# Dynamic Contracting: An Irrelevance Result

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**Abstract**

This paper considers a general, dynamic contracting problem with adverse selection and moral hazard, in which the agent's type stochastically evolves over time. The agent's final payoff depends on the entire history of private and public information, contractible decisions and the agent's hidden actions, and it is linear in the transfer between her and the principal. We transform the model into an equivalent one where the agent's subsequent information is independent in each period. Our main result is that for any fixed decision-action rule implemented by a mechanism, the maximal expected revenue that the principal can obtain is the same *as if* the principal could observe the agent's orthogonalized types after the initial period. In this sense, the dynamic nature of the relationship is *irrelevant*: the agent only receives information rents for her initial private information. We also show that any monotonic decision-action rule can be implemented in a Markov environment satisfying certain regularity conditions.

*Keywords:* asymmetric information, dynamic contracting, mechanism design

## 1 Introduction

We analyze multiperiod principal-agent problems with adverse selection and moral hazard. The principal's per-period decisions and the monetary transfers are governed by a contract signed at the beginning of the relationship, in the presence of some initial informational asymmetry, and the agent's private information stochastically evolves over time. The agent's final payoff can depend, quite generally, on the entire history of private and public information, contractible decisions and the agent's hidden actions, and it is linear in the transfer between her and the principal. We discuss the wide-ranging applications of such models in micro- and macroeconomic modeling below.

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Our main result is an *irrelevance theorem*: In a mechanism implementing a given action-decision rule, the maximal expected revenue that the principal can obtain (and his maximal payoff if it is linear in the transfers) is the same *as if* he could contract on whatever "new" (orthogonal) information is observed by the agent in any future period. Note that in the hypothetical benchmark case where the agent's future orthogonalized types are observable and contractible the parties need not interact beyond the initial period, and the agent has no access to dynamic deviation reporting strategies. In this sense the dynamic nature of the adverse selection problem is *irrelevant*. This irrelevance result holds in a rich environment, with very little assumed about the agent's utility function (no single-crossing or monotonicity assumptions are made), the information structure, and so on.

We also show that monotonic decision rules can be implemented in Markov environments with time-separable payoffs, subject to additional regularity conditions. The regularity assumptions include familiar single-crossing conditions for the agent's utility function, and also assumptions concerning the availability of a contractible public signal about the agent's action and type. If the signal is informative (however imperfectly) about a summary statistic of the agent's hidden action and type, and its distribution is generic, then any monotonic decision rule coupled with any monotonic action rule is approximately implementable.[1] If the contractible signal is uninformative about the agent's action (but the other regularity conditions hold), then monotonic decision rules coupled with agent-optimal actions can be implemented.

The significance of the implementation results is that when they apply, the dynamic contracting problem can indeed be treated as a static one and solved as follows. Consider the benchmark case in which the agent's only private information is her initial type, and the principal can observe her orthogonalized future types. Solve this relaxed case, either by optimizing the action rule as well or taking it to be the agent-optimal one, depending on whether or not a public summary signal about the agent's type and action is available. If the resulting rule is monotonic in the agent's type profile then it can be implemented in the original problem with the same expected payments as in the benchmark, hence it is optimal in the original problem as well.[2] It is important to note that despite the validity of this solution method the original and the benchmark problems are not equivalent: the monotonicity requirement on the decision rule is more stringent, hence the set of implementable decision rules is smaller, in the benchmark. However, under the regularity conditions the optimal solution in the relaxed (benchmark) problem is implementable in the more restrictive original problem.

Models in the class of dynamic contracting problems that we analyze can be, and indeed

[1]The genericity condition and the notion of approximate implementability will be precisely defined in Section 4.

[2]However, Battaglini and Lamba (2012) point out that the regularity conditions guaranteeing the monotonicity of the "pointwise-optimal" decision rule are quite strong.

have been, applied to a wide range of economic problems.[3] The roots of this literature reach back to Baron and Besanko (1984) who used a multi-period screening model to address the issue of regulating a monopoly over time. Courty and Li (2000) studied optimal advance ticket sales, Eso and Szentes (2007a) the optimal disclosure of private information in auctions, Eso and Szentes (2007b) the sale of advice as an experience good. Farhi and Werning (2012), Golosov, Troshkin and Tsivinsky (2011) and Kapička (2013) apply a similar approach to optimal taxation and fiscal policy design, respectively. Pavan, Segal and Toikka (2012) apply their (to date, most general) results on the multi-period pure adverse selection problem to the auction of experience goods (bandit auctions). Garrett and Pavan (2012) use a dynamic contracting model with both adverse selection and moral hazard to study optimal CEO compensation. Such mixed, hidden action – hidden information models could also be applied in insurance problems.

In this paper we also develop three applications in order to illustrate our techniques and new results. The first two examples are dynamic monopoly problems in which the buyer's valuation for the good (her type) stochastically evolves over time. In the second example the valuation also depends on the buyer's hidden, costly action: e.g., she may privately invest in learning how to better enjoy the good. The monopolist cannot observe any signal about the buyer's type and action; all he can do is to offer a dynamic screening contract. We derive the optimal contract and show that our dynamic irrelevance theorem holds: all distortions are due to the buyer's initial private information. The third application is a dynamic principal-agent problem with adverse selection and moral hazard, where the principal is an investor and the agent an investment advisor. The contractible decision is the amount of money invested by the principal with the agent. The agent's type is her ability to generate higher expected returns, whereas her costly action is aimed at picking stocks that conform with the principal's other (e.g., ethical) considerations. Here the principal (investor) observes a summary signal about the agent's (advisor's) type and action, in the form of the principal's flow payoff. We fully solve this problem as well and show that the dynamic irrelevance theorem applies.

In order to formulate the main, irrelevance result of the paper we rely on an idea introduced in our previous work (Eso and Szentes (2007a)): We transform the model into an *equivalent orthogonal representation*, in which the agent's private information in each period is independent of that obtained in earlier periods. The irrelevance theorem obtains by showing that in the original problem (where the agent's orthogonalized future types and actions are *not* observable), in any incentive compatible mechanism, the agent's expected payoff conditional on her initial type are fully determined by her on-path (in the future, truthful) behavior.

[3] Our review of applications is deliberately incomplete; for a more in-depth survey this literature see Krähmer and Strausz (2012) or Pavan, Segal and Toikka (2012).

Therefore, the agent's expected payoff (and payments) coincide with those in the benchmark case, where the orthogonalized future types are publicly observable.

The results on the implementability of monotonic decision rules are obtained in Markovian environments subject to additional regularity conditions. Here, the key step in the derivation (also used in Eso and Szentes (2007a) in a simpler model) is to show that if the agent is untruthful in a given period in an otherwise incentive compatible mechanism, she immediately *undoes her lie* in the following period to make the principal's inference regarding her type correct in all future periods. The explicit characterization of out-of-equilibrium behavior in regular, Markovian environments enables us to pin down the transfers that implement a given monotonic decision rule in a model with adverse selection. The results for models with both moral hazard and adverse selection are obtained by appropriately reducing the general model to ones with only adverse selection, the exact way depending on the assumptions made regarding the observability of a public signal on the agent's type and action.

The technical contributions notwithstanding, we believe the most important message of the paper is the dynamic irrelevance result. The insight that the principal need not pay his agent rents for post-contractual hidden information in a dynamic adverse selection problem has been expressed in previous work (going back to Baron and Besanko (1984)). Our paper highlights both the depth and the limitations of this insight: Indeed the principal that contracts the agent prior to her discovery of new information can limit the agent's rents to the same level as if he could observe the agent's orthogonalized future types; however, we also point out that the two problems are not equivalent.

The paper is organized as follows. In Section 2 we introduce the model and describe the orthogonal transformation of the agent's information. In Section 3 we derive necessary conditions of the implementability of a decision rule and our main, dynamic irrelevance result. Section 4 presents sufficient conditions for implementation in Markov environments. Section 5 presents the applications; Section 6 concludes. Omitted proofs are in the Appendix.

# 2 Model

*Environment.—* There is a single principal and a single agent. Time is discrete, indexed by $t = 0, 1, ..., T$. The agent's private information in period $t$ is $\theta_t \in \Theta_t$, where $\Theta_t = \left[\underline{\theta}_t, \overline{\theta}_t\right] \subset \mathbb{R}$. In period $t$, the agent takes action $a_t \in A_t$ which is not observed by the principal. The set $A_t$ is an open interval of $\mathbb{R}$. Then a contractible public signal is drawn, $s_t \in S_t \subset \mathbb{R}$. After the public signal is observed in period $t$, a contractible decision is made, denoted by $x_t \in X_t \subset \mathbb{R}^n$, which is observed by both parties. Since $x_t$ is contractible, it does not matter whether it is taken by the agent or by the principal. The contract between the principal and the agent is signed at $t = 0$, right after the agent has learned her initial type, $\theta_0$.

We denote the history of a variable through period $t$ by superscript $t$; for example $x^t = (x_0, ..., x_t)$, and $x^{-1} = \{\emptyset\}$. The random variable $\theta_t$ is distributed according to a c.d.f. $G_t\left(\cdot|\theta^{t-1}, a^{t-1}, x^{t-1}\right)$ supported on $\Theta_t$. The function $G_t$ is continuously differentiable in all of its argument, and the density is denoted by $g_t\left(\cdot|\theta^{t-1}, a^{t-1}, x^{t-1}\right)$. The public signal $s_t$ is distributed according to a continuous c.d.f. $H_t\left(\cdot|f_t(\theta_t, a_t)\right)$, where $f_t : \Theta_t \times A_t \to \mathbb{R}$ is continuously differentiable. We may interpret $s_t$ as an imperfect public summary signal about the agent's current type and action; for example, in Application 3 of Section 5 it will be $s_t = \theta_t + a_t + \xi_t$, where $\xi_t$ is noise with a known distribution. In the general model we assume that for all $\theta_t, \widehat{\theta}_t$ and $a_t$ there is a unique $\widehat{a}_t$ such that $f_t(\theta_t, a_t) = f_t(\widehat{\theta}_t, \widehat{a}_t)$.[4]

The agent's payoff is quasilinear in money, and is defined by

$$\widetilde{u}\left(\theta^T, a^T, s^T, x^T\right) - p,$$

where $p \in \mathbb{R}$ denotes the agent's payment to the principal, and $\widetilde{u} : \Theta^T \times A^T \times S^T \times X^T \to \mathbb{R}$ is continuously differentiable in $\theta_t$ and $a_t$ for all $t = 0, ..., T$. We do not specify the principal's payoff. In some applications (e.g., where the principal is a monopoly and the agent its customer) it could be the payment itself, in others (e.g., where the principal is a social planner and the agent the representative consumer) it could be the agent's expected payoff; in yet other applications it could be something different.

A notational convention: We denote partial derivatives with a subscript referring to the variable of differentiation, e.g., $\tilde{u}_{\theta_t} \equiv \partial\tilde{u}/\partial\theta_t$, $f_{t\theta_t} \equiv \partial f_t/\partial\theta_t$, etc.

*Orthogonalization of Information.*— The model can be transformed into an equivalent one where the agent's private information is represented by serially independent random variables. Suppose that at each $t = 0, ..., T$, the agent observes $\varepsilon_t = G_t\left(\theta_t|\theta^{t-1}, a^{t-1}, x^{t-1}\right)$ instead of $\theta_t$. Clearly, $\varepsilon^t$ can be inferred from $\left(\theta^t, a^{t-1}, x^{t-1}\right)$. Conversely, $\theta^t$ can be computed from $(\varepsilon^t, a^{t-1}, x^{t-1})$, that is, for all $t = 0, ..., T$ there is $\psi_t : [0,1]^t \times A^{t-1} \times X^{t-1} \to \Theta_t$ such that

$$\varepsilon_t = G_t\left(\psi_t(\varepsilon^t, a^{t-1}, x^{t-1})|\psi^{t-1}(\varepsilon^{t-1}, a^{t-2}, x^{t-2}), a^{t-1}, x^{t-1}\right), \tag{1}$$

where $\psi^t(\varepsilon^t, a^{t-1}, x^{t-1})$ denotes $\left(\psi_0(\varepsilon_0), \ldots, \psi_t(\varepsilon^t, a^{t-1}, x^{t-1})\right)$. In other words, if the agent observes $\left(\varepsilon^t, a^{t-1}, x^{t-1}\right)$ at time $t$ in the orthogonalized model, she can infer the type history $\psi^t(\varepsilon^t, a^{t-1}, x^{t-1})$ in the original model.

Of course, a model where the agent observes $\varepsilon_t$ for all $t$ is strategically equivalent to the one where she observes $\theta_t$ for all $t$ (provided that in both cases she observes $x^{t-1}$ and

[4]This assumption ensures that the principal cannot resolve the adverse selection problem by requiring the agent to take a certain action and using the public signal to detect the agent's type.

recalls $a^{t-1}$ at $t$). By definition, $\varepsilon_t$ is uniformly distributed on the unit interval[5] for all $t$ and all realizations of $\theta^{t-1}$, $a^{t-1}$ and $x^{t-1}$, hence the random variables $\{\varepsilon_t\}_0^T$ are independent across time. There are many other orthogonalized information structures (e.g., ones obtained by strictly monotonic transformations). In what follows, to simplify notation, we fix the orthogonalized information structure as the one where $\varepsilon_t$ is uniform on $\mathcal{E}_t = [0,1]$.

The agent's gross payoff in the orthogonalized model, $u : \mathcal{E}^T \times A^T \times S^T \times X^T \to \mathbb{R}$, becomes

$$u\left(\varepsilon^T, a^T, s^T, x^T\right) = \widetilde{u}\left(\psi^T\left(\varepsilon^T, a^{T-1}, x^{T-1}\right), a^T, s^T, x^T\right).$$

*Revelation Principle.*— A deterministic mechanism is a four-tuple $\left(Z^T, \mathbf{x}^T, \mathbf{a}^T, p\right)$, where $Z_t$ is the agent's message space at time $t$, $\mathbf{x}_t : Z^t \times S^t \to X_t$ is the contractible decision rule at time $t$, $\mathbf{a}_t : Z^t \times S^{t-1} \to A_t$ is a recommended action at $t$, and $\mathbf{p} : Z^T \times S^T \to \mathbb{R}$ is the payment rule. The agent's reporting strategy at $t$ is a mapping from previous reports and information to a message.

We refer to a strategy that maximizes the agent's payoff as an *equilibrium strategy* and the payoff generated by such a strategy as *equilibrium payoff*. The standard Revelation Principle applies in this setting, so it is without loss of generality to assume that $Z_t = \mathcal{E}_t$ for all $t$, and to restrict attention to mechanisms where telling the truth and taking the recommended action (obedience) is an equilibrium strategy. A direct mechanism is defined by a triple $\left(\mathbf{x}^T, \mathbf{a}^T, \mathbf{p}\right)$, where $\mathbf{x}_t : \mathcal{E}^t \times S^t \to X_t$, $\mathbf{a}_t : \mathcal{E}^t \times S^{t-1} \to A_t$ and $\mathbf{p} : \mathcal{E}^T \times S^T \to \mathbb{R}$. Direct mechanisms in which telling the truth and obeying the principal's recommendation is an equilibrium strategy are called *incentive compatible* mechanisms.

We call a decision-action rule $\left(\mathbf{x}^T, \mathbf{a}^T\right)$ *implementable* if there exists a payment rule, $\mathbf{p} : \mathcal{E}^T \to \mathbb{R}$ such that the direct mechanism $\left(\mathbf{x}^T, \mathbf{a}^T, \mathbf{p}\right)$ is incentive compatible.

*Technical Assumptions.*— We make three technical assumptions to ensure that the equilibrium payoff function of the agent is Lipshitz continuous in the orthogonalized model.

**Assumption 0**.

(i) There exists a $K \in \mathbb{N}$ such that for all $t = 1, ..., T$ and for all $\theta^T, a^T, s^T, x^T$,

$$\widetilde{u}_{\theta_t}\left(\theta^T, a^T, s^T, x^T\right), \widetilde{u}_{a_t}\left(\theta^T, a^T, s^T, x^T\right) < K.$$

(ii) There exists a $K \in \mathbb{N}$ such that for all $t = 1, ..., T$, $\tau < t$, and for all $\theta^t, a^{t-1}, x^{t-1}$,

$$G_{t\theta_t}\left(\theta_t|\theta^{t-1}, a^{t-1}, x^{t-1}\right), \left|G_{t\theta_\tau}\left(\theta_t|\theta^{t-1}, a^{t-1}, x^{t-1}\right)\right| < K.$$

[5] To see this, note that since $\varepsilon_t = G_t\left(\theta_t|\theta^{t-1}, a^{t-1}, x^{t-1}\right)$, the probability that $\varepsilon_t \leq \bar{\varepsilon}$ is $\Pr\left(G_t\left(\theta_t|\theta^{t-1}, a^{t-1}, x^{t-1}\right) \leq \bar{\varepsilon}\right) = \Pr\left(\theta_t \leq G_t^{-1}\left(\bar{\varepsilon}|\theta^{t-1}, a^{t-1}, x^{t-1}\right)\right) = G_t\left(G_t^{-1}\left(\bar{\varepsilon}|\theta^{t-1}, a^{t-1}, x^{t-1}\right)\right) = \bar{\varepsilon}$.

(iii) There exists a $K \in \mathbb{N}$ such that for all $t = 1, ..., T$ and for all $\theta^t, a^t$,

$$\left| \frac{f_{t\theta_t}(\theta_t, a_t)}{f_{ta_t}(\theta_t, a_t)} \right| < K.$$

# 3 The main result

We refer to the model in which the principal never observes the agent's types as the *original model*, whereas we call the model where $\varepsilon_1, ..., \varepsilon_T$ are observed by the principal the *benchmark case*. The contracting problem in the benchmark is a static one in the sense that the principal only interacts with the agent at $t = 0$, and the agent has no access to dynamic deviation reporting strategies. Our *dynamic irrelevance result* is that in any mechanism that implements a given decision-action rule in the original model the principal's maximal expected revenue is the same as it would be the benchmark case.

Specifically, what we show below is that the expected transfer payment of an agent with a given initial type when the principal implements decision-action rule $(\mathbf{x}^T, \mathbf{a}^T)$ in the original problem is the same (up to a type-invariant constant) as it would be in the benchmark. This implies that the principal's maximal expected revenue (and his payoff, in case it is linear in the revenue) when implementing a decision-action rule in the original problem is just as high as it would be in the benchmark. This does not imply, however, that the two problems are equivalent: sufficient conditions of implementability (of a decision rule) are stronger in the original problem than they are in the benchmark. We will turn to the question of implementability in Section 4.

In the next subsection we consider a decision-action rule $(\mathbf{x}^T, \mathbf{a}^T)$ and derive a necesarry condition for the payment rule $\mathbf{p}$ such that $(\mathbf{x}^T, \mathbf{a}^T, \mathbf{p})$ is incentive compatible. This condition turns out to be the same in the benchmark case and in the original model. We then use this condition to prove our main, irrelevance result.

## 3.1 Payment rules

We fix an incentive compatible mechanism $(\mathbf{x}^T, \mathbf{a}^T, \mathbf{p})$ and analyze the consequences of time-0 incentive compatibility on the payment rule, $\mathbf{p}$, in both the original model and the benchmark.

We consider a particular set of deviation strategies and explore the consequence of the non-profitability of these deviations in each case. To this end, let us define this set as follows: If the agent with initial type $\varepsilon_0$ reports $\widehat{\varepsilon}_0$ then (i) she must report $\varepsilon_1, ..., \varepsilon_T$ truthfully, and (ii) for all $t = 0, \ldots, T$, after history $(\varepsilon^t, s^{t-1})$, she must take action $\widehat{\mathbf{a}}_t(\varepsilon^t, \widehat{\varepsilon}_0, s^{t-1})$ such that the distribution of $s_t$ is the same as if the history were $(\widehat{\varepsilon}_0, \varepsilon^t_{-0}, s^{t-1})$ and action $\mathbf{a}_t(\widehat{\varepsilon}_0, \varepsilon^t_{-0}, s^{t-1})$ were taken, where $\varepsilon^t_{-0} = (\varepsilon_1, \ldots, \varepsilon_t)$. Since the distribution of $s_t$ only depends on $f_t(\theta_t, a_t)$, the action $\widehat{\mathbf{a}}_t(\varepsilon^t, \widehat{\varepsilon}_0, s^{t-1})$ is defined by

$$f_t\left(\widehat{\theta}_t, \mathbf{a}_t(\widehat{\varepsilon}_0, \varepsilon_{-0}^t, s^{t-1})\right) = f_t\left(\theta_t, \widehat{\mathbf{a}}_t(\varepsilon^t, \widehat{\varepsilon}_0, s^{t-1})\right), \tag{2}$$

where

$$\begin{aligned}
\widehat{\theta}_t &= \psi_t\left(\widehat{\varepsilon}_0, \varepsilon_{-0}^t, \mathbf{a}^{t-1}\left(\widehat{\varepsilon}_0, \varepsilon_{-0}^{t-1}, s^{t-2}\right), \mathbf{x}^{t-1}\left(\widehat{\varepsilon}_0, \varepsilon_{-0}^{t-1}, s^{t-1}\right)\right), \\
\theta_t &= \psi_t\left(\varepsilon^t, \widehat{\mathbf{a}}^{t-1}\left(\varepsilon^{t-1}, \widehat{\varepsilon}_0, s^{t-2}\right), \mathbf{x}^{t-1}\left(\widehat{\varepsilon}_0, \varepsilon_{-0}^{t-1}, s^{t-1}\right)\right).
\end{aligned}$$

In other words, the deviation strategies we consider require the agent (i) to be truthful in the future about her orthogonalized types, and (ii) to take actions that "mask" her earlier lie so that the principal could not detect her initial deviation based on the public signals, even in a statistical sense.[6] Note that in the benchmark case we only need to impose restriction (ii) since the principal observes $\varepsilon_1, ..., \varepsilon_T$ by assumption. Also note that the strategies satisfying restrictions (i) and (ii) include the equilibrium strategy in the original model because if $\varepsilon_0 = \widehat{\varepsilon}_0$ the two restrictions imply truth-telling and obedience (adherence to the action rule).

We emphasize that we do not claim by any means that after reporting $\widehat{\varepsilon}_0$ it is *optimal* for the agent to follow a continuation strategy defined by restrictions (i) and (ii). Nevertheless, since the mechanism $\left(\mathbf{x}^T, \mathbf{a}^T, \mathbf{p}\right)$ is incentive compatible, none of these deviations are profitable for the agent. We show that this observation enables us to characterize the expected payment of the agent conditional on $\varepsilon_0$ up to a type-invariant constant.

Let $\Pi_0\left(\varepsilon_0\right)$ denote the agent's expected equilibrium payoff conditional on her initial type $\varepsilon_0$ in the incentive compatible mechanism $\left(\mathbf{x}^T, \mathbf{a}^T, \mathbf{p}\right)$. That is,

$$\Pi_0\left(\varepsilon_0\right) = E\left[u\left(\varepsilon^T, \mathbf{a}^T\left(\varepsilon^T, s^{T-1}\right), s^T, \mathbf{x}^T\left(\varepsilon^T, s^T\right)\right) - \mathbf{p}\left(\varepsilon^T\right) \,\middle|\, \varepsilon_0\right], \tag{3}$$

where $E$ denotes expectation over $\varepsilon^T$ and $s^T$.

**Proposition 1** *If the mechanism* $\left(\mathbf{x}^T, \mathbf{a}^T, \mathbf{p}\right)$ *is incentive compatible either in the original model or in the benchmark case, then for all* $\varepsilon_0 \in \mathcal{E}_0$ *:*

$$\begin{aligned}
\Pi_0(\varepsilon_0) = \Pi_0\left(0\right) &+ E\left[\int_0^{\varepsilon_0} u_{\varepsilon_0}\left(y, \varepsilon_{-0}^T, \mathbf{a}^T\left(y, \varepsilon_{-0}^T, s^{T-1}\right), s^T, \mathbf{x}^T\left(y, \varepsilon_{-0}^T, s^T\right)\right) dy \middle| \varepsilon_0\right] \qquad (4) \\
&+ E\left[\int_0^{\varepsilon_0} \sum_{t=0}^{T} u_{a_t}\left(y, \varepsilon_{-0}^T, \mathbf{a}^T\left(y, \varepsilon_{-0}^T, s^{T-1}\right), s^T, \mathbf{x}^T\left(y, \varepsilon_{-0}^T, s^T\right)\right) \widehat{\mathbf{a}}_{t\varepsilon_0}\left(y, \varepsilon_{-0}^t, y, s^{t-1}\right) dy \middle| \varepsilon_0\right],
\end{aligned}$$

[6]Similar ideas are used by Pavan, Segal and Toikka (2012) in a dynamic contracting model without moral hazard and by Garrett and Pavan (2012) in a more restrictive environment with moral hazard.

*where* $\left(y, \varepsilon_{-0}^{t}\right) = (y, \varepsilon_1, ..., \varepsilon_t)$.

Proposition 1 establishes that in an incentive compatible mechanism that implements a particular decision-action rule the expected payoff of the agent with a given (initial) type does not depend on the transfers. Analogous to the necessity part of the Spence–Mirrlees Lemma in static mechanism design (or Myerson's Revenue Equivalence Theorem), necessary conditions similar to (4) have been derived in dynamic environments by Baron and Besanko (1984), Courty and Li (2000), Eso and Szentes (2007a), Pavan, Segal and Toikka (2012), Garrett and Pavan (2012), and others. In our environment, which is not only dynamic but incorporates both adverse selection and moral hazard as well, the real significance of the result is that *the same formula applies* in the original problem and in the benchmark case.

It may be instructive to consider the special case where the principal has no access to a public signal, or equivalently, the distribution of $s_t$ is independent of $(\theta_t, a_t)$. Since the choice of $a_t$ has no impact on $\mathbf{x}^T$ and $\mathbf{p}^T$, the agent chooses $a_t$ to maximize her utility. A necessary condition of this maximization is $E\left[u_{a_t}\left(\varepsilon^T, \mathbf{a}^T(\varepsilon^T, s^{T-1}), s^T, \mathbf{x}^T(\varepsilon^T, s^T)\right)|\varepsilon^t, s^{t-1}\right] = 0$ for all $t$. As a consequence the last term of $\Pi_0(\varepsilon_0)$, i.e. the second line of equation (4), vanishes.

**Proof of Proposition 1** First we express the agent's reporting problem at $t = 0$ in the benchmark case as well as in the original problem subject to restrictions (i) and (ii) discussed at the beginning of this subsection.

In order to do this define

$$U\left(\varepsilon_0, \widehat{\varepsilon}_0\right) = E\left[u\left(\varepsilon^T, \widehat{\mathbf{a}}^T\left(\varepsilon^T, \widehat{\varepsilon}_0, s^{T-1}\right), s^T, \mathbf{x}^T\left(\widehat{\varepsilon}_0, \varepsilon_{-0}^T, s^T\right)\right)|\varepsilon_0\right]$$

and

$$P\left(\widehat{\varepsilon}_0\right) = E\left[\mathbf{p}\left(\widehat{\varepsilon}_0, \varepsilon_{-0}^T, s^T\right)|\varepsilon_0, a^T = \widehat{\mathbf{a}}^T\left(\varepsilon^T, \widehat{\varepsilon}_0, s^{T-1}\right), x^T = \mathbf{x}^T\left(\widehat{\varepsilon}_0, \varepsilon_{-0}^T, s^T\right)\right], \qquad (5)$$

where $\widehat{\mathbf{a}}$ is defined by (2). Recall that the action $\widehat{\mathbf{a}}_t\left(\varepsilon^t, \widehat{\varepsilon}_0, s^{t-1}\right)$ generates the same distribution of $s_t$ as if the agent's true type history was $\left(\widehat{\varepsilon}_0, \varepsilon_{-0}^t\right)$ and the agent had taken $\mathbf{a}_t\left(\widehat{\varepsilon}_0, \varepsilon_{-0}^t, s^{t-1}\right)$. The significance of this is that

$$\begin{aligned} & E\left[\mathbf{p}\left(\widehat{\varepsilon}_0, \varepsilon_{-0}^T, s^T\right)|\varepsilon_0, a^T = \widehat{\mathbf{a}}^T\left(\varepsilon^T, \widehat{\varepsilon}_0, s^{T-1}\right), x^T = \mathbf{x}^T\left(\widehat{\varepsilon}_0, \varepsilon_{-0}^T, s^T\right)\right] \\ = \; & E\left[\mathbf{p}\left(\widehat{\varepsilon}_0, \varepsilon_{-0}^T, s^T\right)|\widehat{\varepsilon}_0, a^T = \mathbf{a}^T\left(\widehat{\varepsilon}_0, \varepsilon_{-0}^T, s^{T-1}\right), x^T = \mathbf{x}^T\left(\widehat{\varepsilon}_0, \varepsilon_{-0}^T, s^T\right)\right], \end{aligned}$$

so the right-hand side of (5) is indeed only a function of $\widehat{\varepsilon}_0$ but not that of $\varepsilon_0$.

In the benchmark case, the payoff of the agent with $\varepsilon_0$ who reports $\widehat{\varepsilon}_0$ and takes action $\widehat{\mathbf{a}}_t\left(\varepsilon^t, \widehat{\varepsilon}_0, s^{t-1}\right)$ at every $t$ is $W\left(\varepsilon_0, \widehat{\varepsilon}_0\right) = U\left(\varepsilon_0, \widehat{\varepsilon}_0\right) - P\left(\widehat{\varepsilon}_0\right)$. Note that $W\left(\varepsilon_0, \widehat{\varepsilon}_0\right)$ is also the payoff of the agent in the original model if her type is $\varepsilon_0$ at $t=0$, reports $\widehat{\varepsilon}_0$ and her continuation strategy is defined by restrictions (i) and (ii) above, that is, she reports truthfully afterterwards and takes action $\widehat{\mathbf{a}}_t\left(\varepsilon^t, \widehat{\varepsilon}_0, s^{t-1}\right)$ after the history $\left(\varepsilon^t, s^{t-1}\right)$.

The incentive compatibility of $\left(\mathbf{x}^T, \mathbf{a}^T, \mathbf{p}\right)$ implies that $\varepsilon_0 \in \arg\max_{\widehat{\varepsilon}_0 \in \mathcal{E}_0} W\left(\varepsilon_0, \widehat{\varepsilon}_0\right)$ both in the benchmark case and in the original model. In addition, $\Pi_0\left(\varepsilon_0\right) = W\left(\varepsilon_0, \varepsilon_0\right)$ and, by Lemma 6 of the Appendix, $\Pi_0$ is Lipshitz continuous. Therefore, Theorem 1 in Milgrom and Segal (2002) implies that

$$\frac{d\Pi_0\left(\varepsilon_0\right)}{d\varepsilon_0} = \left.\frac{\partial U\left(\varepsilon_0, \widehat{\varepsilon}_0\right)}{\partial \varepsilon_0}\right\rfloor_{\widehat{\varepsilon}_0=\varepsilon_0},$$

almost everywhere. Note that

$$\begin{aligned}
\left.\frac{\partial U\left(\varepsilon_0, \widehat{\varepsilon}_0\right)}{\partial \varepsilon_0}\right\rfloor_{\widehat{\varepsilon}_0=\varepsilon_0} &= E\left[\left. u_{\varepsilon_0}\left(\varepsilon^T, \mathbf{a}^T\left(\varepsilon^T, s^{T-1}\right), s^T, \mathbf{x}^T\left(\varepsilon^T, s^T\right)\right)\right| \varepsilon_0\right] \\
&\quad + E\left[\left.\sum_{t=0}^{T} u_{a_t}\left(\varepsilon^T, \mathbf{a}^T\left(\varepsilon^T, s^{T-1}\right), s^T, \mathbf{x}^T\left(\varepsilon^T, s^T\right)\right) \widehat{\mathbf{a}}_{t\varepsilon_0}\left(\varepsilon^t, \widehat{\varepsilon}_0, s^{t-1}\right)\right\rfloor_{\widehat{\varepsilon}_0=\varepsilon_0} \right| \varepsilon_0\right].
\end{aligned}$$

Since $\Pi_0$ is Lipshitz continuous, it can be recovered from its derivative, so the statement of the proposition follows. □

By Proposition 1, for a given decision-action rule, incentive compatibility constraints pin down the expected payments conditional on $\varepsilon_0$ uniquely up to a constant in both the benchmark case and the original model. To see this, note that from (3) and (4) the expected payment conditional on $\varepsilon_0$ can be expressed as

$$\begin{aligned}
E\left[\left.\mathbf{p}\left(\varepsilon^T, s^T\right)\right| \varepsilon_0, \mathbf{a}^T, \mathbf{x}^T\right] &= E\left[\left. u\left(\varepsilon^T, \mathbf{a}^T\left(\varepsilon^T, s^{T-1}\right), s^T, \mathbf{x}^T\left(\varepsilon^T\right)\right)\right| \varepsilon_0\right] - \Pi_0(0) \\
&\quad - E\left[\left.\int_0^{\varepsilon_0} u_{\varepsilon_0}\left(y, \varepsilon_{-0}^T, \mathbf{a}^T\left(y, \varepsilon_{-0}^T, s^{T-1}\right), s^T, \mathbf{x}^T\left(y, \varepsilon_{-0}^T, s^T\right)\right) dy \right| \varepsilon_0\right] \\
- E\left[\left.\int_0^{\varepsilon_0} \sum_{t=0}^{T} u_{a_t}\left(y, \varepsilon_{-0}^T, \mathbf{a}^T\left(y, \varepsilon_{-0}^T, s^{T-1}\right), s^T, \mathbf{x}^T\left(y, \varepsilon_{-0}^T, s^T\right)\right) \widehat{\mathbf{a}}_{t\varepsilon_0}\left(y, \varepsilon_{-0}^t, y, s^{t-1}\right) \right| \varepsilon_0\right]&.
\end{aligned}$$

An immediate consequence of this observation and Proposition 1 is the following

**Remark 1** *Suppose that $\left(\mathbf{x}^T, \mathbf{a}^T, \mathbf{p}\right)$ and $\left(\mathbf{x}^T, \mathbf{a}^T, \overline{\mathbf{p}}\right)$ are incentive compatible mechanisms in the original and in the benchmark case, respectively. Then*

$$E\left[\left.\mathbf{p}\left(\varepsilon^T, s^T\right)\right| \varepsilon_0, \mathbf{a}^T, \mathbf{x}^T\right] - E\left[\left.\overline{\mathbf{p}}\left(\varepsilon^T, s^T\right)\right| \varepsilon_0, \mathbf{a}^T, \mathbf{x}^T\right] = c,$$

*where $c \in \mathbb{R}$.*

### 3.2 Irrelevance of dynamic adverse selection

Now we show that the principal's maximal revenue of implementing a decision rule is the same *as if* he were able to observe the orthogonalized types of the agent after $t = 0$. That is, whenever a decision rule is implementable, the agent only receives information rent for her initial private information. This is our irrelevance result.

To state this result formally, suppose that the agent has an outside option, which we normalize to be zero. This means that any mechanism must satisfy

$$\Pi_0(\varepsilon_0) \geq 0 \tag{6}$$

for all $\varepsilon_0 \in \mathcal{E}_0$. We call the maximum (supremum) of the expected payment of the agent across all the mechanisms that implement $\left(\mathbf{x}^T, \mathbf{a}^T\right)$ and satisfy (6) the principal's maximal revenue from implementing this rule.[7]

**Theorem 1** *Suppose that the decision rule* $\left(\mathbf{x}^T, \mathbf{a}^T\right)$ *is implementable in the original model. Then the principal's maximal revenue from implementing this rule is the same as in the benchmark case.*

**Proof.** Consider first the benchmark case where the principal observes $\varepsilon_1, ..., \varepsilon_T$ and let $\overline{\mathbf{p}}$ denote the payment rule in a revenue-maximizing mechanism. Then the revenue of implementing $\left(\mathbf{x}^T, \mathbf{a}^T\right)$ is just $E\left[\overline{\mathbf{p}}\left(\varepsilon^T, s^T\right)\middle|\,\mathbf{a}^T, \mathbf{x}^T\right]$.

Of course, the principal's revenue in the benchmark case is an upper bound on his revenue in the original model. Therefore, it is enough to show that the principal can achieve $E\left[\overline{\mathbf{p}}\left(\varepsilon^T, s^T\right)\middle|\,\mathbf{a}^T, \mathbf{x}^T\right]$ from implementing $\mathbf{x}^T$ even if the he does not observe $\varepsilon_1, ..., \varepsilon_T$. Suppose that the direct mechanism $\left(\mathbf{x}^T, \mathbf{a}^T, \widehat{\mathbf{p}}\right)$ is incentive compatible. Then, by Remark 1,

$$E\left[\widehat{\mathbf{p}}\left(\varepsilon^T, s^T\right)\middle|\,\varepsilon_0, \mathbf{a}^T, \mathbf{x}^T\right] = E\left[\overline{\mathbf{p}}\left(\varepsilon^T, s^T\right)\middle|\,\varepsilon_0, \mathbf{a}^T, \mathbf{x}^T\right] + c$$

for some $c \in \mathbb{R}$. Define $\mathbf{p}\left(\varepsilon^T, s^T\right)$ to be $\widehat{\mathbf{p}}\left(\varepsilon^T, s^T\right) - c$. Since adding a constant has no effect on incentives, the mechanism $\left(\mathbf{x}^T, \mathbf{a}^T, \mathbf{p}\right)$ is incentive compatible. In addition, $E\left[\mathbf{p}\left(\varepsilon^T, s^T\right)\middle|\,\varepsilon_0, \mathbf{a}^T, \mathbf{x}^T\right] = E\left[\overline{\mathbf{p}}\left(\varepsilon^T, s^T\right)\middle|\,\varepsilon_0, \mathbf{a}^T, \mathbf{x}^T\right]$, that is, the principal's revenue is the same as in the benchmark case.

Finally, notice that the participation constraint of the agent, (6), is also satisfied because

[7] Requiring (6) for all $\varepsilon_0$ implies that we restrict attention to mechanisms where the agent participates irrespective of her type. This is without the loss of generality in many applications where there is a decision which generates a utility of zero for both the principal and the agent. Alternatively, we could have stated our theorem for problems where the participating types in the optimal contract of the benchmark case is an interval.

the agent's expected payoff conditional on her initial type, $\varepsilon_0$, is the same as that in the benchmark case. □

The statement of Theorem 1 is about the revenue of the principal. Note that if the payoff of the principal is also quasi-linear (affine in the payment), then the decision rule and the expected payment fully determines his payoff. Hence, a consequence of Theorem 1 is,

**Remark 2** *Suppose that the decision-action rule $\left(\mathbf{x}^T, \mathbf{a}^T\right)$ is implementable in the original model and the principal's payoff is affine in the payment. Then the principal's maximum (supremum) payoff from implementing $\left(\mathbf{x}^T, \mathbf{a}^T\right)$ is the same as in the benchmark case.*

It is important to point out that our dynamic irrelevance result does not imply that the original problem (unobservable $\varepsilon_1, \ldots, \varepsilon_T$) and the benchmark case (observable $\varepsilon_1, \ldots, \varepsilon_T$) are equivalent. Theorem 1 only states that *if* an decision-action rule is implementable in the original model, *then* it can be done so without revenue loss as compared to the benchmark case. This result was obtained under very mild conditions regarding the stochastic process governing the agent's type, her payoff function and the structure of the public signals. The obvious, next question is what type of decision-action rules can be implemented (under what conditions) in the original problem. We find some answers to this question in the next section.

## 4 Implementation

This section establishes results regarding the implementability of certain decision rules.[8] We restrict attention to a Markov environment with time-separable, regular (monotonic and single-crossing) payoff functions, formally stated in Assumptions 1 and 2 below.

First, we show that in the pure adverse selection model (where there are neither unobservable actions nor public signals) any monotonic decision rule is implementable. Then we turn our attention to the general model with moral hazard. There the set of implementable decision rules depends on the information content of the public signal. If the public signal has no informational content, that is, the distribution of $s_t$ is independent of $f_t\left(\theta_t, a_t\right)$, then naturally the agent cannot be given incentives to choose any action other than the one that maximizes her flow utility in each period. In this case, we show that any decision-action rule can be implemented if $\mathbf{x}^T$ is monotonic and $\mathbf{a}^T$ is determined by the agent's per period maximization problem.

The most interesting (and permissive) implementation result is obtained in the general model with adverse selection and moral hazard in case the signal is informative and its dis-

[8] Throughout this section we require a type-invariant participation constraint for the agent with her outside option normalized to zero payoff, that is, (6) to hold.

tribution satisfies a genericity condition due to McAfee and Reny (1992). This condition requires that the distribution of $s_t$ conditional on any given $y_t = f_t(\theta_t, a_t)$ is not the average of signal distributions conditional on other $\widehat{y}_t \neq y_t$'s such that $\widehat{y}_t = f_t(\widehat{\theta}_t, \widehat{a}_t)$. In this case, we show that *any* monotonic decision-action rule $\left(\mathbf{x}^T, \mathbf{a}^T\right)$ can be *approximately* implemented (to be formally defined below).[9] The result is based on arguments similar to the Full Surplus Extraction Theorem of McAfee and Reny (1992) and exploits the property of the model that $f_t$ is approximately contractible and the agent is risk neutral with respect to monetary transfers. The main result of this section is that in our general model, in a regular Markovian environment with transferable utility and generic public signals, the principal is able to implement any monotonic decision-action rule while not incurring any agency cost apart from the information rent due to the agent's initial private information.

In order to state the regularity assumptions made throughout the section, we return to the model without orthogonalization. Throughout this section, we assume that the public signal does not directly affect the agent's payoff directly and we remove $s^T$ from the arguments of $\widetilde{u}$, that is, $\widetilde{u} : \Theta^T \times A^T \times X^T \to \mathbb{R}$. We make two sets of assumptions regarding the environment. The first set concerns the type distribution, the second one the agent's payoff function.

**Assumption 1.** (Type Distribution)

(i) For all $t \in \{0, ..., T\}$, the random variable $\theta_t$ is distributed according to a continuous c.d.f. $G_t\left(\cdot|\theta_{t-1}\right)$ supported on an interval $\Theta_t = \left[\underline{\theta}_t, \overline{\theta}_t\right]$.

(ii) For all $t \in \{1, ..., T\}$, $G_t\left(\cdot|\theta_{t-1}\right) \geq G_t(\cdot|\widehat{\theta}_{t-1})$ whenever $\theta_{t-1} \leq \widehat{\theta}_{t-1}$.

Part (i) of Assumption 1 states that the agent's type follows a Markov process, that is, the type distribution at time $t$ only depends on the type at $t-1$. In addition, the support of $\theta_t$ only depends on $t$, so any type on $\Theta_t$ can be realized irrespective of $\theta_{t-1}$. Part (ii) states that the type distributions at time $t$ are ordered according to first-order stocahastic dominance. The larger the agent's type at time $t-1$, the more likely it is to be large at time $t$.

**Assumption 2.** (Payoff Function)

(i) There exist $\{\widetilde{u}_t\}_{t=0}^T$, $\widetilde{u}_t : \Theta_t \times A_t \times X^t \to \mathbb{R}$ continuously differentiable, such that

$$\widetilde{u}\left(\theta^T, a^T, x^T\right) = \sum_{t=0}^{T} \widetilde{u}_t\left(\theta_t, a_t, x^t\right).$$

(ii) For all $t \in \{0, ..., T\}$, $\widetilde{u}_t$ is strictly increasing in $\theta_t$.

(iii) For all $t \in \{0, ..., T\}$, $\theta_t \in \Theta_t, a_t \in A_t$: $\widetilde{u}_{t\theta_t}\left(\theta_t, a_t, x^t\right) \geq \widetilde{u}_{t\theta_t}\left(\theta_t, a_t, \widehat{x}^t\right)$ whenever $x^t \geq \widehat{x}^t$.

[9] The approximation can be dispensed with if the public signal is the summary statistic $f_t(\theta_t, a_t)$ itself.

Part (i) of Assumption 2 says that the agent's utility is additively separable over time, such that her flow utility at time $t$ only depends on $\theta_t$ and $a_t$ (and not on any prior information and action) besides all decisions taken at or before $t$. Part (ii) requires the flow utility to be monotonic in the agent's type. Part (iii) is the standard single-crossing property for the agent's type and the contractible decision.

We refer to the model as the one with *pure adverse selection* if $\widetilde{u}_{ta_t} \equiv 0$ for all $t$ and the distribution of $s_t$ is independent of $f_t$. Next, we state our implementation result for this case (Proposition 2). Then, in Sections 4.1 and 4.2 we return to the general model with moral hazard. In both scenarios regarding the informational content of the public signal discussed above we reduce the problem of implementation to that in an appropriately-defined pure adverse selection problem.

**Proposition 2** *Suppose that Assumptions 0,1 and 2 hold in a pure adverse selection model. Then a decision rule,* $\widetilde{\mathbf{x}}^T$, $\widetilde{\mathbf{x}}_t : \Theta^t \to X_t$, *is implementable if* $\widetilde{\mathbf{x}}_t$ *is increasing for all* $t$.

By Corollary 2 of Pavan, Segal and Toikka (2012), Assumptions 1-2 imply their integral monotonicity condition; slight differences in their and our technical assumptions notwithstanding, our Proposition 2 appears to be an implication of their Theorem 2. We present a proof of this result in Section 4.3 relying on the techniques used in Eso and Szentes (2007a).

### 4.1 Uninformative public signal

Suppose that the public signal is uninformative (i.e. $s_t$ is independent of $f_t$). We maintain the assumption that the payoff function of the agent is time-separable and satisfies Assumption 2, but now the flow utility at time $t$ is allowed to vary with $a_t$.

Recall that the action space of the agent at time $t$, $A_t$, was assumed to be an open interval of $\mathbb{R}$ in Section 2. We needed this assumption because we posited that for all $\theta_t, \widehat{\theta}_t$ and $a_t$ there is a unique $\widehat{a}_t$ such that $f_t(\theta_t, a_t) = f_t(\widehat{\theta}_t, \widehat{a}_t)$.[10] Since there is no public signal in the case considered here, we can relax the requirement that $A_t$ is open. In fact, in order to discuss the implementability of allocation rules which may involve boundary actions, we assume that $A_t = [\underline{a}_t, \overline{a}_t]$ is a compact interval throughout this subsection.

**Assumption 3.** For all $t \in \{0, ..., T\}$, for all $\theta_t \in \Theta_t, a_t, \widehat{a}_t \in A_t, x^t, \widehat{x}^t \in X^t$

(i) $\widetilde{u}_{ta_t^2}\left(\theta_t, a_t, x^t\right) \leq 0$,

(ii) $\widetilde{u}_{t\theta_t}\left(\theta_t, a_t, x^t\right) \geq \widetilde{u}_{t\theta_t}\left(\theta_t, \widehat{a}_t, \widehat{x}^t\right)$ whenever $a_t \geq \widehat{a}_t$, and

(iii) $\widetilde{u}_{ta_t}\left(\theta_t, a_t, x^t\right) \geq \widetilde{u}_{ta_t}\left(\theta_t, a_t, \widehat{x}^t\right)$ whenever $x^t \geq \widehat{x}^t$.

[10] If $A_t$ was compact then there would be a pair, $(\theta_t', a_t')$, which maximizes $f_t$. Therefore, if $\widehat{\theta}_t \notin \arg\max_{\theta_t}\left[\max_{a_t} f_t\left(\theta_t, a_t\right)\right]$, then there was no $\widehat{a}_t$ such that $f_t(\theta_t, a_t) = f_t(\widehat{\theta}_t, \widehat{a}_t)$.

Part (i) of the assumption states that the agent's payoff is a concave function of her action. This is satisfied in applications where the action of the agent is interpreted as an effort, and the cost of exerting effort is a convex function of the effort. Part (ii) states that the single-crossing assumption is also satisfied for the action. In the previous application, this means that the marginal cost of effort is decreasing in the agent's type. Part (iii) requires the single-crossing property to hold with respect to actions and decisions.

In what follows, we turn the problem of implementation in this environment with adverse selection and moral hazard into one of pure adverse selection. Since there is no publicly available information about the agent's action, her action maximizes her payoff in each period and after each history. That is, if the agent has type $\theta_t$ and the history of decisions is $x^t$, then she takes an action which maximizes $\widetilde{u}_t\left(\theta_t, a_t, x^t\right)$. Motivated by this observation, let us define the agent's new flow utility function at time $t$, $v_t : \Theta_t \times X^t \to \mathbb{R}$, to be

$$v_t\left(\theta_t, x^t\right) = \max_{a_t} \widetilde{u}_t\left(\theta_t, a_t, x^t\right).$$

We will apply our implementation result for the pure adverse selection case (Proposition 2) to the setting where the flow utilities of the agent are $\{v_t\}_{t=0}^T$ while keeping in mind that the action of the agent in each period $t$ maximizes $\widetilde{u}_t$.

To this end, let $\overline{\mathbf{a}}_t\left(\theta_t, x^t\right)$ denote the generically unique $\arg\max_{a_t} \widetilde{u}_t\left(\theta_t, a_t, x^t\right)$ for all $\theta_t \in \Theta_t$ and $x^t \in X^t$. By part (i) of Assumption 3, if $\overline{\mathbf{a}}_t\left(\theta_t, x^t\right)$ is interior, it is defined by the first-order condition

$$\widetilde{u}_{ta_t}\left(\theta_t, \overline{\mathbf{a}}_t(\theta_t, x^t), x^t\right) = 0. \tag{7}$$

The next lemma states that the flow utilities, $\{v_t\}_0^T$, satisfy the hypothesis of Proposition 2.

**Lemma 1** *Suppose that the functions $\{\widetilde{u}_t\}_{t=0}^T$ satisfy Assumptions 2 and 3. Then the functions $\{v_t\}_{t=0}^T$ satisfy Assumption 2.*

Suppose that the decision-action rule $\left(\mathbf{x}^T, \mathbf{a}^T\right)$ is implementable. Then, since the agent's action maximizes her payoff in each period, $\mathbf{a}_t\left(\theta^t\right) = \overline{\mathbf{a}}_t\left(\theta_t, \mathbf{x}^t\left(\theta^t\right)\right)$. In addition, the decision rule $\mathbf{x}^T$ must be implementable in the pure adverse selection model, where the agent's flow utility functions are $\{v_t\}_{t=1}^T$. Hence, the following result is a consequence of Proposition 2 and Lemma 1.

**Proposition 3** *Suppose that Assumptions 0-3 hold. Then a decision rule, $\left(\widetilde{\mathbf{x}}^T, \widetilde{\mathbf{a}}^T\right)$, $\widetilde{\mathbf{x}}_t : \Theta^t \to X_t$ and $\widetilde{\mathbf{a}}_t : \Theta^t \to A_t$, is implementable if $\widetilde{\mathbf{x}}_t$ is increasing and $\widetilde{\mathbf{a}}_t\left(\theta^t\right) = \overline{\mathbf{a}}_t\left(\theta_t, \widetilde{\mathbf{x}}^t\left(\theta^t\right)\right)$ for all $t \in \{0, ..., T\}$.*

Of course, the statement of this proposition is valid even if the public signal is informative ($s_t$ depends on $f_t$) but the principal ignores it and designs a mechanism which does not condition on $s^T$. However, if $s_t$ is informative about $f_t(\theta_t, a_t)$ the principal can implement more decision rules which is the subject of the next subsection.

### 4.2 Informative public signal

We turn our attention to the case where public signal is informative. The next condition is due to McAfee and Reny (1992); it requires that the distribution of the public signal conditional of any given value of $y_0 = f_t(\theta_t, a_t)$ is *not* the average of the distribution of $s_t$ conditional on other values of $f_t$. This condition is generic.

**Assumption 4.** Suppose that for all $\theta_t \in \Theta_t$ and $a_t \in A_t$, $f_t(\theta_t, a_t) \in Y_t = [\underline{y}_t, \overline{y}_t]$. Then, for all $\mu \in \Delta[\underline{y}_t, \overline{y}_t]$ and $y_0 \in [\underline{y}_t, \overline{y}_t]$, $\mu(\{y_0\}) \neq 1$ implies $h(\cdot|y_0) \neq \int_0^1 h(\cdot|y)\,\mu(dy)$.

Next, we make further assumptions on the agent's flow utility, $\widetilde{u}_t$, and on the shape of the function $f_t$.

**Assumption 5.** For all $t \in \{0, ..., T\}$, for all $\theta_t \in \Theta_t, a_t \in A_t, x^t \in X^t$

(i) $\widetilde{u}_{ta_t}(\theta_t, a_t, x^t) < 0$,

(ii) there exists a $K \in \mathbb{N}$ such that $f_{ta_t}(\theta_t, a_t), f_{t\theta_t}(\theta_t, a_t) > 1/K$,

(iii) $f_{ta_t^2}(\theta_t, a_t) f_{t\theta_t}(\theta_t, a_t) \leq f_{ta_t}(\theta_t, a_t) f_{ta_t\theta_t}(\theta_t, a_t)$, and

(iv) $\widetilde{u}_{t\theta_t x_\tau}(\theta_t, a_t, x^t) f_{ta_t}(\theta_t, a_t) \geq \widetilde{u}_{ta_t x_\tau}(\theta_t, a_t, x^t) f_{t\theta_t}(\theta_t, a_t)$.

Part (i) requires the agent's flow utility to be decreasing in her action. This is satisfied in applications where, for example, the agent's unobservable action is a costly effort from which she does not benefit directly. Part (ii) says that the function $f_t$ is increasing in both the agent's action and type. In many applications, the distribution of the public signal can be ordered according to first-order stochastic dominance. In these applications, part (ii) implies that an increase in either the action or the type improves the distribution of $s_t$ in the sense of first-order stochastic dominance. Part (iii) is a substitution assumption regarding the agent's type and hidden action in the value of $f_t$. It means that an increase in $a_t$, holding the value of $f_t$ constant, weakly decreases the *marginal* impact of $a_t$ on $f_t$.[11] This assumption is satisfied, for example, if $f_t(\theta_t, a_t) = \theta_t + a_t$, but it is clearly more general. As will be explained later part (iv) is a stregthening of the single crossing property posited in part (iii) of Assumption 2. It requires the marginal utility in type to be increasing in the contractible decision while

[11] To see this interpretation, note that the total differential of $f_{ta_t}$ (the change in the marginal impact of $a_t$) is $f_{ta_t^2} da_t + f_{ta_t\theta_t} d\theta_t$. Keeping $f_t$ constant (moving along an "iso-value" curve) means $d\theta_t = (-f_{ta_t}/f_{t\theta_t})da_t$. Substituting this into the total differential of $f_{ta_t}$ yields $(f_{ta_t^2} - f_{ta_t\theta_t} f_{ta_t}/f_{t\theta_t})da_t$. This expression is non-positive for $da_t > 0$ if part (iii) is satisfied.

holding the value of $f_t$ fixed. This assumption is satisfied, for example, if the effort cost of the agent is additively separable in her flow utility.

The key observation is that due to Assumption 4, the value of $f_t$ becomes an approximately contractible object in the following sense. For each value of $f_t$, $y_t$, the principal can design a transfer scheme depending only on $s^T$ that punishes the agent for taking an action which results a value of $f_t$ which is different from $y_t$. Perhaps more importantly, the punishment can be arbitrarily large as a function of the distance between $y_t$ and the realized value of $f_t$. We use this observation to establish our implementation result in two steps. First, we treat $f_t$ (for all $t$) as a contractible object, that is, we add another dimension to the contracible decisions in each period. Since, conditional on $\theta_t$, the value of $f_t$ is determined by $a_t$, we can express the agent's flow utility as a function of $f_t$ instead of $a_t$. These new flow utilities depend only on types and decisions, so we have a pure adverse selection model. We then show that the new flow utilities satisfy the requirements of Proposition 2 and hence, every monotonic rule is implementable. The second step is to construct punishment-transfers mentioned above and show that even if $f_t$ is not contractible, any monotonic decision rule can be approximately implementable.

For each $y_t \in \{f_t(\theta_t, a_t) : \theta_t \in \Theta_t, a_t \in A_t\}$ and $\theta_t \in \Theta_t$, let $\underline{\mathbf{a}}_t(\theta_t, y_t)$ denote the solution to $f_t(\theta_t, a_t) = y_t$ in $a_t$. For each $t = 0, ..., T$, we define the agent's flow utility as a function of $y_t$ as follows:

$$w_t\left(\theta_t, y_t, x^t\right) = \widetilde{u}_t\left(\theta_t, \underline{\mathbf{a}}_t\left(\theta_t, y_t\right), x^t\right).$$

Next, we show that the functions $\{w_t\}_{t=0}^T$ satisfy the hypothesis of Proposition 2.

**Lemma 2** *Suppose that Assumptions 2-5 are satisfied. Then the functions* $\{w_t\}_{t=0}^T$ *satisfy Assumption 2.*

By this lemma and Proposition 2 if the value of $f_t$ was contractible for all $t$, any increasing decision rule was implementable. However, $f_t$ is not contractible; nevertheless we can still implement increasing decisions rules *approximiately* in the sense that by following the principal's recommendation the agent's expected utility is arbitrarily close to her equilibrium payoff. Formally:

**Definition 1** *The decision rule* $\left(\widetilde{\mathbf{x}}^T, \widetilde{\mathbf{a}}^T\right)$ *is approximately implementable if for all* $\delta > 0$ *there exists a payment rule* $\widetilde{\mathbf{p}}$: $\mathbf{\Theta}^T \times S^T \to \mathbb{R}$ *such that for all* $\theta_0 \in \Theta_0$,

$$E_{s^T}\left[\sum_{t=0}^{T} \widetilde{u}_t\left(\theta_t, \widetilde{\mathbf{a}}_t\left(\theta^t\right), \widetilde{\mathbf{x}}^t\left(\theta^t\right)\right) - \widetilde{\mathbf{p}}\left(\theta^T, s^T\right) | \theta_0\right] \geq \Pi_0\left(\theta_0\right) - \delta, \tag{8}$$

*where* $\Pi_0(\theta_0)$ *denotes the agent's equilibrium payoff with initial type* $\theta_0$.

We are ready to state the implementation result of this subsection.

**Proposition 4** *Suppose that Assumptions 0-5 are satisfied. Then a decision rule,* $\left(\widetilde{\mathbf{x}}^T, \widetilde{\mathbf{a}}^T\right)$, $\widetilde{\mathbf{x}}_t : \Theta^t \to X_t$ *and* $\widetilde{\mathbf{a}}_t : \Theta^t \to A_t$*, is approximately implementable if* $\widetilde{\mathbf{x}}_t$ *and* $\widetilde{\mathbf{a}}_t$ *are increasing for all* $t \in \{0, ..., T\}$*.*

### 4.3 The proof of Proposition 2

Since in a pure adverse selection model $\widetilde{u}_{ta_t} \equiv 0$ for all $t$, throughout this section, we remove $a_t$ from the arguments of $\widetilde{u}_t$, that is, $\widetilde{u}_t : \Theta_t \times X^t \to \mathbb{R}$. We first inspect the consequences of Assumptions 1 and 2 on the orthogonalized model. Note that since $\theta_t$ does not depend on $x_{t-1}$, the inference functions defined in equation (1) do not depend on the decisions either, so $\psi_t : \mathcal{E}^t \to \Theta_t$. The time-separability of the agent's payoff (part (i) of Assumption 2) is preserved in the orthogonalized model, except that the flow utility at $t$, $u_t : \mathcal{E}^t \times X^t \to \mathbb{R}$, now depends on the *history of types* up to and including time $t$:

$$u_t\left(\varepsilon^t, x^t\right) = \widetilde{u}_t\left(\psi_t\left(\varepsilon^t\right), x^t\right). \tag{9}$$

Part (iii) of Assumption 1 implies that the larger the type history in the orthogonalized model up to time $t$, the larger is the corresponding period-$t$ type in the original model. This, coupled with part (ii) of Assumption 2 implies that $u_t$ is weakly increasing in $\varepsilon^{t-1}$ and strictly in $\varepsilon_t$. Monotonicy in $x^t$ as well as single-crossing (part (iii) of Assumption 2) are also preserved in the orthogonalized model. We state these properties formally in the following Lemma (see the proof in the Appendix).

**Lemma 3** *(i) For all* $t \in \{0, ..., T\}$ *and* $\widehat{\varepsilon}^t, \varepsilon^t \in \mathcal{E}^t$,

$$\widehat{\varepsilon}^t \leq \varepsilon^t \Rightarrow \psi_t\left(\widehat{\varepsilon}^t\right) \leq \psi_t\left(\varepsilon^t\right), \tag{10}$$

*and the inequality is strict whenever* $\widehat{\varepsilon}_t < \varepsilon_t$*.*

*(ii) The flow utility,* $u_t$ *defined by (9), is weakly increasing in* $\varepsilon^{t-1}$ *and* $x^{t-1}$*, and strictly increasing in* $x_t$ *and* $\varepsilon_t$*.*

*(iii) For all* $t \in \{0, ..., T\}$, $u_{t\varepsilon_t}\left(\varepsilon^t, x^t\right) \geq u_{t\varepsilon_t}\left(\varepsilon^t, \widehat{x}^t\right)$ *whenever* $x^t \geq \widehat{x}^t$*.*

Another important consequence of part (i) of Assumption 1 is that for all $\varepsilon^{t+1}$ and $\widehat{\varepsilon}_t$, there exists a type $\sigma_{t+1}(\varepsilon^{t+1}, \widehat{\varepsilon}_t) \in \mathcal{E}_t$ such that, fixing the principal's past and future decisions as well as the realizations of the agent's types beyond period $t+1$, the agent's utility flow from period $t+1$ on is the *same* with type history $\varepsilon^{t+1}$ as it is with $\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t, \sigma_{t+1}(\varepsilon^{t+1}, \widehat{\varepsilon}_t)\right)$. We will show below that $\sigma_{t+1}$, interpreted in Eso and Szentes (2007) as the agent's "correction of a lie," defines an optimal strategy for the agent at time $t+1$ after a deviation from truthtelling in an incentive compatible direct mechanism at $t$. This is formally stated in the following

**Lemma 4** *For all* $t \in \{0, \ldots, T-1\}$, $\varepsilon^{t+1} \in \mathcal{E}^{t+1}$ *and* $\widehat{\varepsilon}_t \in \mathcal{E}_t$, *there exists a unique* $\sigma_{t+1}\left(\varepsilon^{t+1}, \widehat{\varepsilon}_t\right) \in \mathcal{E}_{t+1}$ *such that for all* $k = t+1, \ldots, T$, *all* $\widehat{\varepsilon}^k \in \mathcal{E}^k$ *and* $\widehat{x}^k \in X^k$,

$$u_k(\varepsilon^{t-1}, \varepsilon_t, \varepsilon_{t+1}, \widehat{\varepsilon}_{t+2}, \ldots, \widehat{\varepsilon}_k, \widehat{x}^k) = u_k(\varepsilon^{t-1}, \widehat{\varepsilon}_t, \sigma_{t+1}, \widehat{\varepsilon}_{t+2}, \ldots, \widehat{\varepsilon}_k, \widehat{x}^k). \quad (11)$$

*The function* $\sigma_{t+1}$ *is increasing in* $\varepsilon_t$, *strictly increasing in* $\varepsilon_{t+1}$ *and decreasing in* $\widehat{\varepsilon}_t$.

The statement of the lemma might appear somewhat complicated at first glance, but its meaning and its intuitive proof are quite straightforward. Part (i) of Assumption 1 requires the support of $\theta_t$ to be independent of $\theta_{t-1}$. Therefore, if the type of the agent is $\psi_t\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t\right)$ at time $t$, there is a chance that the period-$(t+1)$ type will be $\psi^{t+1}\left(\varepsilon^{t+1}\right)$. The type $\sigma_{t+1}\left(\varepsilon^{t+1}, \widehat{\varepsilon}_t\right)$ denotes the orthogonalized information of the agent at $t+1$ which induces the transition from $\psi_t\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t\right)$ to $\psi_{t+1}\left(\varepsilon^{t+1}\right)$, that is,

$$\psi_{t+1}\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t, \sigma_{t+1}\left(\varepsilon^{t+1}, \widehat{\varepsilon}_t\right)\right) = \psi_{t+1}\left(\varepsilon^{t+1}\right).$$

This means that the inferred type in the original model is the same after the histories $\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t, \sigma_{t+1}\left(\varepsilon^{t+1}, \widehat{\varepsilon}_t\right)\right)$ and $\varepsilon^{t+1}$. Part (i) of Assumption 1 and part (ii) of Assumption 2 imply that, given the decisions, the flow utilities in the future only depend on current type which, in turn, imply (11).

The decision rule in the orthogonalized model, $\left\{x_t : \mathcal{E}^t \to X_t\right\}_{t=0}^T$, which corresponds to $\{\widetilde{x}_t\}_{t=0}^T$, is defined by $x_t\left(\varepsilon^t\right) = \widetilde{x}_t\left(\psi^t\left(\varepsilon^t\right)\right)$ for all $t$ and $\varepsilon^t$. Note that, by (10), if $\{\widetilde{x}_t\}_{t=0}^T$ is increasing in type ($\widetilde{x}_t$ is increasing in $\theta^t$ for all $t$) then the corresponding decision rule $\{x_t\}_{t=0}^T$ in the orthogonalized model is also increasing in type.[12]

In fact, the monotonicity of $\{\widetilde{x}_t\}_{t=0}^T$ implies a stronger monotonicity condition on $\{x_t\}_{t=0}^T$. Consider the following two type histories, $\varepsilon^k$ and $\left(\varepsilon_1, ..., \varepsilon_{t-1}, \widehat{\varepsilon}_t, \sigma_{t+1}\left(\varepsilon^{t+1}, \widehat{\varepsilon}_t\right), \varepsilon_{t+2}, ..., \varepsilon_k\right)$. Note that the inferred types in the original model are exactly the same along these histories except at time $t$. At time $t$, the inferred type is smaller after $\varepsilon^t$ if and only if $\varepsilon_t \leq \widehat{\varepsilon}_t$. Since $\widetilde{x}_k$ is increasing in $\theta_t$, the decision is smaller after $\varepsilon^k$ if and only if $\varepsilon_t \leq \widehat{\varepsilon}_t$. Formally,

**Remark 3** *If* $\{\widetilde{x}_t\}_{t=0}^T$ *is increasing then for all* $k = 1, ..., T$, $t < k$, $\varepsilon^k \in \mathcal{E}^k$ :

$$x^k(\varepsilon^k) \leq x^k\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t, \sigma_{t+1}\left(\varepsilon^{t+1}, \widehat{\varepsilon}_t\right), \varepsilon_{t+2}, ..., \varepsilon_k\right) \Leftrightarrow \widehat{\varepsilon}_t \geq \varepsilon_t. \quad (12)$$

[12] To see this, note that if $v^t \geq \widehat{v}^t$ then $x_t\left(\widehat{v}^t\right) = \widetilde{x}_t\left(\psi^t\left(\widehat{v}^t\right)\right) \leq \widetilde{x}_t\left(\psi^t\left(v^t\right)\right) = x_t\left(v^t\right)$, where the inequality follows from the monotonicity of $\{\widetilde{x}_t\}_0^T$ and (10).

To simplify the exposition, we introduce the following notation for $t = 0, \ldots, T$, $k \geq t$:

$$\zeta_t^k(\varepsilon^k, y) = \left(\varepsilon^{t-1}, y, \varepsilon_{t+1}, ..., \varepsilon_k\right),$$
$$\rho_t^k(\varepsilon^k, y, \widehat{\varepsilon}_t) = \left(\varepsilon^{t-1}, \widehat{\varepsilon}_t, \sigma_{t+1}\left(\varepsilon^{t-1}, y, \varepsilon_{t+1}, \widehat{\varepsilon}_t\right), \varepsilon_{t+2}, ..., \varepsilon_k\right).$$

The vectors $\zeta_t^k(\varepsilon^k, y)$ and $\rho_t^k(\varepsilon^k, y, \widehat{\varepsilon}_t)$ are type histories up to period $k$, true or reported, which are different from $\varepsilon^k$ only at $t$ or at $t$ and $t+1$. For $k = t$ these are appropriately truncated, e.g., $\rho_t^t(\varepsilon^t, y, \widehat{\varepsilon}_t) = \left(\varepsilon^{t-1}, \widehat{\varepsilon}_t\right)$.

As we explained, the monotonicity of $\{\widetilde{x}_t\}_{t=0}^T$ implies both the monotonicity of $\{x_t\}_{t=0}^T$ and (12). Therefore, in order to prove Proposition 2, it is sufficient to show that any increasing decision rule in the orthogonalized model which satisfies (12) can be implemented. In what follows, fix a direct mechanism with an increasing decision rule $\{x_t\}_{t=0}^T$ that satisfies (12). Let $\Pi_t(\varepsilon_t|\varepsilon^{t-1})$ denote a truthful agent's expected payoff at $t$ conditional on $\varepsilon^t$. That is,

$$\Pi_t(\varepsilon_t|\varepsilon^{t-1}) = E\left[\sum_{k=0}^{T} u_k\left(\varepsilon^k, x^k(\varepsilon^k)\right) - p(\varepsilon^T)\middle|\, \varepsilon^t\right]. \tag{13}$$

Define the payment function, $p$, such that for all $t = 0, ..., T$ and $\varepsilon^t \in \mathcal{E}^t$,

$$\Pi_t\left(\varepsilon_t|\varepsilon^{t-1}\right) = \Pi_t\left(0|\varepsilon^{t-1}\right) + E\left[\int_0^{\varepsilon_t} \sum_{k=t}^{T} u_{k\varepsilon_t}\left(\zeta_t^k(\varepsilon^k, y), x^k\left(\zeta_t^k(\varepsilon^k, y)\right)\right) dy\middle|\, \varepsilon^t\right]. \tag{14}$$

It is not hard to show that the integral on the right-hand side of (14) exists and is finite because of part (ii) of Assumption 1, part (i) of Assumption 2 and the monotonicity of $x^k$. It should be clear that it is possible to define $p$ such that (14) holds.

In this mechanism, let $\pi_t\left(\varepsilon_t, \widehat{\varepsilon}_t|\varepsilon^{t-1}\right)$ denote the expected payoff of the agent at time $t$ whose type history is $\varepsilon^t$ and has reported $\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t\right)$. This is the maximum payoff she can achieve from using any reporting strategy from $t+1$ conditional on the type history $\varepsilon^t$ and on the reports $\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t\right)$. If the mechanism is incentive compatible then, clearly, $\Pi_t(\varepsilon_t|\varepsilon^{t-1}) = \pi_t(\varepsilon_t, \varepsilon_t|\varepsilon^{t-1})$.

We call a mechanism *IC after time t* if, conditional on telling the truth before and at time $t-1$, it is an equilibrium strategy for the agent to tell the truth afterwards, that is, from period $t$ on. By Lemma 4, the continuation utilities of the agent with type $\varepsilon^{t+1}$ are the same as those of the agent with type $\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t, \sigma_{t+1}\left(\varepsilon^{t+1}, \widehat{\varepsilon}_t\right)\right)$ conditional on the reports and the realization of types after $t+1$. Therefore, if a mechanism is IC after $t+1$, the agent whose type history is $\varepsilon^{t+1}$ and reported $\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t\right)$ up to time $t$ maximizes her continuation payoff by reporting $\sigma_{t+1}\left(\varepsilon^{t+1}, \widehat{\varepsilon}_t\right)$ at time $t+1$ and reporting truthfully afterwards. If this were not the case then the agent with $\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t, \sigma_{t+1}\left(\varepsilon^{t+1}, \widehat{\varepsilon}_t\right)\right)$ would have a profitable deviation after truthful reports up to and including $t$, contradicting the assumption that the mechanism is IC after $t+1$. Therefore, in a mechanism that is IC after $t+1$, we have

$$\begin{aligned}\pi_t\left(\varepsilon_t, \widehat{\varepsilon}_t | \varepsilon^{t-1}\right) &= u_t\left(\varepsilon^t, x^t\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t\right)\right) - u_t\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t, x^t\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t\right)\right) \\ &\quad + \int \Pi_{t+1}\left(\sigma_{t+1}\left(\varepsilon^{t+1}, \widehat{\varepsilon}_t\right) | \varepsilon^{t-1}, \widehat{\varepsilon}_t\right) d\varepsilon_{t+1}.\end{aligned} \tag{15}$$

We use (15) in the following Lemma to characterize the agent's continuation payoff who deviates at $t$ in a mechanism that is IC after $t$.

**Lemma 5** *Suppose that the mechanism is IC after time* $t+1$ *and (14) is satisfied. Then, for all* $\varepsilon^t$ *and* $\widehat{\varepsilon}_t$*,*

$$\pi_t\left(\varepsilon_t, \widehat{\varepsilon}_t | \varepsilon^{t-1}\right) - \pi_t\left(\widehat{\varepsilon}_t, \widehat{\varepsilon}_t | \varepsilon^{t-1}\right) = \sum_{k=t}^{T} E\left[\int_{\widehat{\varepsilon}_t}^{\varepsilon_t} u_{k\varepsilon_t}\left(\zeta_t^k(\varepsilon^k, y), x^k\left(\rho_t^k(\varepsilon^k, y, \widehat{\varepsilon}_t)\right)\right) dy \middle| \varepsilon^t\right]. \tag{16}$$

This lemma is a direct generalization of Lemma 5 of Eso and Szentes (2007a); its proof can be found in the Appendix. We are now ready to prove Proposition 2.

**Proof of Proposition 2.** In order to prove that the transfers defined by (14) implement $\{x_t\}_{t=0}^T$, it is enough to prove that the mechanism is IC after all $t = 0, ..., T-1$. We prove this by induction. For $t = T-1$ this follows from the standard result in static mechanism design with the observation that $x_T$ is monotone and (14) is satisfied for $T$. Suppose now that the mechanism is IC after $t+1$. We show that the mechanism is IC after $t$, that is, the agent has no incentive to lie at $t$ if she has told the truth before $t$.

Consider an agent with type history $\varepsilon^t$ and report history $\varepsilon^{t-1}$ who is contemplating to report $\widehat{\varepsilon}_t < \varepsilon_t$. We have to show that $\pi_t\left(\varepsilon_t, \widehat{\varepsilon}_t | \varepsilon^{t-1}\right) - \pi_t\left(\varepsilon_t, \varepsilon_t | \varepsilon^{t-1}\right) \leq 0$ which can be written as

$$\pi_t\left(\varepsilon_t, \widehat{\varepsilon}_t | \varepsilon^{t-1}\right) - \pi_t\left(\widehat{\varepsilon}_t, \widehat{\varepsilon}_t | \varepsilon^{t-1}\right) + \pi_t\left(\widehat{\varepsilon}_t, \widehat{\varepsilon}_t | \varepsilon^{t-1}\right) - \pi_t\left(\varepsilon_t, \varepsilon_t | \varepsilon^{t-1}\right) \leq 0.$$

By (14) and (16), the previous inequality can be expressed as

$$\begin{aligned}&\sum_{k=t}^{T} E\left[\int_{\widehat{\varepsilon}_t}^{\varepsilon_t} u_{k\varepsilon_t}\left(\zeta_t^k(\varepsilon^k, y), x^k\left(\zeta_t^k(\varepsilon^k, y)\right)\right) dy \middle| \varepsilon^t\right] \\ \geq\ &\sum_{k=t}^{T} E\left[\int_{\widehat{\varepsilon}_t}^{\varepsilon_t} u_{k\varepsilon_t}\left(\zeta_t^k(\varepsilon^k, y), x^k\left(\rho_t^k(\varepsilon^k, y, \widehat{\varepsilon}_t)\right)\right) dy \middle| \varepsilon^t\right].\end{aligned} \tag{17}$$

In order to prove this inequality it is enough to show that the integrand on the left-hand side is larger than the integrand on the right-hand side. By part (iii) of Lemma 3, in order to show this, we only need that $x^k\left(\rho_t^k\left(\varepsilon^k, y, \widehat{\varepsilon}_t\right)\right) \leq x^k\left(\zeta_t^k(\varepsilon^k, y)\right)$ on $y \in [\widehat{\varepsilon}_t, \varepsilon_t]$, which follows from Remark 3. An identical argument can be used to rule out deviation to $\widehat{\varepsilon}_t > \varepsilon_t$.$\square$

From the proof of Proposition 2 it is clear that in the environment satisfying Assumptions 1 and 2 (i.e., with Markov types and a well-behaved agent payoff function), a decision rule $\{\tilde{x}_t\}_{t=0}^T$ is implemented by transfers satisfying (14) if, and only if, condition (17) holds in the orthogonalized model.[13] But (14) is also a necessary condition of implementation (differentiate it in $\varepsilon_t$ and compare that with (4) in Proposition 1), therefore condition (17) is indeed the necessary and sufficient condition of implementability of a decision rule in the regular, Markov environment. Formally, we state

**Remark 4** *Suppose that Assumptions 1 and 2 hold. Then a decision rule, $\{\widetilde{x}_t\}_0^T$, is implementable if, and only if, (17) holds in the model with orthogonalized information.*

*Implementability in the Benchmark Case.*— Suppose that the principal can observe $\varepsilon_1, ..., \varepsilon_T$. Then, using arguments in standard static mechanism design, a decision rule $\{x_t\}_0^T$ can be implemented if, and only if, for all $\widehat{\varepsilon}_0, \varepsilon_0 \in \mathcal{E}_0$, $\widehat{\varepsilon}_0 \leq \varepsilon_0$,

$$E\left[\sum_{k=0}^{T}\int_{\widehat{\varepsilon}_0}^{\varepsilon_0} u_{k\varepsilon_0}\left(y, \varepsilon_{-0}^k, x^k\left(y, \varepsilon_{-0}^k\right)\right) dy \middle| \varepsilon_0\right] \geq E\left[\sum_{k=0}^{T}\int_{\widehat{\varepsilon}_0}^{\varepsilon_0} u_{k\varepsilon_0}\left(y, \varepsilon_{-0}^k, x^k\left(\widehat{\varepsilon}_0, \varepsilon_{-0}^k\right)\right) dy \middle| \varepsilon_0\right].$$

This inequality is obviously a weaker condition than (17), so the principal can implement more allocations in the benchmark case.

# 5 Applications

We present three applications to illustrate how our techniques and results can be applied in substantive economic problems. In each application we first solve the benchmark case, where the the principal can observe the agent's orthogonalized future types. (In the absence of a contractible summary signal about the agent's type and hidden action the action rule is taken to be the agent-optimal one; in the presence of such a signal the action rule is also optimized.) Then we verify the appropriate monotonicity condition regarding the decision-action rule and conclude that the solution is implementable, hence optimal, in the original problem as well.

[13] Note that condition (17) is a joint restriction on $\{x_t\}_0^T$ and the marginal utility of the agent's type, and it is implied by the monotonicity of the decision rule in the environment of Assumptions 1-2.

In all three applications we assume that the agent's type follows the AR(1) process

$$\theta_t = \lambda\theta_{t-1} + (1-\lambda)\varepsilon_t,\ \forall t = 0,\dots,T,$$

where $\theta_{-1} = 0$ and $\varepsilon_0,\dots,\varepsilon_T$ are iid uniform on $[0,1]$. The exact specification is adopted for the sake of obtaining a simple orthogonal transformation of the information structure:

$$\theta_t = (1-\lambda)\lambda^t \textstyle\sum_{k=0}^{t}\lambda^{-k}\varepsilon_k,\ \forall t = 0,\dots,T. \tag{18}$$

The type process is Markovian. Assumption 1 is satisfied except that the support of $\theta_t$ depends on the realization of $\theta_{t-1}$. However, it is easy to make the support of $\theta_t$ the unit interval for all $t$ by mixing the distribution of $\theta_t$ in (18) with the uniform distribution on $[0,1]$; our specification obtains in the limit as the weight on the uniform distribution vanishes.

In all three examples the agent's utility is time-separable, and the flow utility, $\tilde{u}_t(\theta_t, a_t, x_t)$, only depends on the agent's type, hidden action and the contractible decision.[14] Denote the flow utility in the orthogonally transformed model by $u_t(\varepsilon^t, a_t, x_t)$. By Proposition 1, in any incentive compatible mechanism $\left(\mathbf{x}^T, \mathbf{a}^T, p\right)$ the agent's equilibrium payoff can be written as

$$\begin{aligned}\Pi_0(\varepsilon_0) = \Pi_0(0) &+ E\left[\int_0^{\varepsilon_0}\sum_{t=0}^{T} u_{t\varepsilon_0}\left(y, \varepsilon_{-0}^t, a_t(y, \varepsilon_{-0}^t, s^{t-1}), x_t(y, \varepsilon_{-0}^t, s^t)\right) dy \,\middle|\, \varepsilon_0\right] \\ &+ E\left[\int_0^{\varepsilon_0}\sum_{t=0}^{T} u_{ta_t}\left(y, \varepsilon_{-0}^t, a_t(y, \varepsilon_{-0}^t, s^{t-1}), x_t(y, \varepsilon_{-0}^t, s^t)\right) \widehat{a}_{t\varepsilon_0}(y, \varepsilon_{-0}^t, y, s^{t-1})\, dy \,\middle|\, \varepsilon_0\right], \end{aligned} \tag{19}$$

where $\widehat{a}_t(\varepsilon^t, \widehat{\varepsilon}_0, s^{t-1})$, defined by equation (2), is the period-$t$ action of the Agent that "masks" her initial misreport of $\widehat{\varepsilon}_0$ conditional on the history of types and public signals.

Next, we describe Applications 1–3, ordered according to increasing complexity of the agent's payoff function. The first application is a pure adverse selection model; the second one is a variant that includes a hidden action as well, but no contractible signal about the agent's type and action. The third application has both adverse selection and moral hazard, and the agent's type and action generate a contractible summary signal.

**Application 1.** The principal is the seller of an indivisible good; the agent is a buyer with valuation $\theta_t$ in period $t$. The contractible action, $x_t \in [0,1]$, is the probability that the buyer receives the good. The buyer has no hidden action; her flow utility is simply $\tilde{u}_t(\theta_t, x_t) = \theta_t x_t$, or equivalently, in the orthogonalized model, $u_t(\varepsilon^t, x_t) = (1-\lambda)\lambda^t\left(\sum_{k=0}^{t}\lambda^{-k}\varepsilon_k\right)x_t$. Note that Assumption 2 holds and $u_{t\varepsilon_0} = (1-\lambda)\lambda^t x_t$.

[14] Assumption 0 is also satisfied due to the boundedness of all relevant domains and the continuous differentiability of all involved functions.

Since the agent has no hidden action the second line in equation (19) is zero, and so

$$\Pi_0(\varepsilon_0) = \Pi_0(0) + E\left[\int_0^{\varepsilon_0} \sum_{t=0}^{T} (1-\lambda)\lambda^t x_t(y, \varepsilon_{-0}^t)\, dy \,\middle|\, \varepsilon_0\right]. \tag{20}$$

Suppose the buyer's participation is guaranteed if she gets a non-negative payoff; by (20) this is equivalent to $\Pi_0(0) \geq 0$.

In order to compute $E[\Pi_0(\varepsilon_0)]$ we note that by Fubini's Theorem,

$$\int_0^1 \int_0^{\varepsilon_0} x_t(y, \varepsilon_{-0}^t)\, dy d\varepsilon_0 = \int_0^1 \int_y^1 x_t(y, \varepsilon_{-0}^t)\, d\varepsilon_0 dy = \int_0^1 (1-\varepsilon_0) x_t(\varepsilon^t)\, d\varepsilon_0,$$

therefore

$$E[\Pi_0(\varepsilon_0)] = \Pi_0(0) + E\left[\sum_{t=0}^{T} (1-\lambda)\lambda^t (1-\varepsilon_0) x_t(\varepsilon^t)\right]. \tag{21}$$

Assume the seller (principal) maximizes his expected revenue; there is no cost of production. The expected revenue equals the expected social surplus generated by the mechanism less the buyer's expected payoff:

$$\sum_{t=0}^{T} E\left[\theta_t x_t(\varepsilon^t) - (1-\lambda)\lambda^t (1-\varepsilon_0) x_t(\varepsilon^t)\right] - \Pi_0(0),$$

where $\theta_t$ is given by equation (18). Solve the seller's problem by setting $\Pi_0(0) = 0$ and pointwise maximizing the objective in $x_t(\varepsilon^t)$: the solution is found by setting $x_t^*\left(\varepsilon^t\right) = 1$ if and only if $\theta_t \geq (1-\lambda)\lambda^t(1-\varepsilon_0)$ and $x_t^*\left(\varepsilon^t\right) = 0$ otherwise. Equivalently, in the notation of the original model,

$$\tilde{x}^*(\theta^t) = \mathbf{1}_{\theta_t + \lambda^t \theta_0 \geq (1-\lambda)\lambda^t},$$

where $\mathbf{1}$ is the indicator function. This decision rule in monotone in $\theta^t$; therefore, by Proposition 2, it is implementable in the original problem as well as in the benchmark case. Hence it is the optimal solution in both.

In this multi-period trading (single-buyer auction) problem the first-best outcome would be to trade the good whenever $\theta_t \geq 0$. In contrast, in the revenue-maximizing mechanism the good is sold whenever $\theta_t \geq \lambda^t(1-\lambda-\theta_0)$. As in the one-period problem, this decision rule corresponds to setting a reservation price in each period. The reservation price is always non-negative because $\theta_0 \leq 1-\lambda$ by (18). Interestingly, the reservation prices and the distortion that they induce only depend on the buyer's initial information (confirming our dynamic irrelevance result) and disappear over time as $t \to \infty$.

**Application 2.** In this application, as in the previous one, the principal is a seller and the agent a buyer with period-$t$ valuation $\theta_t$. Assume the good is divisible, so $x_t \in [0,1]$ is interpreted as the amount bought by the buyer, and the seller has production cost $x_t^2/2$.

The important difference in this application (as compared to the previous one) is that we assume the buyer takes a costly, hidden action interpreted as investment in every period, which increases her valuation.[15] The buyer's flow utility is $\tilde{u}_t(\theta_t, a_t, x_t) = (\theta_t + a_t)x_t - \psi a_t^2/2$, or equivalently, in the orthogonalized model,

$$u_t(\varepsilon^t, a_t, x_t) = \left[(1-\lambda)\lambda^t \textstyle\sum_{k=0}^t \lambda^{-k}\varepsilon_k + a_t\right] x_t - \tfrac{1}{2}\psi a_t^2.$$

Note that Assumptions 0–3 hold, and $u_{t\varepsilon_0} = (1-\lambda)\lambda^t x_t$ (same as in Application 1).

Assume that the seller cannot observe any signal about the buyer's valuation and investment. Hence the second line in equation (19) is zero, and so $\Pi_0(\varepsilon_0)$ is given by equation (20), and $E[\Pi_0(\varepsilon_0)]$ by equation (21). The seller's (principal's) expected profit is the expected social surplus generated by the mechanism less the buyer's (agent's) expected payoff:

$$\sum_{t=0}^{T} E\left[\left(\theta_t + a_t(\varepsilon^t)\right) x_t(\varepsilon^t) - \tfrac{1}{2}\psi a_t(\varepsilon^t)^2 - \tfrac{1}{2}x_t(\varepsilon^t)^2 - (1-\lambda)\lambda^t(1-\varepsilon_0)x_t(\varepsilon^t)\right] - \Pi_0(0).$$

Since the seller can make no inference about $a_t$, moreover the buyer's future valuations are not affected by her current investment either, $a_t$ is set by the buyer to maximize her current flow utility: $a_t(\varepsilon^t) \equiv x_t(\varepsilon^t)/\psi$. Substituting this into the seller's expected payoff, the first-order condition of pointwise maximization of the seller's objective in $x_t(\varepsilon^t)$ is

$$\theta_t + \frac{x_t(\varepsilon^t)}{\psi} - x_t(\varepsilon^t) - (1-\lambda)\lambda^t(1-\varepsilon_0) = 0. \tag{22}$$

Assuming that the buyer participates with non-negative payoff it is optimal to set $\Pi_0(0) = 0$. Using (18) in rearranging (22) yields, in terms of the original model,

$$\tilde{x}_t^*(\theta^t) = \frac{\psi}{\psi - 1}\left[\theta_t + \lambda^t\theta_0 - (1-\lambda)\lambda^t\right].$$

Assume $\psi > 1$. Then $\tilde{x}_t^*(\theta^t)$ is strictly increasing; by Proposition 3 it is implementable both in the original problem and the benchmark when coupled with investments $\tilde{a}^*(\theta^t) = \tilde{x}_t^*(\theta^t)/\psi$. Therefore this allocation rule is the optimal second-best solution in both problems.

In this application, in the first best (contractible $\theta_t$, $a_t$), the relationship between the

[15] Interpreting $a_t$ as a costly action taken right before $\theta_t$ is realized and shifting the distribution of $\theta_t$, this application can be thought of as a multi-period generalization (of a specific example) of Bergemann and Välimäki (2002). Our focus is on the revenue-maximizing sales mechanism instead of the efficient one.

buyer's investment level and her anticipated purchase (trade) would be the same, $a_t^{FB} \equiv x_t^{FB}/\psi$. However, the first-best level of trade would be $x_t^{FB}(\theta_t) = \psi\theta_t/(\psi-1)$. The distortion, which materializes in the decision rule in the form of less trade, and in the action rule as less investment in comparison to the efficient levels, is again due to the buyer's (agent's) initial private information and it disappears over time.

**Application 3.** The principal is an investor and the agent is an investment advisor (banker); the contractible action $x_t$ is the amount invested. Assume that the agent invests $\kappa x_t$ for herself and $(1-\kappa)x_t$ for the principal. The proportion $\kappa \in [0,1]$ is fixed exogenously; $\kappa > 0$ is realistic but $\kappa = 0$ is an interesting special case. The agent's type $\theta_t$ represents her ability to achieve a greater expected return. Her effort (hidden action $a_t$) is directed at finding assets that fit the principal's other (e.g., ethical) investment goals; it generates a payoff propotional to the invested amount for the principal but imposes an up-front cost on the agent.

Let $\tilde{u}_t = \theta_t \kappa x_t - \psi a_t^2/2$ be the agent's payoff and $v_t = (\theta_t + a_t + \xi_t)(1-\kappa)x_t - r x_t^2/2$ the principal's; in the latter $r x_t^2/2$ represents the principal's (convex) cost of raising funds for investment, and $\xi_t$ is a noise term (e.g., uncertainty in how the advisor's effort affects the investor's non-pecuniary return on investment), included for the sake of generality. Assume that $v_t$ (but not $\theta_t$ nor $a_t$) is contractible, and define $s_t = \theta_t + a_t + \xi_t$ as the contractible public signal. The parties' payoffs are transferable, i.e., they may contract on monetary transfers as well. It is easy to check that in this application Assumptions 0–5 are all satisfied.[16] This is a parametric example of the model discussed in Section 4.2. Garrett and Pavan (2012) solve a related problem where, using the notation of this example, $\kappa = 0$ and $r = 0$; the decision $x_t \in \{0,1\}$ corresponds to whether or not the principal employs the agent instead of a continuous investment decision (which is more meaningful in our example).

In the orthogonalized model

$$u_t(\varepsilon^t, a_t, x_t) = (1-\lambda)\lambda^t \left(\sum_{k=0}^{t} \lambda^{-k}\varepsilon_k\right) \kappa x_t - \tfrac{1}{2}\psi a_t^2,$$

hence $u_{t\varepsilon_0} = \kappa(1-\lambda)\lambda^t x_t$ and $u_{ta_t} = -\psi a_t$. The period-$t$ action of the agent that "masks" her initial misreport of $\widehat{\varepsilon}_0$ conditional on the history of types and prior public signals is $\widehat{a}_t(\varepsilon^t, \widehat{\varepsilon}_0, s^{t-1})$, formally defined by

$$\widehat{\theta}_t + a_t(\widehat{\varepsilon}_0, \varepsilon_{-0}^t, s^{t-1}) \equiv \theta_t + \widehat{a}_t(\varepsilon^t, \widehat{\varepsilon}_0, s^{t-1}),$$

[16] If $\kappa = 0$ then Assumption 2(ii) only holds weakly. However, we will show that the optimal decision rule is continuous in $\kappa$ at $\kappa = 0$, and hence the approximate implementation result holds in the limit.

where $\widehat{\theta}_t = (1-\lambda)\lambda^t \widehat{\varepsilon}_0 + (1-\lambda)\sum_{k=1}^t \lambda^{t-k}\varepsilon_k$, hence

$$\widehat{a}_t(\varepsilon^t, \widehat{\varepsilon}_0, s^{t-1}) = a_t(\widehat{\varepsilon}_0, \varepsilon^t_{-0}, s^{t-1}) + (1-\lambda)\lambda^t(\widehat{\varepsilon}_0 - \varepsilon_0).$$

Note that $\widehat{a}_{t\varepsilon_0}(\varepsilon^t, \widehat{\varepsilon}_0, s^{t-1}) = -(1-\lambda)\lambda^t$.

By equation (19), the agent's expected payoff with initial type $\varepsilon_0$ is

$$\Pi_0(\varepsilon_0) = \Pi_0(0) + E\left[ \int_0^{\varepsilon_0} \sum_{t=0}^{T} (1-\lambda)\lambda^t \left[\kappa x_t(y, \varepsilon^t_{-0}, s^t) + \psi a_t(y, \varepsilon^t_{-0}, s^{t-1})\right] dy \middle| \varepsilon_0 \right].$$

Continue to use $\Pi_0(0) \geq 0$ as the participation constraint. Again, using Fubini's Theorem as in the previous applications we get

$$E\left[\Pi_0(\varepsilon_0)\right] = \Pi_0(0) + E\left[ \sum_{t=0}^{T} (1-\lambda)\lambda^t (1-\varepsilon_0)\left[\kappa x_t(\varepsilon^t, s^t) + \psi a_t(\varepsilon^t, s^{t-1})\right] \right].$$

The principal's ex-ante expected payoff is the difference between the expected social surplus generated by the mechanism and the agent's expected payoff:

$$\sum_{t=0}^{T} E\left[ (\theta_t + a_t + \xi_t)(1-\kappa)x_t - \tfrac{1}{2} r x_t^2 + \theta_t \kappa x_t - \tfrac{1}{2}\psi a_t^2 - (1-\lambda)\lambda^t (1-\varepsilon_0)(\kappa x_t + \psi a_t) \right] - \Pi_0(0), \tag{23}$$

where the arguments of $a_t(\theta^t, s^{t-1})$ and $x_t(\theta^t, s^t)$ are suppressed for brevity.

If the public signal $s_t$ contained no noise term (i.e., in case $\xi_t \equiv 0$), then the principal could infer $a_t$ from the agent's type report and the realized signal and indirectly enforce any action. In this case, the first-order condition of (pointwise) maximization of (23) in $a_t$ is $(1-\kappa)x_t - \psi a_t - (1-\lambda)\lambda^t(1-\varepsilon_0)\psi = 0$, whereas the same with respect to $x_t$ is $\theta_t + (1-\kappa)a_t - r x_t - (1-\lambda)\lambda^t(1-\varepsilon_0)\kappa = 0$. Substitute the former into the latter and write $\theta_0/(1-\lambda)$ for $\varepsilon_0$ to get

$$\tilde{x}^*_t(\theta^t) = \frac{\theta_t + \lambda^t\theta_0 - (1-\lambda)\lambda^t}{r\psi - (1-\kappa)^2}.$$

Assuming $r\psi > (1-\kappa)^2$ the resulting $\tilde{x}^*_t$ is strictly increasing in $\theta^t$, and so is the corresponding optimal $\tilde{a}^*_t$, which is its positive affine transformation. Therefore, by Proposition 4, this decision-action rule is approximately implementable in the original model as well as in the benchmark. It is easy to see that the first-best decision rule would be $x^{FB}_t(\theta^t) = \theta_t / \left[r - (1-\kappa)^2/\psi\right]$. Again, the distortion in $\tilde{x}^*(\theta^t)$ is purely due to the agent's initial private information, illustrating our dynamic irrelevance theorem.

## 6 Conclusions

In this paper we considered a dynamic principal-agent model with adverse selection and moral hazard and proved a dynamic irrelevance theorem: In any implementable decision rule the principal's expected revenue and the agent's payoff are the same *as if* the principal could observe the agent's future, orthogonalized types. We also provided results on the implementability of monotonic decision rules in regular, Markovian environments. The implementation results imply a straightforward method of solving a large class of dynamic principal-agent problems with meaningful economic applications.

The model considered in this paper could be extended in two directions without much difficulty, at the expense of additional notation and technical assumptions. First, it would be possible to accommodate multiple agents in the principal-agent model by replacing the agent's incentive constraints with an appropriate (Bayesian) equilibrium. Second, the model could be extended to have an infinite time horizon. In this case our main theorem still holds assuming time-separable utility, discounting, and uniformly bounded felicity functions.

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# Appendix

**Lemma 6** *If the mechanism $\left(\mathbf{x}^T, \mathbf{a}^T, \mathbf{p}\right)$ is incentive compatible, the equilibrium payoff function of the agent, $\Pi_0$, is Lipshitz continuous.*

**Proof.** Throughout the proof, let $K$ denote an integer such that the inequalities in Assumption 0 are satisfied and, in addition, for all $t = 1, ..., T$, $\tau < t$, and for all $\theta^t, a^t, x^t$

$$\left| \frac{G_{t\theta_\tau}\left(\theta^t | \theta^{t-1}, a^{t-1}, x^{t-1}\right)}{g_t\left(\theta^t | \theta^{t-1}, a^{t-1}, x^{t-1}\right)} \right| < K.$$

First, we show that there exists a $\overline{K} \in \mathbb{N}$ such that $\left|\psi_{t\varepsilon_0}\left(\varepsilon^t, a^{t-1}, x^{t-1}\right)\right| < \overline{K}$. For $t = 0$, $\psi_{0\varepsilon_0}\left(\varepsilon_0\right) = G^{-1}_{0\varepsilon_0}\left(\varepsilon_0\right) = 1/g_0\left(G^{-1}_0\left(\varepsilon_0\right)\right) < K$ by part (ii) of Assumption 0. We proceed by induction and assume that $\psi_{\tau\varepsilon_0}\left(\varepsilon^\tau, a^{\tau-1}, x^{\tau-1}\right) < \overline{K}\left(\tau\right)$ for $\tau = 0, ..., t-1$. Then

$$\begin{aligned}
&\left|\psi_{t\varepsilon_0}\left(\varepsilon^t, a^{t-1}, x^{t-1}\right)\right| = \left|G^{-1}_{t\varepsilon_0}\left(\varepsilon_t | \psi^{t-1}\left(\varepsilon^{t-1}, a^{t-2}, x^{t-2}\right), a^{t-1}, x^{t-1}\right)\right| \\
&= \left| \frac{1}{g_t\left(\psi_t\left(\varepsilon^t, a^{t-1}, x^{t-1}\right) | \psi^{t-1}\left(\varepsilon^{t-1}, a^{t-2}, x^{t-2}\right), a^{t-1}, x^{t-1}\right)} \right| \\
&+ \left| \sum_{\tau=0}^{\tau-1} G^{-1}_{t\psi_\tau}\left(\varepsilon_t | \psi^{t-1}\left(\varepsilon^{t-1}, a^{t-2}, x^{t-2}\right), a^{t-1}, x^{t-1}\right) \psi_{\tau\varepsilon_0}\left(\varepsilon^\tau, a^{\tau-1}, x^{\tau-1}\right) \right| \\
&\leq K + \max_{\tau \leq t} \overline{K}\left(\tau\right) \left| \sum_{\tau=0}^{\tau-1} G^{-1}_{t\psi_\tau}\left(\varepsilon_t | \psi^{t-1}\left(\varepsilon^{t-1}, a^{t-2}, x^{t-2}\right), a^{t-1}, x^{t-1}\right) \right|,
\end{aligned}$$

where the inequality follows from the inductive hypothesis and part (ii) of Assumption 0.

However,

$$\begin{aligned}
& K+\max_{\tau\leq t}\overline{K}\left(\tau\right)\left|\sum_{\tau=0}^{\tau-1}G_{t\psi_{\tau}}^{-1}\left(\varepsilon_{t}|\psi^{t-1}\left(\varepsilon^{t-1},a^{t-2},x^{t-2}\right),a^{t-1},x^{t-1}\right)\right| \\
= \quad & K+\max_{\tau\leq t}\overline{K}\left(\tau\right)\left|\sum_{\tau=0}^{\tau-1}G_{t\psi_{\tau}}\left(\psi_{t}\left(\varepsilon^{t},a^{t-1},x^{t-1}\right)|\psi^{t-1}\left(\varepsilon^{t-1},a^{t-2},x^{t-2}\right),a^{t-1},x^{t-1}\right)\right|\times \\
& \times\frac{1}{g_{t}\left(\psi_{t}\left(\varepsilon^{t},a^{t-1},x^{t-1}\right)|\psi^{t-1}\left(\varepsilon^{t-1},a^{t-2},x^{t-2}\right),a^{t-1},x^{t-1}\right)} \\
\leq \quad & K+\max_{\tau\leq t}\overline{K}\left(\tau\right)K^{2},
\end{aligned}$$

by Assumption 2. So, we can conclude that $\left|\psi_{t\varepsilon_0}\left(\varepsilon^{t},a^{t-1},x^{t-1}\right)\right|<K+\max_{\tau\leq t}\overline{K}\left(\tau\right)K^{2}$.

We are ready to prove that $\Pi_0$ is Lipshitz continuous. Suppose that $\Pi_0\left(\varepsilon_0\right)\geq\Pi_0\left(\widehat{\varepsilon}_0\right)$. Let $\pi_0\left(\widehat{\varepsilon}_0,\varepsilon_0\right)$ denote the payoff of an agent whose initial type is $\widehat{\varepsilon}_0$, reports $\varepsilon_0$, then reports truthfully afterwards and takes action $\widehat{\mathbf{a}}_t\left(\varepsilon^{t},\widehat{\varepsilon}_0,s^{t-1}\right)$ after history $\left(\varepsilon^{t},s^{t-1}\right)$. Since the mechanism $\left(\mathbf{x}^{T},\mathbf{a}^{T},\mathbf{p}\right)$ is incentive compatible, $\pi_0\left(\widehat{\varepsilon}_0,\varepsilon_0\right)<\Pi\left(\widehat{\varepsilon}_0\right)$ and hence,

$$\Pi_0\left(\varepsilon_0\right)-\Pi_0\left(\widehat{\varepsilon}_0\right)<\Pi_0\left(\varepsilon_0\right)-\pi_0\left(\widehat{\varepsilon}_0,\varepsilon_0\right).$$

So, it is enough to prove that

$$\left|\Pi_0\left(\varepsilon_0\right)-\pi_0\left(\widehat{\varepsilon}_0,\varepsilon_0\right)\right|<K\left|\varepsilon_0-\widehat{\varepsilon}_0\right|. \tag{24}$$

In addition,

$$\begin{aligned}
\Pi_0\left(\varepsilon_0\right)-\pi_0\left(\widehat{\varepsilon}_0,\varepsilon_0\right) \quad = \quad & E\left[u\left(\varepsilon^{T},\mathbf{a}^{T}\left(\varepsilon^{T},s^{T-1}\right),s^{T},\mathbf{x}^{T}\left(\varepsilon^{T}\right)\right)\middle|\,\varepsilon_0\right] \\
& -E\left[u\left(\varepsilon^{T},\widehat{\mathbf{a}}^{T}\left(\varepsilon^{T},\widehat{\varepsilon}_0,s^{T-1}\right),s^{T},\mathbf{x}^{T}\left(\widehat{\varepsilon}_0,\varepsilon_{-0}^{T},s^{T}\right)\right)\middle|\,\varepsilon_0\right].
\end{aligned}$$

In order to establish (24) it is sufficient to show that the absolute value of the difference between the terms whose expectations are taken on the right-hand side of the previous equation is smaller than $K\left|\varepsilon_0-\widehat{\varepsilon}_0\right|$. Note that

$$\begin{aligned}
& u\left(\varepsilon^{T},\mathbf{a}^{T}\left(\varepsilon^{T},s^{T-1}\right),s^{T},x^{T}\right)-u\left(\widehat{\varepsilon}_0,\varepsilon_{-0}^{T},\mathbf{a}^{T}\left(\widehat{\varepsilon}_0,\varepsilon_{-0}^{T},s^{T-1}\right),s^{T},x^{T}\right)= \\
& \int_{\widehat{\varepsilon}_0}^{\varepsilon_0}u_{\varepsilon_0}\left(y,\varepsilon_{-0}^{T},\mathbf{a}^{T}\left(y,\varepsilon_{-0}^{T},s^{T-1}\right),s^{T},x^{T}\right) \\
& +\sum_{t=0}^{T}u_{a_t}\left(y,\varepsilon_{-0}^{T},\mathbf{a}^{T}\left(y,\varepsilon_{-0}^{T},s^{T-1}\right),s^{T},x^{T}\right)\widehat{\mathbf{a}}_{t\widehat{\varepsilon}_0}\left(\varepsilon^{t},y,s^{t-1}\right)dy.
\end{aligned}$$

We will show that both terms on the right-hand side of the previous equation is bounded by a constant times $|\varepsilon_0 - \widehat{\varepsilon}_0|$. Note that

$$\begin{aligned}
&\int_{\widehat{\varepsilon}_0}^{\varepsilon_0} u_{\varepsilon_0}\left(y, \varepsilon_{-0}^T, a^T, s^T, x^T\right) dy \\
= &\int_{\widehat{\varepsilon}_0}^{\varepsilon_0} \sum_{t=0}^{T} \widetilde{u}_{\theta_t}\left(\psi^T\left(y, \varepsilon_{-0}^T, a^{T-1}, x^{T-1}\right), a^T, s^T, x^T\right) \psi_{t\varepsilon_0}\left(y, \varepsilon_{-0}^T, a^{T-1}, x^{T-1}\right) dy \leq TK\overline{K}|\varepsilon_0 - \varepsilon|,
\end{aligned}$$

by part (i) of Assumption 0 and since $\psi_{t\varepsilon_0,}\left(\varepsilon^t\right) < \overline{K}$, as shown above. In addition,

$$\begin{aligned}
&\int_{\widehat{\varepsilon}_0}^{\varepsilon_0} \sum_{t=0}^{T} u_{a_t}\left(y, \varepsilon_{-0}^T, \mathbf{a}^T\left(y, \varepsilon_{-0}^T, s^{T-1}\right), s^T, x^T\right) \widehat{\mathbf{a}}_{t\widehat{\varepsilon}_0}\left(\varepsilon^t, y, s^{t-1}\right) dy \\
= &\int_{\widehat{\varepsilon}_0}^{\varepsilon_0} \sum_{t=0}^{T} \widetilde{u}_{a_t}\left(\psi^T\left(y, \varepsilon_{-0}^T, \cdot\right), \mathbf{a}^T\left(y, \varepsilon_{-0}^T, s^{T-1}\right), s^T, x^T\right) \widehat{\mathbf{a}}_{t\widehat{\varepsilon}_0}\left(\varepsilon^t, y, s^{t-1}\right) dy.
\end{aligned} \tag{25}$$

By the Implicit Function Theorem,

$$\widehat{\mathbf{a}}_{t\widehat{\varepsilon}_0}\left(\varepsilon^t, y, s^{t-1}\right) = -\frac{f_{t\theta_t}\left(\psi^t\left(y, \varepsilon_{-0}^t, \cdot\right), \widehat{\mathbf{a}}_t\left(\varepsilon^t, y, s^{t-1}\right)\right)}{f_{ta_t}\left(\psi^t\left(y, \varepsilon_{-0}^t, \cdot\right), \widehat{\mathbf{a}}_t\left(\varepsilon^t, y, s^{t-1}\right)\right)} \psi_{t\widehat{\varepsilon}_0}\left(\varepsilon_{-0}^T, y, a^{T-1}, x^{T-1}\right),$$

which does not exceed $K\overline{K}$ by part (iii) of Assumption 0 and the argument above showing that $|\psi_{t\varepsilon_0}| < \overline{K}$. Hence, (25) is smaller than

$$K\overline{K} \int_{\widehat{\varepsilon}_0}^{\varepsilon_0} \sum_{t=0}^{T} \widetilde{u}_{a_t}\left(\psi^T\left(y, \varepsilon_{-0}^T, \cdot\right), \mathbf{a}^T\left(y, \varepsilon_{-0}^T, s^{T-1}\right), s^T, x^T\right) dy \leq TK^2\overline{K}|\varepsilon_0 - \varepsilon|$$

by part (i) of Assumption 0. □

**Proof of Lemma** 1. Part (i) of Assumption 2 is satisfied by definition. To see part (ii), notice that if $\overline{\mathbf{a}}_t\left(\theta_t, x^t\right)$ is interior then

$$\begin{aligned}
v_{t\theta_t}\left(\theta_t, x_t\right) &= \widetilde{u}_{t\theta_t}\left(\theta_t, \overline{\mathbf{a}}_t\left(\theta_t, x^t\right), x^t\right) + \widetilde{u}_{ta_t}\left(\theta_t, \overline{\mathbf{a}}_t\left(\theta_t, x^t\right), x^t\right) \frac{\partial \overline{\mathbf{a}}_t\left(\theta_t, x^t\right)}{\partial \theta_t} \\
&= \widetilde{u}_{t\theta_t}\left(\theta_t, \overline{\mathbf{a}}_t\left(\theta_t, x^t\right), x^t\right) > 0,
\end{aligned} \tag{26}$$

where the second equality follows from (7), and the inequality follows from part (ii) of Assumption 2. If $\overline{\mathbf{a}}_t\left(\theta_t, x^t\right)$ is not interior then, generically,

$$v_{t\theta_t}\left(\theta_t, x_t\right) = \widetilde{u}_{t\theta_t}\left(\theta_t, \overline{\mathbf{a}}_t\left(\theta_t, x^t\right), x^t\right) > 0, \tag{27}$$

where the inequality again follows from part (ii) of Assumption 2.

It remains to prove that $v^t$ satisfies part (iii) of Assumption 2. To simplify notation, we only prove this claim for the case when the contractible decision is unidimensional in each period, that is, $X_t \subset \mathbb{R}$ for all $t = 0, ..., T$. Suppose first that $\overline{\mathbf{a}}_t\left(\theta_t, x^t\right)$ is interior. Note that for all $\tau \leq t$,

$$\begin{aligned} v_{t\theta_t x_\tau}\left(\theta_t, x_t\right) &= \widetilde{u}_{t\theta_t x_\tau}\left(\theta_t, \overline{\mathbf{a}}_t\left(\theta_t, x^t\right), x^t\right) + \widetilde{u}_{t\theta_t a_t}\left(\theta_t, \overline{\mathbf{a}}_t\left(\theta_t, x^t\right), x^t\right) \frac{\partial \overline{\mathbf{a}}_t\left(\theta_t, x^t\right)}{\partial x_\tau} \\ &= \widetilde{u}_{t\theta_t x_\tau}\left(\theta_t, \overline{\mathbf{a}}_t\left(\theta_t, x^t\right), x^t\right) - \widetilde{u}_{t\theta_t a_t}\left(\theta_t, \overline{\mathbf{a}}_t\left(\theta_t, x^t\right), x^t\right) \frac{\widetilde{u}_{ta_t x_t}\left(\theta_t, \overline{\mathbf{a}}_t\left(\theta_t, x^t\right), x^t\right)}{\widetilde{u}_{ta_t^2}\left(\theta_t, \overline{\mathbf{a}}_t\left(\theta_t, x^t\right), x^t\right)}, \end{aligned}$$

where first equality follows from (26) and the second one follows from (7) and the Implicit Function Theorem. Note that $\widetilde{u}_{t\theta_t x_\tau}$, $\widetilde{u}_{t\theta_t a_t}$ and $\widetilde{u}_{ta_t x_t}$ are all non-negative by part (iii) of Assumption 2 and parts (ii) and (iii) of Assumption 3. In addition, $\widetilde{u}_{ta_t^2}$ is negative by part (i) of Assumption 3. Therefore, $v_{t\theta_t x_\tau}\left(\theta_t, x_t\right) \geq 0$. Suppose now that $\overline{\mathbf{a}}_t\left(\theta_t, x^t\right)$ is not interior. Then, for all $\tau \leq t$, generically,

$$v_{t\theta_t x_\tau}\left(\theta_t, x_t\right) = \widetilde{u}_{t\theta_t x_\tau}\left(\theta_t, \overline{\mathbf{a}}_t\left(\theta_t, x^t\right), x^t\right) \geq 0,$$

where the equality follows from (27) and the inequality follows from part (ii) of Assumption 3. □

**Proof of Lemma 2.** Part (i) of Assumption 2 is satisfied by definition. To see part (ii), notice that

$$w_{t\theta_t}\left(\theta_t, y_t, x^t\right) = \widetilde{u}_{t\theta_t}\left(\theta_t, \underline{\mathbf{a}}_t\left(\theta_t, y_t\right), x^t\right) + \widetilde{u}_{ta_t}\left(\theta_t, \underline{\mathbf{a}}_t\left(\theta_t, y_t\right), x^t\right) \frac{\partial \underline{\mathbf{a}}_t\left(\theta_t, y_t\right)}{\partial \theta_t}. \tag{28}$$

We apply the Implicit Function Theorem for the identity $f_t\left(\theta_t, \underline{\mathbf{a}}_t\left(\theta_t, y_t\right)\right) \equiv y_t$ to get

$$\frac{\partial \underline{\mathbf{a}}_t\left(\theta_t, y_t\right)}{\partial \theta_t} = -\frac{f_{t\theta_t}\left(\theta_t, \underline{\mathbf{a}}_t\left(\theta_t, y_t\right)\right)}{f_{ta_t}\left(\theta_t, \underline{\mathbf{a}}_t\left(\theta_t, y_t\right)\right)},$$

which is negative by part (ii) of Assumption 5. Since $\widetilde{u}_{t\theta_t} > 0$ by part (ii) of Assumption 2 and $\widetilde{u}_{ta_t} < 0$ by part (i) of Assumption 5, we conclude that $w_t$ is strictly increasing in $\theta_t$.

Next, we prove that $w_t$ satisfies part (iii) of Assumption 2. First, we establish the single-crossing property with respect to $\theta_t$ and $y_t$. By (28),

$$\begin{aligned} & w_{t\theta_t y_t}\left(\theta_t, y_t, x^t\right) = \widetilde{u}_{t\theta_t a_t}\left(\theta_t, \underline{\mathbf{a}}_t\left(\theta_t, y_t\right), x^t\right) \frac{\partial \underline{\mathbf{a}}_t\left(\theta_t, y_t\right)}{\partial y_t} + \\ & \widetilde{u}_{ta_t^2}\left(\theta_t, \underline{\mathbf{a}}_t\left(\theta_t, y_t\right), x^t\right) \frac{\partial \underline{\mathbf{a}}_t\left(\theta_t, y_t\right)}{\partial y_t} \frac{\partial \underline{\mathbf{a}}_t\left(\theta_t, y_t\right)}{\partial \theta_t} + \widetilde{u}_{ta_t}\left(\theta_t, \underline{\mathbf{a}}_t\left(\theta_t, y_t\right), x^t\right) \frac{\partial^2 \underline{\mathbf{a}}_t\left(\theta_t, y_t\right)}{\partial \theta_t \partial y_t}. \end{aligned}$$

In order to sign $\partial \underline{\mathbf{a}}_t / \partial y_t$ and $\partial^2 \underline{\mathbf{a}}_t / \partial \theta_t \partial y_t$, we appeal to the Implicit Function Theorem once again,

$$\begin{aligned}
\frac{\partial \underline{\mathbf{a}}_t(\theta_t, y_t)}{\partial y_t} &= \frac{1}{f_{ta_t}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t))} \text{ and} \\
\frac{\partial^2 \underline{\mathbf{a}}_t(\theta_t, y_t)}{\partial \theta_t \partial y_t} &= \frac{f_{ta_t^2}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t)) \frac{f_{t\theta_t}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t))}{f_{ta_t}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t))} - f_{ta_t\theta_t}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t))}{f_{ta_t}^2(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t))}.
\end{aligned}$$

Therefore, $w_{t\theta_t y_t}(\theta_t, y_t, x^t)$ can be rewritten as

$$\begin{aligned}
&\frac{\widetilde{u}_{t\theta_t}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t), x^t)}{f_{ta_t}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t))} + \widetilde{u}_{ta_t^2}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t), x^t) \frac{-f_{t\theta_t}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t))}{f_{ta_t}^2(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t))} + \\
&\widetilde{u}_{ta_t}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t), x^t) \frac{f_{ta_t^2}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t)) \frac{f_{t\theta_t}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t))}{f_{ta_t}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t))} - f_{ta_t\theta_t}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t))}{f_{ta_t}^2(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t))}.
\end{aligned}$$

The first term is positive by part (ii) of Assumption 2 and part (ii) of Assumption 5. The second term is positive by part (i) of Assumption 3 and part (ii) of Assumption 5. The third term is positive by parts (i) and (iii) of Assumption 5. Therefore, we conclude that $w_{t\theta_t y_t} \geq 0$.

It remains to show that the single crossing property in part (iii) of Assumption 2 also holds with respect to $\theta_t$ and $x_\tau$ for all $\tau \leq t$. To simplify notation, we only prove this claim for the case when the contractible decision is unidimensional in each period, that is, $X_t \subset \mathbb{R}$ for all $t = 0, ..., T$. By (28),

$$\begin{aligned}
w_{t\theta_t x_\tau}(\theta_t, y_t, x^t) &= \widetilde{u}_{t\theta_t x_\tau}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t), x^t) + \widetilde{u}_{ta_t x_\tau}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t), x^t) \frac{\partial \underline{\mathbf{a}}_t(\theta_t, y_t)}{\partial \theta_t} \\
&= \widetilde{u}_{t\theta_t x_\tau}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t), x^t) - \widetilde{u}_{ta_t x_\tau}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t), x^t) \frac{f_{t\theta_t}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t))}{f_{ta_t}(\theta_t, \underline{\mathbf{a}}_t(\theta_t, y_t))},
\end{aligned}$$

which is positive by part (iv) of Assumption 5. □

**Proof of Proposition 4.** Fix an increasing decision rule $(\widetilde{\mathbf{x}}^T, \widetilde{\mathbf{a}}^T)$ and a $\delta > 0$. Below, we construct a transfer rule, $\widetilde{\mathbf{p}}$, such that $(\widetilde{\mathbf{x}}^T, \widetilde{\mathbf{a}}^T, \widetilde{\mathbf{p}})$ satisfies (8). To this end, define the function $\widetilde{\mathbf{y}}^T : \Theta^T \to Y^T$ such that $\widetilde{\mathbf{y}}_t(\theta^t) = f_t(\theta_t, \widetilde{\mathbf{a}}_t(\theta^t))$ for all $t$ and $\theta^t$. Since $\widetilde{\mathbf{a}}_t$ is increasing in $\theta^t$ and $f_t$ is strictly increasin in both $\theta_t$ and $a_t$ (see part (ii) of Assumption 5), the function $\widetilde{\mathbf{y}}_t$ is also increasing in $\theta^t$. Therefore, by Lemma 2 and Proposition 2, the decision rule $(\widetilde{\mathbf{x}}^T, \widetilde{\mathbf{y}}^T)$ is implementable in a pure adverse selection model where the agent flow utilities are $\{w_t\}_{t=0}^T$. Let $\overline{\mathbf{p}} : \Theta^T \to \mathbb{R}$ denote a transfer rule which implements $(\widetilde{\mathbf{x}}^T, \widetilde{\mathbf{y}}^T)$.

Fix a $K \in \mathbb{N}$ such that $|\widetilde{u}_{ta_t}| < K$ and $f_{ta_t} > 1/K$. By part (i) of Assumption 0 and part (ii) of Assumption 5, such a $K$ exists. By Theorem 2 of McAfee and Reny (1992), for each

$t = 0, ..., T$, exists a function $p_t : S_t \times Y_t \to \mathbb{R}$ such that $E_{s_t}\left(p_t\left(s_t, y_t\right)|f\left(\theta_t, a_t\right) = y_t\right) = 0$ and

$$E_{s_t}\left(p_t\left(s_t, y_t\right)|f\left(\theta_t, a_t\right) = y_t'\right) \geq K^2|y_t - y_t'| - \frac{\delta}{T+1}. \tag{29}$$

Let us now define $\widetilde{\mathbf{p}} : \Theta^T \times S^T \to \mathbb{R}$ by

$$\widetilde{\mathbf{p}}\left(\theta^T, s^T\right) = \overline{\mathbf{p}}\left(\theta^T\right) + \sum_{t=0}^{T} p_t\left(s_t, \widetilde{\mathbf{y}}_t\left(\theta^t\right)\right). \tag{30}$$

Next, we show that the agent cannot generate an excess payoff of $\delta$ by deviating from truth-telling and obidience in the mechanism $\left(\widetilde{\mathbf{x}}^T, \widetilde{\mathbf{a}}^T, \widetilde{\mathbf{p}}\right)$. First, note that the agent cannot benefit from making her strategy at time $t$ contingent on the history of public signals, $s^{t-1}$, because her continuation payoff does not depend on these variables in the mechanism $\left(\widetilde{\mathbf{x}}^T, \widetilde{\mathbf{a}}^T, \widetilde{\mathbf{p}}\right)$. Therefore, we restrict attention to strategies which do not depend on past realizations of the public signal. Any such strategy induces a mapping from type profile to reports and actions in each period. Let $\rho_t\left(\theta^t\right)$ and $\alpha_t\left(\theta^t\right)$ denote the agent's report and action at time $t$, respectively, conditional on her type history $\theta^t$. Let $\widetilde{\alpha}_t\left(\theta^t\right)$ denote the solution of

$$f_t\left(\theta_t, a_t\right) = f_t\left(\rho_t\left(\theta^t\right), \widetilde{\mathbf{a}}_t\left(\rho_t\left(\theta^t\right)\right)\right) \left[= \widetilde{\mathbf{y}}_t\left(\rho_t\left(\theta^t\right)\right)\right] \tag{31}$$

in $a_t$. In other words, $\widetilde{\alpha}_t\left(\theta^t\right)$ would is the agent's action which generates the same value of $f_t$ conditional on $\theta^t$ as if the agent's true type was $\rho_t\left(\theta^t\right)$ and she took action $\widetilde{\mathbf{a}}_t\left(\rho_t\left(\theta^t\right)\right)$. Then the expected payoff generated by $\left(\rho^T, \alpha^T\right)$, conditional on $\theta_0$, is

$$\begin{aligned}
& E_{\theta^T, s^T}\left[\sum_{t=0}^{T} \widetilde{u}_t\left(\theta_t, \alpha_t\left(\theta^t\right), \widetilde{\mathbf{x}}^t\left(\rho_t(\theta^t)\right)\right) - \widetilde{\mathbf{p}}\left(\rho^T\left(\theta^T\right), s^T\right)|\theta_0\right] \qquad (32) \\
= \; & E_{\theta^T}\left[\sum_{t=0}^{T} \widetilde{u}_t\left(\theta_t, \widetilde{\alpha}_t\left(\theta^t\right), \widetilde{\mathbf{x}}^t\left(\rho_t(\theta^t)\right)\right) - \overline{\mathbf{p}}\left(\rho^T\left(\theta^T\right)\right)|\theta_0\right] \\
& + \sum_{t=0}^{T} E_{\theta^T, s^T}\left[\widetilde{u}_t\left(\theta_t, \alpha_t\left(\theta^t\right), \widetilde{\mathbf{x}}^t\left(\rho_t(\theta^t)\right)\right) - \widetilde{u}_t\left(\theta_t, \widetilde{\alpha}_t\left(\theta^t\right), \widetilde{\mathbf{x}}^t\left(\rho_t(\theta^t)\right)\right) - p_t\left(s_t, \widetilde{\mathbf{y}}_t\left(\theta^t\right)\right)|\theta_0\right],
\end{aligned}$$

where the equality follows from (30).

We first consider the first term on the right-hand side of the previous equality. Note that

$$\begin{aligned} & E_{\theta^T}\left[\sum_{t=0}^{T} \widetilde{u}_t\left(\theta_t, \widetilde{\alpha}_t\left(\theta^t\right), \widetilde{\mathbf{x}}^t\left(\rho_t(\theta^t)\right) - \overline{\mathbf{p}}\left(\rho^T\left(\theta^T\right)\right) | \theta_0\right]\right. \\ = \; & E_{\theta^T}\left[\sum_{t=0}^{T} w_t\left(\theta_t, \mathbf{y}_t\left(\rho_t\left(\theta^t\right)\right), \widetilde{\mathbf{x}}^t\left(\rho_t(\theta^t)\right) - \overline{\mathbf{p}}\left(\rho^T\left(\theta^T\right)\right) | \theta_0\right]\right. \\ \leq \; & E_{\theta^T}\left[\sum_{t=0}^{T} w_t\left(\theta_t, \mathbf{y}_t\left(\theta^t\right), \widetilde{\mathbf{x}}^t\left(\theta^t\right)\right) - \overline{\mathbf{p}}\left(\theta^T\right) | \theta_0\right] = E_{\theta^T}\left[\sum_{t=0}^{T} \widetilde{u}_t\left(\theta_t, \widetilde{\mathbf{a}}_t\left(\theta^t\right), \widetilde{\mathbf{x}}^t\left(\theta^t\right)\right) - \overline{\mathbf{p}}\left(\theta^T\right) | \theta_0\right], \end{aligned} \tag{33}$$

where the inequality follows from the assumption that the transfer rule $\overline{\mathbf{p}}$ implements $\left(\widetilde{\mathbf{x}}^T, \widetilde{\mathbf{y}}^T\right)$ if the flow utilities are $\left\{w^t\right\}_{t=0}^{T}$. Also note that

$$\begin{aligned} & \widetilde{u}_t\left(\theta_t, \alpha_t\left(\theta^t\right), \widetilde{\mathbf{x}}^t\left(\rho_t(\theta^t)\right) - \widetilde{u}_t\left(\theta_t, \widetilde{\alpha}_t\left(\theta^t\right), \widetilde{\mathbf{x}}^t\left(\rho_t(\theta^t)\right) - E_{s^T}\left[p_t\left(s_t, \widetilde{\mathbf{y}}_t\left(\rho_t\left(\theta^T\right)\right)\right) | \theta_t, \alpha_t\left(\theta^t\right)\right]\right.\right. \\ \leq \; & K|\widetilde{\alpha}_t\left(\theta^t\right) - \alpha_t\left(\theta^t\right)| - E_{s^T}\left[p_t\left(s_t, \widetilde{\mathbf{y}}_t\left(\rho_t\left(\theta^T\right)\right)\right) | \theta_t, \alpha_t\left(\theta^t\right)\right] \\ \leq \; & K^2|f_t\left(\theta_t, \widetilde{\alpha}_t\left(\theta^t\right)\right) - f_t\left(\theta_t, \alpha_t\left(\theta^t\right)\right)| - E_{s^T}\left[p_t\left(s_t, \widetilde{\mathbf{y}}_t\left(\rho_t\left(\theta^T\right)\right)\right) | \theta_t, \alpha_t\left(\theta^t\right)\right] \\ = \; & K^2|\widetilde{\mathbf{y}}_t\left(\rho_t\left(\theta^T\right)\right) - f_t\left(\rho_t\left(\theta^t\right), \alpha_t\left(\theta^t\right)\right)| - E_{s^T}\left[p_t\left(s_t, \widetilde{\mathbf{y}}_t\left(\rho_t\left(\theta^T\right)\right)\right) | \theta_t, \alpha_t\left(\theta^t\right)\right] \leq \frac{\delta}{T+1}, \end{aligned}$$

where the first and second inequalities follow from $|\widetilde{u}_{ta_t}| < K$ and $f_{ta_t} > 1/K$, the equality follows from (31), and the last inequality follows from (29). Summing up these inequalities for $t = 0, ..., T$ and taking expactation with respect to $\theta^T$,

$$\sum_{t=0}^{T} E_{\theta^T, s^T}\left[\widetilde{u}_t\left(\theta_t, \alpha_t\left(\theta^t\right), \widetilde{\mathbf{x}}^t\left(\rho_t(\theta^t)\right) - \widetilde{u}_t\left(\theta_t, \widetilde{\mathbf{a}}_t\left(\rho_t\left(\theta^t\right)\right), \widetilde{\mathbf{x}}^t\left(\rho_t(\theta^t)\right) - p_t\left(s_t, \widetilde{\mathbf{y}}_t\left(\theta^t\right)\right) | \theta_0\right] \leq \delta.\right.\right. \tag{34}$$

Therefore, plugging (33) and (34) into (32) we get that

$$\begin{aligned} & E_{\theta^T, s^T}\left[\sum_{t=0}^{T} \widetilde{u}_t\left(\theta_t, \alpha_t\left(\theta^t\right), \widetilde{\mathbf{x}}^t\left(\rho_t(\theta^t)\right) - \widetilde{\mathbf{p}}\left(\rho^T\left(\theta^T\right), s^T\right) | \theta_0\right]\right. \\ \leq \; & E_{\theta^T}\left[\sum_{t=0}^{T} \widetilde{u}_t\left(\theta_t, \widetilde{\mathbf{a}}_t\left(\theta^t\right), \widetilde{\mathbf{x}}^t\left(\theta^t\right)\right) - \overline{\mathbf{p}}\left(\theta^T\right) | \theta_0\right] + \delta \\ = \; & E_{\theta^T, s^T}\left[\sum_{t=0}^{T} \widetilde{u}_t\left(\theta_t, \widetilde{\mathbf{a}}_t\left(\theta^t\right), \widetilde{\mathbf{x}}^t\left(\theta^t\right)\right) - \widetilde{\mathbf{p}}\left(\rho^T\left(\theta^T\right), s^T\right) | \theta_0\right] + \delta, \end{aligned}$$

where the equality follows from $E_{s_t}\left[p_t\left(s_t, y_t\right) | f(\theta_t, a_t) = y_t\right] = 0$. This implies that the agent cannot gain more than $\delta$ by deviating from truth-telling and obedience in the mechanism $\left(\widetilde{\mathbf{x}}^T, \widetilde{\mathbf{a}}^T, \widetilde{\mathbf{p}}\right)$. □

**Proof of Lemma 3.** Part (i) $\varepsilon_t = H_t^{-1}(\theta_t|\theta_{t-1})$, therefore $\psi_0(\varepsilon_0) = H_0^{-1}(\varepsilon_0)$ and $\psi_t$ for $t > 0$ is defined recursively by $\psi_t(\varepsilon^t) = H_t^{-1}(\varepsilon^t|\psi^{t-1}(\varepsilon^{t-1}))$. We prove the statement of this part by induction. For $t = 0$, we have $H_0^{-1}(\varepsilon_0) \geq H_0^{-1}(\widehat{\varepsilon}_0)$ whenever $\varepsilon_0 \geq \widehat{\varepsilon}_0$ and the inequality is strict if $\varepsilon_0 > \widehat{\varepsilon}_0$.

Suppose that (10) holds for $t$, that is, $\psi_t(\widehat{\varepsilon}^t) \leq \psi_t(\varepsilon^t)$ whenever $\widehat{\varepsilon}^t \leq \varepsilon^t$ and the inequality is strict whenever $\widehat{\varepsilon}^t < \varepsilon_t$. Note that $\psi_{t+1}(\widehat{\varepsilon}_{t+1}) = H_{t+1}^{-1}(\widehat{\varepsilon}_{t+1}|\psi_t(\widehat{\varepsilon}^t))$ and $\psi_{t+1}(\varepsilon^{t+1}) = H_{t+1}^{-1}(\varepsilon_{t+1}|\psi_t(\varepsilon^t))$. Since $\psi_t(\widehat{\varepsilon}^t) \leq \psi_t(\varepsilon^t)$ by the inductive hypothesis, part (ii) of Assumption 1 implies that $\psi_{t+1}(\widehat{\varepsilon}^{t+1}) \leq \psi_{t+1}(\varepsilon^{t+1})$. In addition, if $\varepsilon_{t+1} > \widehat{\varepsilon}_{t+1}$ then $H_{t+1}^{-1}(\varepsilon_{t+1}|\psi_t(\varepsilon^t)) > H_{t+1}^{-1}(\varepsilon'_{t+1}|\psi_t(\widehat{\varepsilon}^t))$.

Part (ii): The function $u_t$ is strictly increasing in $x_t$ and weakly increasing $x^{t-1}$ because of part (ii) Assumption 2 and (9). Equalities (9) and (10) imply that $u_t$ is strictly increasing in $\varepsilon_t$ and weakly increasing in $\varepsilon^{t-1}$.

Part (iii): Fix a $t \in \{0, ..., T\}$ and note that by (9),

$$u_{t\varepsilon_t}(\varepsilon^t, x^t) = \widetilde{u}_{t\theta_t}(\psi_t(\varepsilon^t), x^t)\frac{\partial \psi_t(\varepsilon^t)}{\partial \varepsilon_t}.$$

The result follows from (10) and part (iii) of Assumption 2. □

**Proof of Lemma 4.** Fix a $t \in \{0, \ldots, T-1\}$, $\varepsilon^{t+1} \in \mathcal{E}^{t+1}$ and $\widehat{\varepsilon}_t \in \mathcal{E}_t$. Let

$$\sigma_{t+1} = H_{t+1}(\psi_{t+1}(\varepsilon^{t+1})|\psi_t(\varepsilon^{t-1}, \widehat{\varepsilon}_t)). \tag{35}$$

By the full support assumption in part (i) of Assumption 1, it follows that

$$\psi_{t+1}(\varepsilon^{t+1}) = \psi_{t+1}(\varepsilon^{t-1}, \widehat{\varepsilon}_t, \sigma_{t+1}),$$

that is, the computed time-$(t+1)$ type of the original model is the same after $\varepsilon^{t+1}$ and $(\varepsilon^{t-1}, \widehat{\varepsilon}_t, \sigma_{t+1})$. Therefore the inferred type in the original model is also the same after any future observations, that is,

$$\psi_k(\varepsilon^{t-1}, \varepsilon_t, \varepsilon_{t+1}, \widehat{\varepsilon}_{t+2}, \ldots, \widehat{\varepsilon}_k) = \psi_k(\varepsilon^{t-1}, \widehat{\varepsilon}_t, \sigma_{t+1}, \widehat{\varepsilon}_{t+2}, \ldots, \widehat{\varepsilon}_k),$$

for all $k = t+1, \ldots, T$, all $\widehat{\varepsilon}^k \in \mathcal{E}^k$. This equality and (9) imply (11). Also note that $\sigma_{t+1}(\varepsilon^{t+1}, \widehat{\varepsilon}_t)$, defined by (35), is increasing in $\varepsilon_t$, strictly increasing in $\varepsilon_{t+1}$ by part (i) of Lemma 3 and decreasing in $\widehat{\varepsilon}_t$ by part (i) of Lemma 3 and part (iii) of Assumption 1.

It remains to show that there does not exist any other $\sigma_{t+1}$ which satisfies (11). This follows from part (ii) of Lemma 3, which states that $u_{t+1}$ is strictly increasing in $\varepsilon_{t+1}$, which implies that (11) with $k = t+1$ cannot hold for two different $\sigma_{t+1}$'s. □

**Proof of Remark 3.** Recall from the proof of Lemma 4 that for all $k = t+1, ..., T$,

$$\psi_k\left(\varepsilon^{t+1}, \varepsilon_{t+2}..., \varepsilon_k\right) = \psi_k\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t, \sigma_{t+1}\left(\varepsilon^{t+1}, \widehat{\varepsilon}_t\right), \varepsilon_{t+2}, ..., \varepsilon_k\right).$$

By (10), $\psi_t\left(\varepsilon^t\right) \le \psi_t\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t\right)$ if and only if $\widehat{\varepsilon}_t \ge \varepsilon_t$. Then (12) follows from the monotinicity of $\{\widetilde{x}_t\}_0^T$ and the definition of $\{x\}_0^T$. □

**Proof of Lemma 5.** Let $\gamma_t^k\left(\varepsilon^k, \widehat{\varepsilon}_t, y\right)$ denote $\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t, y, \varepsilon_{t+2}, ..., \varepsilon_k\right)$ for $k = t+1, ..., T$. Suppose first that $\varepsilon_t > \widehat{\varepsilon}_t$. Then $\sigma_{t+1}\left(\varepsilon^{t+1}, \widehat{\varepsilon}_t\right) > \varepsilon_{t+1}$, and

$$\begin{aligned}
\pi_t\left(\varepsilon_t, \widehat{\varepsilon}_t | \varepsilon^{t-1}\right) &= u_t\left(\varepsilon^t, x^t\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t\right)\right) - u_t\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t, x^t\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t\right)\right) \\
&\quad + \int \Pi_{t+1}\left(\sigma_{t+1}\left(\varepsilon^{t+1}, \widehat{\varepsilon}_t\right) | \varepsilon^{t-1}, \widehat{\varepsilon}_t\right) d\varepsilon_{t+1} \\
&= u_t\left(\varepsilon^t, x^t\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t\right)\right) - u_t\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t, x^t\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t\right)\right) + \Pi_t\left(\widehat{\varepsilon}_t | \varepsilon^{t-1}\right) \\
&\quad + \sum_{k=t+1}^{T} \int ... \int \int_{\varepsilon_{t+1}}^{\sigma_{t+1}\left(\varepsilon^{t+1}, \widehat{\varepsilon}_t\right)} u_{\varepsilon_{t+1}}^k\left(\gamma_t^k\left(\varepsilon^k, \widehat{\varepsilon}_t, y\right), x^k\left(\gamma_t^k\left(\varepsilon^k, \widehat{\varepsilon}_t, y\right)\right)\right) dy d\varepsilon_{t+1} ... d\varepsilon_k \\
&= u_t\left(\varepsilon^t, x^t\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t\right)\right) - u_t\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t, x^t\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t\right)\right) + \pi_t\left(\widehat{\varepsilon}_t, \widehat{\varepsilon}_t | \varepsilon^{t-1}\right) \\
&\quad + \sum_{k=t+1}^{T} \int ... \int \int_{\varepsilon_{t+1}}^{\sigma_{t+1}\left(\varepsilon^{t+1}, \widehat{\varepsilon}_t\right)} u_{k\varepsilon_{t+1}}\left(\gamma_t^k\left(\varepsilon^k, \widehat{\varepsilon}_t, y\right), x^k\left(\gamma_t^k\left(\varepsilon^k, \widehat{\varepsilon}_t, y\right)\right)\right) dy d\varepsilon_{t+1} ... d\varepsilon_k
\end{aligned}$$

where the first equality is just (15), the second one follows from (14), and the third one from $\Pi_t\left(\widehat{\varepsilon}_t | \varepsilon^{t-1}\right) = \pi_t\left(\widehat{\varepsilon}_t, \widehat{\varepsilon}_t | \varepsilon^{t-1}\right)$. So, in order to prove (16), we only need to show that

$$u_t\left(\varepsilon^t, x^t\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t\right)\right) - u_t\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t, x^t\left(\varepsilon^{t-1}, \widehat{\varepsilon}_t\right)\right) = \int_{\widehat{\varepsilon}_t}^{\varepsilon_t} u_{t\varepsilon_t}\left(\varepsilon_{-t}^t, y, x^t\left(\rho_t^t(\varepsilon^t, \widehat{\varepsilon}_t)\right)\right) dy. \quad (36)$$

and

$$\begin{aligned}
&\sum_{k=t+1}^{T} \int ... \int \int_{\varepsilon_{t+1}}^{\sigma_{t+1}\left(\varepsilon^{t+1}, \widehat{\varepsilon}_t\right)} u_{k\varepsilon_{t+1}}\left(\gamma_t^k\left(\varepsilon^k, \widehat{\varepsilon}_t, y\right), x^k\left(\gamma_t^k\left(\varepsilon^k, \widehat{\varepsilon}_t, y\right)\right)\right) dy d\varepsilon_{t+1} ... d\varepsilon_k \\
&\qquad = \sum_{k=t+1}^{T} \int ... \int \int_{\widehat{\varepsilon}_t}^{\varepsilon_t} u_{t\varepsilon_t}\left(\zeta_t^k(\varepsilon^k, y), x^k\left(\rho_t^k(\varepsilon^k, y, \widehat{\varepsilon}_t)\right)\right) dy d\varepsilon_{t+1} ... d\varepsilon_k. \quad (37)
\end{aligned}$$

Equation (36) directly follows from the Fundamental Theorem of Calculus. We now turn our attention to (37). By Lemma 4, $\sigma_{t+1}$ is continuous and monotone. The image of $\sigma_{t+1}\left(\varepsilon^{t+1}, y\right)$ on $y \in [\widehat{\varepsilon}_t, \varepsilon_t]$ is $\left[\varepsilon_{t+1}, \sigma_{t+1}\left(\varepsilon^{t+1}, \widehat{\varepsilon}_t\right)\right]$. Hence, after changing the variables of integration, for all $k = t+1, ..., T$:

$$\int_{\varepsilon_{t+1}}^{\sigma_{t+1}\left(\varepsilon^{t+1},\widehat{\varepsilon}_t\right)} u_{k\varepsilon_{t+1}}\left(\gamma_t^k\left(\varepsilon^k,\widehat{\varepsilon}_t,y\right),x^k\left(\gamma_t^k\left(\varepsilon^k,\widehat{\varepsilon}_t,y\right)\right)\right)dy =$$
$$\int_{\widehat{\varepsilon}_t}^{\varepsilon_t} u_{k\varepsilon_{t+1}}\left(\gamma_t^k\left(\varepsilon^k,\widehat{\varepsilon}_t,\sigma_{t+1}\left(\varepsilon^{t-1},y,\varepsilon_{t+1},\widehat{\varepsilon}_t\right)\right),x^k\left(\gamma_t^k\left(\varepsilon^k,\widehat{\varepsilon}_t,\sigma_{t+1}\left(\varepsilon^{t-1},y,\varepsilon_{t+1},\widehat{\varepsilon}_t\right)\right)\right)\right)$$
$$\frac{\partial\sigma_{t+1}\left(\varepsilon^{t-1},y,\varepsilon_{t+1},\widehat{\varepsilon}_t\right)}{\partial y}dy. \quad (38)$$

Recall that by (11) the following is an identity in $y$:

$$u_k\left(\varepsilon^{t-1},y,\varepsilon_{t+1},...,\varepsilon_k,x^k\right) \equiv u_k\left(\gamma_t^k\left(\varepsilon^k,\widehat{\varepsilon}_t,\sigma_{t+1}\left(\varepsilon^{t-1},y,\varepsilon_{t+1},\widehat{\varepsilon}_t\right)\right),x^k\right),$$

so, by the Implicit Function Theorem,

$$u_{k\varepsilon_t}\left(\varepsilon^{t-1},y,\varepsilon_{t+1},...,\varepsilon_k,x^k\right)$$
$$= u_{k\varepsilon_{t+1}}\left(\varepsilon^{t-1},\widehat{\varepsilon}_t,\sigma_{t+1}\left(\varepsilon^{t-1},y,\varepsilon_{t+1},\widehat{\varepsilon}_t\right),...,\varepsilon_k,x^k\right)\frac{\sigma_{t+1}\left(\varepsilon^{t-1},y,\varepsilon_{t+1},\widehat{\varepsilon}_t\right)}{\partial y}. \quad (39)$$

Plugging (39) into (38) and noting that $\gamma_t^k\left(\varepsilon^k,\widehat{\varepsilon}_t,\sigma_{t+1}\left(\varepsilon^{t-1},y,\varepsilon_{t+1},\widehat{\varepsilon}_t\right)\right) = \rho_t^k\left(\varepsilon^k,y,\widehat{\varepsilon}_t\right)$ yields (37).

An identical argument can be used to deal with the case where $\widehat{\varepsilon}_t > \varepsilon_t$.$\square$