Copulas and bivariate risk measures : an application to hedge funds

Rihab BEDOUI [*][†] and Makram BEN DBABIS [‡]

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Abstract

With hedge funds, managers develop risk management models that mainly aim to play on the effect of decorrelation. In order to achieve this goal , companies use the correlation coefficient as an indicator for measuring dependencies existing between (i) the various hedge funds strategies and share index returns and (ii) hedge funds strategies against each other. Otherwise, copulas are a statistic tool to model the dependence in a realistic and less restrictive way, taking better account of the stylized facts in finance. This paper is a practical implementation of the copulas theory to model dependence between different hedge fund strategies and share index returns and between these strategies in relation to each other on a "normal" period and a period during which the market trend is downward. Our approach based on copulas allows us to determine the bivariate VaR level curves and to study extremal dependence between hedge funds strategies and share index returns through the use of some tail dependence measures which can be made into useful portfolio management tools.

JEL classification : C13; C14; C15; G23.

Keywords : Hedge fund strategies, share index, dependence, copula, tail dependence, bivariate Value at Risk.

[*] EconomiX, University of Paris Ouest Nanterre La Défense, 200 avenue de la République, 92000 Nanterre Cedex France. E-mail: Rihab.bedoui@u-paris10.fr. Tel:+33140977820, Fax:+33140975973

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[‡] SDFI, University of Paris Dauphine and CNP Insurance, Place du Maréchal de Lattre de Tassigny, 75775 PARIS CEDEX 16 France. E-mail: makram.bendbabis@cnp.fr. Tel:+33142188279

1 Introduction

Academic research began to focus on sources of returns and risk of hedge funds in 1997 with the pioneering work of Fung and Hsieh (1997). Brown, Goetzmann and Ibbotson (1999) studied a sample of offshore hedge funds between 1989 and 1995 and found a positive risk adjusted. Their results did not highlight neither the effect of managers talent, nor the persistence of the performance of some others. Other studies have focused on the performance of hedge funds without taking into account the different factors, styles and characteristics associated with abnormal returns.
Capocci (2004) argued that many hedge fund strategies are designed to be weakly correlated with share index returns and bond index returns and he also studied the correlation between various hedge fund strategies. He showed that the correlation between hedge fund strategies and share and bond indexes varies greatly from one strategy to another and between the various strategies. The high yield index is more strongly correlated with most indexes. To investigate the role of the period under study, Capocci (2004) studied the correlations on the sub-period January 2000-December 2002 (bear market) and he noticed that correlations generally tend to decrease comparing to the "normal" period even if this is not true for all indexes (for instance short selling or emerging market, see Capocci and Hübner (2003) and Capocci and Mahieu (2003)). Capocci (2004) found that the short selling index is negatively correlated with all other strategies when it was not with the share index. Moreover, the correlations between individual strategies remain relatively the same for the two periods. All strategies (without considering the short selling strategy) are positively correlated with the share index but the mortgage backed indexes, equity and multi-market strategy are only very weakly correlated as well as these strategies have little or no correlation with the bond index and the raw material index. However, this dependence coefficient which is often used by practitioners has several disadvantages. Indeed, the correlation coefficient is not defined if the second order moments of random variables are not finite. Besides, a correlation coefficient of zero does not necessarily imply independence between the variables studied. In addition, we can measure only linear correlation using this coefficient. Furthermore, the linear correlation coefficient is not invariant by a continuous and increasing function (such as the logarithm function, Embrechts et al (1999)) and it does not take into account the dependence of extremal values.

The study of comovements between hedge fund strategies on the one hand and comovements between them and the share index on the other hand, has been made using the correlation coefficient. This paper proposes to use a more appropriate methodology based on copulas theory. Our aim is firstly to model the structure of dependence between the returns of different hedge fund strategies and secondly between returns of each strategy and the market on two periods: a "normal" period and a period representing the occurrence of a rare and extreme event (when the market trend is downward) as well as modelling tail dependence between these variables through the use of extremal dependence coefficients. Moreover, we are interested in determining the level curves of the bivariate Value at Risk (VaR) between (i) different hedge funds strategies and the marginal rate of substitution (MRS) between the VaR of two hedge funds strategies and (ii) the VaR of a particular hedge funds strategy and the share index for a given risk level by using copulas theory. Hence, our methodology enables us specifically to take into account the extreme values and to study the impact of dependence on various measures of risk without making restrictive assumptions about the linearity and monotonocity of these series.

Copulas functions are a statistical tool which has many advantages. First, copulas make it possible to determine the nature of dependence of the series, be it linear or not, monotone or not. In addition to the fact that they offer a great flexibility in the implementation of the multivariate analysis, copulas authorize a wider selection of the marginal distributions of the financial series. Second, they allow a less banal representation of the statistical dependence in finance based on the traditional correlation measure (see Embrechts et al. (1999)). Third, they authorize less restrictive univariate probability distributions which make it possible to better accounting for the stylized facts in finance (leptokurticity, asymmetry, tail dependence). Fourth, they consider very general multivariate distributions, independently of the laws of the marginal ones which can have different laws and be unspecified. Furthermore, the copulas approach enables us to ease the implementation of multivariate models. Indeed, this approach allows the decomposition of the multidimensional law into its univariate marginal functions and a dependence function that would make possible extensions of some results obtained in the univariate case to the multivariate case. Hence, copula is an exhaustive statistic of the dependence.

The structure of the paper is as follows. Section 2 describes the concept of copulas and their basic properties. Section 3 presents the empirical aspect of our work. We present the hedge funds data and highlight a certain number of dependencies between the share index and the various hedge funds strategies, as well as dependences existing between the various strategies. We identify the copulas that model these dependences over two different periods: one "normal" period and a period during which the market trend is downward. Moreover, this modeling of the dependences will enable us to determine the level curves of the bivariate VaR between the hedge funds strategies. Lastly, we evaluate tail dependences between the market index, the hedge funds index and the strategy dedicated shorts through the use of various extremal dependence coefficients. Section 4 is devoted to the conclusion.

2 Dependent Models : a copula approach

A copula is **a multidimensional uniform distribution**. It is a relatively old statistical tool introduced by Sklar (1959), brought up to date by Genest and Mackay (1986).
As defined by Nelsen (1998), *"Copulas are functions that join or couple multivariate functions to their one dimensional margins. Copulas are distribution functions whose one dimensional margins are uniform"*.
In what follows and with reference to the work of Nelsen (1998), Genest & MacKay (1986), Denuit & Charpentier (2004),we attempt to give a more precise definition to copulas and to present some main properties.

2.1 Definition of a bivariate copula

A bivariate copula C is $2-$*increasing* function on the unit 2-cube $I^2 = [0,1] \times [0,1]$. Formally, it is a function $C : I^2 \to I$ with the following properties: :

- For all $u, v \in I$,
$$C(u,0) = 0 = C(0,v) \textit{ and } C(u,1) = u,\ C(1,v) = v.$$

- For all u_1, u_2, v_1 and $v_2 \in I$ such as $u_1 \le u_2$ and $v_1 \le v_2$:
$$C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1) \ge 0.$$

  Thus:
$$\Delta_{u_1}^{u_2} \Delta_{v_1}^{v_2} C(x,y) \ge 0.$$

2.2 Basic properties of Copula function

2.2.1 Sklar's theorem

Let H be a bivariate cumulative density function with univariate marginal cumulative density functions F and G. There exists a copula C such that for all $x, y \in \overline{R}$:

$$H(x, y) = C(F(x), G(y)).$$

If F are G continuous, then C is unique. This theorem shows that any bivariate cumulative density function H can be written in the form of a copula function. Therefore, it is possible to construct a wide range of multivariate distributions by choosing the marginal distributions and an apropriate copula.

2.2.2 Property of Invariance

Let X_1 and X_2 be two continuous random variables with margins F_1 and F_2, linked by a copula C. Let h_1 and h_2 two strictly increasing functions,then :

$$C(h_1(X_1), h_2(X_2)) = C(X_1, X_2).$$

2.2.3 Theorem 1 (Fréchet bounds)

Let C be a bivariate copula, then for all $(u, v) \in Dom\ C$:

$$max(u + v - 1, 0) \leq C(u, v) \leq min(u, v).$$

Fréchet (1951) shows that there exists upper and lower bounds for a copula function. In two dimensions, both of the Fréchet bounds are copulas themselves, but as soon as the dimension increases, the Fréchet lower bound is no longer an n-increasing function.

3 hedge funds and study of dependences

This section is devoted to the empirical part of our work. After presenting the data, we seek to highlight a number of dependencies between the share index and the hedge fund index on the one hand, and the different hedge funds strategies on the other hand. We select suitable copulas representing these dependencies over two different periods: a "normal" period and a period during which the market trend is downward. Besides, the copulas approach enables us not only to determine the level curves of the bivariate VaR between the hedge funds strategies but also to evaluate tail dependences between the market index, the hedge funds index and the strategy dedicated short via extremal dependence coeficients.

3.1 Data

The period under study extends from January 1994 to December 2006. Regarding the data of alternative assets, we use the historical returns of the 13 monthly indexes divided into 2 categories: an overall index of hedge funds for all strategies, and a set of indexes representing the 12 strategies of hedge funds that make up the database CSFB / TREMONT.

According to Fung and Hsieh (2000), these indexes are less affected by the survivor bias than the individual funds data. Unlike other indexes, CSFB / Tremont indexes take into account the return of the committee weighted by the size of funds in the basket of funds. To characterize the share market, we use the international index share MSCI World whose returns are taken from the TASS database.

3.2 Modeling dependence over the entire period

We carry out the study of the dependence structure between the hedge fund indexes and share index. We consider two periods, a "normal" period and a period during which the market trend is downward, in order to analyse the deformation of the structure of dependence.
A goodness of fit test allows us to validate the choice of copula selected. The search of dependencies focuses on both hedge funds strategies and share index as well as on hedge fund strategies in relation to each other.
Through the application of various tests of adjustment, it appears that the normal distribution provides the best adjustment for the considered variables. Figure 1 shows the adjustment of the distribution of share and hedge funds indexes to a Gaussian distribution using two graphic approaches : respectively hitograms'goodness of fit and Quantile-Quantile Plot (QQPlot). We

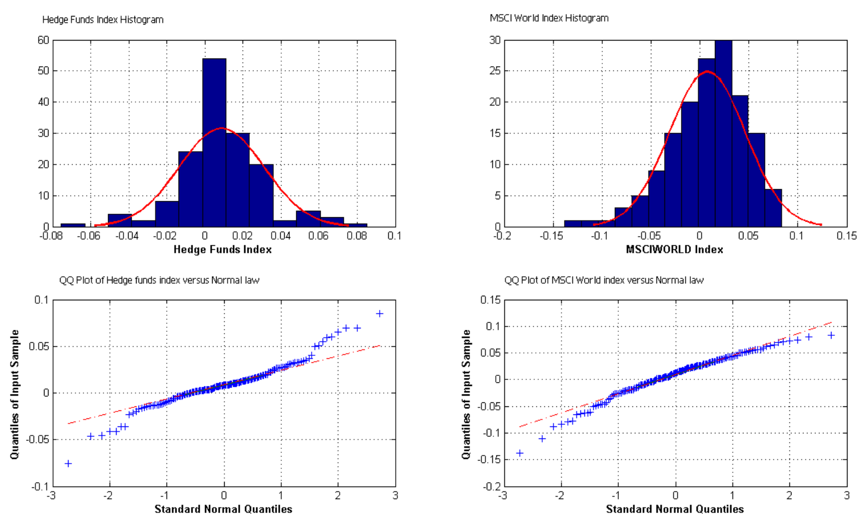

Figure 1: Adjustement of the share index distribution and the hedge funds distribution to a Gaussian distribution

remark more "picked" central values and heavier left tails of share and hedge funds indexes distributions than the Gaussian density function. In fact, the empirical distribution presents rare observations with a decrease slower than the exponential decrease of the normal distribution. However the use of the Jarque-Bera test of the null hypothesis that the sample in the empirical vector comes from a normal distribution with unknown mean and variance, against the alternative that it does not come from a normal distribution, leads us to assume a normal distribution for the share and hedge funds indexes.

3.2.1 Graphical analysis

The choice of the suitable copula is the first difficulty in the implementation of modeling dependence.

Figures 2, 3 and 4 make it possible to apprehend the form of the dependences which exist on the one hand between the share index, the hedge funds index and the hedge funds strategies and the structure of dependence between the different strategies hedge funds on the other hand.

In addition to the sign and the intensity of the dependences, these graphs provide us a first indication on the tail dependence which we will treat thereafter.

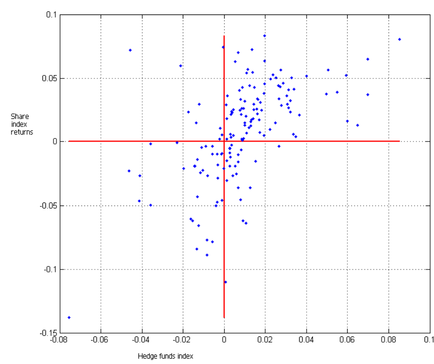

Figure 2: Scatter plot of the hedge funds index returns and the share index returns

The set of points is very close to the first bisector, illustrating a positive dependence between the share index and the hedge funds index. Moreover, the set of points is very concentrated on the 2 dials of the first bisector. Positive values for one variable coincide with positive values for the other variable. This result seems contradictory to what is expected since the funds managers urge the lack of correlation between the share market and hedge funds returns.

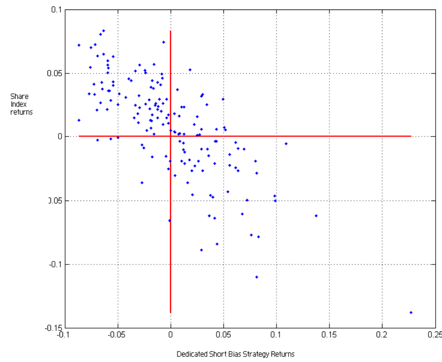

Figure 3: Scatter plot of the Dedicated Short strategy and the share index

The set of points is very close to the second bisector and the upper tail of the distribution is highly concentrated. The largest values in the upper tail for one variable coincide with large values of the same sign for the other variable. We note that the dependence between the share index and the dedicated short index is negative. This is not surprising since this strategy aims to maintain a short position on assets.

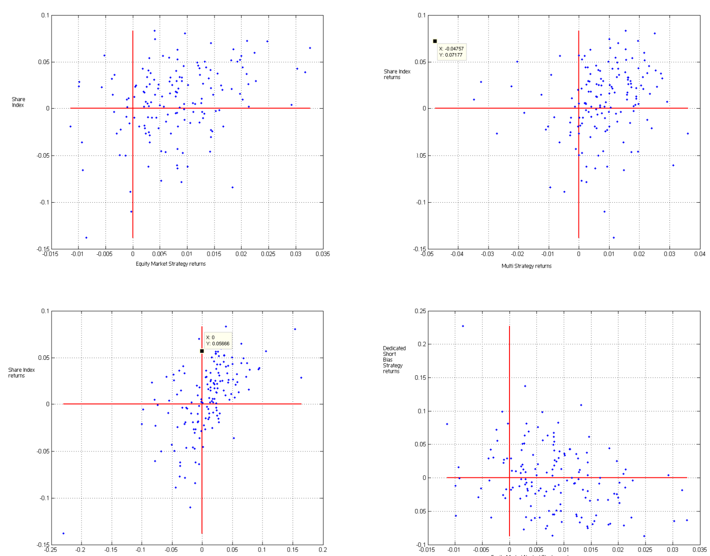

Figure 4: Scatter plot of the share index, the Dedicated Short strategy and other hedge funds strategies

Figure 4 shows that there exists positive dependences between share index and the Equity Market strategy, the Multi-strategy and the Emerging Market strategy as well as a negative dependence between the Dedicated Short Bias strategy and the Equity Market Strategy.

Relying on this graphical approach (Figures 2, 3 and 4) as a first step, the normal copula seems to be the most suitable copula that models the form of dependence between the share index and hedge fund index and between hedge fund strategies in relation to each other. We will use different goodness-of-fit tests to choose the adequate copula to our variables.

3.2.2 Estimation of copulas parameters

The Normal copula is defined as follows. Let Φ^{-1} be the inverse function of a standard normal distribution, f_i is the density function of the $i-$ th strategy returns and H_α the distribution function of the bivariate normal distribution with correlation coefficient α.
The normal copula $C_\alpha^N is$:
$C_\alpha^N(u,v) = H_\alpha\left(\Phi^{-1}(u), \Phi^{-1}(v)\right).for\ \ \mathrm{u, v} \in [0,1],\ \ \alpha \in [-1,1].$
There is no explicit form for the distribution function of the bivariate normal law.

To estimate the parameters of copulas which model the dependencies between these variables, we use the parametric method Inference Functions for Margins (IFM). This method was introduced by Shih & Louis (1995). It can reduce the estimation problem in two steps :
- The estimation of the parameters $\theta_1, \ldots, \theta_n$ of the margins.
- The estimation of the parameter θ_C of the copula.

Let us denote :

$$\theta = (\theta_1, \ldots, \theta_n, \theta_C)$$

We begin by determining the maximum likelihood estimators of margins:

$$\widehat{\theta_i} = argmax_{\theta_i} \sum_{k=1}^{n} f_i\left(x_i^k, \theta_i\right).$$

We then introduce these estimators in the copula part of the log-likelihood function, which leads to:

$$\theta_C = argmax_{\theta_C} \sum_{k=1}^{n} ln\left(c\left(F_1\left(x_1^k, \widehat{\theta_1}\right), \ldots, F_1\left(x_n^k, \widehat{\theta_n}\right), \theta_C\right)\right).$$

Other procedures can be applied, such as non-parametric estimation of margins followed by maximum likelihood estimation for the parameter of the copula. We refer the readers to Genest et al. (1995) and Shih and Louis (1995) procedures. There are other non-parametric estimation methods such as empirical copula (see Deheuvels (1979)) and parametric estimation methods like the moments method, the maximum likelihood method and the CML Canonical Maximum Likelihood (see Bouye and al. (2000)).
The IFM method we use has the advantage of being based on calculations less cumbersome than maximum likelihood procedure which consists in estimating the margins and copula parameters simultanously. Nevertheless, the determination of Godambe information matrix can be very complicated because it creates multiple derived calculations. For the same reasons as for the maximum likelihood method, a possible error in margins estimation in this method may lead to erroneous

estimator of the copula parameter.

Tables 1 to 3 include the values of the estimated parameters of copulas and table 4 illustrates the estimators of Gaussian margins parameters.

Table 1 : Estimated copulas parameters of the share index and hedge funds strategies (January 1994 − December 2006)

	The share index
hedge funds index	0.5516
Dedicated short strategy	-0.7421
Equity market strategy	0.3161
Multi Strategy	0.2560
emerging markets strategy	0.5781

Table 2 :Estimated copulas parameters of the dedicated short strategy and the other hedge funds strategies (January 1994 − December 2006)

	Dedicated short strategy
Equity market strategy	-0.2976
multi Strategy	-0.111
emerging markets strategy	-0.5522

Table 3 : Estimated copulas parameters of hedge funds strategies (January 1994 − December 2006)

	multi Strategy
Dedicated short strategy	-0.5522
Equity market strategy	0.3170

Table 4 : Estimated margins parameters of the share index and hedge funds strategies (January 1994 − December 2006)

	μ	σ
The share index	$7.68\ 10^{-3}$	$1.5\ 10^{-3}$
hedge funds index	$8.78\ 10^{-3}$	$4.94\ 10^{-4}$
Dedicated short strategy	$-5.44\ 10^{-4}$	$4.9\ 10^{-2}$
Equity market strategy	$8\ 10^{-3}$	$7.01\ 10^{-5}$
Multi Strategy	$5.55\ 10^{-3}$	$1.22\ 10^{-3}$
emerging markets strategy	$8.2\ 10^{-3}$	$4.64\ 10^{-2}$

We note a positive dependence between the share index and the hedge funds index on the one hand and the share index with the equity market strategy and multi-strategy on the other hand. However, the strength of this dependence differs from one strategy to another. There is a negative dependence between the share index and dedicated short strategy. The upper tail of the distribution is very concentrated. Our results also show a negative dependence between dedicated short strategy and the other hedge funds strategies and a positive dependence between the equity market strategy and multi-strategy. This means that returns of share index coincide with the returns of the same sign for the hedge funds index, the equity market strategy and multi-strategy. Therefore losses of first strategy and gains of the second one occur simultaneously. By contrast, returns of the dedicated short strategy coincide with the returns of the opposite sign for the share index, hedge funds index, the equity market strategy and multi-strategy. Thus, losses of the dedicated short strategy occur simultaneously with gains of the share index, hedge funds index, the equity market strategy and multi-strategy. These observations offer useful portfolio management tools.

3.2.3 Goodness of fit tests

Choosing the suitable copulas that model the dependencies between our random variables is of paramount importance. The question is the following : what is the best structure of dependence that can be adapted to the phenomenon studied?
Goodness of fit tests for copulas are relatively recent. There are few papers on this issue, but the field is in constant development.

Let consider a sample of random vectors iid $X_i \in \mathbb{R}^p$ as $X_i = \left(X_{1,i}, \ldots, X_{p,i}\right) \sim X$ iid. Let H be the distribution function of X and C the copula of X, thus :

$$H\left(x_1, \ldots, x_p\right) = C\left(F_1\left(x_1\right), \ldots, F_p\left(x_p\right)\right).$$

Generally, a statistic test distinguishes between two assumptions :

Null hypothesis $H = H_0$ or $C = C_0$
Alternative hypothesis $H \neq H_0$ or $C \neq C_0$

A first solution is to compare the empirical copula defined as :

$$\widehat{C}\left(\frac{k_1}{n}, \ldots, \frac{k_p}{n}\right) = \frac{1}{n} card\left\{i \mid R_{1,i} \leq k_1, \ldots, R_{p,i} \leq k_p\right\}.$$

to the estimated parametric copula C, where $R_{j,i},\ j = 1, \ldots, p \ \&\ i = 1, \ldots, n$ denotes the rank of $X_{j,i}$ among the observations $X_{j,1}, \ldots, X_{j,n}$. Dependency accepted is one that ensures that C is as close as possible to $\widehat{C}$.

Let $\{(x_k, y_k)\}_{k=1}^n$ a sample of realizations of the random vector (X, Y) . The empirical copula is the function C_n defined as :[1]

$$C_n\left(\frac{i}{n}, \frac{j}{n}\right) = \frac{\text{Number of pairs } (x, y) \text{ in the sample as } x \leq x_i,\ y \leq y_j}{n}.$$

The empirical density function of the copula C is given by:

$$c_n\left(\frac{i}{n}, \frac{j}{n}\right) = \begin{cases} \frac{1}{n} \text{ if } \left(x_i, y_j\right) \text{is an element of the sample.} \\ 0 \text{ otherwise} \end{cases}$$

The link between C_n and c_n is defined as follows :[2]:

[1]The empirical copulas were originally introduced by (Deheuvels (1979)).
[2]To demonstrate this proposition, see Nelsen (1998).

$$C_n\left(\frac{i}{n},\frac{j}{n}\right)=\sum_{p=1}^{i}\sum_{q=1}^{j}c_n\left(\frac{p}{n},\frac{q}{n}\right).$$

$$c_n\left(\frac{i}{n},\frac{j}{n}\right)=C_n\left(\frac{i}{n},\frac{j}{n}\right)-C_n\left(\frac{i-1}{n},\frac{j}{n}\right)-C_n\left(\frac{i}{n},\frac{j-1}{n}\right)+C_n\left(\frac{i-1}{n},\frac{j-1}{n}\right).$$

The empirical copulas are useful to provide non-parametric estimators of measures of dependence such as the ρ of Spearman, and the τ of Kendall. Indeed, these two measures are determined empirically as follows:

$$\widehat{\rho}=\frac{12}{n^2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(C_n\left(\frac{i}{n},\frac{j}{n}\right)-\frac{i}{n}-\frac{j}{n}\right).$$

$$\widehat{\tau}=\frac{2n}{n-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{p=1}^{i-1}\sum_{q=1}^{j-1}\left(c_n\left(\frac{i}{n},\frac{j}{n}\right)c_n\left(\frac{p}{n},\frac{q}{n}\right)-c_n\left(\frac{i}{n},\frac{q}{n}\right)c_n\left(\frac{p}{n},\frac{j}{n}\right)\right).$$

Figure 5 presents the empirical copula that models the dependence between the hedge fund index (IHF). It also reports the parametric copula allowing the adjustment of the empirical copula to the normal copula of parameter $alpha = 0.5516$ estimated by the IFM method. We note a good fit between the empirical copula and the normal one. This validates the choice of the normal copula to model dependence between share and hedge fund indexes.

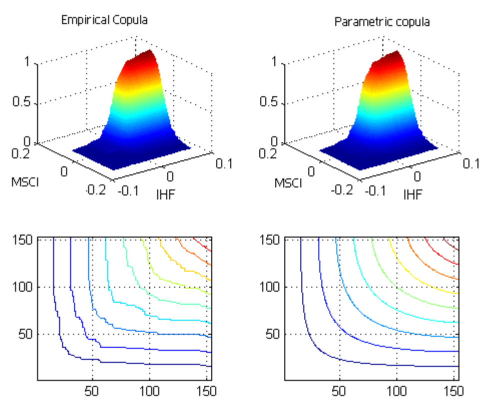

Figure 5: Empirical copula versus parametric copula for the hedge funds index (IHF) and the share index

Figure 6 shows the density of the copula that models dependence between the hedge funds index and the share index. There is a strong concentration in the upper tail of the distribution, and to a lesser extent in the lower tail distribution which indicates that positive largest values from the share index and hedge funds index occur together . By contrast, the probability of the simultanous occurence of negative largest values is smaller than the probability of the positive ones.

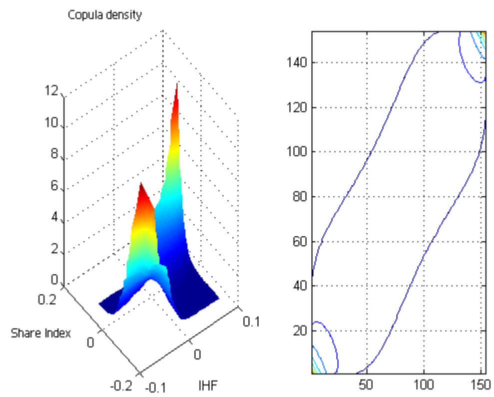

Figure 6: Density function of the copula hedge funds index (IHF)- share index

In addition to these graphical methods, we use the goodness of fit test of Genest, Quessy and Remillard (2006) to validate the choice of copula selected.
Genest, Quessy & Rémillard (2006) have expanded the work of Wang and Wells (2000) providing alternative statistics given by :

$$S_n = \int_0^1 \mid \mathbb{K}_n\,(t) \mid^2 k\,(\theta_n, t)\, dt \quad \text{where } k\,(\theta_n, t) \quad \text{is the density function associated to } K\,(\theta, t)\,.$$

and

$$T_n = sup_{0 \leq t \leq 1} \mid \mathbb{K}_n\,(t) \mid .$$

$K\,(\theta_n, t)$ is a kandell process defined by Genest and Rivest (1993) as follows :

$$K\,(\theta_n, t) = \sqrt(n)(k_n(t) - k(t)),$$

where k is a kendall function defined by $k(t) = P(F(x_1, ...x_p \leq t)) = P(C(x_1, ...x_p \leq t)$ and k_n is the empirical version of k defined by $k_n(t) = \frac{1}{t} \sum_{i=1}^{n} 1_{(F_n(x_1,...x_p \leq t))}$ with F_n is the empirical ditribution function.

S_n is based on the distance of Cramèr Von Mises, while the statistic T_n is based on that of Kolmogorov Smirnov. It is important to note that these statistics have several advantages:

- Some simple formulas are available for S_n and T_n in terms of the rank of obsevations, which is not the case with the original statistics of Wang and Wells (2000).

- The selection procedures are not influenced by an external constant whose selection and influence on the limit distribution of the test statistic were not considered by Wang and Wells (2000).

- S_n and T_n distributions can be determined not only for archimedian bivariate copulas but also for copulas with dimension greater than 2 and for copulas satisfying the general condition of regularity.

- The parametric bootstrap method is valid and can be used to approximate the P-values associated with functional $\mathbb{K}_n$ and in particular with S_n and T_n.

Performing a parametric bootstrap for statistics S_n and T_n, it is possible to obtain approximate thresholds associated with these statistical assumptions of type $C \in (C_\theta)$. In the paper of Genest and Rémillard (2006), one confirms the validity of this bootstrap approach for both types of the most common goodness of fit tests : (i) tests where we compare the distance between a multivariate empirical distribution and parametric estimation under the null hypothesis or (ii) those where we compare the distance between the empirical estimators and parametric univariate pseudoobservations such as W_i, obtained through the integral transformation of probability.

Genest et al. (2006) showed that :

$$S_n = \frac{n}{3} + n \sum_{j=1}^{n-1} K_n^2\left(\frac{j}{n}\right)\left(K\left(\theta_n, \frac{j+1}{n}\right) - K\left(\theta_n, \frac{j}{n}\right)\right).$$

$$-n \sum_{j=1}^{n-1} K_n\left(\frac{j}{n}\right)\left(K^2\left(\theta_n, \frac{j+1}{n}\right) - K^2\left(\theta_n, \frac{j}{n}\right)\right).$$

and

$$T_n = \sqrt{n}\, max_{i=0,1\ \&\ 0\le j\le n-1}\left(\mid K_n\left(\frac{j}{n}\right) - K\left(\theta_n, \frac{j+1}{n}\right)\mid\right).$$

The test concludes to the rejection of the null hypothesis $H_0 : C = C_0$ when the observed values S_n and T_n are above the quantile of order $1-\alpha$ of their distributions under the assumption H_0.

After applying this test, the normal copula is the suitable copula modeling the dependence between the share index and the different hedge funds strategies and between these strategies in relation to each other. Indeed, the table 5 shows the goodness of fit statistics and results.

Table 5 : Goodness of fit test results for the period "January 1994 − December 2006"

	Share index			
	VaR(S)	S_n	VaR(T)	T_n
Hedge fund index	0.0578	0.048	0.4709	0.4474
Dedicated short strategy	0.6501	0.6312	1.0688	0.9879
Equity market strategy	0.1002	0.0821	0.6035	0.5667
multi Strategy	0.1064	0.08994	0.6977	0.6195
emerging markets strategy	0.0538	0.04879	0.5270	0.45

Note that there exists other goodness of fit tests : the parametric bootstrap method in the approach of Wang and Wells (2000), the parametric bootstrap method based on C_{θ_n} (see Genest and Rémillard (2005)), the goodness of fit test of copulas based on the transformation of Rosemblatt (old fashion) (see Breymann and al. (2003)) and the goodness of fit test of copulas based on the transformation of Rosemblatt (new fashion) (see Klugman and Parsa (1999), Ghoudi and Rémillard (2004), Genest et al. (2008).

3.3 Modeling dependence on the subperiod January 2000-December 2002

In order to try to answer the question of the dependence structure variation when the market trend is downward, a first possibility is to estimate the parameters of bivariate copulas existing between the share index, hedge funds index and hedge funds strategies via the Inference Functions for Margins method.

As previously, we find that the normal copula is the suitable copula that models the dependence structure between the share index, hedge funds strategies and hedge fund index. Moreover, this modeling of dependence provides extra information on the evolution of the degree of dependence in relation with the market trends.

Tables 6, 7, 8 and 9 show the different values of the estimated copulas parameters in a crisis period (January 2000-December 2002) and the estimated margins parameters.

Table 6 : Estimated copulas parameters of the share index and hedge funds strategies in a crisis period (subperiod January 2000-December 2002)

	Share index
Hedge fund index	0.4010
Dedicated short strategy	-0.8819
Equity market strategy	-0.1517
multi Strategy	0.5166
emerging markets strategy	0.7949

Table 7 : Estimated copulas parameter of the dedicated short strategy and the emerging market strategy in a crisis period (subperiod January 2000-December 2002)

	Dedicated short strategy
emerging markets strategy	-0.80

Table 8 : Estimated copulas parameter of the equity market strategy and the multi strategy in a crisis period (subperiod January 2000-December 2002)

	Equity market strategy
Multi Strategy	-0.3571

Table 9 : Estimated margins parameters of the share index and hedge funds strategies in a crisis period (subperiod January 2000-December 2002)

	μ	σ
The share index	-1.32 10^{-1}	2.36 10^{-3}
hedge funds index	3.51 10^{-3}	1.81 10^{-1}
Dedicated short strategy	9.18 10^{-3}	3.06 10^{-3}
Equity market strategy	0.839 10^{-3}	4.07$10^{-5}$
Multi Strategy	6.18 10^{-3}	7.47 10^{-5}
emerging markets strategy	2.55 10^{-3}	3.45 10^{-1}

These results show that the dependence parameter has increased for the normal copulas that model (i) dependence between the share index and the strategies dedicated short , multi strategy and emerging market and (ii) the dependencies between the hedge funds strategies. In other words, the higher the copula parameter, the greater the dependency is. This means that the share index, the dedicated short strategy and emerging market strategy have become more dependent when the market trend is downward.

However, during the "normal" period, the dependence between the share index and the equity market strategy was positive but the sign of dependence has becoming negative, when an extreme event had occurred (September 11, 2001). It is the same for the sign of dependence between the equity market strategy and the multi-strategy, which indicates that losses and, respectively, gains of the share index have a higher probability to coincide with gains and, respectively, losses of the dedicated short strategy and emeregent market strategy when the market trend is downward than in the "normal" period. While in the "normal" period, the returns of the equity market strategy coincide with returns of the same sign for the share index and the multi-strategy, when the market trend is downward the returns of the equity market strategy occur simultanously with returns of the opposite sign for the share index and the multi-strategy.

3.4 Copulas and bivariate VaR

Value at Risk (VaR) has become the standard measure that financial analysts use to asses market risk. The VaR is defined as the maximum potential loss due to adverse market movements for a given probability.

The use of copulas allows us to determine the level curves of the bivariate VaR and examine for a given threshold level, the marginal rate of substitution between the VaR of two univariate risks .

Indeed, since we have marginal distributions of returns of different hedge fund strategies, and the share index, it is possible to trace the level curves corresponding to the minimum copula (anti-monotonicity), maximum copula (comonotonicity) and the independence copula.
Let r_A and r_B be the returns of the series A and B. Let F_A and F_B be the univariate distribution functions of returns of respectively r_A and r_B . We have, for any threshold $\alpha \in [0,1]$:

$$\{(r_A, r_B) \,; max\left(F_A\left(r_A\right) + F_B\left(r_B\right) - 1, 0\right) = \alpha\}\,, anti - monotonicity;$$

$$\{(r_A, r_B) \,; F_A\left(r_A\right) .F_B\left(r_B\right) = \alpha\}\,, independence;$$

$$\{(r_A, r_B) \,; min\left(F_A\left(r_A\right), F_B\left(r_B\right)\right) = \alpha\}\,, comonotonicity.$$

The level curves from the empirical copula are given by :

$$\{(r_A, r_B) \,; C\left(F_A\left(r_A\right), F_B\left(r_B\right)\right) = \alpha\}\,.$$

The level curves are used to determine the marginal rate of substitution between the two univariate VaR. The more the empirical curve is high approaching the case anti-monotonicity, the more is the dependence between the returns A et B and the more is the compensation effect. However, the more the curves are close to their lower limit, corresponding to the case of comonotonicity or positive dependence, the more the returns tend to move in the same direction, the dependence between losses (correlation) is therefore very high. Regarding the curves of multiplication, they correspond to diversification .

We determine the 95% level curves of the bivariate VaR between the share index and the hedge funds index (Figure 7) and the level curves of bivariate VaR of different hedge funds strategies (Figures 8 and 9).

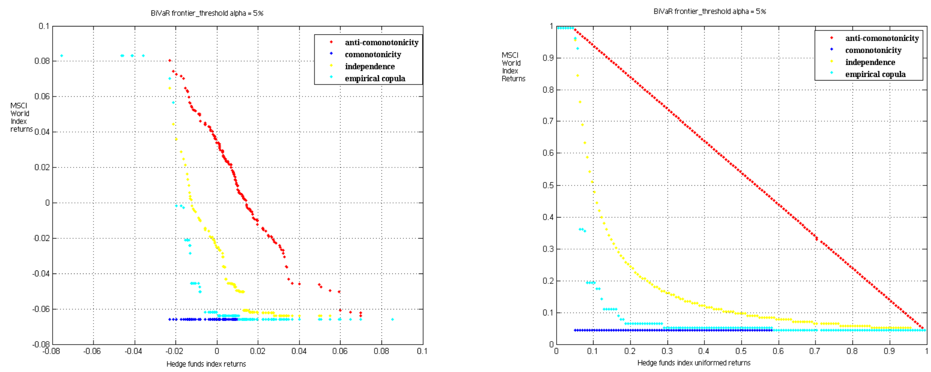

Figure 7: Bivariate VaR of share index and hedge funds index

We note from the Figure 7 that the 95% level curve of the empirical copula is closer to that corresponding to the case co-monotony (the lower limit) or positive dependence. It follows that the returns of the share index and the hedge funds index operate in the same direction,"the correlation"between the losses is therefore high. As a consequence, it is preferably not to put these two elements into a single portfolio.

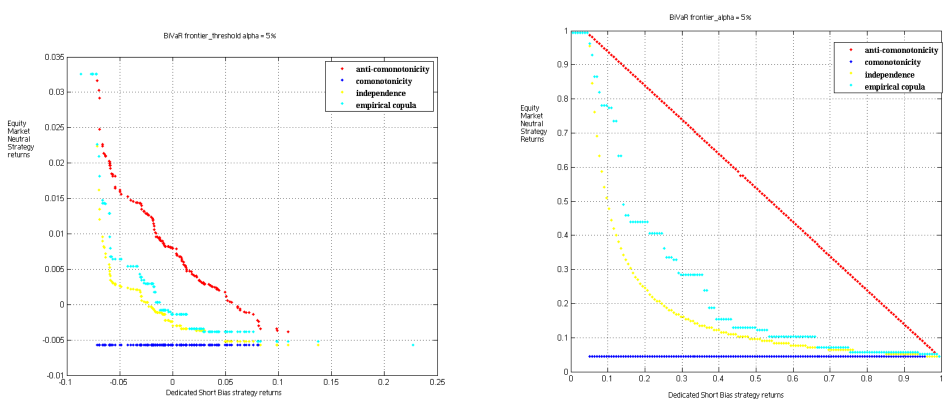

Figure 8: Bivariate VaR of Dedicated short strategy and Equity Market Neutral strategy

We remark according to the Figure 8 that the 95% level curve of the empirical copula is closer to the level curve of multiplication or independence which is the case of diversification: the losses of the Dedicated short strategy and the Equity Market Neutral strategy are not correlated. In order to guarantee diversification, it is best to put these two strategies in the same portfolio. Finally, we note from Figure 9 that the 95% level curve of the empirical copula is high and closer to that of the

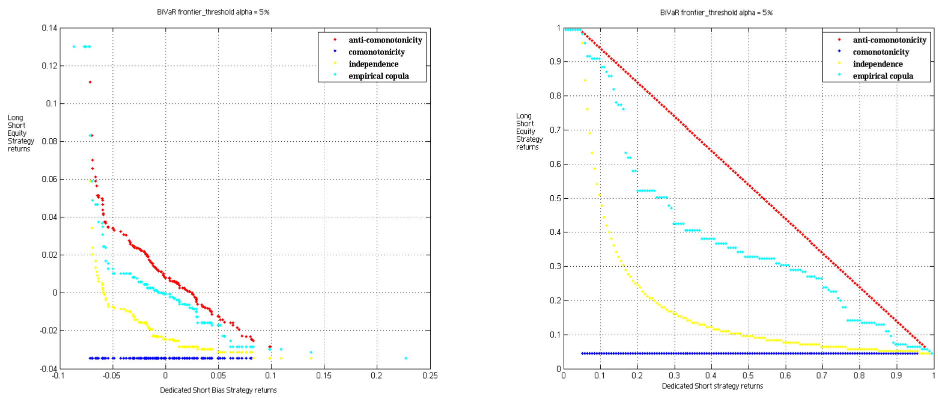

Figure 9: Bivariate VaR of Dedicated Short strategy and the Long Short Equity strategy

anti-monotonicity case. Consequently, the dependence between the returns of the Dedicated Short strategy and the Long Short Equity strategy is negative and the effect of adjustment and compensation will take place. To ensure a better allocation in its portfolio, it is advisable to combine these two strategies in the same portfolio of funds.

We note that the maximum potential loss, calculated through the risk measure VaR is higher in the dependence case compared to the case of independence. This means that assuming the non-correlation between hedge fund strategies and share index underestimate the portfolio risk measured by VaR.

3.5 Tail dependence : extremal coefficients χ and $\overline{\chi}$

The study of the tail dependence allows us to describe the dependence in the tails of distribution and to examine the simultaneous occurrence of extreme values. We use two coefficients of tail dependence enabling us to measure the asymptotic dependence between hedge funds strategies and the share index.

The dependence measures χ and $\overline{\chi}$ were introduced by Coles et al. (1999). After transformation of (X,Y)which are the two series to study into (U, V) having uniform marginal distributions, we get :

$$Pr(V > u \middle| U > u) = 2 - \frac{1 - Pr\left(U < u, V < u\right)}{1 - Pr\left(U < u\right)} \approx 2 - \frac{logC\left(u, u\right)}{log\left(u\right)} for \;\; 0 \leq u \leq 1.$$

The dependence measure $\chi\left(u\right)$ is defined as

$$\chi\left(u\right) = 2 - \frac{logC\left(u, u\right)}{log\left(u\right)} for \;\; 0 \leq u \leq 1.$$

The function $\chi\left(u\right)$ is thus a quantile-dependent measure of dependence. The sign of $\chi\left(u\right)$ determines whether the variables are positively or negatively associated to the quantile level u.

- $\chi(u)$ is bounded as follows :

$$2-\frac{log(2u-1)}{log(u)} \leq \chi(u) \leq 1.$$

The lower bound is interpreted as $-\infty$ for $u \leq 1/2$, and 0 for $u = 1$.
A single parameter measure of extremal dependence is given by

$$\chi = lim_{u\to 1}\chi(u).$$

Loosely stated, χ is the probability of one variable being extreme given that the other is extreme.
- In the case $\chi = 0$, the variables are asymptotically independent. Thus, we require a complementary dependence measure to assess extremal dependence within the class of asymptotically independent variables. By analogy with the definition of $\chi(u)$, comparison of joint and marginal survivor functions of (U, V) leads to :

$$\overline{\chi}(u) = \frac{2logPr(U>u)}{logPr(U>u, V>u)} - 1 = \frac{2log(1-u)}{log\overline{C}(u,u)} - 1 \ \ for \ \ 0 \leq u \leq 1,$$

where $-1 < \overline{\chi}(u) < 1$ for all $0 \leq u \leq 1$.
To focus on extremal characteristics, we also define :

$$\overline{\chi} = lim_{u\to 1}\overline{\chi}(u).$$

The measures χ and $\overline{\chi}$ are related to the Ledford & Tawn (1996, 1998) characterisation of the joint tail behavior, through η, the tail dependence coefficient and $L(t)$, the relative strength of limiting dependence :

$$\overline{\chi} = 2\eta - 1.$$

$$\chi = \begin{cases} c \ \ if \ \ \overline{\chi}=1 \ \ and \ \ L(t)\to c>0 \ \ as \ \ t\to\infty, \\ 0 \ \ if \ \ \overline{\chi}=1 \ \ and \ \ L(t)\to 0 \ \ as \ \ t\to\infty, \ \ and \ \ if \ \ \overline{\chi}<1. \end{cases}$$

We have the following classification :
- $\chi \in [0, 1]$: the set $(0, 1]$ corresponds to asymptotic dependence;
- $\overline{\chi} \in [-1, 1]$; the set $[-1; 1)$ corresponds to asymptotic independence.
Thus the complete pair $(\chi, \overline{\chi})$ is required as a summary of extremal dependence:
- $(\chi > 0; \overline{\chi} = 1)$ implies asymptotic dependence, in which case the values of χ and $\overline{\chi}$ determine the strength of dependence within the class;
- $(\chi = 0; \overline{\chi} < 1)$ implies asymptotic independence.
In practice, we first assess $\overline{\chi}$:
- $\overline{\chi} < 1 \Rightarrow$ asymptotic independence ;
- $\overline{\chi} = 1 \Rightarrow$ asymptotic dependence, here we must also estimate χ.

We calculate the extremal dependence coefficient $\overline{\chi}$ in order to examine the simultaneous occurrence of extreme values and to measure the asymptotic dependence of the minimum and the maximum between the share index and the hedge funds index as well as the asymptotic dependence between the share index and the strategy Dedicated Short.

Table 10 : Measure $\overline{\chi}$

	Share index
hedge funds index	-0.015
dedicated short Strategy	-0.0568

The results which are grouped in Table 10, lead us to conclude that the value of $\overline{\chi}$ is lower than 1 for the share index and the hedge funds index and for the share index and strategy Dedicated Short. Therefore, there is an asymptotic independence between the share index and the hedge funds index on the one hand and between the share index and strategy Dedicated Short on the other hand. Loosely speaking, knowing extreme loss for the share index, there is zero probability that loss with a comparable intensity to take place concurrently for the hedge fund index and the Dedicated Short strategy.

4 Conclusion

There is a regular expansion of the hedge funds debate towards the financial institutions. Faced with the ever-changing environment, managers of hedge funds are expected to adapt by modeling the structure of dependence between different hedge fund strategies in relation to each other and in relation to the stock market.

This paper provides elements and techniques on this debate by relying on the theory of copulas. We have proposed a study of dependence and tail dependence of the different hedge funds strategies and the share index on one hand and of the different hedge funds strategies in relation to each other on the other hand. We also assess the risk of different hedge funds strategies in various cases of dependence through the use of bivariate Value at Risk.

Contrary to what the funds managers preconize, our results highlighted a certain number of dependences between the share index and some hedge funds strategies and between these various strategies themselves. The sign and the intensity of dependence increase for most hedge funds strategies in times of crisis . The share index, the dedicated short strategy and the emerging market strategy are more dependent when the market is bear. For some other strategies, the sign of dependence changes from a positive into a negative value in a crisis period. This is the case for the equity market strategy and the share index on the one hand, and the equity market strategy and multi-strategy on the other.

A note worthy finding is that taking into account the dependencies between the different hedge funds strategies via the copula theory has a significant impact on the risk measures such as VaR. Furthermore, the determination of the level curves of the bivariate VaR and study of the simultaneous occurrence of extreme values using the coefficients of tail dependence could provide to alternative fund managers a more precise estimation of risk and a better allocation of their portfolio in both the absence and presence of extreme movements.

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