [33] by Thomson and Kayi.

2. As mentioned in Section 1, our axioms and results do not extend to the homothetic domain. There Resource Monotonicity, Efficiency, and the very mild Fair Share Guarantee property[5], are incompatible ([20]), therefore RM fails for the Egalitarian *and* Competitive rules. Moreover RS and ILB are not defined in economies where marginal rates of substitution are not constant.

3. Leontief preferences assume perfectly complementary goods,[6] so they are "diametrally opposed" to linear ones. Such preferences come up naturally when agents compete for cloud computing resources. Remarkably in this domain the Egalitarian rule is not only Envy-Free like the Competitive rule, it also has very strong incentives properties, that the Competitive rule lacks: it

[5] Everyone ends up no worse than receiving $1/n$-th of each good.

[6] So i's utility for allocation z_i is $\min_a\{\frac{z_{ia}}{v_{ia}}\}$ where the v_{ia}-s are fixed rates of complementarity.

is Strategyproof (truthful report of one's preferences is a dominant strategy), even Groupstrategyproof (see [**?**] and [16]). Although neither rule is Resource Monotonic, the Egalitarian one is now the hero while the Competitive one has no known normative advantage

4. A stream of research in algorithmic mechanism design focuses on the fair division of *indivisible* objects, with the aim to emulate, approximately, the results of the divisible goods model. The *Nash Product Maximizer* (NMP) still plays a central role in the discussion, although it loses its competitive interpretation and becomes hard to compute. In the general domain of "combinatorial preferences" (specifying the utility of each subset of objects) Ramezani and Endriss ([23]) approximate the NMP allocation, and Lipton et al. ([17]) find efficient allocations that are approximately envy-free. In the additive domain the NMP allocation is envy-free "up to at most one object" (Caragianis et al. [5]). It is still hard to compute ([15]), but finding an approximation is easier ([7]).

Finally Budish ([3]) follows a different approximation route for problems with a large number of copies of several object-types: a little flexibility in the number of available copies ensures the existence of a "quasi-CEEI" allocation.

3 The model and two division rules

3.1 Basic definitions

The finite set of agents is N with generic element i. We assume $|N| = n \geq 2$. The finite set of (divisible) goods is A with generic element a. The manna consists of 1 unit of each object.

Agent i's allocation (or share) is some $z_i \in [0,1]^A$; the profile $z = (z_i)_{i \in N}$ is a feasible allocation if $\sum_N z_i = e^A$, where all coordinates of e^A in $\mathbb{R}_+^A$ are 1. The set of feasible allocations is $\Phi(N, A)$.

Each agent is endowed with linear preferences over $[0,1]^A$, represented for convenience by a vector $u_i \in \mathbb{R}_+^A$ (a utility function). We keep in mind that only the *ordinal* preferences matter, i. e., for any $\lambda > 0$, u_i and λu_i carry the same information. Given an allocation z we write i's corresponding utility as $U_i = u_i \cdot z_i = \sum_A u_{ia} z_{ia}$.

A **division problem** is a triple $\mathcal{Q} = (N, A, u)$ and the corresponding set of feasible utility profiles is $\Psi(\mathcal{Q})$. Note that we may have *useless* goods ($u_{ia} = 0$ for all i) or *uninterested* agents ($u_{ia} = 0$ for all i). Otherwise we speak of *useful* goods and *interested* agents.

We use two equivalent definitions of a division rule, in terms of utility profiles or of feasible allocations. When we rescale each u_i as $\lambda_i u_i$ the new profile of utilities is written $\lambda * u$.

Definition 1
i) A division rule F associates to every problem $\mathcal{Q} = (N, A, u)$ a utility profile $F(\mathcal{Q}) = U \in \Psi(\mathcal{Q})$. Moreover $F(N, A, \lambda * u) = \lambda * U$ for any rescaling λ where $\lambda_i > 0$ for all i.

ii) A division rule f associates to every problem $\mathcal{Q} = (N, A, u)$ a subset $f(\mathcal{Q})$ of $\Phi(N, A)$ such that for some $U \in \mathbb{R}_+^A$:

$$f(\mathcal{Q}) = \{z \in \Phi(N, A) | (u_i \cdot z_i)_{i \in N} = U\}$$

Moreover $f(N, A, \lambda * u) = f(\mathcal{Q})$ for any rescaling λ where $\lambda_i > 0$ for all i.

The one-to-one mapping from F to f is clear. Definition 1 makes no distinction between two allocations with identical welfare consequences. For instance a useless good can be divided arbitrarily, and, if the rule is efficient, an uninterested agent can only consume positive amounts of useless goods.

Efficient allocations have a particular structure in the linear domain, a very important fact for several of our results. In particular this explains why in most efficient allocations z most entries of the matrix $[z_{ia}]$ are nil.

For any $z \in \Phi(N, A)$ define the bipartite $N \times A$ *consumption graph* $\Gamma(z) = \{(i, a) | z_{ia} > 0\}$. Write $\Psi^{eff}(\mathcal{Q})$ for the set of efficient utility profiles at $\mathcal{Q}$.

Lemma 1
a) Fix a problem $\mathcal{Q} = (N, A, u)$. If $U \in \Psi^{eff}(\mathcal{Q})$ then there is some $z \in \Phi(N, A)$ representing U such that $\Gamma(z)$ is a forest (an acyclic graph). For such allocation z the matrix $[z_{ia}]$ has at least $(n-1)(p-1)$ zeros.

b) Fixing N, A, on an open dense subset $\mathcal{U}^*(N, A)$ of utility profiles $u \in \mathbb{R}_+^{N \times A}$, every efficient utility profile $U \in \Psi^{eff}(N, A, u)$ is achieved by a single allocation z. At such profiles an efficient division rule f is single-valued. The definition of $\mathcal{U}^*(N, A)$ is given in Subsection 6.1.

3.2 The Competitive rule

Definition 2 *Given a problem $\mathcal{Q} = (N, A, u)$ we say that the feasible allocation $z \in \Phi(N, A)$ is a Competitive Equilibrium with Equal Incomes (or simply a* ***competitive*** *allocation) if there is a price $p \in \mathbb{R}_+^A$ such that $\sum_A p_a = n$ and*

$$z_i \in \arg\max_{y_i \in \mathbb{R}_+^A} \{u_i \cdot y_i | p \cdot y_i \leq 1\} \text{ for all } i$$

We write $f^c(\mathcal{Q})$ for the set of all such allocations.

Proposition 1 *([9]): Fix a problem $\mathcal{Q} = (N, A, u)$ where all agents are interested, $u_{iA} > 0$ for all i. Then the Competitive rule f^c selects precisely all the feasible allocations maximizing the product of utilities:*

$$f^c(\mathcal{Q}) = \arg\max_{\Phi(N,A)} \Pi_N u_i \cdot z_i \tag{1}$$

Note that the above maximization is trivial if there is at least one uninterested agent. And if all agents are interested we have $u_i \cdot z_i > 0$ for all i and $z \in f^c(\mathcal{Q})$.

The Proposition implies that f^c is a bona fide division rule in the sense of Definition 1: the set $f^c(\mathcal{Q})$ is non empty, and the corresponding utility profile $F^c(\mathcal{Q})$ maximizes over the convex compact set $\Psi(\mathcal{Q})$ the strictly quasi-concave function $\Pi_N U_i$, therefore it is unique and efficient.[7] The system of first order optimality conditions of this convex program (known as the *KKT conditions*) plays a central role in the entire paper.

Lemma 2 *Fix a problem $\mathcal{Q} = (N, A, u)$ where N^* is the set of interested agents, and A^* that of useful goods. The two following statements are equivalent:*
i) $U = F^c(\mathcal{Q})$
ii) $U = (u_i \cdot z_i)_{i \in N}$ for some $z \in \Phi(N, A^*)$ such that $z_i = 0$ if and only if $i \notin N^*$, and for all $i \in N^*$ we have

$$\text{for all } a\text{: } z_{ia} > 0 \Longrightarrow \{\frac{u_{ia}}{U_i} \geq \frac{u_{ja}}{U_j} \text{ for all } j \in N^*\} \tag{2}$$

(in particular $U_i > 0$ if $i \in N^*$)

3.3 The Egalitarian rule and a sharp example

For a vector $W \in \mathbb{R}_+^N$ we write $W^* \in \mathbb{R}_+^n$ its order statistics, where the coordinates of W are reordered increasingly. Then the *leximin ordering* $\succeq_{lxmn}$ of $\mathbb{R}_+^N$ compares W^1 and W^2 exactly like the lexicographic ordering of $\mathbb{R}_+^n$ compares W^{1*} and W^{2*}. Recall that over any convex compact subset of $\mathbb{R}_+^N$ the leximin ordering has a unique maximum.

Given a problem $\mathcal{Q}$ and a feasible utility profile $U \in \Psi(\mathcal{Q})$, we write $\widetilde{U}$ for its normalization where eating all the goods has utility 1 for everyone:

$$\widetilde{U} = \lambda * U \text{ where } \lambda_i = \frac{1}{u_i \cdot e^A}$$

Definition 3 *Given a problem $\mathcal{Q} = (N, A, u)$ where all agents are interested ($u_{iA} > 0$ for all i), the egalitarian utility profile $U = F^e(\mathcal{Q})$ is such that $\widetilde{U}$ maximizes the leximin ordering over $\widetilde{\Psi}(\mathcal{Q}) = \{\widetilde{V} | V \in \Psi(\mathcal{Q})\}$.*

The egalitarian utility profile is clearly efficient. It equalizes the normalized utilities as much as permitted by efficiency. If all agents care for all goods, $u_{ia} > 0$ for all $i \in N, a \in A$, it is easy to check all coordinates of $\widetilde{U}$ are equal, in other words the egalitarian utility profile is defined by efficiency plus the equality of normalized utilities:

$$\frac{U_i}{u_i \cdot e^A} = \frac{U_j}{u_j \cdot e^A} \text{ for all } i, j \tag{3}$$

If some u_{ia} are zero this equality may be incompatible with efficiency. For instance two goods a, b, agents 1 likes only a and agents 2, 3 like only b.

Our canonical n-person example is one where the difference between the competitive and egalitarian allocations is largest.[8] We have $n-1$ goods, $(n-1)$

[7] We omit the proof of the Proposition for brevity. A simple presentation is in [28].
[8] We conjecture that in this example the ℓ_∞ and ℓ_1 distances between the profiles of normalized utilities at the two allocations are the largest possible for a fixed n.

single-minded agents $i \in \{1, \cdots, n-1\}$ who like only good i, and one *flexible* agent n who likes equally all goods.

The competitive price is $\frac{n}{n-1}$ for every good, and each agent $i \in \{1, \cdots, n-1\}$ buys $\frac{n-1}{n}$ units of "his" good while agent n gets a $\frac{1}{n}$-th share of each good. The egalitarian solution splits each good i equally between agent i and agent n so that everyone ends up with a share worth one half of the entire manna. For instance

$$\begin{array}{cccc} & a & b & c \\ u_1 & 1 & 0 & 0 \\ u_2 & 0 & 1 & 0 \\ u_3 & 0 & 0 & 1 \\ u_4 & 1 & 1 & 1 \end{array} \quad \text{Competitive:} \quad \begin{array}{cccc} & a & b & c \\ z_1^c & \frac{3}{4} & 0 & 0 \\ z_2^c & 0 & \frac{3}{4} & 0 \\ z_3^c & 0 & 0 & \frac{3}{4} \\ z_4^c & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{array} \quad \text{Egalitarian:} \quad \begin{array}{cccc} & a & b & c \\ z_1^e & \frac{1}{2} & 0 & 0 \\ z_2^e & 0 & \frac{1}{2} & 0 \\ z_3^e & 0 & 0 & \frac{1}{2} \\ z_4^e & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}$$

The Egalitarian rule focuses on the (relative) benefits of consuming each good i, and in this example shares it equally between the two relevant agents, i and n. The Competitive rule is much more generous to the single-minded agents: everyone is equal owner of each good i, and so is entitled to an equal share of the the competitive surplus it generates.

The two rules meet the oldest test of the fair division literature ([26]): everyone is guaranteed, welfarewise, at least her "fair share" of the entire manna:

Fair Share Guarantee: $U_i = u_i \cdot z_i \geq u_i \cdot (\frac{1}{n} e^A)$ for all i

Notice that at the Competitive allocation z^c above, the flexible agent n gets exactly her fair share, while everybody else gets strictly more ($(n-1)$ times more !). By contrast an Egalitarian allocation, *here and always*, gives strictly more than fair share to *everybody*, unless *nobody* can get more than fair share.[9] This is clearly a valid objection to the Competitive rule: in this example it does not reward the flexibility of agent n's preferences.

On the other hand we submit that the Egalitarian allocation gives *too much* to agent n in the same example : she gets (much) more than her fair share of *every* good. By contrast at a Competitive allocation, *here and always*, everyone gets *at most* a $\frac{1}{n}$-th share of *at least* one good:[10]

$$\min_{a \in A} z_{ia} \leq \frac{1}{n} \text{ for all } i \tag{4}$$

Property (4) is in the same spirit as the familiar test

No Envy: $u^i \cdot z^i \geq u^i \cdot z^j$ for all i, j

that inspired the Competitive rule in the first place. There is no systematic logical relation between these two properties: in the example the Egalitarian allocation is envy-free as well.

[9] Suppose at the egalitarian U we have $U_i = u_i \cdot (\frac{1}{n} e^A)$ for $i \leq k$ and $U_j > u_j \cdot (\frac{1}{n} e^A)$ for $j \geq k+1$. Then no i agent can like any good a consumed by some j agent at some egalitarian allocation z: else we could transfer some of a from this j to all i-s and improve the leximin ordering. Let B be the set of goods liked by the i-s: since the j-s do not eat any B at z, the i-s can divide B equally and get each $u_i \cdot (\frac{1}{k} e^B) > u_i \cdot (\frac{1}{n} e^A)$, without affecting the j-s. Contradiction.

[10] If $z_{ia} > \frac{1}{n}$ for all a the competitive price must be parallel to u_i and the equal budget $p \cdot z_i = p \cdot (\frac{1}{n} e^A)$ gives $u_i \cdot z_i = u_i \cdot (\frac{1}{n} e^A)$, contradiction.

4 Three axioms

4.1 Resource Monotonicity

More goods to divide should not be bad news to anyone: all agents "own" the goods equally and welfare should be comonotonic to ownership. This simple normative property has played a major role in the modern fair division literature ([30]). Its incentive interpretation: if it fails, someone has an incentive to sabotage the discovery of additional "manna", or destroy parts of it.

In the following definition we write $u_{[B]}$ for the restriction of the utilities in $\mathbb{R}_+^A$ to $\mathbb{R}_+^B$:

Proposition 1 *The Competitive rule meets, and the Egalitarian rule fails, Resource Monotonicity (RM):*

$$\text{for all } \mathcal{Q} = (N, A, u) \text{ and all } B \subset A : \; F(N, B, u_{[B]}) \leq F(\mathcal{Q}) \tag{5}$$

Recall that in the general Arrow-Debreu preference domain, no efficient division rule can be resource monotonic and meet Fair Share Guarantee ([20]): we see that some complementarity between goods is essential to that result.

It is easy and instructive to show why the Egalitarian violates (5) as soon as $n \geq 3$. We compare two problems with $B = \{a, b, c\}$and $A = \{a, b, c, d\}$ respectively:

$$\begin{array}{cccc} & a & b & c \\ u_1 & 3 & 1 & 1 \\ u_2 & 1 & 3 & 1 \\ u_3 & 1 & 1 & 3 \end{array} \quad \text{and} \quad \begin{array}{ccccc} & a & b & c & d \\ u_1 & 3 & 1 & 1 & 0 \\ u_2 & 1 & 3 & 1 & 4 \\ u_3 & 1 & 1 & 3 & 4 \end{array}$$

The B-problem is symmetric. Any efficient and symmetric rule allocates goods "diagonally": agent 1 gets all of a and so on; normalized utilities are $\frac{3}{5}$. In the A-problem the natural idea is to keep the same allocation of a, b, c and divide d equally between agents 2 and 3, because agent 1 does not care for d. This is what the Competitive rule recommends (prices are $(1, \frac{3}{5}, \frac{3}{5}, \frac{4}{5})$). But the normalized utilities at this allocation are $(\frac{3}{5}, \frac{5}{9}, \frac{5}{9})$, so the Egalitarian rule must compensate agents $2, 3$ for the *loss* in normalized utilities caused by the *gain* of some new good! Equality is restored at the allocation

$$z^e = \begin{array}{cccc} a & b & c & d \\ 55/59 & 0 & 0 & 0 \\ 2/59 & 1 & 0 & 1/2 \\ 2/59 & 0 & 1 & 1/2 \end{array}$$

thus agent 1's welfare decreases when d is added to the manna.

Remark 1 Another popular solidarity property in the fair division literature is Population Monotonicity (PM): if a new agent joins the beneficiaries while the manna is unchanged, the welfare of all the old agents decreases weakly. The Egalitarian rule meets PM for fully general preferences (strictly monotonic and continuous) and in the linear domain it does as well (this is clear by (3) if all agents care for all goods; we omit the argument when some entries u_{ia} are zero

). *On the full domain the Competitive rule fails PM (ChiThom) but in the linear domain, it does. This follows easily from RM and the celebrated Consistency property ([31], [32]).*[11]

4.2 Responsive Shares

If I raise my bid (my reported utility) on a certain good, ceteris paribus, I expect my share of that good to increase, at least weakly. Otherwise the rule will have the perverse effect that, by liking a good more relative to all others, we sometimes end up consuming less of it: the relation between my reported utility and my shares becomes confusing, and invites participants to go beyond sincere reporting.

Under the Competitive rule shares are responsive to bids in the sense just described, but under the Egalitarian rule they are not. Consider the simple example with two goods a, b, two agents and compute the egalitarian allocation in the following two problems

$$\begin{array}{ccc} & a & b \\ u_1 & 2 & 1 \\ u_2 & 2 & 2 \end{array} \Longrightarrow z^e = \begin{array}{cc} a & b \\ \frac{6}{7} & 0 \\ \frac{1}{7} & 1 \end{array} \;;\; \begin{array}{ccc} & a & b \\ u_1 & 3 & 1 \\ u_2 & 2 & 2 \end{array} \Longrightarrow z^e = \begin{array}{cc} a & b \\ \frac{4}{5} & 0 \\ \frac{1}{5} & 1 \end{array}$$

where agent 1 eats less good a when it becomes more attractive relative to good b.

Definition 4 *The rule f has Responsive Shares (RS) if for any N, A, any $u, u' \in \mathbb{R}_+^{N \times A}$ that only differ in the coordinate i, a,* we have:

$$u_{ia} < u'_{ia} \Longrightarrow z_{ia} \leq z'_{ia} \text{ for any } z \in f(u) \text{ and } z' \in f(u') \tag{6}$$

Recall that $f(u)$ can be multivalued: we require that all allocations in $f(u')$ give weakly more good a to i than all in $f(u)$.

Proposition 2 *The Competitive rule has Responsive Shares, but the Egalitarian rule does not.*

4.3 Independence of Lost Bids

Under the Egalitarian rule if an agent i has a good idea of the profile of other bids, and in particular if she knows that at the true preference profile, she will not consume a certain good a, $z_{ia} = 0$, she has a transparent strategic manipulation: by raising her *losing bid* u_{ia} to u'_{ia} while making sure that this new bid remains losing, $z'_{ia} = 0$. Here is a two-person, three-good example

$$\begin{array}{cccc} & a & b & c \\ u_1 & 6 & 3 & 1 \\ u_2 & 1 & 3 & 6 \end{array} \rightarrow z^e = \begin{array}{ccc} a & b & c \\ 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \end{array} \;;\; \begin{array}{cccc} & a & b & c \\ u'_1 & 6 & 3 & 3 \\ u_2 & 1 & 3 & 6 \end{array} \rightarrow z'^e = \begin{array}{ccc} a & b & c \\ 1 & \frac{8}{11} & 0 \\ 0 & \frac{3}{11} & 1 \end{array}$$

In general if the egalitarian allocation z meets (3) and we raise u_{ia} to u'_{ia} the numerator of $\frac{U_i}{u_i \cdot e^A} = \frac{u_i \cdot z_i}{u_i \cdot e^A}$ does not change while the denominator increases;

[11] While dividing the manna ω between $n+1$ agents, say that the shares of the first n agents are z_i. After dropping agent $n+1$, by Consistency the n agents still share $\omega - z_{n+1}$ as z_i. And by RM they are all weakly better off when they share ω.

therefore to restore equality of $\frac{U_i}{u_i'\cdot e^A}$ and $\frac{U_j}{u_j\cdot e^A}$ we must increase i's utility U_i. As i still eats no a after this move, her new allocation increases her true welfare, and decreases that of every other agent. The statement is still true if some entries of the utility matrix are zero and the egalitarian profile of relative welfares maximizes the leximin ordering (we omit the details).

By contrast under the Competitive rule, raising or lowering a bid on a "lost good" has no effect at all on the allocation, therefore it allows no profitable manipulation. This follows from the KKT conditions (2) characterizing the competitive allocation. Fix u, u' that only differ in that $u_{ia} > u'_{ia}$ and assume that $z_{ia} = 0$ at some $z \in f^c(u)$. In the KKT system at z the only inequalities involving u_{ia} are $\frac{u_{ja}}{U_j} \geq \frac{u_{ia}}{U_i}$ for any j eating some a at z: they still hold at u'_{ia}, therefore the same allocation z meets (2) at u'.

For our characterization result, we only need to consider the lowering of a lost bid to zero.

Definition 5 *The rule f is Independent of Lost Bids (ILB) if for any N, A, and $u, u' \in \mathbb{R}_+^{N\times A}$ that only differ in a single entry ia where $u_{ia} > u'_{ia} = 0$, we have*

$$\forall z \in f(N, A, u) : z_{ia} = 0 \Longrightarrow z \in f(N, A, u') \tag{7}$$

Proposition 3 *The Competitive rule meets, and the Egalitarian rule fails, Independence of Lost Bids (ILB).*

Unlike Propositions 1,2, of which the proof requires some work (see Section 6), Proposition 3 follows at once from the KKT conditions (2) as explained above.

Interestingly for an efficient rule f, the ILB implies two related properties of which the interpretation is somewhat more intuitive. For any $u, u' \in \mathbb{R}_+^{N\times A}$ that only differ by $u_{ia} > u'_{ia} = 0$, we have (with a simplified notation)

$$\{\forall z \in f(u) : z_{ia} = 0\} \Longrightarrow f(u) = f(u') \tag{8}$$

Indeed we already checked that under the premises of (8) $f(u) \subseteq f(u')$; moreover for all z such that $z_{ia} = 0$ we have $u' \cdot z = u \cdot z$ hence $F^c(u) = F^c(u')$. Fix now some $z' \in f(u')$ and observe that $z'_{ia} = 0$: at least one agent other than i likes a, otherwise i would eat all a at u by efficiency; so at u' it is inefficient for i to eat any a. This implies $u' \cdot z' = u \cdot z'$ and in turn $u \cdot z' = F^c(u') = F^c(u)$ gives $z' \in f(u)$, and completes the proof.

Property (8) further implies that nothing changes when *several* agents simultaneously change their bids on goods they do not consume before or after the move: adjustments of bids that do not change the support of the allocation matrices in $f(u)$ are inconsequential. *For any N, A and $u, u' \in \mathbb{R}_+^{N\times A}$ we have*

$$\{\forall i, a : u_{ia} \neq u'_{ia} \Longrightarrow \{\forall z \in f(u), z' \in f(u') : z_{ia} = z'_{ia} = 0\}\} \Longrightarrow f(u) = f(u') \tag{9}$$

Indeed if u and u' differ in several entries where $u_{ia} > u'_{ia} = 0$, repeated application of (8) implies again $f(u) = f(u')$. Now if u, u' are as in the premises of (9) define $\widetilde{u}$ by $\widetilde{u}_{ia} = 0$ whenever u and u' differ, and equal to u, u' otherwise:

then $f(u)$ and $f(u')$ are both equal to $f(\widetilde{u})$. Conversely if f is efficient property (9) implies (8), because if u, u' are as in (8) we saw above that $z'_{ia} = 0$ for all $z' \in f(u')$.

Property (9) has an incentive interpretation as well. Suppose a coalition of agents change their bids on goods they do not consume before or after the shift. Then they cannot jointly benefit from this move (i., e., if one of them benefits strictly from the coordinated misreport, then another one must be hurt). To check this is equivalent to (9) is easy and omitted for brevity.

5 Characterizing the Competitive rule

We use four axioms in addition to ILB. The first two are standard

- **Efficiency** *(EFF): $F(\mathcal{Q}) \in \Psi^{eff}(\mathcal{Q})$ for all $\mathcal{Q} = (N, A, u)$*
- **Symmetry** *(SYM)**:** (the label of agents and goods does not matter) F is invariant with respect to permutations of N, and of A*

The next two axioms come from the fair division context, and they are very mild.

We say that problem $\mathcal{Q} = (N, A, u)$ is *partitioned* in two subproblems (N^k, A^k, u^k), $k = 1, 2$, if N^1, N^2 partition N, A^1, A^2 partition A, and for $\{k, l\} = \{1, 2\}$ no agent in N^k likes any object in A^l.

- **Partition** *(PAR): the rule solves each subproblem of a partitioned problem separately*

(i. e., $F(\mathcal{Q})$ is the concatenation of $F(N^k, A^k, u^k)$ for $k = 1, 2$)

We say finally that two goods a, b are *equivalent* in problem $\mathcal{Q} = (N, A, u)$ if

$$u_{ia} \cdot u_{jb} = u_{ib} \cdot u_{ja} \text{ for all } i, j \in N$$

To *merge* goods a and b means to replace the problem $(\mathcal{Q})$ by (N, A^*, u^*) where a, b become a single good a^* with utilities $u^*_{ia^*} = u_{ia} + u_{ib}$ for all i, while utilities for other goods are unchanged.

- **Equivalent Goods** *(EG)**:** if we merge two equivalent goods a, b then $F(N, A^*, u^*) = F(\mathcal{Q})$*

Note that a useless good a, $u_{ia} = 0$ for all i, is equivalent to every other good, hence EG says that it can be merged with any other good without affecting the utility $F(\mathcal{Q})$: this is a way of saying that useless object are irrelevant.

All four axioms are met by many rules, such as the one-dimensional family of *welfarist* rules F^q, for $-\infty \leq q < +\infty$. If q is finite and non zero, $F^q(\mathcal{Q})$ maximizes in $\Psi(\mathcal{Q})$ the additive welfare $W^q(U) = sign(q) \sum_{j \in N} (\frac{U_j}{u_i \cdot e^A})^q$. The limit of F^q for $q \to 0$ maximizes $\sum_{j \in N} \ln(\frac{U_j}{u_i \cdot e^A})$, so it is the Competitive rule. When $q \to -\infty$ the limit of F^q is the Egalitarian rule.

Theorem: *The Competitive rule is characterized by Efficiency, Symmetry, Partition, Equivalent Goods, and Independence of Lost Bids.*

We stress that the only fairness axiom in the Theorem is Symmetry. No Envy is not used.

6 Proofs

6.1 Lemma 1

6.1.1 $a)$ the consumption forest at efficient allocations

Assume first that all goods are useful. Pick $\mathcal{Q}$ and z representing $U \in \Psi^{eff}(\mathcal{Q})$ and assume there is a K-cycle in $\Gamma(z)$: $z_{ka_k}, z_{ka_{k-1}} > 0$ for $k = 1, \cdots, K$, where $z_{1a_{1-1}} = z_{ia_K}$. Then $u_{ka_k}, u_{ka_{k-1}}$ are positive for all k: if $u_{ka_k} = 0$ efficiency and $u_{Na_k} > 0$ imply $z_{ka_k} = 0$.
Assume now

$$\frac{u_{1a_1}}{u_{2a_1}} \cdot \frac{u_{2a_2}}{u_{3a_2}} \cdot \cdots \cdot \frac{u_{(K-1)a_{K-1}}}{u_{Ka_{K-1}}} \cdot \frac{u_{Ka_K}}{u_{1a_K}} > 1 \tag{10}$$

Then we can pick arbitrarily small positive numbers ε_k such that

$$\frac{u_{1a_1} \cdot \varepsilon_1}{u_{1a_K} \cdot \varepsilon_K} > 1, \frac{u_{2a_2} \cdot \varepsilon_2}{u_{2a_1} \cdot \varepsilon_1} > 1, \cdots, \frac{u_{Ka_K} \cdot \varepsilon_K}{u_{Ka_{K-1}} \cdot \varepsilon_{K-1}} > 1 \tag{11}$$

and the corresponding transfer to each agent k of ε_k units of good k against ε_{k-1} units of good $k-1$ is a Pareto improvement, contradiction. Therefore (10) is impossible; the opposite strict inequality is similarly ruled out so we conclude

$$\frac{u_{1a_1}}{u_{2a_1}} \cdot \frac{u_{2a_2}}{u_{3a_2}} \cdot \cdots \cdot \frac{u_{(K-1)a_{K-1}}}{u_{Ka_{K-1}}} \cdot \frac{u_{Ka_K}}{u_{1a_K}} = 1 \tag{12}$$

Now if we perform a transfer as above where

$$\frac{u_{1a_1} \cdot \varepsilon_1}{u_{1a_K} \cdot \varepsilon_K} = \frac{u_{2a_2} \cdot \varepsilon_2}{u_{2a_1} \cdot \varepsilon_1} = \cdots = \frac{u_{Ka_K} \cdot \varepsilon_K}{u_{Ka_{K-1}} \cdot \varepsilon_{K-1}} = 1$$

the utility profile U is unchanged. If we choose the numbers ε_k as large as possible for feasibility, this will bring at least one entry (k, a_k) or (k, a_{k-1}) to zero, so in our new representation z' of U the graph $\Gamma(z')$ has fewer edges. We can clearly repeat this operation until we eliminate all cycles of $\Gamma(z)$.

Now if some goods are useless we give them all to an arbitrary agent and the statement still holds.

The last statement follows at once from the fact that a forest with $n + p$ edges contains at most $n + p - 1$ edges.

6.1.2 *b*) at almost all profiles each efficient utility profile is achieved by a single allocation

We let $\mathcal{U}^*(N,A)$ be the open and dense subset of $\mathbb{R}_+^{N\times A}$ such that for any cycle $\mathcal{C}=\{1,a_1,2,a_2,\cdots,a_K,1\}$ in the bipartite graph $N\times A$ we have $\pi(\mathcal{C})=\frac{u_{1a_1}}{u_{2a_1}}\cdot\frac{u_{2a_2}}{u_{3a_2}}\cdot\cdots\cdot\frac{u_{(K-1)a_{K-1}}}{u_{Ka_{K-1}}}\cdot\frac{u_{Ka_K}}{u_{1a_K}}\neq 1$ (property (12) fails) and moreover $u_{ia}>0$ for all i,a. It is clearly an open dense subset of $\mathbb{R}_+^{N\times A}$.

We pick a problem $\mathcal{Q}$ with $u\in\mathbb{R}_+^{N\times A}$, fix $U\in\Psi^{eff}(\mathcal{Q})$ and assume there is two different $z,z'\in\Phi(N,A)$ such that $u\cdot z=u\cdot z'=U$. There must be some pair $1,a_1$ such that $z_{1a_1}>z'_{1a_1}$. Because a_1 is eaten in full there is some agent 2 such that $z_{2a_1}<z'_{2a_1}$ and because $u_2\cdot z_2=u_2\cdot z'_2$ there is some good a_2 such that $z_{2a_2}>z'_{2a_2}$. Continuing in this fashion we build a sequence $1,a_1,2,a_2,3,a_3,\cdots$, such that $\{z_{ka_{k-1}}<z'_{ka_{k-1}}$ and $z_{ka_k}>z'_{ka_k}\}$ for all $k\geq 2$. This sequence must cycle, i. e., we must reach K,a_K such that $z_{Ka_K}>z'_{Ka_K}$ and $z_{\widetilde{k}a_K}<z'_{\widetilde{k}a_K}$ for some $\widetilde{k},1\leq\widetilde{k}\leq K-1$. Without loss we label $\widetilde{k}$ as 1, and the corresponding cycle as $\mathcal{C}$.

If $\pi(\mathcal{C})>1$ we pick as in the above proof small positive numbers ε_k meeting (11) and we add ε_k of good k to z'_{ka_k} while taking it away from $z'_{(k+1)a_k}$ (with the convention $K+1=1$): then every agent k in the cycle improves strictly upon z'_k without affecting other agents' allocation, contradicting efficiency. If $\pi(\mathcal{C})<1$ we construct similarly a Pareto improvement of z for the agents in the cycle without affecting other agents.

We conclude that $\pi(\mathcal{C})=\infty$ so that $u\in\mathbb{R}_+^{N\times A}\diagdown\mathcal{U}^*(N,A)$ as desired.

6.2 Lemma 2: KKT conditions for the Competitive rule

It is clearly enough to prove the result when all goods are useful and all agents interested.

ii) $\Longrightarrow$ **i)**. Take z representing U s. t. $\Gamma(z)$ is a forest (Lemma 1). Then if $z_{ia}>0$ and we transfer some small amount of a from i to any agent j, the inequality (2) ensures this does not increases the Nash product.

i) $\Longrightarrow$ **ii).** We check that the system $\{\frac{u_{ia}}{U_i}\geq\frac{u_{ja}}{U_j}$ for all $(i,a)\in\Gamma(z)$ and all $j\}$ is precisely the KKT conditions for problem (1). The Lagrangien is

$$\mathcal{L}(z,\delta,\theta)=\sum_N\ln(u_i\cdot z_i)-\sum_A\delta_a(z_{Na}-1)+\sum_{i,a}\theta^+_{ia}z_{ia}+\theta^-_{ia}(1-z_{ia})$$

where $\theta\geq 0$ and the sign of each δ_a is arbitrary. The conditions $\frac{\partial\mathcal{L}}{\partial z}(z,\delta,\theta)=0$ amount to

$$\frac{u_{ia}}{U_i}-\delta_a+\theta^+_{ia}-\theta^-_{ia}=0\text{ for all }i,a\tag{13}$$

If $z_{ia}=1$ then $z_{ja}=0$ for all $j\neq i$, and system (13) gives $\frac{u_{ia}}{U_i}\geq\delta_a\geq\frac{u_{ja}}{U_j}$. If $0<z_{ia}<1$ then $\frac{u_{ia}}{U_i}=\delta_a$, and for another agent j we have $z_{ja}<1$ hence $\frac{u_{ja}}{U_j}\leq\delta_a$.

6.3 Proposition 1: the Competitive rule is Resource Monotonic

We can assume without loss of generality that all agents are interested and all goods are useful.

We first generalize the definition of F^c, f^c to problems where the endowment ω_a of each good is arbitrary, and we check that the KKT conditions capturing the optimal allocations $f^c(N, A, \omega, u)$ are unchanged. Then we fix N, A, u, ω, ω' such that $\omega \leq \omega'$. For $\lambda \in [0, 1]$ we write $\omega^\lambda = (1 - \lambda)\omega + \lambda\omega'$, and for every forest Γ in $N \times A$ we define

$$\mathcal{B}(\Gamma) = \{\lambda \in [0, 1] | \exists z \in f^c(N, A, \omega^\lambda, u) : \Gamma(z) = \Gamma\}$$

Note that $\mathcal{B}(\Gamma)$ can be empty or a singleton, but if it is not, then it is an interval. To see this take $z \in f^c(\omega^\lambda), z' \in f^c(\omega^{\lambda'})$ such that $\Gamma(z) = \Gamma(z')$. For any $\omega'' = (1 - \mu)\omega^\lambda + \mu\omega^{\lambda'}$ the allocation $z'' = (1 - \mu)z + \mu z'$ is feasible, $z^{''} \in \Phi(N, A, \omega'')$, the forest $\Gamma(z'')$ is unchanged, and the KKT system (2), which holds at z and z', also holds at z''. Thus $z'' \in f^c(\omega'')$ and the claim is proven.

Next we check that inside an interval $\mathcal{B}(\Gamma)$ the rule F^c is resource monotonic. The forest Γ is a union of trees. If a tree contains a single agent i, she eats (in full) the same subset of goods for any λ in $\mathcal{B}(\Gamma)$, hence her utility increases weakly in λ. If a subtree of Γ connects the subset S of agents, then system (2) fixes the direction of the utility profile $(U_i)_{i \in S}$, because along a path of Γ the equalities $\frac{u_{ia}}{U_i} = \frac{u_{ja}}{U_j}$ ensure that all ratios $\frac{U_i}{U_j}$ are independent of λ in $\mathcal{B}(\Gamma)$. As λ increases in $\mathcal{B}(\Gamma)$ the agents in S together eat the same subset of goods, therefore the U_i-s increase weakly by efficiency.

Finally Lemma 1 implies that the finite set of intervals $\mathcal{B}(\Gamma)$ cover $[0, 1]$. On each true interval (not a singleton) the utility profile $U^\lambda = F(N, A, \omega^\lambda, u)$ and there is at most a finite set of isolated points not contained in any true interval.Moreover the mapping $\lambda \rightarrow U^\lambda$ is continuous because $\omega \rightarrow U(\omega)$ is (an easy consequence of Berge Theorem). The desired conclusion $U(\omega) \leq U(\omega')$ follows. ■

6.4 Proposition 2: the Competitive rule has Responsive Shares

The long proof takes three steps. We show RS when the consumption forest $\Gamma(z)$ does not change (Step 1); then that this forest does not change on an interval of the utility parameter (Step 2); and finally in Step 3 we provide a local version of RS at those points where the consumption forest does change.

We fix throughout the proof N, A, a pair i, a and a profile $u \in \mathbb{R}_+^{N \times A \smallsetminus (i,a)}$. We write $u[\lambda] \in \mathbb{R}_+^{N \times A}$ the utility matrix where $\lambda \in \mathbb{R}_+$ replaces u_{ia} and the rest is unchanged.

Step 1 *Pick λ, λ' such that $\lambda < \lambda'$ and $z \in f^c(u[\lambda])$, $z' \in f^c(u[\lambda'])$. If $\Gamma(z) = \Gamma(z')$ then $z_{ia} \leq z'_{ia}$.*

This is clear if $z_{ia} = 0$, and if $z_{ia} = 1$, as i remains the sole consumer of a at z'. We assume now $0 < z_{ia} < 1$, and let K be the set of agents k in the same tree of Γ (the common forest) as i and on the other side of a. It contains at least any other agent eating some a. The remaining agents in the tree of i, if any, are connected to i by a path of Γ not containing a: their set is T.

Note first that agents in a different tree of Γ than i are unaffected by the shift from $u[\lambda]$ to $u[\lambda']$. The easiest way to see this uses the Consistency property: together the agents in $N \diagdown (K \cup T \cup \{i\})$ are allocated the same set of goods and have the same preferences in both problems, and the competitive rule allocates those goods to them as in this constant sub-problem.

We turn to the agents in $K \cup T \cup \{i\}$, for whom the net utility goes from $U_j = u_j[\lambda] \cdot z_j$ to $U'_j = u_j[\lambda'] \cdot z_j$. The KKT conditions (2), before and after, imply:

$$\frac{U'_k}{U_k} \text{ is independent of } k \in K \text{ ; } \frac{U'_i}{U_i} = \frac{U'_t}{U_t} \text{ for all } t \in T \tag{14}$$

Within the $T \cup \{i\}$-subtree (with all goods they eat except a) the equalities $\frac{u_{td}}{U_t} = \frac{u_{sd}}{U_s}$ give the right-hand equalities by the same propagation argument as in the previous proof. If K^* is the set of $k \in K$ eating some a we have

$$\frac{u_{ka}}{U_k} = \frac{\lambda}{U_i} \text{ and } \frac{\lambda'}{U'_i} = \frac{u_{ka}}{U'_k} \tag{15}$$

so that $\frac{U'_k}{U_k}$ is constant in K^*. Any other k is connected in Γ to some $k^* \in K^*$ so that $\frac{U'_k}{U_k} = \frac{U'_{k^*}}{U_{k^*}}$ by the usual argument.

Observe now that $U'_i \leq U_i$ and $\lambda < \lambda'$ imply $\frac{\lambda}{U_i} < \frac{\lambda'}{U'_i}$ so by the KKT conditions above we get $U'_k < U_k$ for k sharing a with i, then everywhere in K by (14). And $U'_t \leq U_t$ for $t \in T$ follows from the second statement of (14). Moreover

$$u_i[\lambda'] \cdot z'_i \leq u_i[\lambda] \cdot z_i < u_i[\lambda'] \cdot z_i$$

As the agents in $K \cup T \cup \{i\}$ are allocated the same goods before and after, we see that at the utility matrix $u[\lambda']$ the restriction of z to those goods is more efficient than z', a contradiction of the efficiency of z'.

We have shown $U_i < U'_i$, and $U_t < U'_t$ for all $t \in T$.

Write now D for the goods d allocated to $T \cup \{i\}$, other than a. We write $\Delta = \sum_{d \in D} u_{id} z_{id}$, so that $U_i = \lambda z_{ia} + \Delta$, and we compare Δ and $\Delta' = \sum_{d \in D} u_{id} z'_{id}$. If $T = \varnothing$ then $\Delta = \Delta'$ because D is a set of goods that i alone consumes, before and after. It $T \neq \varnothing$ we know $U_t < U'_t$ for all t, so $\Delta \leq \Delta'$ would mean that the good in D are assigned more efficiently at z' than at z, impossible. Therefore $\Delta \geq \Delta'$.

Finally we assume $z'_{ia} \leq z_{ia}$ and derive a contradiction. Consider the total allocation to the agents in K: they eat more of a after than before, and the other goods allocated to K are unchanged. On the other hand $\frac{U'_k}{U_k}$ is independent of k, so it can only go up when K eats a bigger total share: $U'_k \geq U_k$ for all k. Back to (15) we conclude $\frac{\lambda'}{U'_i} \leq \frac{\lambda}{U_i}$ and in turn

$$\frac{U_i}{\lambda} \leq \frac{U'_i}{\lambda'} \Longleftrightarrow z_{ia} + \frac{\Delta}{\lambda} \leq z'_{ia} + \frac{\Delta'}{\lambda'}$$

and the desired contradiction.

We also proved $z'_{ia} \leq z_{ia}$ and $U'_k < U_k$ for all $k \in K$

Step 2 *Under the same premises as in Step 1 we show that for all* $\mu, \lambda < \mu < \lambda'$, *there is some* $z^* \in f^c(u[\mu])$ *such that* $\Gamma(z^*) = \Gamma(z)$.

By Step 1 if the competitive utility profile U^* at $u[\mu]$ obtains by some allocation z^* such that $\Gamma(z^*) = \Gamma(z)$, then $U^*_j = U_j$ for j outside the tree containing i, and for the agents in this tree, utilities are determined by two positive numbers δ, ε such that

$$\frac{U_i^*}{U_i} = \frac{U_t^*}{U_t} = 1 + \delta \text{ for all } t \in T \ ; \ \frac{U'_k}{U_k} = 1 - \varepsilon \text{ for all } k \in K \tag{16}$$

We show how δ, ε are computed for any μ in $]\lambda, \lambda'[$. Property (15) implies, for any k in K who eats some a:

$$u_{ka} = \lambda \frac{U_k}{U_i} = \mu \frac{U_k^*}{U_i^*} \Longrightarrow \lambda(1+\delta) = \mu(1-\varepsilon) \tag{17}$$

Moreover the gain $U_t^* - U_t$ is linear in δ therefore so is the loss $\Delta^* - \Delta = \sum_{d \in D} u_{id} z^*_{id} - \sum_{d \in D} u_{id} z_{id}$. Similarly the gain $z^*_{ia} - z_{ia}$ is linear in ε because each loss $U^*_k - U_k$ is too. Thus there exist positive numbers A, B such that

$$\Delta^* = \Delta - A\delta \ ; \ z^*_{ia} = z_{ia} + B\varepsilon$$

$$\Longrightarrow \frac{U_i^*}{U_i} = \frac{\mu(z_{ia} + B\varepsilon) + \Delta - A\delta}{\lambda z_{ia} + \Delta} = 1 + \frac{(\mu - \lambda)z_{ia} + B\mu\varepsilon - A\delta}{\lambda z_{ia} + \Delta}$$

and the first equality in (16) gives

$$(\lambda z_{ia} + \Delta + A)\delta - B\mu\varepsilon = (\mu - \lambda) z_{ia}$$

while from (17) we get

$$\lambda\delta + \mu\varepsilon = \mu - \lambda$$

This system in δ, ε is non singular, and its solutions take the form

$$\delta = (\mu - \lambda)D \ ; \ \varepsilon = \frac{\mu - \lambda}{\mu} E$$

for some positive constants D, E. Both δ, ε increase in μ, so the utility profile U^* is between U and $U'/$

Now we construct an allocation z^* achieving the utility profile U^*. As the KKT system (2) holds at (z, U) and (z', U') it holds at (z^*, U^*) as well: for instance if b is eaten by j outside the tree containing i, we have for $k \in K$

$$\frac{u_{jb}}{U_j} \geq \frac{u_{kb}}{U_k}, \frac{u_{kb}}{U'_k} \Longrightarrow \frac{u_{jb}}{U_j} \geq \frac{u_{kb}}{U_k^*}$$

and the same holds true for agents in T and for i. We let the reader check that all the KKT conditions hold, so that $z^* \in f^c(u[\mu])$ as desired.

Step 3 For $r = 1, 2, \cdots$, pick a strictly increasing sequence μ^r and a strictly decreasing sequence ν^r that both converge to λ. Pick also $z^r \in f^c(u[\mu^r])$ and $\widetilde{z}^r \in f^c(u[\nu^r])$. We show

$$\limsup_r z_{ia}^r \leq w_{ia} \leq \liminf_r \widetilde{z}_{ia}^r \text{ for all } w \in f^c(u[\lambda]) \tag{18}$$

By contradiction: suppose there is a subsequence of z^r, also denoted z^r for simplicity, such that $\alpha = \lim_r z_{ia}^r > w_{ia}$.

The correspondence $\mu \to f^c(u[\mu])$ is upper hemi continuous (by Berge Theorem) so there is another subequence such that z^r converges to $z \in f^c(u[\lambda])$. As $\alpha = z_{ia} > w_{ia}$, and z, w yield the same utility profiles, we can construct a cycle $i = i_1, a = a_1, i_2, a_2, \cdots, a_L, i_{L+1} = i_1$ such that

$$z_{i_1 a_1} > w_{i_1 a_1}; \cdots ; z_{i_\ell a_\ell} > w_{i_\ell a_\ell}; w_{i_{\ell+1} a_\ell} > z_{i_{\ell+1} a_\ell}; \cdots ; w_{i_{L+1} a_L} > z_{i_{L+1} a_L} \tag{19}$$

(as $z_{ia} > w_{ia}$ there is some j such that $w_{ja} > z_{ja}$ by feasibility, and so on). For r large enough these inequalities hold with z^r replacing z. Suppose now that the following inequality holds for the entries of the matrix $u[\mu^r]$:

$$\frac{\mu^r}{u_{i_2 a_1}} \cdot \frac{u_{i_2 a_2}}{u_{i_3 a_2}} \cdot \cdots \cdot \frac{u_{i_{L-1} a_{L-1}}}{u_{i_L a_{L-1}}} \cdot \frac{u_{i_L a_L}}{u_{i_1 a_L}} < 1$$

Then we can pick positive numbers ε_ℓ such that

$$\frac{\mu^r \cdot \varepsilon_1}{u_{i_1 a_L} \cdot \varepsilon_L} < 1, \frac{u_{i_2 a_2} \cdot \varepsilon_2}{u_{i_2 a_1} \cdot \varepsilon_1} < 1, \cdots, \frac{u_{i_L a_L} \cdot \varepsilon_L}{u_{i_L a_{L-1}} \cdot \varepsilon_{L-1}} < 1$$

and when these numbers are small enough they allow a Pareto improving shift from z^r at $u[\mu^r]$: agent i_ℓ gets $\varepsilon_{\ell-1}$ extra units of good $\ell - 1$ and gives up ε_ℓ units of good ℓ. This contradiction means that we have

$$\frac{\mu^r}{u_{i_2 a_1}} \cdot \frac{u_{i_2 a_2}}{u_{i_3 a_2}} \cdot \cdots \cdot \frac{u_{i_{L-1} a_{L-1}}}{u_{i_L a_{L-1}}} \cdot \frac{u_{i_L a_L}}{u_{i_1 a_L}} \geq 1$$

for r large enough. Because μ^r increases strictly, we conclude

$$\frac{\lambda}{u_{i_2 a_1}} \cdot \frac{u_{i_2 a_2}}{u_{i_3 a_2}} \cdot \cdots \cdot \frac{u_{i_{L-1} a_{L-1}}}{u_{i_L a_{L-1}}} \cdot \frac{u_{i_L a_L}}{u_{i_1 a_L}} > 1$$

Now we can pick positive numbers ε_ℓ such that

$$\frac{\lambda \cdot \varepsilon_1}{u_{i_1 a_L} \cdot \varepsilon_L} > 1, \frac{u_{i_2 a_2} \cdot \varepsilon_2}{u_{i_2 a_1} \cdot \varepsilon_1} > 1, \cdots, \frac{u_{i_L a_L} \cdot \varepsilon_L}{u_{i_L a_{L-1}} \cdot \varepsilon_{L-1}} > 1$$

and we construct a Pareto improvement of w at $u[\lambda]$.

We omit the symmetrical proof of the other inequality in (18).

Step 4

We define the closed interval $\boldsymbol{\psi}(\lambda) = \{z_{ia} | \text{ for some } z \in f^c(u[\lambda])\}$. The correspondence $\lambda \to \boldsymbol{\psi}(\lambda)$ is upper hemi continuous and (6) means that is increases weakly: $\lambda < \lambda' \Longrightarrow \boldsymbol{\psi}(\lambda) \leq \boldsymbol{\psi}(\lambda')$, which we must prove.

For any forest Γ and $\lambda < \lambda'$ we define $\Theta(\Gamma; \lambda, \lambda') = \{\mu \in [\lambda, \lambda'] | \Gamma(z) = \Gamma$ for some $z \in f^c(u[\mu])\}$. By Step 2 $\Theta(\Gamma; \lambda, \lambda')$ is an interval (possibly empty or a singleton), and these intervals cover $[\lambda, \lambda']$ so some of them are non trivial (not a singleton). We call the corresponding forest non trivial as well.

Let $\theta(\lambda, \lambda')$ be the number of non trivial forests in $[\lambda, \lambda']$. We prove by induction on q the following property $\mathcal{P}_q(\lambda, \lambda')$

$$\{\lambda < \lambda' \text{ and } \theta(\lambda, \lambda') \leq q\} \Longrightarrow \boldsymbol{\psi}(\lambda) \leq \boldsymbol{\psi}(\lambda')$$

which, for q large enough, is the desired conclusion.

Step 1 implies $\mathcal{P}_1$. If Γ be the unique non trivial forest in $[\lambda, \lambda']$, then $\Theta(\Gamma; \lambda, \lambda')$ contains $]\lambda, \lambda'[$. By (18) (Step 3) we have $\boldsymbol{\psi}(\lambda) \leq \liminf_r \widetilde{z}^r_{ia}$ for any sequence $\widetilde{z}^r \in f^c(u[\lambda + \frac{1}{r}])$; similarly $\boldsymbol{\psi}(\lambda') \geq \limsup_r \widetilde{\widetilde{z}}^r_{ia}$ for a sequence $\widetilde{\widetilde{z}}^r \in f^c(u[\lambda' - \frac{1}{r}])$. We can choose these sequences with $\Gamma(\widetilde{z}^r) = \Gamma(\widetilde{\widetilde{z}}^r) = \Gamma$, and the desired conclusion follows from Step 1.

Assume now $\mathcal{P}_{q'}$ holds for all $q' \leq q$ and pick λ, λ' such that $\theta(\lambda, \lambda') = q + 1$. Let Γ_0 be a non trivial forest in $[\lambda, \lambda']$, and J be the closure of $\Theta(\Gamma_0; \lambda, \lambda')$, a non trivial subinterval $[\mu, \mu']$ of $[\lambda, \lambda']$.

For any $\nu \in]\lambda, \mu[$ we have $\theta(\lambda, \nu) \leq q$, so the induction assumption ensures $\boldsymbol{\psi}(\lambda) \leq \boldsymbol{\psi}(\nu)$. For the same reason $\boldsymbol{\psi}(\nu) \leq \boldsymbol{\psi}(\nu')$ for $\nu \leq \nu' < \mu$. Applying now Step 3 to a sequence $z^r \in f^c(u[\mu - \frac{1}{r}])$ we see that the sequence z^r_{ia} is increasing therefore (18) gives $\psi(\mu - \frac{1}{r}) \leq \psi(\mu)$, and ψ increases in $[\lambda, \mu]$. That it increases in $[\mu, \mu']$ is clear by the same argument proving $\mathcal{P}_1$ above, and in $[\mu', \lambda']$ we mimick the argument of the previous sentence.

6.5 Characterization Theorem

Step 0 *The rule f^c meets all the axioms.*

Efficiency, Symmetry and Partition require no proof. To check EG we fix (N, A, u) where goods a, b are equivalent and let $z^c \in f^c(\mathcal{Q})$. Assume first that both a and b are useful and all agents are interested. Define the allocation z^* in the merged problem (N, A^*, u^*), identical to z^c on goods other than a, b, and such that for all i

$$z^*_{ia^*} = \frac{u_{ia}}{u_{ia} + u_{ib}} z^c_{ia} + \frac{u_{ib}}{u_{ia} + u_{ib}} z^c_{ib} \text{ if } u_{ia} + u_{ib} > 0 \text{ ; } z^*_{ia^*} = 0 \text{ if } u_{ia} = u_{ib} = 0$$

Clearly $z^*_{Na^*} = 1$ because if $u_{ia} = u_{ib} = 0$ efficiency implies $z^c_{ia} = z^c_{ib} = 0$. And $u^*_i \cdot z^*_i = u_i \cdot z^c_i \Longleftrightarrow U^*_i = U^c_i$ is equally clear for all i. It remains to check that z^* is the competitive allocation in the merged problem. The KKT conditions (2) at any good other than a^* are unchanged, and at a^* we must check for all i, j:

$$z^*_{ia^*} > 0 \Longrightarrow \frac{u^*_{ia^*}}{U^c_i} \geq \frac{u^*_{ja^*}}{U^c_j}$$

From $z^*_{ia^*} > 0$ one of z^c_{ia}, z^c_{ib} is positive, say $z^c_{ia} > 0$, and by efficiency $u_{ia} > 0$ as well. Applying (2) to u gives $\frac{u_{ia}}{U^c_i} \geq \frac{u_{ja}}{U^c_j}$. The equivalence of a and b implies

$\frac{u_{ja}}{u_{ia}} = \frac{u_{jb}}{u_{ib}}$ if $u_{ib} > 0$ and $u_{jb} = 0$ if $u_{ib} = 0$: in both cases we get

$$\frac{u_{ia}}{U_i^c} \geq \frac{u_{ja}}{U_j^c} \Longleftrightarrow \frac{u_{ib}}{U_i^c} \geq \frac{u_{jb}}{U_j^c} \Longleftrightarrow \frac{u_{ia} + u_{ib}}{U_i^c} \geq \frac{u_{ja} + u_{jb}}{U_j^c}$$

cpmpleting the proof of EG when both a and b are useful and all agents are interested. The cases where one of a, b is useless, and/or some agents are uninterested are left to the motivated reader.

Converse statement: The set N is fixed throughout and written $[n] = \{1, \cdots, n\}$. The set A varies and in view of the Symmetry property is always written $[m]$. A problem is described by a non negative $n \times m$ *utility matrix* $u = [u_{ia}]_{i \in [n], a \in [m]}$. We write $\mathcal{U}(n, m)$ the set of such matrices, and $\Phi(n, m)$ the set of feasible allocation matrices $z = [z_{ia}]$. A rule F maps $\mathcal{U}(n, m)$ into $\mathbb{R}_+^{[n]}$ and f is a correspondence into $\Phi(n, m)$. Recall from Definition 1 that F, f are *scale invariant*: multiplying a certain row of u by a (strictly) positive constant does not change the image of f, and multiplies the corresponding coordinate of $F(u)$ by the constant.

In Step 1 we define a family denoted $\mathcal{D}(n, n+1)$ of relatively simple problems, for which we compute explicitly the competitive allocation. Then in Steps 2,3 we show that a rule f meeting all the axioms in the Theorem must coincide with f^c on $\mathcal{D}(n, n+1)$.

Step 1 For each $\ell \in [n]$ and $w \in \mathbb{R}_+^{[n]}$ we define the following matrix $u^{\ell, w} \in \mathcal{U}(n, n+1)$, with the help of the familiar symbol δ : $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$:

$$u_{ij}^{\ell, w} = \delta_{ij} w_i \text{ for all } i, j \leq n \tag{20}$$

$$u_{i(n+1)}^{\ell, w} = 1 \text{ if } i \leq \ell \text{ ; } u_{i(n+1)}^{\ell, w} = 0 \text{ if } i \geq \ell + 1 \tag{21}$$

and we write $\mathcal{D}(n, n+1)$ the subset of matrices $u^{\ell, w}$ for $\ell \in [n]$, $w \in \mathbb{R}_+^{[n]}$. Each good $j, j \leq n$ is liked by agent j only (or is useless if $w_j = 0$), while good $n+1$ is liked by the first ℓ agents only.

Define now for any $\ell \in [n]$ and $t \in \mathbb{R}_+^{[\ell]}$ such that $\sum_1^\ell t_i = 1$ the allocation matrix $z^{\ell, t} \in \Phi(n, n+1)$:

$$z_{ij}^{\ell, t} = \delta_{ij} \text{ for all } i \text{ and } j \leq n \text{ ; } z_{i(n+1)}^{\ell, t} = t_i \text{ if } i \leq \ell, \ = 0 \text{ if } i \geq \ell + 1 \tag{22}$$

Clearly $z^{\ell, t}$ is efficient in the problem $([n], [n+1], u^{\ell, w})$; conversely if all goods are useful, $w \gg 0$, then all efficient allocations in this problem are of this form. We write $\mathcal{ED}(n, n+1)$ the set of allocations $z^{\ell, t}$ for all $\ell \in [n]$, $t \in \mathbb{R}_+^{[\ell]}$ s. t. $\sum_1^\ell t_i = 1$.

Now we compute by Lemma 2 the allocation $f^c(u^{\ell, w}) = z^{\ell, t} \in \mathcal{ED}(n, n+1)$ at an arbitrary $u^{\ell, w} \in \mathcal{D}(n, n+1)$.[12] Relabel the rows so that the sequence w_k

[12] Strictly speaking $f^c(u^{\ell, w})$ is a singleton only if all goods are useful, i. e., $w \gg 0$.

is weakly increasing in $[\ell]$ (and arbitrary after ℓ). The inequality

$$w_i \leq \frac{1}{i}(1+\sum_{j=1}^{i} w_j) \tag{23}$$

holds for $i=1$ and we let i^* be the largest integer in $[\ell]$ such that it does. Thus if $i^* \leq \ell-1$ we must have $\frac{1}{i'}(1+\sum_{j=1}^{i'} w_j) < w_{i'}$ for $i^*+1 \leq i' \leq \ell$.

The vector t defining $z^{\ell,t}$, and the utility profile $F^c(u^{\ell,w}) = U^{\ell,w}$, are

$$t_i = \frac{1}{i^*}(1+\sum_{j=1}^{i^*} w_j) - w_i \text{ for } 1 \leq i \leq i^*, \text{ and } t_i = 0 \text{ for } i^*+1 \leq i \leq \ell$$

$$U_i^{\ell,w} = \frac{1}{i^*}(1+\sum_{j=1}^{i^*} w_j) \text{ for } 1 \leq i \leq i^*, \text{ and } U_i^{\ell,w} = w_i \text{ for } i^*+1 \leq i \leq n \tag{24}$$

as one checks easily by the KKT conditions (2).

We fix from now on a rule f as in the statement of the Theorem and show in Step 3 that it coincides with f^c on $\mathcal{D}(n,n+1)$. To that end we need a key reduction result which is our next step.

Step 2 *Fix $u \in \mathcal{U}(n,m)$, a good $a \in [m]$ useful only to the first ℓ agents, $\ell \geq 1$, and an allocation $z \in f(u)$. Consider the matrix $u^{\ell,w} \in \mathcal{D}(n,n+1)$, where*

$$w_i = \frac{u_i \cdot z_i}{u_{ia}} - z_{ia} \text{ for } 1 \leq i \leq \ell \tag{25}$$

$$w_i = u_i \cdot z_i \text{ for } i \geq \ell+1 \tag{26}$$

Then we have $f(u^{\ell,w}) = z^{\ell,t}$ (or $z^{\ell,t} \in f(u^{\ell,w})$ if some goods are useless), with $t_i = z_{ia}$ for all $i \leq \ell$.

Proof of the claim. Set $\mathcal{Q} = ([n],[m],u)$ and $U = F(\mathcal{Q})$ so $U_i = u_i \cdot z_i$. We note that EO is about merging equivalent goods, but it also allows us to split any good $b \in [m]$ into equivalent goods provided the corresponding split utilities keep the same sum. We apply this remark to transform problem $\mathcal{Q}$ into a new problem ${}^1\mathcal{Q} = ([n],[{}^1m],{}^1u)$ where ${}^1m = 1+n\cdot(m-1)$, each good $b \in [m]\diagdown a$ (recall a is fixed in the statement of the claim) is split into n *equivalent* goods b^i, one for each $i \in [n]$, and the utility of agent j for good b^i is ${}^1u_{jb^i} = z_{ib}u_{jb}$. For instance

$$\begin{array}{cccc} & a & b & c \\ u_1 & 5 & 1 & 1 \\ u_2 & 6 & 3 & 1 \\ u_3 & 1 & 4 & 4 \end{array} \quad \rightarrow \quad \begin{array}{cccc} & a & b & c \\ z_1^c & \frac{2}{3} & 0 & 0 \\ z_2^c & \frac{1}{3} & \frac{2}{3} & 0 \\ z_3^c & 0 & \frac{1}{3} & 1 \end{array} \quad \rightarrow \quad \begin{array}{cccccccc} & a & b^1 & b^2 & b^3 & c^1 & c^2 & c^3 \\ {}^1u_1 & 5 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 1 \\ {}^1u_2 & 6 & 0 & 2 & 1 & 0 & 0 & 1 \\ {}^1u_3 & 1 & 0 & \frac{8}{3} & \frac{4}{3} & 0 & 0 & 4 \end{array}$$

(in the example we use the competitive allocation but the construction aplies to any allocation z)

As the matrix z contains many zeros (when $\Gamma(z)$ is a tree which is the normal pattern), the problem ${}^1\mathcal{Q}$ contains many useless goods.

As all n fragments of good b are equivalent EO implies $F({}^1\mathcal{Q}) = U$, therefore $f({}^1\mathcal{Q})$ contains the allocation ${}^1z$ that for each $b \in [m] \diagdown a$ gives all good b^i to i (${}^1z_{ib_j} = \delta_{ij}$) and such that ${}^1z_{ia} = z_{ia}$ for all i. For $j \neq i$ agent j does not eat any b^i at ${}^1z$, hence if we lower to zero the term ${}^1u_{jb^i} = z_{ib}u_{jb}$ while every other entry of ${}^1u$ is unchanged, ILB tells us that ${}^1z$ remains chosen by f in the new matrix. Lowering successively every entry $j, b^i, j \neq i$, in ${}^1u$ and each time invoking ILB we obtain the following matrix ${}^2u \in \mathcal{U}(n, [{}^1m])$

$$ {}^2u_{ja} = u_{ja} \ , \ {}^2u_{jb^i} = \delta_{ij} z_{ib} u_{jb} $$

such that ${}^1z \in f({}^2u)$ and $F([n], [{}^1m], {}^2u) = U$. In the above example

	a	b^1	b^2	b^3	c^1	c^2	c^3
${}^2u_1$	5	0	0	0	0	0	0
${}^2u_2$	6	0	2	0	0	0	0
${}^2u_3$	1	0	0	$\frac{4}{3}$	0	0	4

	a	b^1	b^2	b^3	c^1	c^2	c^3
${}^1z_1$	$\frac{2}{3}$	0	0	0	0	0	0
${}^1z_2$	$\frac{1}{3}$	0	1	0	0	0	0
${}^1z_3$	0	0	0	1	0	0	1

Fix now agent i and consider goods b and c in $[m] \diagdown a$: if ${}^2u_{ib^i}$ and ${}^2u_{ic^i}$ are both positive the goods b^i and c^i are equivalent at ${}^2u$ because i likes them both and others dislike them both; if ${}^2u_{ib^i}$ and/or ${}^2u_{ic^i}$ is zero they are still equivalent because a useless good is equivalent to any other good. Thus if we merge b^i and c^i into a good for which i's utility is ${}^2u_{ib^i} + {}^2u_{ic^i} = z_{ib}u_{ib} + z_{ic}u_{ic}$, EO implies that in the new problem the rule still picks the same utility profile U.

Now we merge successively all goods b^i, $b \in [m] \diagdown a$, into a single good labeled i, for which the utilities are now ${}^3u_{ii} = \sum_{[m] \diagdown a} z_{ib}u_{ib} = U_i - z_{ia}u_{ia}$ and ${}^3u_{ji} = 0$ for $j \neq i$. We do this for all agents and reach a problem ${}^3\mathcal{Q} = ([n], [n+1], {}^3u)$ where only agent i may like good i. Still, $F({}^3\mathcal{Q}) = U$ and, upon labeling good a as $n+1$, $f({}^3\mathcal{Q})$ contains the allocation $z^{\ell,t}$, where $t_i = z_{ia}$ for all i. In the example $\ell = 3$ and

	a	1	2	3
${}^3u_1$	5	0	0	0
${}^3u_2$	6	0	2	0
${}^3u_3$	1	0	0	$\frac{16}{3}$

$$ z^{3,t} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ z_1^{3,t} & 1 & 0 & 0 & \frac{2}{3} \\ z_2^{3,t} & 0 & 1 & 0 & \frac{1}{3} \\ z_3^{3,t} & 0 & 0 & 1 & 0 \end{array} $$

To reach the format $u^{\ell,w}$ as in (20) (21), with w in (25),(26), we divide each row $i, i \leq \ell$, of ${}^3u$ by ${}^3u_{ia} = u_{ia}$ and leave alone the rows $j, j \geq \ell+1$, so their only non zero term is U_i. E. g.,

$$ u^{3,w} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ u_1^{3,t} & 0 & 0 & 0 & 1 \\ u_2^{3,t} & 0 & \frac{1}{3} & 0 & 1 \\ u_3^{3,t} & 0 & 0 & \frac{16}{3} & 1 \end{array} $$

This rescaling does not affect the allocations selected by f (Definition 1), and the proof of the claim is complete.

Step 3 *f and f^c coincide on $\mathcal{D}(n, n+1)$: proof by induction on n*

Step 3.1 $n = 2$. A problem in $\mathcal{D}(2,3)$ with $\ell = 1$ is one where each good is liked by one agent (at most) so all efficient rules coincide. A problem $\mathcal{Q}$ with $\ell = 2$ is

$$u^\circ = \begin{matrix} w_1 & 0 & 1 \\ 0 & w_2 & 1 \end{matrix}$$

For any number $\gamma, 0 < \gamma \leq 1$ consider the problem

$$u = \begin{matrix} w_1 & 0 & \gamma & 1 \\ 0 & w_1 & 1 & \gamma \end{matrix}$$

By EFF and SYM $F(u) = (w_1 + 1) \cdot (1, 1)$, and agent 1 (on the top row) gets the 4th good (4th column) in $f(u)$ (for sure if $\gamma < 1$, and in one $z \in f(u)$ if $\gamma < 1$). By Step 2 this agent gets the 3rd good in

$$u' = \begin{matrix} w_1 & 0 & 1 \\ 0 & \frac{w_1+1}{\gamma} & 1 \end{matrix}$$

For any w_1, w_2 such that $w_1 + 1 \leq w_2$ we can choose γ so that $\frac{w_1+1}{\gamma} = w_2$. We conclude that whenever $|w_1 - w_2| \geq 1$ in problem $\mathcal{Q}$, the "low utility" agent eats all good 3, and $F(u^\circ) = (w_1 + 1, w_2)$. This is also what the Competitive rule does in this case.

Conversely we assume that in $\mathcal{Q}$ agent 1 eats all good 3 and show that $w_1 + 1 \leq w_2$. Fix $\varepsilon > 0$ and split good 3 in two as follows

$$u'' = \begin{matrix} w_1 & 0 & \varepsilon & 1-\varepsilon \\ 0 & w_2 & \varepsilon & 1-\varepsilon \end{matrix}$$

By EO the rule still gives goods 3 and 4 to agent 1, so by Step 2 this agent eats the 3rd good in

$$u''' = \begin{matrix} \frac{w_1+1-\varepsilon}{\varepsilon} & 0 & 1 \\ 0 & \frac{w_2}{\varepsilon} & 1 \end{matrix}$$

Now if $w_2 < w_1 + 1$ we have $\frac{w_2}{\varepsilon} + 1 < \frac{w_1+1-\varepsilon}{\varepsilon}$ for small enough ε, therefore agent 2 should get good 3, contradiction.

We pick now $\mathcal{Q}$ such that $|w_1 - w_2| < 1$: we just proved that both agents must eat some of good 3 for instance $z_{13} = \lambda, z_{23} = 1 - \lambda$. We split good 3 in $\mathcal{Q}$ as follows

$$u^{\circ\circ} = \begin{matrix} w_1 & 0 & \lambda & 1-\lambda \\ 0 & w_2 & \lambda & 1-\lambda \end{matrix}$$

and note that $f(u^{\circ\circ})$ contains the allocation where agent 1 eats all good 3 and none of good 4. By Step 2 agent 1 still eats all good 3 in

$$u^{\circ\circ} = \begin{matrix} \frac{w_1}{\lambda} & 0 & 1 \\ 0 & \frac{w_2+1-\lambda}{\lambda} & 1 \end{matrix}$$

therefore $\frac{w_1}{\lambda} + 1 \leq \frac{w_2+1-\lambda}{\lambda} \iff w_1 + \lambda \leq w_2 + 1 - \lambda$. Exchanging the roles of agents 1 and 2 gives the opposite inequality so we conclude $w_1 + \lambda = w_2 + 1 - \lambda$, i.e., $F(u^\circ) = \frac{1}{2}(w_1 + w_2 + 1) \cdot (1, 1)$ just like the Competitive rule.

Step 3.2 induction argument. We assume F is the Competitive rule in $\mathcal{D}(m, m+1)$ for $m \leq n - 1$, and pick a problem $u^{\ell,w} \in \mathcal{D}(n, n+1)$ as in (20) (21). If $\ell \leq n - 1$ we can partition the problem into $N^1 = [\ell], A^1 = [\ell] \cup \{n + 1\}$ and $N^2 = \{\ell + 1, \cdots, n\}, A^2 = \{\ell + 1, \cdots, n\}$: the PAR property and the inductive assumption ensure that F distributes A^1 to N^1 exactly like Competitive, and A^2 to N^2 in the obvious efficient way, so F and F^c coincide on $u^{\ell,w}$.

Assume now $\ell = n$ so all agents like good $n+1$. By EFF $f(u^{n,w}) = z^{n,t}$ where t, the allocation t of good $n+1$, is unique. Without loss we label the agents so that t_i is weakly decreasing in i. We let i^* be the largest i such that $t_i > 0$. If $i^* \leq n-1$, we invoke PAR: f allocates the goods in $[i^*] \cup \{n+1\}$ exactly like in the smaller problem with those i^*+1 goods and the first i^* agents, so by the inductive assumption, exactly like CEEI, and we are done.

We are left with the case where $t_i > 0$ for all $i \in [n]$. We split now good $n+1$ in two

$$u = \begin{matrix} w_1 & \cdots & 0 & t_1 & 1-t_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & w_n & t_n & 1-t_1 \end{matrix}$$

By EO $F(u^{n,w}) = F(u)$, and $f(u)$ contains an allocation where all good $n+1$ goes to agent 1 while the shares of good $n+2$ are $(0, \frac{t_2}{1-t_1}, \cdots, \frac{t_n}{1-t_1})$. Upon partitioning problem u into $N^1 = \{1\}, A^1 = \{1, n+1\}$ and $N^2 = \{2, \cdots, n\}, A^2 = \{2, \cdots, n\} \cup \{n+2\}$, PAR implies that in the reduced problem (N^2, A^2) the shares of good $n+2$ are $(\frac{t_2}{1-t_1}, \cdots, \frac{t_n}{1-t_1})$ as well. After normalizing utilities we see that at the following problem $\widetilde{u}^{n-1,\widetilde{w}}$

$$\widetilde{u}^{n-1,\widetilde{w}} = \begin{matrix} \frac{w_2}{1-t_1} & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \frac{w_n}{1-t_1} & 1 \end{matrix}$$

our rule f shares the last good as $(\frac{t_2}{1-t_1}, \cdots, \frac{t_n}{1-t_1})$. But by the inductive assumption this is also the choice of the Competitive rule: as all shares are strictly positive, we conclude by (24) that all utilities are equal, i. e., $w_2+t_2 = \cdots = w_n+t_n$. The choice of agent 1 in the above argument was arbitrary, so by repeating it with another agent, we conclude that $F(u^{n,w})$ gives the same utility $w_i + t_i = \frac{1}{n}(1 + \sum_{j=1}^n w_j)$ to all agents. Moreover w_i increases weakly in i, and $w_n \leq \frac{1}{n}(1 + \sum_{j=1}^n w_j)$ as required by (23). The proof that $F(u^{n,w}) = F^c(u^{n,w})$ is complete.

Step 4 We fix an arbitrary $u \in \mathcal{U}(n,m)$ with associated utility profile $U = F(u)$; we also choose an allocation $z \in f(u)$ and an arbitrary good $a \in [m]$. We need to show that the KKT inequalities (2) corresponding to good a: for all $i \in [n]$ such that $z_{ia} > 0$ we have $\frac{u_{ia}}{U_i} \geq \frac{u_{ja}}{U_j}$ for all j.

If good a is liked by exactly ℓ agents, and exactly i^* of those consume some a at z, relabel the latter as the first i^* in u, followed by the $\ell - i^*$ who like a but do not eat any of it. Let $u^{\ell,w} \in \mathcal{U}(n, n+1)$ be defined as in Step 2 by (25),(26): for $i \in [\ell]$ we have $F_i(u^{\ell,w}) = w_i + z_{ia} = \frac{U_i}{u_{ia}}$. And by step 3 $F(u^{\ell,w}) = F^c(u^{\ell,w})$ is given by (24): the first i^* agents end up with the same utility $\frac{1}{i^*}(1+\sum_{j=1}^{i^*} w_j)$ and every agent in $\{i^*+1, \cdots, \ell\}$ with a higher utility w_i (it does not matter for this statement that the w_i are not ordered increasingly before or after i^*). Hence $\frac{U_i}{u_{ia}}$ is constant for each i such that $z_{ia} > 0$, and all ratios $\frac{U_i}{u_{ia}}$ for each for $i^*+1 \leq i \leq \ell$ are larger. And for $i \geq \ell+1$ we have $\frac{u_{ia}}{U_i} = 0$. This proves (2) for good a as desired.

7 Two open questions

We noticed in Section 2 that neither *Responsive Shares* nor *Invariance to Lost Bids* has a clear counterpart with non additive utilities, so our results are strictly limited to the additive domain. Extensions and variants of our results may be investigated in two other directions.

The first one is the cake division model, where the resource is a compact subset in an euclidian space and utilities are non atomic non negative measures on this cake. Technical difficulties notwithstanding, we suspect that our arguments can be extended to that model.

The second, equally difficult, extension comes from dropping symmetry in N. The focal rule becomes *Competitive Equilibrium with Fixed income Shares*, which still maximizes the corresponding weighted Nash product. This rule meets all other axioms in our Theorem axioms, but it is not clear if, conversely, these axioms characterize this new family of Competitive rules.

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