Weighing Experts, Weighing Sources: The Diversity Value[1]

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Abstract

A decision maker has to come up with a probability, preference or other judgment based on the judgments of a number of different information sources. To do so, he needs to assign weights to each source reflecting their assessed reliability. We argue that, crucially, reliability is to be understood as an attribute of sets of sources, not of sources in isolation. Specifically, we propose to view reliability as "valued diversity", reflecting both individual source quality and similarity among sources. Intuitively, larger weight should be assigned to sources of greater quality and greater dissimilarity from the others. The main contribution of this paper is to propose and axiomatize a particular weighting rule, the Diversity Value, that captures these desiderata. The Diversity Value is defined by a logarithmic scoring criterion and can be characterized as a weighted Shapley value, in which source weights are determined endogenously. Due to the central role of source similarity, the Diversity Value frequently violates Reinforcement and exhibits the No-Show Paradox.

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1 Introduction

Subjective Probability is, in many ways, an attractive, even compelling framework for the representation of uncertainty and the grounding of decision-making. However, even the best extant justifications such as Savage's axiomatic foundation are silent on the question of where subjective probabilities come from.

For strong personalists, this is just fine; yet many others have questioned the apparent arbitrariness of subjective probability, and have felt that "shooting from the hip" is not enough. In practical decision and prediction contexts, the existence of a (subjectively) sound basis for subjective probability judgments matters for the applicability and effectiveness of subjective probability as an analytical tool and decision aid. From a foundational perspective, the potential arbitrariness can give rise to questions about the rationality of SEU itself, as witnessed, for example, by many contributions to the literature on decision-making under ignorance and ambiguity. Here, we will show how the room for arbitrariness can be narrowed down substantially, perhaps even completely removed, in situations in which the would-be Bayesian decision maker has a lot of help in the form of the input of multiple "experts", or, more generally and abstractly, "sources".

There are two principal approaches to combining probability judgments in the literature, the *judgment aggregation* and the *belief revision* approaches; see Genest and Zidek (1986) for a classical and Clemen and Winkler (2007) for a recent survey.

The judgment aggregation approach. The judgment aggregation approach – which we shall pursue in this paper – takes the source judgments at face value, and aims to distill a best pooled judgment from the content of source judgments itself, from what the sources "say". If the expert judgments agree, this is unambiguous, and the agent can simply take the shared expert judgment as his own. If the expert judgments conflict, the decision maker needs to determine the optimum balance of argument by weighing the judgments of the various experts and aggregate them (average them) using appropriate weights. For example, if two experts estimate the probability of some event at 30% and 70%, and the two experts are viewed as equally reliable, then, on balance, the best-supported estimate for the DM is 50%. The question at the heart of this paper is how to generalize this simple syllogism to multiple, heterogeneous sources with possibly different reliabilities. This is a non-trivial task; for example, we shall argue that equally reliable sources should not necessarily be weighed equally.

The aggregation approach seems very intuitive. It recommends itself on both practical and theoretical grounds from the point of view of *informational minimalism*: it leaves the material judgment on the uncertainty at issue entirely to the sources. Hence, the aggregating agent ("aggregator") has to materially evaluate only the sources themselves in terms of their "reliability" (competence, credibility, expected accuracy etc.) regarding the matter at hand. The aggregating agent *defers* to the sources who may be viewed as his *epistemic intermediaries*. The aggregation approach is informationally minimalist since, evidently, *some* information or judgments regarding source reliability have to enter in any approach; one can hardly demand less from the decision-maker. Informational minimalism helps secure accessibility of the judgment task, hence, hopefully, leads to sound judgments. One would also expect well-designed minimalist approaches ("simple, but not too simple") to perform robustly.[4]

The Supra-Bayesian approach. We do not argue that the aggregation approach is the only or necessarily the best way to make use of a set of experts. In particular, an alternative approach that is common in the literature (and frequently referred to as "Supra-Bayesian") operates by belief revision via Bayesian updating. The belief revision approach asks the decision-maker to produce a prior over the uncertain true state $\omega \in \Omega$ under consideration and the profile of expert judgments, and to *derive* the target probability judgment on Ω via updating this prior on the actual profile of expert judgments observed.

The Supra-Bayesian approach does not take the input judgments at face value, but tries to look "under the hood". It is extremely flexible, since, in terms of principle, it treats the expert judgments as pieces of information just like any other pieces of information. The Supra-Bayesian approach thus *allows* to accommodate all sorts of information and hunches that a decision-maker may want to use in assessing the experts and the matter at hand directly. On the flip side, the Supra Bayesian approach *demands* a lot more of the decision maker than the aggregation approach. It may sometimes be feasible to satisfy these demands, or at least sensible to try, but many times this would appear to be a challenge. To begin with, the would-be Bayesian has

[4] Below, we derive some basic robustness properties of the solution concept developed here (the diversity value). Peter Klibanoff has rightly pointed to out us in personal discussion that arguing that informational parsimony is not enough for 'workability' if it is not accompanied by robustness of the derived judgments/decisions in the required inputs. We believe that the diversity value promises to perform strongly on both counts, but more work and experience in applications is needed to confirm this expectation.

to come up with a prior over a much larger state space than the one describing the value of interest. This is not just computationally more effortful, but would appear in many cases *hard* to do in a way that would or should inspire much confidence in the result.[5] In particular, the Supra-Bayesian approach requires a well-informed prior over the distribution of expert opinions conditional on the true state. Is this very plausible in situations in which the uncertainty at issue is unfamiliar and the experts disagree about how to interpret the available empirical evidence?
The Supra-Bayesian approach is sometimes referred to as "the" Bayesian approach (e.g. Clemen and Winkler 2007), suggesting that the aggregation approach is somehow less than fully Bayes-rational. But such a conclusion would be premature and misleading. In particular, the aggregation approach is fully consistent with subjective expected utility. Yet, rather than using Bayes rule, it constructs aggregator probabilities conditional on the profile of source judgments directly, leaving the joint distribution on judgment profiles undetermined. If needed, the aggregator is free, of course, to fill in this gap on his own.

"Experts" are but one example of "sources" of judgment formation. More generally, the different sources can be in the agent's own head, so to speak, as different models, theories, heuristics, frames, perspectives etc. Source aggregation will be relevant if the agent has not been able to come up with an encompassing way to integrate all of the available pieces of evidence directly, but acknowledges the partial validity of a multiplicity of them. For instance, laymen can be valued as relevant sources besides the "experts" in the ordinary sense; they may, for example, contribute "common sense" that the experts may not necessarily have. Similarly, the aggregator may use his own personal "gut judgment" as one source along with the judgment of others.
There is no set minimum threshold for the quality of the cognitive input generating the source judgments. Source aggregation can be applied, in principle, to whatever hunches and epistemic crutches the decision maker has at his disposal, as long as they can be cast in probabilistic form. Our model can then be viewed as explaining how the DM can construct minimally sensible probability judgments without engaging in demanding forms of cognitive integration.
On the flip side, one might be inclined to think that source aggregation by weighted averaging, while sensible, is perhaps too unambitious, and that, ordinarily, decision-makers are equipped with a rich sensorium of personal judgment that could exploit source inputs in far more sophis-

[5] Indeed, it is well-known that the Supra-Bayesian approach suffers from potential robustness problems; see, for example, Winkler and Clemen (1992).

ticated and accurate way than mere averaging can. Would that it were so! For in practice, it has proven very hard to demonstrate that personal probability judgment can beat even simple statistics, as shown by the influential contributions by Meehl (1955) and Dawes (1979). Tetlock (2005) provides a recent, extremely thorough and rich study of this in the context of expert political judgment.

Determining Source Weights. To make the aggregation approach operational, two things are needed: an aggregation rule and aggregation weights. The literature, in particular the literature on the aggregation of probabilities, has focused on the characterization of various aggregation rules, most prominently on the linear and logarithmic "opinion pools", see the surveys by Genest and Zidek (1986), Clemen and Winkler (2007), and Dietrich and List (2014) for references. Both of these can be viewed as given by weighted averages of source judgments taking values in some convex subset of some vector space, see in particular Rubinstein and Fishburn (1986), who obtain these as instances of the same 'abstract' aggregation theorem. Which of these rules, if any, is "the right" one, remains controversial. We sidestep this important issue here, but assume that the aggregator is committed to some class of aggregation rules that can be expressed as a weighted average.

Instead, we address here the second important issue which has remained largely unaddressed in the literature, the Weighting Problem: how are the source weights themselves to be determined? To get a sense of what a sound and helpful answer might look like, we need to ask why it is worth aggregating at all? Why not simply put all weight on the best one (or ones)? To see this, consider two experts, Ann and Bob, one of whom You view as slightly more reliable than the other. Why not put all the weight on the superior expert Ann? Sometimes this might indeed be the best, and, arguably, would be best if Bob was simply an inferior copy of Ann. Yet, frequently, it would be preferable to shift some weight to Bob if Bob's judgment reflects something that Ann might have missed. Even if Bob has missed more than Ann has, his opinion seems worth taking into account at least to some extent. This emerges even more clearly in the case in which You view Bob as equally reliable to Ann. Then it seems strictly better to give both equal weight, rather than arbitrarily plump for one of them. By continuity, if Bob is inferior but has something of his own to add, You would want to give him positive weight. More broadly, reliability is based on "valued diversity": aggregation helps because it allows to (optimally) combine the disparate, and unequally valuable, perspectives of the different sources.

Reliability as valued diversity must be understood as an attribute of sets of sources, not of singleton sources by themselves. It is not enough to think of reliability in terms of the quality of individual sources, and take weights to reflect relative source quality directly. Such a "fixed-weight" approach would assign each potential source $i \in I$ a non-negative real number q_i measuring its "quality". In any subset of available sources $J \subseteq I$, the relative weights of two sources $\frac{w_i^{[J]}}{w_j^{[J]}}$ are given by the ratio of qualities $\frac{q_i}{q_j}$; so $w_i^{[J]}$ is given by $\frac{q_i}{\sum_{j \in J} q_j}$. But, we submit, that this is too simple, and too restrictive[6].

To see the problem, consider the case in which each of the n experts is viewed as equally reliable in pairwise comparison. Thus, by assumption, if the aggregator has to base his judgment on the judgments of just two of them, he deems it best to weigh them equally. Does this imply that if the judgments of all n experts are available, each of them should be assigned weight $\frac{1}{n}$?

The fixed-weight approach answers this question in the affirmative: assigning equal weights in pairwise comparisons reveals equal underlying quality, which in turn implies equal weight assignments in any set. But this seems far too strong: equal quality does not necessarily imply equal weights. To the contrary, the similarity of sources – in the epistemically relevant ways – matters crucially as well. To take an extreme case, consider a situation with three experts Ann, Bob and Rob such that two of them, Bob and Rob, are epistemically indistinguishable ("clones"). In such a situation, the judgment of Rob merely duplicates that of Bob. Hence, from the point of view of the aggregator, the availability of Rob does not add anything; the weight of the non-cloned expert Ann should thus remain at $1/2$, rather than drop to $1/3$. More generally, if Bob and Rob are more similar to each other than they are to Ann, Ann's weight should be the largest among the three, since she is the most distinct expert. We shall call this the *Similarity Effect* .

As a second example, consider a situation in which one expert a "dominates" the other, b: the aggregator not only deems b to be inferior to a, but b's judgment does not add anything to a's judgment. For instance, b may be a popularizer of a or a disciple of a, or b could be yesterday's somewhat less informed or less reflected version of a. Thus, if only a and b are available, the aggregator would simply follow a's judgment ignoring b's, i.e., $w_a^{[\{a,b\}]} = 1$ and $w_b^{[\{a,b\}]} = 0$. What does this imply for the relative weights in pairwise aggregation in the presence of a third,

[6] Much of the literature on expert aggregation, especially aggregation of probability judgments, reads that way, but this is hard to pin down. The reason is that the basic characterization results are based on variable profiles (of judgments) but a constant set of experts, so they do not pin down whether and how relative weights change with the set of available sources.

unrelated source c? The fixed-weights approach implies that, if c receives positive weight in the presence of a, $w_c^{[\{a,c\}]} > 0$, then it must receive all the weight in the presence of b, $w_c^{[\{b,c\}]} = 1$, and b none. Again, this seems far too strong. Arguably, all that one can validly infer is that b's weight should be smaller than a's, $w_b^{[\{b,c\}]} < w_a^{[\{a,c\}]}$; after all, b, considered in isolation, may be almost as reliable as a while being dominated. We call this the *Dominance Effect.*

The Similarity and Dominance effects demonstrate that the fixed-weights approach is too restrictive. "Reliability" is thus not simply an attribute of singleton sources, but needs to be thought of as applying to sets of sources. We thus propose to describe sources in terms of a "*reliability function"* $v : 2^{|I|} \to \mathbb{R}_+$. This set function will typically be non-additive, reflecting the fact that the increase in reliability due to a given source will typically depend on the presence of other sources.[7]

The Diversity Value. As v is generally non-additive, one needs to derive additive reliance weights from non-additive reliability measures. Formally, this is analogous to determining the value of a cooperative game. Yet, perhaps not very surprisingly, off-the shelf solution concepts from the literature such as the Shapley value do not work here. Thus, this paper sets out to axiomatically derive an appropriate weighting rule, the "Diversity value", and study some of its properties. In particular, we show that the Diversity Value naturally accounts for the Similarity and Dominance Effects (see Section 5 for formal results).

In order to define and characterize the Diversity Value, we need to assume that reliability functions have the formal structure of diversity functions as introduced in Nehring-Puppe (2002), henceforth NP. Principally, this requires submodularity, which means that the marginal reliability of any source decreases (weakly) in the presence of other sources, corresponding to a view of sources as epistemic substitutes. As argued in more detail in Section 2, this assumption of sources as epistemic substitutes follows in turn naturally from the aggregator's deference to sources as epistemic intermediaries which is central to the aggregation approach. The restriction of reliability functions to diversity functions has an important potential side-benefit, as it makes possible a 'quasi-objective' explanation of reliability functions in terms of the diversity of sources in relevant epistemic characteristics, in particular their richness and (dis)similarity.

[7] This provides room for the Similarity and Dominance Effects. The Similarity Effect arises in situations where all sources are equally reliable as singletons, but where some pairs are deemed less reliable by the aggregator than others, reflecting their greater similarity. The Dominance Effect, on the other hand, arises in situations in which one source adds to the reliability of some sources but not to that of others, i.e., $v(b,c) > v(c)$, while $v(b) < v(a) = v(a,b)$.

Loosely speaking, the Diversity Value can be viewed as the "best approximation" of the non-additive diversity function by an additive weighting function: if the diversity function is additive, the Diversity Value coincides with it. Thus, the fixed-weight approach is embedded as a special case. Formally, the diversity value is based on the maximization of a logarithmic scoring criterion and can be viewed as an extension of logarithmic scoring rules for probabilities (additive, non-negative set functions) to diversity functions, see for example Savage (1971) and Lindley (1982). We also characterize the Diversity Values as version of the Shapley value in which source weights are determined endogeneously.[8]

Potential Applications. The Diversity value has a wide range of potential applications. We have motivated it above in terms of the aggregation of probabilities, but it is relevant to the aggregation of judgments derived from multiple sources quite broadly. For instance, a consumer may evaluate a product based on the product ratings on different websites that may use different algorithms and/or different data inputs.

But even outside source aggregation, the Diversity value may be applicable in many contexts in which weighted averages are taken, such as in the construction of many index numbers. Often, such indices are constructed based on multiple criteria or sub-indices. As there typically is no canonical "head-count" of criteria, criteria need to be weighed explicitly or implicitly, and similarity effects frequently matter. As a classical example from the realm of sports, take the decathlon in track-and-fields which serves to determine the best all-around track-and-fields athlete. An equal weighting of Olympic disciplines would result in a composite weights of $9 : 4 : 3$ for running, throwing and jumping respectively. By contrast, the actual weights are $4 : 3 : 3$, reflecting the greater similarity among running disciplines.[9] There are many economic applications. For example, one question that has attracted recent attention is the imputation of a preference ordering in the presence of observed choice inconsistencies. One natural approach to this is to evaluate how well a hypothetical preference ordering explains a particular choice in a particular choice situation, and to form some weighted average over choice situations, see in particular Apesteguia and Ballester (2013) and Chambers and Hayashi (2012) for relevant axiomatics. For this method to produce a sensible measure, it will have to appropriately reflect potential (dis)similarities between choice situations.

[8] While the Shapley value itself displays the Similarity Effect, it does not accomodate the Dominance Effect.

[9] In further recognition of similarity effects, the running distances are well spaced (100m, 400m, 1500m, plus 110m hurdles) if, for obvious pragmatic reasons, a bit tilted towards the short end.

Another pertinent example is the case-based decision theory (CBDT) as developed by Gilboa and Schmeidler (2001) and in a long line of works. There, past cases take the role of sources, and the judgment (prediction) regarding the case at hand takes the role of the aggregator's target judgment. In broad terms, the Diversity value looks appealing here; in particular, the Similarity Effect would amount to a discounting of the joint weight of a set of similar cases relative to their individual weight. This seems sensible and relevant in many applications.

While formally related, case-based induction addresses a different problem than source-based judgment aggregation (SBJA): whereas the latter concerns the resolution of differences in judgment, the former concerns the issue of how to learn from past experience encoded in a data-base. There is no counterpart in source-based judgment aggregation to the guiding idea of CBDT that "From causes that appear similar we expect similar effects. This is the sum of all of our experimental conclusions", Hume (1748, Section IV). Also, while CBDT seems more naturally applicable in some environments than others, the structure of the uncertain environment has no obvious bearing on the applicability of SBJA (since here the structure of the uncertainty at issue is captured within the judgments of the sources on whom the aggregator relies). So CBDT and SBJA bear on quite different problems, and need to be evaluated independently on their own terms.

At the same time, there are interesting points of contact between SBJA and CBDT. In particular, the main result of Billot et al. (2005) (henceforth BGSS) can be read as a result on SBJA, specifically as a characterization of the linear opinion pool in probability aggregation with fixed weights. Their central axiom is a version of the reinforcement axiom of Young (1975) and Myerson (1995) introduced to characterize scoring rules in voting theory. We have pointed out the limitations of the fixed-weights-assumption above; the role of the reinforcement axiom within the present framework and its close relation to the class of additive reliability functions are spelled out in more detail in Section 5 below.

"Similarity" plays a key role in both CBDT and SBJA, but in rather different ways. BGSS interpret their fixed weights as similarities between past cases and the case at hand. While attractive from a case-based perspective, this notion of similarity has little heuristic value from a judgment-aggregation perspective. By contrast, in SBJA, similarity plays an important role as similarity *between sources*, and thus as a key determinant of the context-dependence of weights. Yet similarities between past cases (cases within the data base) should be quite relevant for

CBDT as well; we thus plan to devote future work to explore to the applicability of the Diversity value to CBDT.

The rest of the paper is organized as follows. In Section 2, we define a class of reliability functions, which describe the perceived reliability of a set of sources. We then impose some basic requirements on how the perceived reliability should be reflected in the weights used for aggregation. Section 3 introduces the Diversity value as a way to determine the weights assigned to sources, discusses its properties and provides some examples. Section 4 provides the axioms characterizing the Diversity value and states the main representation theorem. In Section 5, we explain how the perceived similarity between sources (as captured by the reliability function) affects the weight assignment and the aggregate judgment. Section 6 concludes. All proofs are collected in the Appendix.

2 From Source Reliability to Source Weights

2.1 General Setting

Let I be a finite set of information sources (e.g., experts, models), each of which supplies a judgment to the aggregator. The source judgments are to be aggregated using appropriate weights for each source. In turn, the determination of weights is to be based on the aggregator's assessment of the reliability of various subsets of sources. This assessment is summarized by a *reliability function* $v : 2^{|I|} \to \mathbb{R}$.

"Reliability" is introduced here as technical term like "utility" and "probability". Heuristically, the reliability of a set of sources $J \subseteq I$ reflects the aggregator's assessment of its informativeness on the assumption that his judgment on an issue or set of issues is to be wholly based on the material judgments of exactly these sources. Technically, reliability functions can be viewed as von-Neumann Morgenstern "utility" functions representing rankings of lotteries (probability distributions) of sets of sources in terms of their expected reliability, as in NP; reliability functions thus have an affine scale. For the following, it is useful to normalize $v(\varnothing) = 0$, thus leaving v with a ratio-scale (unique up to positive multiplication).

In the next subsection, we shall describe which set functions $v : 2^{|I|} \to \mathbb{R}$ should count as admissible reliability functions. Let their set be given by $\mathbf{V}$. For each $v \in \mathbf{V}$, we would like to find a vector $w \in \Delta^{|I|-1}$ which assigns to each source $i \in I$ a weight w_i. These weights can then be used to aggregate the judgments submitted by the sources, e.g., by taking the weighted average, the median, etc. Hence, we are looking for a mapping Ψ which assigns to each $v \in \mathbf{V}$ a (set of) weight(s) $w \in \Delta^{|I|-1}$ in a consistent way. Such a mapping $\Psi : \mathbf{V} \rightrightarrows \Delta^{|I|-1}$ will be called a *value*.

The weights specified by the value Ψ are to be used in the aggregation of source judgments. In the simplest, paradigmatic case, source judgments are given by a real number, representing, for example, the expected value of some random variable; if that random variable is an indicator function $\mathbf{1}_E$, source judgments represent the subjective probability of the event E. In these cases, Ψ induces *aggregation scheme* $\widehat{\Psi} : \mathbb{R}^{|I|} \times \mathbf{V} \to \mathbb{R}$ given by

$$\widehat{\Psi}\left(x_1, ..., x_n; v\right) = \{x \mid x = \sum_{i \in I} w_i x_i, w \in \Psi\left(v\right)\}. \tag{1}$$

Note that the potential set-valuedness of Ψ may spill over into set-valuedness of $\widehat{\Psi}$. We will show, however, that this is not an issue for the Diversity value which is set-valued only in rather degenerate cases.

The Diversity value applies to much more general aggregation schemes than those in the form of (1). First of all, the notion of weighted averaging applies not just to the real line, but to general convex subsets of finite or infinite-dimensional vector spaces. Second, weights need not be used via averaging. A substantially more general form of aggregation schemes is based on the weighted scoring functions in the sense of Myerson (1995); see also Pivato (2013) for a recent study. To define these, let $\mathfrak{X}$ be an abstract space of judgments, and $\ell : \mathfrak{X} \times \mathfrak{X} \to \mathbb{R}_+$ denote a loss function with the property that $\ell\left(x, x\right) = 0$ for all $x \in \mathfrak{X}$. Loss functions might be given by metrics, but need not satisfy symmetry or the triangle-inequality in general. The loss function ℓ induces the aggregation scheme $\widehat{\Psi} : \mathfrak{X}^{|I|} \times \mathbf{V} \to \mathbb{R}$ given by

$$\widehat{\Psi}\left(x_1, ..., x_n; v\right) = \bigcup_{w \in \Psi(v)} \arg\min_{x \in \mathfrak{X}} \sum_{i \in I} w_i \ell\left(x_i, x\right). \tag{2}$$

The weighted arithmetic average in (1) is obtained by taking ℓ to be the square Euclidean distance; by contrast, the weighted median is obtained by taking ℓ to be Euclidean distance itself. The functional form (2) is extremely flexible. If $\mathfrak{X} = \{0, 1\}$, it gives rise to weighted supermajority-voting. If $\mathfrak{X}$ is the set of linear orderings on a finite set of alternatives, then different loss-functions ℓ induce the weighted Borda and weighted Kemeny rules. In this example, the judgments submitted by the sources could be viewed as their recommendations on the best course of action in terms of an ordinal ranking.

2.2 The Structure of Reliability Functions

Sources as Epistemic Substitutes. Which set functions v are meaningful reliability functions? First of all, reliability is monotone: additional sources cannot harm the aggregator (in his own assessment), since he is free to ignore any source that he deems as wholly irrelevant or useless. To go further, we rely on a concept of sources as *epistemic substitutes.* This concept makes sense within the aggregation approach within which the agent defers to the sources as epistemic intermediaries and is thus confined to making only 'schematic' use of the source judgments by weighted averaging. By contrast, outside the aggregation approach, sets of pieces of information need not be epistemic substitutes at all. As it may take two keys to unlock a door while one would be useless, it may take two witnesses to establish the guilt of an accused, while the testimony of one would be inconclusive. But such complementarities require a material interpretation of the content of the witness judgments, not merely an averaging of their individual stand-alone assessments of the defendants' chance of being guilty.

The concept of sources as epistemic substitutes motivates the formal assumption that v should have the structure of a diversity function in the sense of NP. We will refer to this assumption as the Diversity Postulate. It implies in particular that v is *submodular*: for all $S \subseteq S'$ and $i \notin S$,

$$v(S \cup \{i\}) - v(S) \geq v(S' \cup \{i\}) - v(S').$$

As a property of measures of diversity, this is very intuitive. The larger the extant set already is, the harder it becomes for an element to add further diversity. For reliability functions, it also makes sense to postulate that the larger the extant set S, the (weakly) less any given source adds to its reliability, as expressed by submodularity.

Diversity functions strengthen this by assuming *total submodularity* which may be understood as requiring 'uniformly decreasing' marginal reliability. To expand, Monotonicity can be understood as requiring first-order "derivatives" $v(S \cup \{i\}) - v(S)$ to be non-negative; likewise, submodularity can be understood as requiring second-order "derivatives" $v(S \cup \{j\} \cup \{i\}) - v(S \cup \{j\}) - v(S \cup \{i\}) + v(S)$ to be non-positive. Generalizing this, total submodularity amounts to requiring that all higher-order derivatives can be signed, with alternating sign. [10]

Existence of a multi-attribute representation. As elaborated in detail in NP, a key benefit

[10] Formally, it is shown in NP that the attribute measure λ defined below is non-negative if and only if v is totally submodular, i.e., ν is monotone ($S \subset S'$ implies $\nu(S) \leq \nu(S')$) and satisfies

of the assumption of total rather than plain submodularity is the existence of a multi-attribute representation $\lambda : 2^{|I|} \backslash \{\varnothing\} \rightarrow \mathbb{R}$ such that

$$v(S) = \sum_{\{A | A \cap S \neq \varnothing\}} \lambda_A, \tag{3}$$

with $\lambda_A \geq 0$ for all $A \subseteq I$.

In this representation, the reliability of a set S, $v(S)$, is the sum of the values λ_A of all "attributes" A possessed by some element in S. Any set function v has a (unique) representation λ satisfying (3); it is given by the inverse of the linear map $\lambda \mapsto v$, its "Conjugate Moebius Inverse", as follows:

$$\lambda_A = \sum_{\{S | S \subseteq A\}} (-1)^{|A \backslash S|+1} v(S^c), \tag{4}$$

where S^c denotes the complement of S in I. The substantive restriction on reliability functions is thus the non-negativity constraint on the implied attribute values λ_A. The reliability function v and the *attribute measure* λ are thus two dual ways to describe the same object.

Diversity as evidential basis for reliability. At one level, one can take λ to be merely a formal, mathematically illuminating "dual" redescription of v. From this perspective, λ_A can be understood as the implied contribution to reliability that is specific to the set of sources A. The dual λ proves an excellent reasoning tool and intuition pump. Indeed, it is essential for the very definition and characterization of the Diversity value to begin with.

At another level, one can take the multi-attribute representation λ to describe identifiable features of the sources, as deemed relevant and valued by the aggregator. From this "quasi-objective" viewpoint, λ measures the distribution of valuable epistemic features across sets of sources. Examples of such features are "being an ecologist", "having a doctoral degree from MIT", "having published a paper in *Nature"*, etc.; in our formalism, the attribute associated with a feature is simply the set of sources having that feature. The numeric value of an attribute λ_A can then be viewed as the total measure of the *reliable expertise* specific to the set A. Often, only a subset of the set of conceivable attributes will be realized at all in the sense of having

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the condition: for all collections of subsets of I, $\{S_j\}_{j \in J}$,

$$\nu(\cap_{j \in J} S_j) \leq \sum_{\{K | \varnothing \neq K \subseteq J\}} (-1)^{|K|+1} \nu(\cup_{j \in K} S_j).$$

For the interpretation of total submodularity in terms of higher-order differences, see Nehring and Puppe (2004) following Choquet (1953). The former paper also contains a formal argument (Theorem 3) supporting the claim that the gap between submodularity and total submodularity is "small". For related work on 'totally supermodular' set functions, see Crama et al. (1989).

strictly positive value $\lambda_A > 0$. This set of "relevant attributes" will be denoted by Λ. The reliability of $v(S)$ is then the total amount of reliable expertise possessed by some source in S. On this quasi-objective interpretation, the reliability of a singleton source i is simply the amount of reliable expertise possessed by i. A source i offers high marginal reliability to S if i possesses distinct valuable attributes possessed by no source in S :

$$v(S \cup \{i\}) - v(S) = \lambda(\{A \mid i \in A, A \cap S = \varnothing\}).$$ [11]

Further, the attribute measure λ yields natural notions of dissimilarity $d(i,j)$ and similarity $s(i,j)$ between sources.

Definition 2.1 ***(Dissimilarity)*** *The dissimilarity between i and j, $d(i,j)$ is the marginal reliability i adds to j:*

$$d(i,j) =: v(\{i,j\}) - v(\{j\}) = \lambda(\{A \mid i \in A, j \notin A\}). \tag{5}$$

Dissimilarity need not be symmetric; it is symmetric iff all singletons have equal value, which we will refer to as the uniform case. Whether symmetric or not, d satisfies the triangle inequality.

Likewise, one can define a natural notion of similarity as the overlap in valued attributes:

$$s(i,j) := \lambda(\{A \mid i \in A, j \in A\}).$$

Evidently, for any $i, j \in I$, $v(i) = d(i,j) + s(i,j)$. By construction, s is symmetric; interestingly, any non-negative, symmetric function $s : I \times I \to \mathbb{R}_+$ is the similarity function for some attribute metric λ. The multi-attribute model of diversity thus links up naturally with intuitive, everyday notions of dissimilarity and similarity.

As in our everyday understanding, it makes sense to think of dissimilarity, similarity and diversity judgments as saying something (coarsely) about the world. It is this aboutness which allows one to think of the attribute metric λ as an evidential basis for the reliability function v. At the same time, those judgments are not simply statements of 'raw facts', but necessarily incorporate a subjective, 'judgmental' element. In particular, (dis)similarity judgments are meaningful only relative to a context of "relevance" which determines which attributes matter and how much. Relevance of experts, for example, is determined by their epistemic competence in regards to the judgment task at hand.[12]

[11] This meaures notation is often convenient; we write $\lambda(\mathcal{A})$ for $\sum_{A \in \mathcal{A}} \lambda_A$.

[12] We believe that the potential objectifve interpretation of the central primitive, the reliability function, is an important strength of the proposed model. At the same time, it is not essential to its conceptual coherence, and can be viewed as a more-or-less appealing piece of rhetorics.

2.3 Some Terminology

We now state some properties of the reliability (diversity) function which will be useful in the sequel. The first one, uniformity, expresses equal reliability of all sources.

Definition 2.2 *A diversity function v is called **uniform** if $v(i) = v(j)$ for all $i, j \in I$.*

Intuitively, uniformity captures the idea that, taken in isolation, each of the sources is equally informative.

The second property strengthens the requirement that v is monotone to strict monotonicity:

Definition 2.3 *A diversity function v is **strictly monotone** if for any $S \subset S'$, $v(S) < v(S')$.*

Note that, by submodularity, v is strictly monotone iff the condition is satisfied for $S' = I$ and all S such that $S = \{I \setminus \{i\}\}$ for some $i \in I$. Condition (4) in turn implies that this is equivalent to[13] $\lambda_i = v(I) - v(I \setminus \{i\}) > 0$ for all $i \in I$, i.e., each of the sources possesses a specific relevant attribute not shared by any other source.

Definition 2.4 *A diversity function v is **additive** if for all $S \subseteq I$, $v(S) = \sum_{i \in S} v(i)$.*

Clearly, v is additive if all relevant attributes are singletons.

We can also use the properties of the reliability function to describe the type of information sources available. We can, e.g., identify sources which have no specific attributes on their own. Such sources are called inessential:

Definition 2.5 *A source i is **inessential** if $v(I \setminus \{i\}) = v(I)$.*

Obviously, a source i is inessential iff $\lambda_i = 0$. Hence, a reliability function is strictly monotone iff all sources are essential.

A special cases of an inessential source is a null source, which does not have any attribute with a positive value, i.e., $\lambda_A = 0$ whenever $i \in A$:

Definition 2.6 *A source i is **null** if $v(S \cup \{i\}) = v(S)$ for all $S \subset I$.*

Clones, or sources which are epistemical copies of each other present another example of inessential sources:

Definition 2.7 *Sources i and j are **clones** if $v(S \cup \{i\}) = v(S \cup \{j\}) = v(S \cup \{i,j\})$ for all $S \subset I$ such that $i, j \notin S$.*

[13] In the sequel, we slightly abuse notation by omitting the curly brackets and writing λ_i for $\lambda_{\{i\}}$, λ_{ijk} for $\lambda_{\{i,j,k\}}$, as well as $v(i)$, $v(i,j)$ instead of $v(\{i\})$, $v(\{i,j\})$ etc.

It is easy to see that i and j are clones iff they possess the same set of attributes.

Below, we will encounter and discuss in detail a further instance of inessential sources, that of dominated sources.

A value Ψ is defined for a fixed set of sources I. However, Ψ also determines how weights depend on which subset of sources J is effectively available; such availability restrictions can be modeled simply by considering the effective restriction $v^{[J]} \in \mathbf{V}$ of v to a subset $J \subseteq I$.

Definition 2.8 ***(Effective Restriction to J)*** *For any* $S, J \subseteq I$, $v^{[J]} := v(S \cap J)$.

Conceptually, the reliability of any subset of J is simply the reliability of available sources within J. The sources in $I \backslash J$ are evidently null under $v^{[J]}$. Note also that Conjugate Moebius inverse of $v^{[J]}$, $\lambda^{[J]}$ is given by

$$\lambda_A^{[J]} = \sum_{\{B|B \cap J = A\}} \lambda_B,$$

for any $A \subseteq I$. Thus, the reliable expertise specific to A within J is the total value of reliable expertise of any superset of A in I whose members in J are just A.

2.4 From Source Reliability to Source Weights: The Two Fundamental Axioms

Given a reliability assessment $v \in \mathbf{V}$, what are the optimal weights $w \in \Delta^{|I|-1}$? A particular concept of optimality will be described by a, possibly multi-valued, mapping $\Psi : \mathbf{V} \rightrightarrows \Delta^{|I|-1}$ from the set of reliability assessments to the set of weight vectors. Formally, we are looking for the "value" of a cooperative game. While this analogy is heuristically somewhat useful, one cannot expect to be able to take solution concepts from the literature off the shelf.

Simply, if loosely, the question is this: how does reliance follow from reliability? We consider two axioms in particular as fundamental, which we shall call the Core and the Dominance axioms. They provide lower and upper bounds for reliance weights given a particular reliability function.

The guiding rough intuition behind both axioms is the idea that reliance weights of sources should be aligned with their marginal contribution to the overall reliability of the set of available sources. Recall that, by submodularity, the marginal reliability of any subset is decreasing in the size of the set of co-available sources. Thus, for any pair of sets J, J',

$$\frac{v(I) - v(J^c)}{v(J')} = \min_{\{S,T|S \cap J = \varnothing, T \cap J' = \varnothing\}} \frac{v(J \cup S) - v(S)}{v(J' \cup T) - v(T)}$$

provides a lower bound on the ratios of marginal reliabilities. This in turn provides a natural lower bound on the reliance weights of sets of sources.

Axiom (Core) For all $w \in \Psi(v)$, and any sets $J, J' \subseteq I$, $\frac{w(J)}{w(J')} \geq \frac{v(I)-v(J^c)}{v(J')}$.

Simple algebra reveals that this is equivalent to requiring that, for all $w \in \Psi(v)$ and all $J \subseteq I$,

$$w(J) \geq \frac{v(I) - v(J^c)}{v(I)}. \tag{6}$$

In particular, if $v(I) = 1$, then one can express this dually in terms of attributes as

$$w(J) \geq \lambda(\{A \mid A \subseteq J\}).$$

Thus, the Core axiom requires that the value of all attributes unique to sources in J be allocated to these sources.[14]

Remark 2.1 *We have used the term "core" for what is technically the game-theoretic core of the 'conjugate' game $S \mapsto v(I) - v(S^c)$. Hopefully, the core terminology triggers helpful associations. In particular, it may help to think of every subset of sources as being 'entitled' to its marginal contribution to the reliability of the others.*

Remark 2.2 *In the axiomatization to follow, we need only a special, boundary case of the Core axiom applied to additive reliability functions. In that case, the axiom requires the optimal weights to be proportional to reliabilities. Note that additivity of a reliability function is equivalent to the absence of any similarity, i.e. to the condition that $s(i,j) = 0$ for all $i, j \in I$; hence, the term "Additive Core" for the following axiom:*

Axiom (Additive Core) For all additive $v \in \mathbf{V}$, $\Psi(v) = \left\{ \left(\frac{v(i)}{v(I)}\right)_{i \in I} \right\}$.

The second axiom applies to situations in which one source "dominates" another.

Definition 2.9 *Source i* ***dominates*** *source j if, for all S s.t. $i, j \notin S$, $v(S \cup \{j\}) \leq v(S \cup \{i\}) = v(S \cup \{i,j\})$, with at least one strict inequality.*

Note that, by submodularity, i dominates j if and only if $v(j) < v(i) = v(i,j)$. Equivalently, i dominates j if and only if i belongs to a strictly larger set of valued attributes, i.e. iff

$$\{A \in \Lambda \mid i \in A\} \supset \{A \in \Lambda \mid j \in A\}.$$

If i dominates j, j's judgment adds nothing to that of i, hence, it should be ignored without loss.

Axiom (Dominance) If i dominates j, $w_j = 0$ for all $w \in \Psi(v)$.

A more fine-grained argument can be given. Consider any hypothetical assignment of weights $w \in \Delta^{|I|-1}$ with $w_j > 0$. Then the weight assignment $w' = \left(w_{-\{i,j\}}, w_i + w_j, 0\right)$ with $w'_j = 0$, $w'_i = w_i + w_j$, and $w'_k = w_k$ for $k \neq i, j$, is superior to the weight assignment w as it replaces the

[14] This is reminiscent of Shafer's (1976) notion of allocating probabilty mass in the context of non-additive probability theory.

reliance on source j with weight w_j by more extensive reliance on the dominating source i. It follows that any optimal assignment of weights w must involve zero reliance on the dominated source j.

Note that, for this argument to work, j needs to be dominated by a specific source i to which the weight can be shifted in an improving manner. It is not enough that j merely be *inessential*, i.e. that $v(I) = v(I \backslash j)$. The argument for Dominance fails to extend to this case, because while j may appear not to add anything, it is not clear how weights should be shifted away from j in an improving manner. Indeed, note that one cannot even coherently require inessential sources to carry zero weight in all cases, since there exist reliability functions $v \in \mathbf{V}$ for which all sources are inessential.

While the Core axiom is standard in cooperative game theory, the Dominance appears novel. Indeed, we are not aware of a cooperative value that satisfies it. For example, while the Shapley value satisfies the Core axiom, it violates Dominance. To see this, observe that, for any reliability function v, by monotonicity of v, the Shapley value assigns zero weight to a source i if and only if that source is a null source, i.e. $v(J) = v(J \backslash \{i\})$ for all $J \subseteq I$; this is equivalent to i being dominated by all other sources, not just by some source as assumed by Dominance.

3 The Diversity Value: Definition, Examples and Normative Properties

The central task of the paper is to define a value that satisfies the Core and Dominance axioms (and is otherwise well-behaved), and to characterize this value axiomatically. That value will be referred to as the Diversity value. Intuitively, the Diversity value tries to allocate the weight of sources in line with their distinct marginal contribution to the overall reliability of the available set. Formally, the diversity value is based on the maximization of a logarithmic scoring criterion and can be viewed as an extension of logarithmic scoring rules for probabilities (additive, non-negative set functions) to diversity functions, see for example Savage (1971) and Lindley (1982).

Specifically, $\Psi^{DIV}(v)$ is the set of maximizers of

$$\sum_{A \subseteq I} \lambda_A \phi\left(w\left(A\right)\right), \tag{7}$$

where $\phi(\cdot) = \ln(\cdot)$[15].

Definition 3.1 ***(Diversity value)*** *For a given v with associated λ,*

$$\Psi^{DIV}(v) = \arg \max_{w \in \Delta^{|I|-1}} \sum_{A \in \Lambda} \lambda_A \ln \left(\sum_{i \in A} w_i \right). \tag{8}$$

The functional form (7) with a strictly monotone ϕ ensures Dominance, while the specification $\phi(\cdot) = \ln(\cdot)$ guarantees the Core property. We shall refer to the class of values defined in (7) as *score-based values,* Ψ_ϕ^{SCV}. The 'scoring function' ϕ 'rewards' a larger weight assignment to any attribute A. Since the total weight mass to be assigned is fixed, the additively separable functional form (7) rewards large weight assignments to those sources i that possess attributes $A \ni i$ with large distinct reliability λ_A.

To see that score-based values Ψ_ϕ^{SCV} satisfy Dominance, consider a hypothetical weight assignment w violating that condition, i.e., assigning positive weight $w_j > 0$ to some source j that is dominated by some other source i. Consider the weight assignment

$$w' = \left(w'_i = w_i + w_j, w'_j = 0, (w'_k = w_k)_{k \notin \{i,j\}} \right).$$

By construction, for all $A \in \Lambda$, $w'(A) \geq w(A)$, and for some $A \in \Lambda$ containing i but not j, $w'(A) > w(A)$. By strict monotonicity of ϕ,

$$\sum_{A \in \Lambda} \lambda_A \phi(w'(A)) > \sum_{A \in \Lambda} \lambda_A \phi(w(A)),$$

thus $w \notin \Psi_\phi^{SCV}(v)$.

However, in general, score-based values Ψ_ϕ^{SCV} do not satisfy the Core property, indeed not even the Additive Core property. For example, if ϕ is linear, Ψ_ϕ^{SCV} puts weight only on those sources i that have maximal value $v(i)$ as singletons. Indeed, Ψ_ϕ^{SCV} satisfies the Additive Core property if and only if ϕ is logarithmic, i.e. if and only if $\Psi_\phi^{SCV} = \Psi_{\ln}^{SCV} = \Psi^{DIV}$. (The simple calculus argument for this is actually standard in the above-mentioned literature on probabilistic scoring rules). Satisfaction of the Core property for general v's can be shown by inspection of the first-order conditions of (8), see the proof of Part (iii) of Proposition 3.1 in the Appendix.

The following is a list of a number of properties satisfied by the Diversity value.

Proposition 3.1 *The Diversity value Ψ^{DIV} satisfies the following properties:*

(i) *for any v, $\Psi^{DIV}(v)$ is a closed, convex set;*

(ii) *$\Psi^{DIV}(v)$ is single-valued whenever the set of vectors $\{\mathbf{1}_A \mid A \in \Lambda \cup \{I\}\}$ has full rank; in particular, $\Psi^{DIV}(v)$ is single-valued whenever v is strictly monotone;*

[15] Note that since $\lambda_A > 0$ iff $A \in \Lambda$, the summation can be taken over Λ.

(iii) Ψ^{DIV} *satisfies the Core property, i.e., for any* $v \in \mathbf{V}$, $w \in \Psi^{DIV}(v)$ *and* $S \subseteq I$, $\sum_{i \in S} w_i \geq \frac{v(I)-v(I \setminus S)}{v(I)}$;

(iv) Ψ^{DIV} *satisfies Dominance;*

(v) Ψ^{DIV} *satisfies Clone Invariance, i.e., whenever* i *has a clone in* v, $\Psi(v) \supseteq \Psi\left(v^{[I \setminus \{i\}]}\right)$;

(vi) Ψ^{DIV} *is upper hemicontinuous; i.e. for any sequence of* v^n *converging to* v *and a sequence of weights* $w^n \in \Psi^{DIV}(v^n)$ *converging to* w, $w \in \Psi^{DIV}(v)$.

(vii) Ψ^{DIV} *is robust, i.e., for any* v *and any* $w \in \Psi^{DIV}(v)$, *there is a sequence of strictly monotone* v^n *converging to* v *and a sequence of weights* $w^n \in \Psi^{DIV}(v^n)$ *converging to* w.

$(viii)$ Ψ^{DIV} *is scale invariant, i.e. for any* $c > 0$ *and any* $v \in \mathbf{V}$, $\Psi^{DIV}(cv) = \Psi^{DIV}(v)$.

(ix) Ψ^{DIV} *is shift-invariant, i.e. for any* $v, v' \in \mathbf{V}$ *and any* $b \in \mathbb{R}$ *such that* $v'(J) = v(J) + b$ *for all* $J \neq \varnothing$, $\Psi^{DIV}(v') = \Psi^{DIV}(v)$.

Part (i) is a direct consequence of defining the Diversity value as a solution of the concave maximization problem in (8). While, in general, this solution is not unique, Part (ii) of Proposition 3.1 asserts that for strictly monotone reliability functions v, and more generally, for reliability functions such that the set of vectors $\{\mathbf{1}_A \mid A \in \Lambda \cup I\}$ has full rank, Ψ^{DIV} is single-valued. Intuitively, uniqueness is guaranteed whenever the reliability function provides sufficient information to identify the distinct contribution of each source. This is the case, for example, if each source has a unique attribute, i.e., $\lambda_i > 0$ for all i.

A simple instance of non-uniqueness arises when all sources are clones, $v(S) = v(I)$ for all $S \subset I$ and $\Psi^{DIV}(v) = \Delta^{|I|-1}$. In this case, the aggregator has no basis for assigning specific weights. Accordingly, the Diversity value implies that mass can be shifted across sources in arbitrary fashion. If the sources have submitted different judgments, the multiplicity of weights entails a multiplicity of judgments. One might wish to posit that clones must necessarily submit identical judgments, and that, therefore, such multiplicity would not arise. We do not need to take a stance on this matter here. We note though that if the decision maker feels that he has *some* basis for assigning weights, he is free to recognize this information by incorporating it into the reliability function. Note in particular that by Part (ii) of Proposition 3.1, a small perturbation of the diversity function v is sufficient to eliminate such indeterminacy. In particular, choosing v_ϵ in such way that $v_\epsilon(I) - v_\epsilon(I \setminus \{i\}) = \epsilon_i$ and $v_\epsilon(I) - v_\epsilon(I \setminus \{j\}) = \epsilon_j$ implies that $\frac{w_i}{w_j} = \frac{\epsilon_i}{\epsilon_j}$ for all $w \in \Psi^{DIV}(v_\epsilon)$. Hence, adding an individual attribute to each of the clones helps resolve indeterminacy. Evidently, the weight ratio will depend crucially on the perturbation itself, i.e., on the ratio of ϵ_i and ϵ_j. Indeed, Part (vii) of Proposition 3.1, robustness, implies that by choosing the perturbation in an appropriate way, it is possible to approximate any of the elements of $\Psi^{DIV}(v)$. Thus, the potential multiplicity of the Diversity value at the boundary can be fully attributed to its continuity asserted in Part (vi).

We have already discussed the Core and Dominance properties. Invariance to Cloning says that any cloned source can be discarded without affecting the weight of the remaining sources. For example, if all sources are clones, Cloning implies that, for all $i \in I$, $\mathbf{1}_i \in \Psi^{DIV}(v)$ for all i. By convex-valuedness as in (i), thus $\Psi^{DIV}(v) = \Delta^{|I|-1}$.

By $(viii)$, only ratios of reliabilities need to be meaningful. Recalling that v is normalized at $v(\varnothing) = 0$, v can thus be viewed as expressing a ranking over lotteries of sets of sources as in NP's axiomatization of diversity functions. By (ix), that ranking may in fact be restricted to a ranking over lotteries of *non-empty* sets of sources. Note that, in the multi-attribute representation, the presupposition of (ix) amounts to the condition that $\lambda_A = \lambda'_A$ for all $A \neq I$. Thus (ix) can be read as saying that the amount of 'expertise' common to *all* sources does not matter to the assignment of weights.

The reader may miss a characterization of when a source i receives a strictly positive weight. By the Core property, it is *sufficient* that a source is essential, i.e. that $\lambda_{\{i\}} > 0$. By Dominance, it is *necessary* that a source be undominated. Call a reliability function v *regular* if every source is either strictly dominated or essential. For example, this property is satisfied by reliability functions for which Λ is closed under taking intersections. Intuitively, this means that if the subsets of sources A and B are deemed to make a distinct reliability contribution, so does their intersection. This is frequently natural. Note that all strictly monotone reliability functions are regular. If v is regular, the support of the weight vector w coincides with the support of v. On the other hand, for irregular v, no simple, qualitative characterization is available. In particular, reliability functions v and v' with the same set of valued attributes Λ need not assign strictly positive weight to the same set of sources.

For illustration of how the Diversity value works, we will consider a few simple examples.

Example 3.1 ***(Two sources)*** *Consider the case of two sources that are not clones. Here source weights are proportional to their marginal reliability contributions, by the Additive Core and shift-invariance properties,*

$$\Psi^{DIV}(v) \propto (v(1,2) - v(2), v(1,2) - v(1)),$$

or, equivalently, to the values of the source-specific attributes,

$$\Psi^{DIV}(v) = \left\{ \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}, \frac{\lambda_2}{\lambda_1 + \lambda_2} \right) \right\}.$$

Example 3.2 ***(Chain)*** *Suppose that any subset of sources is no more reliable than its most reliable member, i.e. that, for any $S \subseteq I$, $v(J) = \max_{i \in J} v(i)$. This is easily seen to correspond to the ordering of valued attributes as in a chain, in that, for all A, $B \in \Lambda$, $A \supseteq B$ or $B \subseteq A$, with more reliable sources having strictly larger sets of attributes. By Dominance, $\Psi^{DIV}(v^{[J]})$ concentrates all weight on the most reliable sources:*

$$\Psi^{DIV}\left(v^{[J]}\right) = \left\{ w \mid supp\,(w) = \arg\max_{j \in J} \{v\,(j)\} \right\}.$$

Example 3.3 ***(Type Partitions)*** *Consider a set of sources I which can be partitioned into mutually exclusive types, $T_1 ... T_K$ with $T_k \cap T_m = \varnothing$ for $k \neq m$ and $\cup_{k=1}^K T_k = I$. Types can, e.g., refer to experts from different disciplines such as economics, ecology, geophysics, etc., whose opinion about the consequences of climate change will be sampled. We assume that experts of different types share no valuable common attributes beyond those shared by all sources. Since the latter can be ignored by shift-invariance (Part (ix) of Proposition 3.1), w.l.o.g. one can assume the reliability function to be additive across types. This separability assumption rules out the existence of shared factors of reliability among subsets of types, e.g. among ecologists and geophysicists but excluding economists, for example due to their natural-science education and outlook.*

Furthermore, we assume that experts within a given type are mutually exchangeable, hence, the reliability function depends only on the number of experts of a given type available, but not on their identities. These two assumptions imply that the reliability function can be written as:

$$v\,(S) = \sum_{k=1}^K v\,(S \cap T_k) = \sum_{k=1}^K f_k\,(|S \cap T_k|)\,,$$

where $f_k(n)$ describes the reliability assigned to a given number n of experts of type T_k.

For reliability functions with this structure, the aggregate judgment is given as an average of type judgments $\widehat{X}_k$ by[16]*:*

$$\begin{aligned} \widehat{\Psi}\,(x_1, .., x_n; v) &= \sum_{k=1}^K \frac{v\,(T_k)}{\sum_{k'=1}^K v\,(T_{k'})} \widehat{X}_k = \sum_{k=1}^K \frac{f_k\,(|T_k|)}{\sum_{k'=1}^K f_{k'}\,(|T_{k'}|)} \widehat{X}_k, \text{ where} \\ \widehat{X}_k &= \sum_{i \in T_k} \frac{1}{|T_k|} x_i. \end{aligned}$$

Hence, the weight assigned to each type's judgment is proportional to the reliability of the set of sources belonging to this type in I. In turn, each type's judgment is a (non-weighted) average of the sources belonging to this type.

To flesh the model out further, consider the functional form $f_k(m) = \theta_k \left(1 - e^{-\frac{m}{m_k}}\right)$. Here θ_k describes the "maximal" weight assigned to the type when it is sufficiently represented in the available sources, and m_k determines how many individual sources it takes to obtain this "maximal" weight approximately. For example, if $m \geq 3m_k$, then $0.95\theta_k \leq f_k(m) \leq \theta_k$. Thus,

[16] This follows from the Core property Ψ and its neutrality towards sources. As an aside, note that, by consequence, the Diversity value and Shapley values coincide for reliability functions associated with type partitions, a rather exceptional phenomenon.

if all types are sufficiently represented in this sense,

$$\widehat{\Psi}(x_1,..,x_n;v) \approx \sum_{k=1}^{K} \frac{\theta_k}{\sum_{k'=1}^{K}\theta_{k'}} \widehat{X}_k. \tag{9}$$

Note that (9) may obtain even for a small number of sources in a given type when the 'saturation levels' are low. E.g., if $m_k \leq \frac{1}{3}$ for all k, (9) holds even for $T_k = 1$ for all k. If the approximation in (9) applies, the weight of an individual expert is approximately given by the weight of his discipline divided by the number of all sampled experts from the same discipline.

Different types (e.g. different disciplines) may have different weights if the expertise of some disciplines is more relevant than that of others. Different types may also have larger 'saturation levels' m_k accounting for the fact that some disciplines are broader or more dispersed than others. If saturation levels differ, it may well be that an individual representative from a narrower discipline is more reliable on his own than an individual representative from another discipline ($v(i) > v(j)$), while the overall reliability of the discipline is smaller ($v(T_{k_i}) < v(T_{k_j})$).

4 Axiomatizing the Diversity Value

To provide a characterization of the Diversity value, we will use the following axioms:

Axiom 1 **(Additive Core)** If v is additive, then $\Psi(v) = \left\{\frac{v(i)}{v(I)}\right\}$.

Axiom 2 **(Dominance)** If a source i is dominated in v, then $w_i = 0$ for all $w \in \Psi(v)$.

Axiom 3 **(Exclusion)** If for a source i, $w_i = 0$ for all $w \in \Psi(v)$, then $\Psi(v) = \Psi\left(v^{[I\setminus\{i\}]}\right)$.

Axiom 4 **(Betweenness)** If $\Psi(v) \cap \Psi(v') \neq \varnothing$, then $\Psi(\alpha v + (1-\alpha)v') = \Psi(v) \cap \Psi(v')$.

Axiom 5 **(Convex-Valuedness)** For all $v \in \mathbf{V}$, $\Psi(v)$ is convex.

Axiom 6 **(Continuity)** $\Psi(v)$ is upper hemicontinuous.

The first two axioms have already been discussed in Section 2. Axiom 3, Exclusion states that those sources, whose weight is zero under any assignment can be excluded from the set of sources without changing the weights assigned to the remaining sources.

Axiom 4, Betweenness states that if the reliability assessment provided by v and that provided v' both entail that the optimal weight assignment is given by w, then so should any reliability assessment v'' which is "between" the two, i.e., $v'' = \alpha v + (1-\alpha)v'$ for some $\alpha \in (0;1)$. Importantly, Betweenness is preserved under independent rescaling of the reliability functions v and v'. It is thus meaningfully defined for rankings over lotteries of subsets of sources as in NP[17].

Axiom 5, Convex-Valuedness requires that whenever w and w' are optimal weights, any weight

[17] Indeed, v'' is between v and v' if and only if the preference over lotteries associated with v'', $\succsim_{v''}$ extends the intersection of $\succsim_v$ and $\succsim_{v'}$. That is, whenever $\succsim_v$ and $\succsim_{v'}$ agree on any comparison of lotteries, $\succsim_{v''}$ agrees with them as well.

vector w'' "between" the two, i.e., $w'' = \alpha w + (1-\alpha) w'$ for some $\alpha \in (0;1)$ is also optimal. Finally, Axiom 6 requires the value to be upper hemicontinuous: small changes in the reliability assessment should lead only to small changes in the assigned weights.

Axioms 1–6 characterize the Diversity value defined above:

Theorem 4.1 *For a given finite set I, Ψ satisfies Axioms 1–6 iff $\Psi(v) = \Psi^{DIV}(v)$ for all $v \in \mathbf{V}$.*

To understand how the different axioms narrow down the choice of weights to those identified by the Diversity value, it is useful to consider some alternative rules.

Score-Based Values Consider the class of score-based values Ψ_{ϕ}^{SCV} defined in (7). As noted above, these satisfy Dominance, but not Additive Core, unless ϕ is logarithmic. If ϕ is concave and thus, by monotonicity, also continuous, Ψ_{ϕ}^{SCV} satisfies all of the remaining axioms.

Take-the-Best An interesting special case obtains when ϕ is linear. The resulting rule

$$\Psi^{TTB}(v) = \{w \mid supp\,(w) = \arg\max_{i \in I} v\,(i)\}$$

can be referred to as “Take-the-Best”, since it puts all weight on the set of those sources which are most reliable when considered in isolation. Ψ^{TTB} is thus completely indifferent to the potential diversification benefits of taking into account somewhat less reliable, but undominated sources. By contrast, $\Psi^{DIV}(v)$ concentrates all weight on a single source only if that source dominates all others.

Take-the-Best is named after the eponymous rule suggested by Gigerenzer and Goldstein (1996) in the related context of drawing inferences from a number of different cues. Gigerenzer and Goldstein (1996) argue that, despite its provocative simplicity, this rule works well in practice in many empirical contexts, see Czerlinski, Goldstein and Gigerenzer (1999) and Graefe and Armstrong (2012). On the other hand, in the context of expert aggregation, many decision-makers mistakenly try to align their probability judgment with the judgment of the putatively best expert, reflecting a mistaken belief that the quality of an average of judgments is equal to the average of the quality of judgments, see Larrick and Soll (2006).

Shapley Value The Shapley value is the most widely used solution concept for cooperative games and can be directly applied to the class of reliability functions v upon appropriate normalization. It is given by:

$$\Psi^{SV}(v) := \left\{ \left(\frac{1}{v\,(I)} \sum_{\{A \mid i \in A\}} \frac{\lambda_A}{|A|} \right)_{i \in I} \right\}.$$

It is easy to see that the Shapley value satisfies all axioms except for Axiom 2, Dominance[18]. Indeed, under the Shapley value, the value of each attribute λ_A is divided equally among all sources sharing this attribute, regardless of what other attributes these sources possess. Hence, a dominated, but non-null source will receive $\frac{1}{|A|}$ of the value λ_A of each of its attributes A.[19]

Shapley Fixpoint Value We can modify the Shapley value to satisfy Dominance on the class

[18] Conversely, the Diversity value satisfies all of the axioms defining the Shapley value except for Additivity.
[19] For the same reason, the Shapley value also fails to satisfy Cloning Invariance.

of regular v's by introducing weights into the determination of the value and endogenizing them in fix-point manner by equating them to the resulting weight assignment. For $w \in \Delta^{|I|-1}$, the w-weighted Shapley value Ψ_w^{WSV} is given by

$$\Psi_w^{WSV}(v) := \left\{ \left(\frac{1}{v(I)} \sum_{\{A|i \in A\}} \frac{w_i}{\sum_{i \in A} w_i} \lambda_A \right)_{i \in I} \right\}.$$

Just like the regular Shapley value, the weighted Shapley value satisfies all axioms except Dominance. Endogenizing the weights by considering the fixpoints of the mapping $w \mapsto \Psi_w^{WSV}(v)$ for a given v defines the Shapley Fixpoint value:

$$\Psi^{SFX}(v) := \left\{ w \mid w \in \Psi_w^{WSV}(v) \right\}.$$

The Shapley Fixpoint value always contains the Diversity value and coincides with it on the set of regular diversity functions v:

Proposition 4.2 [20]*The Shapley Fixpoint value Ψ^{SFX} on $\mathbf{V}$ has the following properties:*

(i) *for every $v \in \mathbf{V}$, $\Psi^{SFX}(v) \supseteq \Psi^{DIV}(v)$;*

(ii) *if v is regular, $\Psi^{SFX}(v)$ is a singleton and $\Psi^{SFX}(v) = \Psi^{DIV}(v)$; in particular, $\Psi^{SFX}(v) = \Psi^{DIV}(v)$ for strictly monotone v's;*

(iii) *for any $v \in \mathbf{V}$, Ψ^{DIV} is the unique upper hemicontinuous and convex-valued selection of Ψ^{SFX} on $\mathbf{V}$.*

Note that it is straightforward from their definition that weighted Shapley values and Shapley fixpoints are in the core. By the first part of this Proposition, this extends immediately to the Diversity value, something that is not obvious from the latter's definition.

The Shapley Fixpoint value is problematic on irregular v's. Consider v with three sources given by $\lambda_{12} = \lambda_{23} = \frac{1}{2}$, and $\lambda_A = 0$ otherwise. Note that v is irregular since the only undominated source, 2, is inessential. Here Dominance alone requires that $\Psi(v) = \{(0,1,0)\}$, however, $\Psi^{SFX}(v) = \{(0,1,0), (\frac{1}{2}, 0, \frac{1}{2})\}$. Thus, Ψ^{SFX} violates Axioms 2 and 5, Dominance and Convex-Valuedness.

5 The Effect of Similarity on Weights and Aggregate Judgment

In this section, we examine how the assessment that some sources are more similar than others affects the weights assignments and the eventual aggregate judgment. We will show how the Diversity value incorporates the idea that higher weights should be assigned to more "distinct" sources. We will also show that aggregate judgments may exhibit a No-Show paradox, and

[20] The fact that the Shapley Fixpoint Value coincides with the Diversity Value on the set of strictly monotone diversity functions turns out to be useful in the proof of the main representation theorem. For this reason, in the Appendix, the proof of this Proposition precedes the proof of Theorem 4.1.

argue that this apparent 'paradox' is not a flaw but a design feature of the our model reflecting the crucial role of similarity considerations.

Similarity and Weights. A natural requirement on the weighting scheme used is that it assign higher weight to more distinct sources of information. Our next example illustrates how the Diversity value captures this intuition.

Example 5.1 *Consider the case of three experts, an economist* (1) *and two ecologist (*2 *and* 3*), who are asked for an estimate of the probability that the economic cost of global warming will not exceed 3% of the world GDP annually by 2100. Assume that all experts are considered equally reliable, i.e. that* $v(1) = v(2) = v(3)$*. Assume further that the two ecologists are "more similar" to each other than to the economist. This can be stated either in terms of attributes, as* $\lambda_{23} > \lambda_{12} = \lambda_{13}$*, or, equivalently, in terms of dissimilarities, as* $d(1,2) = d(1,3) > d(2,3)$*. Both statements are also equivalent to saying that the* $\text{pair } \{2,3\}$ *is the least reliable among the three pairs:*

$$v(2,3) < v(1,2) = v(1,3)\,.$$

While, by uniformity, any two experts should be given equal weight when only these two experts are available, in the triple, the most distinct source arguably makes the largest individual (marginal) contribution to the overall reliability of the triple, hence it should be given the largest weight. This is the "Similarity Effect" discussed in the Introduction, and, in this example, it is exhibited by the Diversity value since $\Psi^{DIV}(v) = \{w\}$ *with* $w_1 > \frac{1}{3} > w_2 = w_3$*.*[21]

Our next proposition generalizes this Similarity Effect to arbitrary uniform triples:

Proposition 5.1 *Suppose that* v *is a uniform diversity function on* I *with* $|I| = 3$*. Unless sources* 1 *and* 2 *are clones, the following are equivalent:*

(i) $v(1,3) \geq v(2,3)$*;*

(ii) $\lambda_1 \geq \lambda_2$*;*

(iii) $d(1,3) \geq d(2,3)$*;*

(iv) *for all* $w \in \Psi^{DIV}(v)$*,* $w_1 \geq w_2$*.*

The proof of Proposition 5.1 makes use of the fact that uniformity within a triple I implies the existence of a most distinct source i, i.e., a source i such $d(i,k) \geq d(j,k)$ and $d(i,j) \geq d(j,k)$ for $j,k \neq i$, as well as that of a least distinct source i' satisfying the reverse inequalities. The equivalence of (ii), (iii) and (iv) implies that the Similarity Effect can be restated to say that

[21] Indeed, denoting by $b := 1 - \lambda_1 - 4\lambda_{12}$ and exploiting the symmetry between sources 2 and 3, one can solve the polynomial equations defining the Diversity value explicitly. We obtain:

$$\Psi^{DIV}(v) = \left\{ w = \left(w_1 = \frac{\sqrt{b^2 + 4\lambda_1^2} - b}{2}, w_2 = w_3 = \frac{2 + b - \sqrt{b^2 + 4\lambda_1^2}}{4} \right) \right\}$$

weights among sources are ordered according to their distinctness, or according to the value of their unique attributes (within the triple). Furthermore, the total weight of pairs is ordered in line with the ordering of their reliabilities: in fact, by the equivalence of (i) and (iv), for all $J, J' \subseteq I$,

$$w(J) \geq w(J') \text{ if and only if } v(J) \geq v(J'). \tag{10}$$

This congruence of the weight measure and the reliability functions does not extend, in general, to more than three sources, or to three non-uniform sources. The former is clear from the contrast between the additivity of w and the non-additivity of v, and is easily verified, for example, in the type partition model. The latter is a bit more involved.[22]

Similarity and Aggregate Judgment. We now consider how similarity affects the aggregate judgment.

Example 5.2 *Consider again the three experts from Example 5.1 and assume that they estimate the probability that the cost of climate change would not exceed* 3% *of the world GDP annually by 2100 as follows:* $x_1 = 0.8$, $x_2 = 0.2$ *and* $x_3 = 0.5$. *If only the economist and the first ecologist are available, the best aggregate estimate is* $\hat{\Psi}^{DIV}\left((x_i)_{i\in\{1,2\}}, v^{[1,2]}\right) = \{0.5\}$, *since both are viewed as equally reliable. Now, the estimate of the second ecologist becomes available as well, and it turns out to be equal to the "standing" aggregate estimate, namely* 0.5. *Should this be viewed as* reinforcing *this standing estimate, i.e. should* $\hat{\Psi}^{DIV}\left((x_i)_{i\in\{1,2,3\}}, v\right) = \{0.5\}$ *as well?*

While this conclusion might appear attractive superficially, we submit that one should reject it. Heuristically, the opinion of the second ecologist should more strongly counterbalance ('correct') the opinion of the more similar source (the first ecologist) than that of the less similar source (the economist). The aggregator "learns" that the first ecologist might have made a mistake underestimating the probability as compared to the "common view" held by ecologists. Hence, rather than reflecting the "wisdom" of ecology, some of the initial disagreement between 0.8 *and* 0.2 *must reflect the idiosyncratic understanding of the first ecologist. Thus, one would expect the best aggregate* $\hat{\Psi}^{DIV}\left((x_i)_{i\in\{1,2,3\}}, v\right)$ *to shift towards that of the economist, and thus to* exceed 0.5.

This is indeed what the Diversity value does, and what the Similarity Effect implies. Simply note that, as shown in Example 5.1, for $\lambda_{23} > \lambda_{12}$, $w_1 > w_2 = w_3$, *and, hence,* $\hat{\Psi}^{DIV}\left((x_i)_{i\in 1,2,3}, v\right) > 0.5$. *Note also that by Continuity, the aggregate estimate will increase even if the second ecologist's opinion is slightly lower than the standing aggregate.*

In social choice theory, the phenomenon that an aggregate judgment or vote moves away from the judgment of an individual that joins the poll is known as the No-Show paradox, see Fishburn

[22] Consider the following family of reliability functions v^ϵ on $I = \{1,2,3\}$ given by $\lambda_{12} = \lambda_{13} = \frac{1}{2}$, $\lambda_{23} = 0$, $\lambda_1 = \lambda_2 = \lambda_3 = \epsilon$. By Dominance, $\Psi^{DIV}\left(v^0\right) = \{(1,0,0)\}$. Hence, by Continuity, for small but positive ϵ and $\{w^\epsilon\} = \Psi^{DIV}(v^\epsilon)$, w_1^ϵ is close to, but strictly less than 1, and thus $w_1^\epsilon > w_2^\epsilon + w_3^\epsilon$ while $v^\epsilon(1) < v^\epsilon(2,3)$.

and Brams (1983) and Moulin (1988). It is viewed as 'paradoxical' there because the individual is made worse off by joining the poll ('showing up'). But this is not an issue here, because source welfare plays no role, and may indeed be a meaningless concept.[23] Furthermore, in the standard social choice setting anonymity is usually taken for granted. By contrast, here it is, as total symmetry across sources, rather exceptional. In general, there are similarities among sources, and similarities are typically asymmetric. Indeed, in the special cases in which the reliability function is totally symmetric across sources ("exchangeable"), the Diversity value will assign equal weights to all available sources, and will thus never exhibit the No-Show paradox. On the other hand, in the typical situations of asymmetric sources, the Similarity Effect suggests that the No-Show paradox is likely to occur for some profiles of source judgments.

To understand a bit better when it materializes, it is illuminating to flesh out Example 5.2 within the Type Partition model, with types $\{1\}$ and $\{2,3\}$. Specifically, for $\alpha \in [0,1]$, let v^{α} be given by $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 1-\alpha$, $\lambda_{23} = \alpha$. Here α describes the similarity between the two ecologists. An easy computation yields

$$\Psi^{DIV}\left(v^{\alpha}\right) = \left(\frac{1}{3-\alpha}, \frac{1}{3-\alpha}\frac{2-\alpha}{2}, \frac{1}{3-\alpha}\frac{2-\alpha}{2}\right).$$

Intuitively, the more similar the two ecologists are, the smaller is their aggregate weight. Consider the pattern of judgments described in Example 5.2, with the property

$$x_1 > \frac{1}{2}x_1 + \frac{1}{2}x_2 \geq x_3.$$

When the second ecologist joins the poll, there are two countervailing effects: on the one hand, the aggregate opinion of the ecologists shifts up (becomes more moderate), increasing the overall estimate; on the other hand, the aggregate weight placed on the ecologists' opinion increases (from $\frac{1}{2}$ to $\frac{2-\alpha}{3-\alpha}$), since, the ecologists' aggregate reliability is now increased by $\lambda_2 = (1-\alpha)$. The No-Show paradox occurs if the first effect dominates the second, so that the resulting aggregate is even further away from x_3 than the initial one. Some straightforward algebra establishes that this will be the case iff

$$x_3 > \frac{(1-\alpha)\,x_1 + x_2}{2-\alpha}.$$

For example, if $\alpha = \frac{1}{2}$ and $x_1 = 0.8$ and $x_2 = 0.2$, the No-Show paradox will occur iff $0.5 \geq x_3 > 0.4$. Thus, for a given α, the estimate of the second ecologist needs to be sufficiently

[23] A No-Show paradox may also give rise to incentive-compatibility issues, but these are not directly relevant here, since the question here is how to aggregate given 'true' source judgments. Incentive-compatibility issues may make it difficult for an aggregator to determine what these judgments are, and entail uncertainty about them.

close to the standing average, and thus, sufficiently different from the first ecologist's estimate. The exact magnitudes depend on their similarity α; in the limiting case of the two ecologists being clones, ($\alpha = 1$), the No-Show paradox will occur whenever the second ecologists estimate is just slightly higher than that of the first ecologist.

We will now show that the potential occurrence of the No-Show paradox is the rule rather than the exception in our model. In social choice theory, a strong form of the negation of the No-Show paradox is given by the Reinforcement axiom due to Young (1975); see also Myerson (1995). Formally,

Definition 5.1 *An aggregation scheme $\hat{\Psi}$ satisfies **Reinforcement** at v if for all $J \subseteq I$ and all judgment profiles $x = (x_i)_{i \in I}$,*

$$\hat{\Psi}\left((x_i)_{i\in I}, v\right) = \hat{\Psi}\left((x_i)_{i\in I}, v^{[J]}\right) \cap \hat{\Psi}\left((x_i)_{i\in I}, v^{[I\setminus J]}\right)$$

if the intersection on the r.h.s. is non-empty.

Reinforcement says that if the aggregation of the sources in J and the aggregation of those in $I\setminus J$ leads to the same judgment, then so does the aggregation of their union I. The No-Show paradox corresponds to the negation of Reinforcement when either J or $I\setminus J$ is a singleton.

When does the Diversity value satisfy Reinforcement? The argument will proceed in two steps, focussing on the case of a strictly monotone v. First, we will show that Reinforcement is tantamount to a Fixed Weights property corresponding to the Fixed Weights model discussed in the introduction. Second, we demonstrate that this property is satisfied by the Diversity value only for exceptional reliability functions.

Definition 5.2 *(**Fixed Weights**) Let Ψ be single-valued at v, with $\Psi(v) = \{w\}$ and $w >> 0$. Ψ satisfies **Fixed Weights** at v if for all $J \subset I$,*

$$\Psi\left(v^{[J]}\right) = \left\{ \left(\left(\frac{w_j}{\sum_{k\in J} w_k} \right)_{j\in J}, \underbrace{(0...0)}_{i\in I\setminus J} \right) \right\}. \tag{11}$$

The following result establishes that, as a matter of simple arithmetic, Reinforcement is tantamount to Fixed Weights. Note that the result applies to weighting rules generally, and is not restricted to the Diversity value[24].

Proposition 5.2 *For $|I| \geq 3$ and $v \in \mathbf{V}$, let $\Psi\left(v^{[J]}\right)$ be single-valued for all $J \subseteq I$ and let $\{w\} = \Psi(v)$ satisfy $w >> 0$. Ψ satisfies Reinforcement at v iff Ψ satisfies Fixed Weights at v.*

[24] Note that, for the Diversity value, the premise is satisfied for all strictly monotone v. This will be the case for many sensible values (such as the Shapley value). An exception is Take-The-Best which satisfies Reinforcement at all v. This is the exception that proves the rule, inasmuch as TTB fails to obey the Similarity Principle.

Note that as an immediate consequence of the Additive Core axiom, Fixed Weights and thus Reinforcement holds for Ψ^{DIV} at any additive reliability function, i.e. in the absence of any similarity whatsoever. On the other hand, generalizing the discussion of Example 5.2, for non-additive, strictly monotone reliability functions, one would expect the Fixed Weight property to hold only by coincidence, if the similarities among sources just happen to balance each other in the needed way. This is shown by our following result.

Definition 5.3 *v is quasi-additive if $v(J \cup J') = v(J) + v(J') - v(J \cap J')$ for all non-disjoint J and J'.*

It is immediate to see that v is quasi-additive if and only if $\Lambda \subseteq \{\{i\}\}_{i \in I} \cup \{I\}$.

Proposition 5.3 *Let $|I| \geq 3$ and consider the set of strictly monotone reliability functions v, $\mathbf{V}_M$. Ψ^{DIV} violates Reinforcement on an open and dense subset of $\mathbf{V}_M$. Indeed, for any $v \in \mathbf{V}_M$, if v satisfies Reinforcement and is not quasi-additive, there exist arbitrarily close $v' \in \mathbf{V}_M$ such that v' violates Reinforcement and has the same set of relevant attributes as v, $\Lambda' = \Lambda$.*

In the special case of uniform reliability functions, we can provide an exact characterization of Reinforcement.

Proposition 5.4 *Ψ^{DIV} satisfies Reinforcement at a uniform reliability function v if and only if v is exchangeable.*

Note that, for the special case of triples (but not for the general case), this Proposition is a straightforward corollary to Proposition 5.1 above.[25]

6 Conclusion

In this paper, we have shown how diverse sources of judgment can be aggregated in a methodical, well-grounded fashion on the basis of assessments of their reliability. We have argued that diverse sources can be more or less similar, and that the marginal contribution of sources to the overall reliability can therefore differ significantly and systematically from their stand-alone reliability. As illustrated by the Similarity Effect, optimally assigned weights need to reflect the marginal rather than the standalone contribution of sources. The relative weights between

[25] There are also nonuniform reliability functions v with the Fixed Weight property. For example, for $|I| = 3$, let $\lambda_1 = \frac{1}{18}$, $\lambda_2 = \lambda_3 = \frac{1}{12}$, $\lambda_{13} = \lambda_{12} = \frac{1}{3}$, $\lambda_{23} = \frac{1}{9}$. This satisfies Fixed Weights with $\Psi(v) = \left\{w = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)\right\}$. One may want to look for an exact characterization of Reinforcement for general reliability functions. However, this example suggests that an interesting characterization of this property might not exist.

sources will thus depend critically on which other sources are available as well. The Diversity value proposed here pins down this context-dependence in a transparent, axiomatically grounded manner. Superficially paradoxical-looking behaviors such as the No Show paradox, in which the aggregate judgment moves in opposite direction to the input of new sources taken at face vale, are not really paradoxical but a key design feature of the theory.

As a leading example of source aggregation, we have focused in much of the exposition on the aggregation of expert judgments, in particular judgments of probability. Of course, the formal framework of source aggregation is much broader than this, and so indeed is the weighting problem. The main formal restriction of the framework is the assumption that the target judgments is of the same kind (belongs to the same space) as the input judgments so that the target judgment can be obtained by some kind of "weighted averaging".

"Sources" might, in principle, be anything a decision maker is willing to base his own judgment on. Sources need not be outside experts but could be alternative models a decision maker is prepared to take seriously. Also, the components of an index on some matter might be viewed as sources 'by construction'. There are many potential applications, and in many of them, context-dependence based on similarity effects appears natural. The diversity value holds substantial promise for these applications, and it should be very interesting to develop these applications in greater detail.

Of course, for some potential applications, the formal analogy may prove to be more apparent than real. For example, are "cases" in the sense of case-based decision theory usefully viewed as "sources" or not? On the face of it, the problems of learning and induction are quite different from problems of aggregation, so one might be skeptical. On the other hand, as in BGSS, one might identify cases with probability judgments derived from them, and take the probability of a target case to be the weighted average of such case-based probabilities. "Classical" case-based decision theory focuses on the relationship between individual cases and the target case; the source-based model, by contrast, focuses on the relationship among individual sources/cases. The resulting context-dependence and similarity effects seem very natural and highly relevant in a case-based setting as well. We thus take the application of the diversity value to case-based decision theory to be one line of significant interest for future research.

Once the general approach and solution proposal of this paper appeal, many follow-up questions arise. How should a decision maker assess the reliability of sources? Can one flesh out the intuition of “reliability as valued diversity” in an instructive manner, or is one left with an unexplained subjectivity, as in much of the tradition of subjective expected utility? If one takes a third-person perspective, what testable implications does the diversity value have if a decision-makers’ reliability assessments cannot be observed directly? To what extent can the reliability assessments be inferred from observable aggregate judgments and choices?

Finally, and more directly related to the central task of this paper, is the diversity value normatively uniquely privileged as a weighting rule? Does it have serious competitors? If so, what virtues does the competitor have that the diversity value is missing?

Appendix A. Proofs

Proof of Proposition 3.1:

Define $H(w,\lambda) =: -\sum_{A\in\Lambda}\lambda_A \ln\left(\sum_{i\in A} w_i\right)$.

Part (i): Follows from the continuity and quasi-concavity of $-H(w,\lambda)$.■

Part (ii)**:** Denote by $w_A =: \sum_{i\in A} w_i$ and note that $-H(w,\lambda)$ is strictly concave in w_A. Hence, if it has a maximum w.r.t. w_A, it is unique. It follows that for all $w, w' \in \Psi^{DIV}(v)$, $w_A = w'_A$ for all A such that $\lambda_A > 0$ and (since $w, w' \in \Delta^{|I|-1}$), also for $A = I$, which is equivalent to the system of equations $(w_A - w'_A = 0)_{\{A|A\in\Lambda\cup I\}}$. Since $\{\mathbf{1}_A \mid A \in \Lambda \cup I\}$ has full rank, we conclude that $w = w'$.■

Part (iii)**:** Note that the condition in (iii) is equivalent to: for any v and the respective λ, any $w \in \Psi^{DIV}(v)$ and any $A \subseteq I$, $w_A =: \sum_{i\in A} w_i$ satisfies $w_A \geq \sum_{A'\subseteq A} \frac{\lambda_{A'}}{v(I)}$. Since $w_A \geq 0$, the statement of the Lemma is non-trivial only if there is an $A' \subseteq A$ with $\lambda_{A'} > 0$. Assume such an A' exists. If $w_A = 0$, the entropy defined in (8) is infinitely negative. Hence $w_A = 0$ cannot hold in the optimum. It follows that $w_A > 0$, whenever $\lambda_{A'} > 0$ for some $A' \subseteq A$, and, thus, there is an $I' \subseteq A$, such that $w_i > 0$ for all $i \in I'$ and $w_i = 0$ for $i \in A\backslash I'$, i.e., $w_A = \sum_{i\in I'} w_i$.

If I' is a singleton, i.e., $I' = \{i'\}$, and if $w_{i'} = 1$, then $w_A = 1 \geq \frac{\sum_{A'\subseteq A}\lambda_{A'}}{v(I)}$. Otherwise, for all $i \in I'$, $w_i \in (0,1)$. The f.o.c. w.r.t. $i' \in I$ then implies:

$$w_{i'} = \sum_{\{A|i'\in A\}} \frac{w_{i'}\frac{\lambda_A}{v(I)}}{\sum_{j\in A} w_j}$$

and summing over all $i' \in A$, we obtain

$$w_A = \sum_{i'\in A}\sum_{\{A'|i'\in A'\}} \frac{w_{i'}\frac{\lambda_{A'}}{v(I)}}{\sum_{j\in A'} w_j} = \frac{\sum_{A'\subseteq A}\lambda_{A'}}{v(I)} + \sum_{i'\in A}\sum_{\{A'|i'\in A', A'\nsubseteq A\}} \frac{w_{i'}\frac{\lambda_{A'}}{v(I)}}{\sum_{j\in A'} w_j} \geq \frac{\sum_{A'\subseteq A}\lambda_{A'}}{v(I)}.■$$

Part (iv): Suppose that i strictly dominates j. By definition, $w \in \Psi^{DIV}(v)$ if it solves the system of equations[26]

$$\left(\sum_{\{A\in\Lambda|i'\in A\}}\frac{\lambda_A}{\sum_{j'\in A}w_{j'}}-v\left(I\right)+\underline{\gamma}_{i'}=0\right)_{i'\in I}$$

for some coefficients $\underline{\gamma}_{i'}\geq 0$ with $\underline{\gamma}_{i'}>0$ implying $w_{i'}=0$. Since i strictly dominates j in v, for any such w, we have:

$$\begin{aligned}\sum_{\{A|i\in A\}}\frac{\frac{\lambda_A}{v(I)}}{\sum_{k\in A}w_k}-1+\frac{\underline{\gamma}_i}{v\left(I\right)} &= \sum_{\{A|j\in A\}}\frac{\frac{\lambda_A}{v(I)}}{\sum_{k\in A}w_k}+\sum_{\{A|i\in A,j\notin A\}}\frac{\frac{\lambda_A}{v(I)}}{\sum_{k\in A}w_k}-1+\frac{\underline{\gamma}_i}{v\left(I\right)}\\ &= \sum_{\{A|j\in A\}}\frac{\frac{\lambda_A}{v(I)}}{\sum_{k\in A}w_k}-1+\frac{\underline{\gamma}_j}{v\left(I\right)}.\end{aligned}$$

Since $\sum_{\{A|i\in A,j\notin A\}}\frac{\frac{\lambda_A}{v(I)}}{\sum_{k\in A}w_k}>0$, this implies that $\underline{\gamma}_j>0$ (regardless of whether $\underline{\gamma}_i>0$ or $\underline{\gamma}_i=0$ holds), or, $w_j=0$.■

Part (v): Let $C\left(i\right)$ denote the maximal set of clones of i in v. Note that for any $A\in\Lambda$, either $C\left(i\right)\subseteq A$, or $C\left(i\right)\cap A=\varnothing$. Hence, $w\in\Psi^{DIV}\left(v\right)$, iff it satisfies the following conditions:

$$\begin{aligned}\sum_{\{A|k\in A\}}\frac{\frac{\lambda_A}{v(I)}}{\sum_{j'\in A\setminus C(i)}w_{j'}+w_{C(i)}}-1+\gamma_k &= 0 \text{ for } k\notin C\left(i\right)\\ \sum_{\{A|C(i)\subseteq A\}}\frac{\frac{\lambda_A}{v(I)}}{\sum_{j'\in A\setminus C(i)}w_{j'}+w_{C(i)}}-1+\gamma_{C(i)} &= 0\\ \sum_{j'\in C(i)}w_{j'} &= w_{C(i)}\end{aligned}\tag{A-1}$$

Similarly, $w'\in\Psi^{DIV}\left(v^{[I\setminus\{j\}]}\right)$ for some $j\in C\left(i\right)$ iff it satisfies:

$$\begin{aligned}\sum_{\{A|k\in A\}}\frac{\frac{\lambda_A}{v(I)}}{\sum_{j'\in A\setminus C(i)}w'_{j'}+w'_{C(i)}}-1+\gamma_k &= 0 \text{ for } k\notin C\left(i\right)\\ \sum_{\{A|C(i)\subseteq A\}}\frac{\frac{\lambda_A}{v(I)}}{\sum_{j'\in A\setminus C(i)}w'_{j'}+w'_{C(i)}}-1+\gamma_{C(i)} &= 0\\ \sum_{j'\in C(i)\setminus\{j\}}w'_{j'} &= w'_{C(i)},w'_j=0.\end{aligned}\tag{A-2}$$

It is obvious that if w' is a solution to (A-2), then $\left(\left(w'_k\right)_{k\notin C(i)},w'_{C(i)}\right)$ satisfies the first two conditions in (A-1). Furthermore, we have that $\sum_{j'\in C(i)}w'_{j'}=w'_{C(i)}=w_{C(i)}$, hence $w'\in\Psi\left(v\right)$ and, hence, invariance to cloning is satisfied.■

Part (vi)**:** The proof of this statement proceeds by proving three consecutive claims. Let $\{v^n\}$, the corresponding $\{\lambda^n\}$ and $\{w^n\}$ be sequences converging to $\bar{v}$, the corresponding $\overline{\lambda}$ and $\overline{w}$, respectively, such that, for all $n\in\mathbb{N}$, $w^n\in\Psi^{DIV}\left(v^n\right)$ .We need to show that $\overline{w}\in\Psi^{DIV}\left(\bar{v}\right)$, which is equivalent to:

[26] It can be shown that the Lagrangian multiplier for the condition $\sum_{i\in I}w_i=1$ is $v\left(I\right)$, see the proof of Proposition 4.2 below.

for any $z \in \Delta^{|I|-1}$, $H\left(\overline{w}, \overline{\lambda}\right) \leq H\left(z, \overline{\lambda}\right)$. Note that if $H\left(z, \overline{\lambda}\right) < H\left(\overline{w}, \overline{\lambda}\right)$, then, by convexity of H in w, $H\left(\frac{1}{2}z + \frac{1}{2}\overline{w}, \overline{\lambda}\right) < H\left(\overline{w}, \overline{\lambda}\right)$ and

$$\begin{aligned} H\left(\frac{1}{2}z + \frac{1}{2}\overline{w}, \overline{\lambda}\right) - H\left(\overline{w}, \overline{\lambda}\right) &= \left[H\left(w^n, \lambda^n\right) - H\left(\overline{w}, \overline{\lambda}\right)\right] \\ &\quad + \left[H\left(\frac{1}{2}z + \frac{1}{2}w^n, \lambda^n\right) - H\left(w^n, \lambda^n\right)\right] \\ &\quad + \left[H\left(\frac{1}{2}z + \frac{1}{2}\overline{w}, \overline{\lambda}\right) - H\left(\frac{1}{2}z + \frac{1}{2}w^n, \lambda^n\right)\right]. \end{aligned}$$

Hence,

$$\begin{aligned} H\left(\frac{1}{2}z + \frac{1}{2}\overline{w}, \overline{\lambda}\right) - H\left(\overline{w}, \overline{\lambda}\right) &\geq \liminf_{n\to\infty} \left[H\left(w^n, \lambda^n\right) - H\left(\overline{w}, \overline{\lambda}\right)\right] \\ &\quad + \liminf_{n\to\infty} \left[H\left(\frac{1}{2}z + \frac{1}{2}w^n, \lambda^n\right) - H\left(w^n, \lambda^n\right)\right] \\ &\quad + \liminf_{n\to\infty} \left[H\left(\frac{1}{2}z + \frac{1}{2}\overline{w}, \overline{\lambda}\right) - H\left(\frac{1}{2}z + \frac{1}{2}w^n, \lambda^n\right)\right]. \end{aligned}$$

We now proceed to prove that the three terms on the r.h.s. of this inequality are greater or equal to 0, in contradiction to the assumption of $H\left(z, \bar{\lambda}\right) < H\left(\bar{w}, \bar{\lambda}\right)$. We do so by proving the following three claims:

Claim 1 $\lim_{n\to\infty}\left[H\left(w^n, \lambda^n\right) - H\left(\overline{w}, \overline{\lambda}\right)\right] = 0.$

<u>Proof of Claim 1</u>:

Decompose

$$H\left(w^n, \lambda^n\right) - H\left(\overline{w}, \overline{\lambda}\right) = \sum_{A\in\bar{\Lambda}} \left[\overline{\lambda}_A \ln \overline{w}_A - \lambda_A^n \ln w_A^n\right] + \sum_{A\in\bar{\Lambda}^c} \left[\overline{\lambda}_A \ln \overline{w}_A - \lambda_A^n \ln w_A^n\right].$$

First note that, by the proof of Part (iii), for all $A \in 2^{|I|}$, since $w_A^n \geq \lambda_A^n$ for all n, we have $\overline{w}_A \geq \overline{\lambda}_A$. Thus,

$$\lim_{n\to\infty}\left(\sum_{A\in\bar{\Lambda}} \left[\overline{\lambda}_A \ln \overline{w}_A - \lambda_A^n \ln w_A^n\right]\right) = 0 \tag{A-3}$$

by a straightforward continuity argument. Further,

$$\overline{\lambda}_A \ln \overline{w}_A = 0 \tag{A-4}$$

for all $A \in \bar{\Lambda}^c$ by the definition of $\bar{\Lambda}$. Finally, in view of Part (iii), for any A such $\overline{\lambda}_A = 0$,

$$0 \geq \lim_{n\to\infty}\left(\lambda_A^n \ln w_A^n\right) \geq \lim_{n\to\infty}\left(\lambda_A^n \ln \lambda_A^n\right) = 0,$$

hence,

$$\lim_{n\to\infty}\left(\lambda_A^n \ln w_A^n\right) = 0. \tag{A-5}$$

The Claim follows combining (A-3), (A-4) and (A-5). □

Claim 2 $\lim_{n\to\infty}\left[H\left(\frac{1}{2}z + \frac{1}{2}\overline{w}, \overline{\lambda}\right) - H\left(\frac{1}{2}z + \frac{1}{2}w^n, \lambda^n\right)\right] = 0.$

<u>Proof of Claim 2</u>:

Analogously to the proof of Claim 1, decompose

$$H\left(w^{n},\lambda^{n}\right)-H\left(\overline{w},\overline{\lambda}\right) = \sum_{A\in\bar{\Lambda}}\left[\overline{\lambda}_{A}\ln\overline{w}_{A}-\lambda_{A}^{n}\ln w_{A}^{n}\right]+\sum_{A\in\bar{\Lambda}^{c}}\left[\overline{\lambda}_{A}\ln\overline{w}_{A}-\lambda_{A}^{n}\ln w_{A}^{n}\right].$$

$$H\left(\frac{1}{2}z+\frac{1}{2}\overline{w},\overline{\lambda}\right)-H\left(\frac{1}{2}z+\frac{1}{2}w^{n},\lambda^{n}\right) = \sum_{A\in\bar{\Lambda}}\left[\lambda_{A}^{n}\ln\left(\frac{1}{2}z+\frac{1}{2}w_{A}^{n}\right)-\overline{\lambda}_{A}\ln\left(\frac{1}{2}z+\frac{1}{2}\overline{w}_{A}\right)\right]+ \sum_{A\in\bar{\Lambda}^{c}}\left[\lambda_{A}^{n}\ln\left(\frac{1}{2}z+\frac{1}{2}w_{A}^{n}\right)-\overline{\lambda}_{A}\ln\left(\frac{1}{2}z+\frac{1}{2}\overline{w}_{A}\right)\right].$$

Since $w_{A}^{n}\rightarrow\overline{w}_{A}>0$ for all $A\in\bar{\Lambda}$,

$$\lim_{n\rightarrow\infty}\sum_{A\in\bar{\Lambda}}\left[\lambda_{A}^{n}\ln\left(\frac{1}{2}z+\frac{1}{2}w_{A}^{n}\right)-\overline{\lambda}_{A}\ln\left(\frac{1}{2}z+\frac{1}{2}\overline{w}_{A}\right)\right]=0. \tag{A-6}$$

Also, by the definition of $\bar{\Lambda}$,

$$\overline{\lambda}_{A}\ln\left(\frac{1}{2}z+\frac{1}{2}\overline{w}_{A}\right)=0\text{ for all }A\in\bar{\Lambda}^{c}. \tag{A-7}$$

Finally, in view of Claim 1, for all $A\in\bar{\Lambda}^{c}$,

$$0\geq\lim_{n\rightarrow\infty}\left(\lambda_{A}^{n}\ln\left(\frac{1}{2}z+\frac{1}{2}w_{A}^{n}\right)\right)\geq\lim_{n\rightarrow\infty}\left(\lambda_{A}^{n}\left(\ln\frac{1}{2}+\ln\lambda_{A}^{n}\right)\right)=0,$$

hence,

$$\lim_{n\rightarrow\infty}\left(\lambda_{A}^{n}\ln\left(\frac{1}{2}z+\frac{1}{2}w_{A}^{n}\right)\right)=0. \tag{A-8}$$

The Claim follows combining (A-6), (A-7) and (A-8). □

Claim 3 $\liminf_{n\rightarrow\infty}\left[H\left(\frac{1}{2}z+\frac{1}{2}w^{n},\lambda^{n}\right)-H\left(w^{n},\lambda^{n}\right)\right]\geq 0.$

<u>Proof of Claim 3</u>:

This is straightforward from the assumption that $w^{n}\in\Psi^{DIV}\left(v^{n}\right)$, which implies that $H\left(\frac{1}{2}z+\frac{1}{2}w^{n},\lambda^{n}\right)-H\left(w^{n},\lambda^{n}\right)\geq 0$ for all n.□

Putting together Claims 1, 2 and 3 implies the upper hemicontinuity of Ψ^{DIV}.■

Part (vii): Note that Robustness is vacuously satisfied for any strictly monotone v. Hence, consider a v with a corresponding λ such that $\lambda_{i}=0$ for all $i\in I_{0}\subseteq I$ and $\lambda_{i}>0$ for all $i\notin I_{0}$. Let $w\in\Psi^{DIV}\left(v\right)$ and define $\tilde{\lambda}$ and $\hat{\lambda}$ as:

$$\tilde{\lambda}_{A} =: \begin{cases} w_{i} & \text{if } A=\{i\}\text{ for some } i\in I \\ 0 & \text{else}\end{cases}$$

$$\hat{\lambda}_{A} =: \begin{cases} \frac{1}{|I|} & \text{if } A=\{i\}\text{ for some } i\in I \\ 0 & \text{else}\end{cases}$$

For $\xi^{n}\in(0,1)$ and $\epsilon^{n}\in(0,1)$, let $\bar{\lambda}\left(\xi^{n}\right):=\xi^{n}\tilde{\lambda}+\left(1-\xi^{n}\right)\lambda$ and $\lambda^{n}=:\epsilon^{n}\hat{\lambda}+\left(1-\epsilon^{n}\right)\bar{\lambda}\left(\xi^{n}\right)$. Then[27], $\Psi^{DIV}\left(\bar{\lambda}\left(\xi^{n}\right)\right)=\{w\}$. We now prove the following Claim:

Claim 4 For each n, $\Psi^{DIV}\left(\lambda^{n}\right)$ is a singleton and $\lim_{\epsilon^{n}\rightarrow\infty}\Psi^{DIV}\left(\lambda^{n}\right)=\{w\}$.

[27] This follows from the fact that Ψ^{Div} satisfies Betweenness, see Axiom 4 in Section 4 and the proof of Theorem 4.1.

<u>Proof of Claim 4</u>:

Note that the function

$$\epsilon^n \sum_{i\in I} \frac{1}{|I|} \ln \tilde{w}_i + (1-\epsilon^n) \sum_{A\in\bar{\Lambda}} \bar{\lambda}_A(\xi^n) \ln \sum_{i\in A} \tilde{w}_i$$

is strictly concave for each $\epsilon^n > 0$. Hence, it has a unique optimum and, therefore, $\Psi^{DIV}(\lambda^n)$ is indeed a singleton. By Part (iii) of the Proposition, if $\hat{w}^n \in \Psi^{DIV}(\lambda^n)$, then $\hat{w}_i^n > 0$ for all $i \in I$ and all n. From the f.o.c., for any $i \in I$,

$$\frac{\epsilon^n}{|I|} \frac{1}{\hat{w}_i^n} + (1-\epsilon^n) \sum_{\{A|i\in A\}} \frac{\bar{\lambda}_A(\xi^n)}{\sum_{j\in A} \hat{w}_j^n} = 1, \text{ or}$$

$$(1-\epsilon^n) \left(\sum_{\{A|i\in A\}} \frac{\bar{\lambda}_A(\xi^n)\, \hat{w}_i^n}{\sum_{j\in A} \hat{w}_j^n} - \hat{w}_i^n \right) = \hat{w}_i^n \frac{(|I|-1)\,\epsilon^n}{|I|}.$$

Since $\hat{w}_i^n < 1$, we have:

$$\left(\sum_{\{A|i\in A\}} \frac{\bar{\lambda}_A(\xi^n)\, \hat{w}_i^n}{\sum_{j\in A} \hat{w}_j^n} - \hat{w}_i^n \right) \leq \frac{|I|-1}{|I|} \frac{\epsilon^n}{1-\epsilon^n}$$

and we conclude that:

$$\sum_{A|i\in A} \frac{\bar{\lambda}_A(\xi^n) \lim_{\epsilon^n\to 0} \hat{w}_i^n}{\sum_{j\in A} \lim_{\epsilon^n\to 0} \hat{w}_j^n} - \lim_{\epsilon^n\to 0} \hat{w}_i^n = 0.$$

Since w is the unique Diversity value of $\bar{\lambda}(\xi^n)$ and since, Ψ^{DIV} is upper hemicontionuous, it follows that $\lim_{\epsilon^n\to 0} \hat{w}_i^n = w_i$ for all $i \in I$, or $\lim_{\epsilon^n\to 0} \Psi^{DIV}(\lambda^n) = \{w\}$.□

From Claim 4, it follows that for a given ξ^n and any $\varepsilon > 0$, there exists an $\bar{\epsilon} > 0$ such that $\|\hat{w}^n - w\| \leq \varepsilon$, whenever $\epsilon^n < \bar{\epsilon}$. Take two sequences ξ^n and ε^n such that $\lim_{n\to\infty} \xi^n = \lim_{n\to\infty} \varepsilon^n = 0$. Let $\bar{\epsilon}^n$ be such that $\|\hat{w}^n - w\| \leq \varepsilon^n$, whenever $\epsilon^n \leq \bar{\epsilon}^n$. Construct the sequence $\epsilon^n = \frac{\bar{\epsilon}^n}{n}$ for each n. Since $\bar{\epsilon}^n$ is bounded above, $\lim_{n\to\infty} \epsilon^n = 0$. We then have $\lim_{n\to\infty} \lambda^n = \lambda$ and for each n, $\|\hat{w}^n - w\| \leq \varepsilon^n$, or $\lim_{n\to\infty} \hat{w}^n = w$.■

Part $(viii)$**:** Multiplication of v by a constant c is equivalent to multiplication of all $(\lambda_A)_{A\in\Lambda}$ by c. Clearly, the solution of the optimization problem in (8) remains unchanged.■

Part (ix)**:** Condition (4), which defines λ for a given v implies that if $v'(J) = v(J) + b$ for every $J \subseteq I$, $J \neq \varnothing$, then $\lambda_A = \lambda'_A$ for all $A \neq I$ and $\lambda'_I = \lambda_I + b$. Hence,

$$\begin{aligned}
\Psi^{DIV}(v') &= \arg \max_{\tilde{w}\in\Delta^{|I|-1}} \sum_{A\in\Lambda} \lambda'_A \ln \left(\sum_{i\in A} \tilde{w}_i \right) \\
&= \arg \max_{\tilde{w}\in\Delta^{|I|-1}} \left(\sum_{A\in\Lambda} \lambda_A \ln \left(\sum_{i\in A} \tilde{w}_i \right) + b \ln \sum_{i\in I} \tilde{w}_i \right) \\
&= \arg \max_{\tilde{w}\in\Delta^{|I|-1}} \sum_{A\in\Lambda} \lambda_A \ln \left(\sum_{i\in A} \tilde{w}_i \right) = \Psi^{DIV}(v).■
\end{aligned}$$

Proof of Proposition 4.2:

Note that the definition of the Shapley Fixpoint value can be rewritten as:

$$w_i = \sum_{\{A|i\in A\}} \frac{w_i \frac{\lambda_A}{v(I)}}{\sum_{j\in A} w_j} \text{ for all } i \in I. \tag{A-9}$$

Part (i)**:** We have to show that for a given v and the corresponding λ, the solution of the following optimization problem:

$$\max_{(w_i)_{i\in I}\in\Delta^{|I|-1}} \left\{ \sum_A \lambda_A \ln \sum_{i\in A} w_i \right\} \tag{A-10}$$

is an element of Ψ^{SFX}, i.e., it satisfies condition (A-9). Since the function in (A-10) is continuous and since $\Delta^{|I|-1}$ is a compact and convex set, a maximizer always exists and it satisfies the f.o.c. of (A-10):

$$\left(\sum_{\{A|i\in A\}} \frac{\lambda_A}{\sum_{j\in A} w_j} - \kappa + \underline{\gamma}_i = 0 \right)_{i\in I},$$

where κ is the Lagrangian multiplier corresponding to the condition $\sum_{i\in I} w_i = 1$, whereas $\underline{\gamma}_i$ is the multiplier associated with $w_i \geq 0$. Consider two cases:

Case 1: $\underline{\gamma}_i = 0$ for all i and, hence, $w_i \in (0, 1)$ for all $i \in I$. Then the condition reduces to:

$$\sum_{\{A|i\in A\}} \frac{\lambda_A}{\sum_{j\in A} w_j} - \kappa = 0 \text{ for all } i \in I. \tag{A-11}$$

Since $w_i > 0$ for all $i \in I$, this is equivalent to

$$\sum_{\{A|i\in A\}} \frac{\lambda_A w_i}{\sum_{j\in A} w_j} - \kappa w_i = 0 \text{ for all } i \in I.$$

Summing over all i's then gives:

$$\begin{aligned}
\sum_{i\in I} \sum_{\{A|i\in A\}} \frac{\lambda_A w_i}{\sum_{j\in A} w_j} - \kappa &= \sum_A \sum_{i\in A} \frac{\lambda_A w_i}{\sum_{j\in A} w_j} - \kappa = \sum_A \frac{\lambda_A \sum_{i\in A} w_i}{\sum_{j\in A} w_j} - \kappa \\
&= \sum_A \lambda_A - \kappa = v(I) - \kappa = 0,
\end{aligned}$$

hence, $\kappa = v(I)$ and (A-11) is equivalent to (A-9). It follows that $w \in \Psi^{SFX}(v)$.

Case 2: $\underline{\gamma}_i > 0$ for all $i \in I' \subset I$ and, hence, $w_i = 0$ for all $i \in I'$. In this case, (A-11) only holds for $i \notin I'$. Just as in Case 1, $\kappa = v(I)$. Hence,

$$\sum_{\{A|i\in A\}} \frac{\frac{\lambda_A}{v(I)} w_i}{\sum_{j\in A} w_j} = w_i$$

is satisfied for all $i \in I$, or $w \in \Psi^{SFX}(v)$.$\blacksquare$

Part (ii) Since all sources in v are either strictly dominated, or essential, by Parts (iii) and (iv) of Proposition 3.1, under the Diversity value, $w_i > 0$ for all essential sources, whereas $w_i = 0$ for all dominated ones. Note that under Ψ^{SFX}, a source which is dominated by an essential source receives a 0-weight. To see this, assume that i is essential and dominates j given v. Then, the definition of Ψ^{SFX}

implies that $w_i > 0$ and by dominance,

$$\sum_{\{A|i\in A\}} \frac{\lambda_A}{\sum_{j'\in A} w_{j'}} - \kappa = 0 > \sum_{\{A|j\in A\}} \frac{\lambda_A}{\sum_{j'\in A} w_{j'}} - \kappa.$$

Therefore, $\underline{\gamma}_j > 0$ and $w_j = 0$ under the Shapley Fixpoint value. Hence, both Ψ^{SFX} and Ψ^{DIV} assign a weight of 0 to all dominated sources.

Consider the system of equations constrained to the undominated / essential sources:

$$\left(\sum_{\{A|i\in A\}} \frac{\lambda_A}{\sum_{j'\in A} w_{j'}} - \kappa = 0\right)_{i\text{ - undominated}}$$

As shown above, the solution to this system is unique and, together with $w_j = 0$, whenever j is dominated, is equivalent to the condition specifying the Shapley Fixpoint value. It also corresponds to the condition specifying the Diversity value. Hence, $\Psi^{SFX}(v) = \Psi^{DIV}(v)$.

Since for a strictly monotone v, all sources are essential, the same result applies.■

Part (iii): We will prove the following two claims, which combined imply the result.

Claim 5 Any upper hemicontinuous selection Ψ of Ψ^{SFX} must satisfy $\Psi(v) \supseteq \Psi^{DIV}(v)$ for all $v \in \mathbf{V}$

Proof of Claim 5: Follows from the robustness of Ψ^{DIV}.□

Claim 6 For any $v \in \mathbf{V}$, $\Psi^{DIV}(v)$ is the only convex subset of $\Psi^{SFX}(v)$ which contains $\Psi^{DIV}(v)$.

Proof of Claim 6:

Note that both $\Psi^{DIV}(\lambda)$ and $\Psi^{SFX}(\lambda)$ are invariant up to normalizing $v(I) = 1$. Hence, for the purposes of this proof, we set $v(I) =: 1$. We start with the following observation. Suppose that $w \in \Psi^{DIV}(\lambda)$ and $supp(w) =: J$. Then

$$\left\{w \in \Psi^{SFX}(\lambda) \mid supp(w) = J\right\} = \left\{w \in \Psi^{DIV}(\lambda) \mid supp(w) = J\right\}.$$

To see this, suppose that $w \in \Psi^{DIV}(\lambda)$ and $supp(w) =: J$. It follows that

$$(w_i)_{i\in J} \in \arg\max_{(w_i)_{i\in J}\in\Delta^{|J|-1}} \sum_A \lambda_A \ln\left(\sum_{i\in A\cap J} w_i\right),$$

and since $w_i > 0$ for all $i \in J$, we conclude that the following f.o.c. hold:

$$\sum_{\{A|i\in A\}} \frac{\lambda_A}{\sum_{j\in A\cap J} w_j} = 1 \text{ for all } i \in J$$

and since the function to be maximized is concave, all of the solutions are maximizers. It follows that

$$\left\{w \in \Psi^{SFX}(\lambda) \mid supp(w) = J\right\} = \left\{w \in \Psi^{DIV}(\lambda) \mid supp(w) = J\right\}.$$

Hence, for any $J \subseteq I$, we have that: if there is a $w \in \Psi^{DIV}(\lambda)$ s.t. $supp(w) = J$, then

$$\left\{w \in \Psi^{SFX}(\lambda) \mid supp(w) = J\right\} = \left\{w \in \Psi^{DIV}(\lambda) \mid supp(w) = J\right\}.$$

Furthermore, if $supp(w) = J$ and $supp(w') = J'$ for $w, w' \in \Psi^{DIV}(\lambda)$, then (by Convexity) there is a $w'' \in \Psi^{DIV}(\lambda)$ such that $supp\,(w'') = J \cup J'$.

If for each $J \subseteq I$ such that there is a $w \in \Psi^{SFX}(\lambda)$ with $supp\,(w) = J$, there exists a $w' \in \Psi^{DIV}(\lambda)$ with $supp\,(w') = J$, then $\Psi^{SFX}(\lambda) = \Psi^{DIV}(\lambda)$ and the statement of the Proposition is true. Hence, suppose that there is a $J \subseteq I$ such that there is a $w \in \Psi^{SFX}(\lambda)$ with $supp\,(w) = J$, but no $w' \in \Psi^{DIV}(\lambda)$ with $supp\,(w') = J$. We want to show that there is a $w' \in \Psi^{DIV}(\lambda)$ and an $\alpha \in (0,1)$ such that $\alpha w + (1-\alpha) w' \notin \Psi^{SFX}(\lambda)$.

Case 1: Let there exist a $w' \in \Psi^{DIV}(\lambda)$ with $supp\,(w') = J' \supset J$. Then, $supp\,(\alpha w + (1-\alpha) w') = J'$. Hence, if $\alpha w + (1-\alpha) w' \in \Psi^{SFX}(\lambda)$ for all $\alpha \in (0,1)$, then $\alpha w + (1-\alpha) w' \in \Psi^{DIV}(v)$ for all $\alpha \in (0,1)$. But $\Psi^{DIV}(\lambda)$ is u.h.c., hence, we should have $w \in \Psi^{DIV}(v)$, a contradiction.

Case 2: Suppose that there is no $w' \in \Psi^{DIV}(\lambda)$ with $supp\,(w') \supseteq J$. Suppose, however, that there is a $w' \in \Psi^{DIV}(\lambda)$ with $supp\,(w') = J' \subset J$. We will show that there is an α such that $\alpha w + (1-\alpha) w' \notin \Psi^{SFX}(\lambda)$. Write down the conditions defining $w \in \Psi^{SFX}(\lambda)$ with support J and those defining $w' \in \Psi^{DIV}(\lambda)$ with support J' as follows:

$\sum_{\{A|i\in A\}} \frac{\lambda_A}{\sum_{j\in A} w_j} = 1$ for all $i \in J$

$\sum_{\{A|i\in A\}} \frac{\lambda_A}{\sum_{j\in A} w_j} > 1$ for some $i \notin J$ (this follows from the fact that $w \notin \Psi^{DIV}(\lambda)$)

$\sum_{\{A|i\in A\}} \frac{\lambda_A}{\sum_{j\in A} w'_j} = 1$ for all $i \in J'' \supseteq J'$ (here, we take into account that for some i s.t. $w'_i = 0$, $\underline{\gamma}_i = 0$ can obtain)

$\sum_{\{A|i\in A\}} \frac{\lambda_A}{\sum_{j\in A} w'_j} < 1$ for all $i \notin J''$

Take a $i \notin J''$. Then, since the function $\frac{\lambda_A}{\sum_{j\in A} w'_j}$ is convex in w, we obtain that for small enough α's,

$$\sum_{\{A|i\in A\}} \frac{\lambda_A}{\sum_{j\in A}\left[\alpha w_j + (1-\alpha) w'_j\right]} \leq \sum_{\{A|i\in A\}} \alpha \frac{\lambda_A}{\sum_{j\in A} w_j} + \sum_{\{A|i\in A\}} (1-\alpha) \frac{\lambda_A}{\sum_{j\in A} w'_j} \leq 1. \quad \text{(A-12)}$$

Now take a $i \in J'' \cap J$. We have:

$$\sum_{\{A|i\in A\}} \frac{\lambda_A}{\sum_{j\in A}\left[\alpha w_j + (1-\alpha) w'_j\right]} \leq \sum_{\{A|i\in A\}} \alpha \frac{\lambda_A}{\sum_{j\in A} w_j} + \sum_{\{A|i\in A\}} (1-\alpha) \frac{\lambda_A}{\sum_{j\in A} w'_j} = 1.$$

Hence, if $\sum_{\{A|i\in A\}} \frac{\lambda_A}{\sum_{j\in A} w_j} > 1$ only if $i \notin J''$, we have that for sufficiently small α's,

$$\sum_{\{A|i\in A\}} \frac{\lambda_A}{\sum_{j\in A}\left[\alpha w_j + (1-\alpha) w'_j\right]} \leq 1$$

for all $i \in I$. Hence, if $\alpha w + (1-\alpha) w' \in \Psi^{SFX}(\lambda)$, then $\alpha w + (1-\alpha) w' \in \Psi^{DIV}(\lambda)$, in contradiction to the assumption that there is no $w \in \Psi^{DIV}(\lambda)$ such that $supp\,(w) \supseteq J$. It follows that for small α's, $\alpha w + (1-\alpha) w' \notin \Psi^{SFX}(\lambda)$.

Suppose therefore that $\sum_{\{A|i\in A\}} \frac{\lambda_A}{\sum_{j\in A} w_j} > 1$ for some $i \in J''$, i.e., $J'' \supset J$. Note that for $i \in J'' \backslash J$, $w'_i = 0$ and, hence, for each A such that $i \in A$, there must be a j such that $j \in J' \subset J$ (i.e., $w_j > 0$ and $w'_j > 0$) and $j \in A$. For this j to be in the support of the Fixpoint value defined by $\alpha w + (1-\alpha) w'$, the weak equality in (A-12) has to hold with equality, which, by the Jensen's inequality requires that

$\sum_{j' \in A} w_{j'} = \sum_{j' \in A} w'_{j'}$ for all A s.t. $j \in A$ and in particular, for each A, s.t. $i, j \in A$. It follows that for each $i \in J'' \backslash J$ and each $A \in \Lambda$ s.t. $i \in A$, we have

$$\sum_{j' \in A} w_{j'} = \sum_{j' \in A} w'_{j'},$$

and since

$$\sum_{\{A|i \in A\}} \frac{\lambda_A}{\sum_{j \in A} w'_j} = 1,$$

we obtain

$$\sum_{\{A|i \in A\}} \frac{\lambda_A}{\sum_{j \in A} w_j} = 1,$$

as well, in contradiction to the assumption made above.

Case 3: Suppose that there is no $w' \in \Psi^{DIV}(\lambda)$ with $supp(w') \supseteq J$. Suppose, however, that there is a $w' \in \Psi^{DIV}(\lambda)$ with $supp(w') = J'$ such that neither $J' \subset J$, nor $J' \supset J$. We will show that there is an α such that $\alpha w + (1-\alpha) w' \notin \Psi^{SFX}(\lambda)$.

The conditions defining $w \in \Psi^{SFX}(\lambda)$ with support J and those defining $w' \in \Psi^{DIV}(\lambda)$ with support J' are just as above. We also have that $supp(\alpha w + (1-\alpha) w') = J \cup J'$. Hence, if for each $\alpha \in (0,1)$, $(\alpha w + (1-\alpha) w') \in \Psi^{SFX}(\lambda)$, we could choose a $\alpha \in (0,1)$ and $\alpha w + (1-\alpha) w' \in \Psi^{SFX}(\lambda)$. If $\alpha w + (1-\alpha) w' \in \Psi^{DIV}(\lambda)$, we have a contradiction to the assumption made above. If $\alpha w + (1-\alpha) w' \notin \Psi^{DIV}(\lambda)$, then we have $supp(\alpha w + (1-\alpha) w') \neq J \cup J' \supset J'$, just as in Case 2. Hence, applying the same reasoning, we obtain that there is a β sufficiently close to 0 such that $\beta(\alpha w + (1-\alpha) w') + (1-\beta) w' \notin \Psi^{SFX}(\lambda)$, in contradiction to the assumption that $\tilde{\alpha} w + (1-\tilde{\alpha}) w' \in \Psi^{SFX}(\lambda)$ for each $\tilde{\alpha} \in (0,1)$.

We have thus shown that

(i) if for each $J \subseteq I$ such that there is a $w \in \Psi^{SFX}(\lambda)$ with $supp(w) = J$, there exists a $w' \in \Psi^{DIV}(\lambda)$ with $supp(w') = J$, then $\Psi^{SFX}(\lambda) = \Psi^{DIV}(\lambda)$;

(ii) if there is a $J \subseteq I$ such that there is a $w \in \Psi^{SFX}(\lambda)$ with $supp(w) = J$, but no $w' \in \Psi^{DIV}(\lambda)$ with $supp(w') = J$, then for any $w' \in \Psi^{DIV}(\lambda)$, there exists an $\alpha \in (0,1)$ such that $\alpha w + (1-\alpha) w' \notin \Psi^{SFX}(\lambda)$.

Combining (i) and (ii), we conclude that the set $\Psi^{DIV}(\lambda)$ is the only convex subset of $\Psi^{SFX}(\lambda)$ containing $\Psi^{DIV}(\lambda)$.□

Claims 5 and 6 imply the result of Part (iii). ■

Proof of Theorem 4.1:

The intuition behind the proof can be summarized as follows: for the subclass of diversity functions, which are additive, except for clones, we can show that $\Psi(v) \subseteq \Psi^{DIV}(v)$. We then show that any strictly monotone diversity function v with Diversity value $\Psi^{DIV}(v) = \{w\}$ can be represented as a linear combination of its Diversity value w and diversity functions which are additive except for clones and such that w belongs to their Diversity values as well. If the so-constructed linear combination is also a convex combination, Betweenness implies that w is the unique value in $\Psi(v)$. If not, we can show that there exists an $\alpha \in (0;1)$ such that $\alpha w + (1-\alpha)v$ can be represented as such a convex combination for sufficiently large α's and, by Betweenness, $\{w\} = \Psi(\alpha w + (1-\alpha)v) = \Psi^{DIV}(\alpha w + (1-\alpha)v)$. Furthermore, for strictly monotone v, $w \in \Psi^{DIV}(\alpha w + (1-\alpha)v)$ implies $w \in \Psi^{DIV}(v)$. It follows that for strictly monotone v, $\Psi(v) \subseteq \Psi^{DIV}(v)$, which in turn in a singleton, hence, $\Psi(v) = \Psi^{DIV}(v)$ in the interior of $\mathbf{V}$. The last part of the proof uses the continuity of Ψ as well as the robustness of Ψ^{DIV} to extend Ψ to the set of all diversity functions.

By Proposition 3.1, Ψ^{DIV} satisfies Axioms 1, 2, 5 and 6. We now prove that Ψ^{DIV} also satisfies the remaining axioms.

Lemma A.1 Ψ^{DIV} *satisfies Exclusion and Betweenness.*

Proof of Lemma A.1:

To show that Ψ^{DIV} satisfies Exclusion, let j be such that $w_j = 0$ for all $w \in \Psi^{DIV}(v)$. Hence, any such w solves the system of equations:

$$\begin{aligned}
\sum_{\{A\in\Lambda'|i\in A\}} \frac{\frac{\lambda_A}{v(I)}}{\sum_{j'\in A} w_{j'}} - 1 + \underline{\gamma}_i &= 0 \text{ for all } i \in I \text{, which is equivalent to} \\
\sum_{\{A\in\Lambda'|i\in A\}} \frac{\frac{\lambda_A}{v(I)}}{\sum_{\substack{j'\in A \\ j'\neq j}} w_{j'}} - 1 + \underline{\gamma}_i &= 0 \text{ for all } i \in I \backslash \{j\} \\
\sum_{\{A\in\Lambda'|j\in A\}} \frac{\frac{\lambda_A}{v(I)}}{\sum_{\substack{j'\in A \\ j'\neq j}} w_{j'}} - 1 &\leq 0
\end{aligned}$$

The second set of equations is exactly equivalent to the condition defining the Diversity value of $v^{[I\backslash\{j\}]}$, hence, $\Psi^{DIV}(v) \subseteq \Psi^{DIV}\left(v^{[I\backslash\{j\}]}\right)$. Now suppose that there is a $\tilde{w} \in \Psi^{DIV}\left(v^{[I\backslash\{j\}]}\right)$ which does not satisfy the last inequality, i.e.,

$$\begin{aligned}
\sum_{\{A\in\Lambda'|i\in A\}} \frac{\frac{\lambda_A}{v(I)}}{\sum_{\substack{j'\in A \\ j'\neq j}} \tilde{w}_{j'}} - 1 + \underline{\gamma}_i &= 0 \text{ for all } i \in I \backslash \{j\} \\
\sum_{\{A\in\Lambda'|j\in A\}} \frac{\frac{\lambda_A}{v(I)}}{\sum_{\substack{j'\in A \\ j'\neq j}} \tilde{w}_{j'}} - 1 &> 0
\end{aligned}$$

and thus, $\tilde{w} \notin \Psi^{DIV}(v)$, but $\tilde{w} \in \Psi^{SFX}(v)$. If $\tilde{w}$ has the same support as an element $w \in \Psi^{DIV}(v)$, then $\tilde{w} \in \Psi^{DIV}(v)$, see the proof of Part (iii) of Proposition 4.2, a contradiction. Hence, there is no element in $\Psi^{DIV}(v)$ with the same support as $\tilde{w}$. It follows that, see the proof of Part (iii) of Proposition 4.2, there is $w' \in \Psi^{DIV}(v)$ and an $\alpha \in (0;1)$ such that $\alpha\tilde{w} + (1-\alpha)w' \notin \Psi^{SFX}(v)$ and hence, $\alpha\tilde{w} + (1-\alpha)w' \notin \Psi^{DIV}\left(v^{[I\backslash\{j\}]}\right)$. Since however, $\tilde{w}$ and $w' \in \Psi^{DIV}\left(v^{[I\backslash\{j\}]}\right)$ and

$\Psi^{DIV}\left(v^{[I\setminus\{j\}]}\right)$ is convex, we obtain a contradiction.

To see that Ψ^{DIV} satisfies Betweenness, note that if for λ and λ',

$$\begin{aligned} w &\in \arg\max_{\tilde{w}\in\Delta^{|I|-1}} \sum \lambda_A \ln \sum_{i\in A} \tilde{w}_i \\ w &\in \arg\max_{\tilde{w}\in\Delta^{|I|-1}} \sum \lambda'_A \ln \sum_{i\in A} \tilde{w}_i, \end{aligned}$$

then

$$w \in \arg\max_{\tilde{w}\in\Delta^{|I|-1}} \sum \left[\alpha\lambda + (1-\alpha)\lambda'_A\right] \ln \sum_{i\in A} \tilde{w}_i.$$

It follows that $\Psi\left(\alpha v + (1-\alpha) v'\right) \supseteq \Psi^{DIV}(v) \cap \Psi^{DIV}(v')$. Let $w \notin \Psi^{DIV}(v) \cap \Psi^{DIV}(v')$. Since $\Psi^{DIV}(v) \cap \Psi^{DIV}(v') \neq \varnothing$,

$$\begin{aligned} \sum_A \left(\alpha\lambda_A + (1-\alpha)\lambda'_A\right) \ln \sum_{i\in A} w_i &< \max_{w\in\Delta} \sum_A \left(\alpha\lambda_A + (1-\alpha)\lambda'_A\right) \ln \sum_{i\in A} w_i \\ &= \alpha \max_{w\in\Delta} \sum_A \lambda_A \ln \sum_{i\in A} w_i + (1-\alpha) \max_{w\in\Delta} \sum_A \lambda'_A \ln \sum_{i\in A} w_i, \end{aligned}$$

Hence, $w \notin \Psi\left(\alpha v + (1-\alpha) v'\right)$. We conclude that $\Psi\left(\alpha v + (1-\alpha) v'\right) = \Psi^{DIV}(v) \cap \Psi^{DIV}(v')$.□

We now show that the Axioms 1–5 imply Ψ^{DIV}. We first provide the dual statements of the axioms in terms of λ:

Axiom 1A **(Additive Core)** If λ is additive, then $\Psi(\lambda) = \left\{\frac{\lambda_i}{\sum_{i\in I}\lambda_i}\right\}$.

Axiom 2A **(Dominance)** If a source i is dominated in λ, then $w \in \Psi(\lambda)$ implies $w_i = 0$.

Axiom 3A **(Exclusion)** If for a source i, $w_i = 0$ for all $w \in \Psi(\lambda)$, then $\Psi(\lambda) = \Psi\left(\lambda^{[I\setminus\{i\}]}\right)$.

Axiom 4A **(Betweenness)** If $\Psi(\lambda) \cap \Psi(\lambda') \neq \varnothing$, then $\Psi(\alpha\lambda + (1-\alpha)\lambda') = \Psi(\lambda) \cap \Psi(\lambda')$..

Axiom 5A **(Convex-Valuedness)** For all λ, $\Psi(\lambda)$ is convex.

Axiom 6A **(Continuity)** For any sequence λ^n such that $\lim_{n\to\infty} \lambda^n = \lambda$ and any sequence of $w^n \in \Psi(\lambda^n)$ for all n, $\lim_{n\to\infty} w^n \in \Psi(\lambda)$.

It is obvious that $\Psi(v)$ satisfies Axioms 1–6 iff $\Psi(\lambda)$ satisfies Axioms 1A–6A. Hence, from now on, we will work with λ and $\Psi(\lambda)$ instead of v and $\Psi(v)$. For a given $v(I)$, consider the class of all λ such that $\sum_{A\in\Lambda} \lambda_A = v(I)$. For an arbitrarily chosen $v(I)$, we now prove the Theorem on this class of λ's.

Lemma A.2 *If Ψ satisfies Axioms 2A, 5A and 6A (Dominance, Convex-Valuedness and Continuity), then for λ's such that $\lambda_A = v(I)$ for some $A \subseteq I$,*

$$\Psi(\lambda) = \left\{ w \mid \sum_{i\in A} w_i = 1, w_i = 0 \text{ for all } i \notin A \right\}. \tag{A-13}$$

<u>Proof of Lemma A.2</u>:

The statement is true by Axiom 2A if $|A| = 1$. Thus, consider the case of $|A| > 1$. Take a λ satisfying the condition of the Lemma and note that all sources $i \notin A$ are strictly dominated by the sources in A.

Hence, by Axiom 2A, $w_i = 0$ for all $i \notin A$ and all $w \in \Psi(\lambda)$. Take a $j \in A$ and consider a sequence λ^ξ such that $\lambda_A^\xi = v(I) - \xi$, $\lambda_j^\xi = \xi$. Since each $i \neq j$ is strictly dominated by j, by Axiom 2A, $w_i^\xi = 0$ for all $i \neq j$ and all $w \in \Psi(\lambda^\xi)$. It follows that $\Psi(\lambda^\xi) = (w_j = 1, w_i = 0 \text{ for all } i \neq j)$. Obviously, $\lim_{\xi\to 0} \lambda^\xi = \lambda$ and $\lim_{\xi\to 0} w^\xi = (w_j = 1, w_i = 0 \text{ for all } i \neq j)$. Thus, by Axiom 6A, we must have $(w_j = 1, w_i = 0 \text{ for all } i \neq j) \in \Psi(\lambda)$ for all $j \in A$. By Axiom 5A, $\Psi(\lambda)$ is convex and, thus contains the convex hull of all these points, which is exactly (A-13).□

Furthermore,

Lemma A.3 *If Ψ satisfies Axioms 1A—3A, 5A and 6A, then for any $B \subseteq I$ and λ such that $\lambda_A > 0$ only if $A \in \left\{\{i\}_{i\notin B}; B\right\}$,*

$$\Psi(\lambda) \supseteq \left\{ w \mid w_i = \lambda_i \text{ for } i \notin B \text{ and } \sum_{i\in B} w_i = \lambda_B \right\}. \tag{A-14}$$

<u>**Proof of Lemma A.3**</u>:

By Axiom 1A, the statement is true for $|B| = 1$. Thus, consider the case of $|B| > 1$. Take a $j \in B$ and consider a sequence λ^ξ such that $\lambda_k^\xi = \lambda_k$ for $k \notin B$, $\lambda_B^\xi = \lambda_B - \xi$, $\lambda_j^\xi = \xi$. Since each $i \in B$, $i \neq j$ is strictly dominated by j, by Axiom 2A, $w_i^\xi = 0$ for all $i \in B$, $i \neq j$ and all $w \in \Psi(\lambda^\xi)$. Define $\lambda_j'^\xi = \lambda_B$, $\lambda_k'^\xi = \lambda_k$ for $k \notin B$ and note that by iterative application of Axiom 3A, and by Axiom 1A,

$$\Psi\left(\lambda^\xi\right) = \Psi\left(\lambda'^\xi\right) = \left\{ w^\xi \mid w_k^\xi = \lambda_k \text{ for } k \notin B,\, w_j^\xi = \lambda_B,\, w_i^\xi = 0 \text{ for all } i \in B,\, i \neq j \right\}.$$

Since $\lim_{\xi\to 0} \lambda^\xi = \lambda$ and

$$\lim_{\xi\to 0} w^\xi = (w_k = \lambda_k \text{ for } k \notin B,\, w_j = \lambda_B,\, w_i = 0 \text{ for all } i \in B,\, i \neq j),$$

by Axiom 6A, $\Psi(\lambda^\xi) \subseteq \Psi(\lambda)$ for any choice of $j \in B$. By Axiom 5A, $\Psi(\lambda)$ is convex and, thus contains the convex hull of all these points, which is exactly the r.h.s. of (A-14).□

Proposition A.4 *If Ψ satisfies Axioms 1A – 6A, then for any strictly monotone v, i.e., for any λ such that $\lambda_i > 0$ for all $i \in I$, we have:*

$$\Psi(\lambda) = \Psi^{DIV}(\lambda).$$

<u>**Proof of Proposition A.4**</u>:

For a strictly monotone v, i.e., $\lambda_i > 0$ for all $i \in I$, implies $w_i > 0$ for all $w \in \Psi^{DIV}(\lambda)$. Furthermore, the unique w such that $\{w\} = \Psi^{DIV}(\lambda)$ satisfies the system of equations:

$$\left(\sum_{\{A|i\in A\}} \frac{\frac{\lambda_A}{v(I)}}{\sum_{j\in A} w_j} = 1 \right)_{i\in I}. \tag{A-15}$$

For each A such that $\lambda_A > 0$, $A \notin \left\{\{i\}_{i\in I}\right\}$, construct λ^A in the following way: $\lambda_i^A = v(I)\, w_i$ for all $i \notin A$, $\lambda_A^A = v(I) \sum_{i\in A} w_i$ and $\lambda_{A'}^A = 0$ for all $A' \notin \left\{\{i\}_{i\notin A}, A\right\}$. Let $\bar{\lambda}$ be defined as: $\bar{\lambda}_i = v(I)\, w_i$ for all $i \in I$ and $\bar{\lambda}_A = 0$ for all $A \notin \left\{\{i\}_{i\in I}\right\}$. For each A with $\lambda_A > 0$ and $A \notin \left\{\{i\}_{i\in I}\right\}$, let

$\beta_A =: \frac{\frac{\lambda_A}{v(I)}}{\sum_{j\in A} w_j}$. Then, for any such A, we have:

$$\sum_{\{A|\lambda_A>0,\, A\notin\{\{i\}_{i\in I}\}\}} \beta_A \lambda_A^A = \frac{\lambda_A}{\sum_{j\in A} w_j} \sum_{k\in A} w_k = \lambda_A.$$

Furthermore, for any $i \in I$,

$$\sum_{\{A|\lambda_A>0,\, A\notin\{\{j\}_{j\in I}\}\}} \beta_A \lambda_i^A + \left(1 - \sum_{\{A|\lambda_A>0,\, A\notin\{\{j\}_{j\in I}\}\}} \beta_A\right) \bar{\lambda}_i \qquad \text{(A-16)}$$

$$= \sum_{\{A|i\notin A\}} \frac{\lambda_A}{\sum_{k\in A} w_k} w_i + \left(v(I) - \sum_{\{A|\lambda_A>0,\, A\notin\{\{j\}_{j\in I}\}\}} \frac{\lambda_A}{\sum_{k\in A} w_k}\right) w_i$$

$$= w_i \left(v(I) - \sum_{\{A|\lambda_A>0,\, i\in A,\, A\neq\{i\}\}} \frac{\lambda_A}{\sum_{k\in A} w_k}\right) = w_i \frac{\lambda_i}{w_i} = \lambda_i$$

It follows that

$$\sum_{\{A|\lambda_A>0,\, A\notin\{\{i\}_{i\in I}\}\}} \beta_A \lambda^A + \left(1 - \sum_{\{A|\lambda_A>0,\, A\notin\{\{i\}_{i\in I}\}\}} \beta_A\right) \bar{\lambda} = \lambda. \qquad \text{(A-17)}$$

To prove the remainder of Proposition A.4, we state the following Lemmata:

Lemma A.5 *For any λ, and any $\tilde{w} \in \Delta^{|I|-1}$ such that*

$$supp\,(\tilde{w}) = \{i \in I,\, i \text{ is not null in } \lambda\}\,,$$

there exists an $\alpha \in (0;1)$ such that $\Psi(\alpha v(I)\tilde{w} + (1-\alpha)\lambda) = \Psi^{SFX}(\alpha v(I)\tilde{w} + (1-\alpha)\lambda)$.

Proof of Lemma A.5:

Let $\tilde{w} \in \Delta^{|I|-1}$ be such that $supp\,(\tilde{w}) = \{i \in I, i \text{ is not null in } \lambda\}$. For such $\tilde{w}$, and any $\alpha \in (0;1)$, we will show that, by Part (ii) of Proposition 4.2, there is a unique $w^\alpha \in \Psi^{SFX}(\alpha v(I)\tilde{w} + (1-\alpha)\lambda)$. To see this, note that if i is null, then $\sum_{\{A|i\in A\}} (\alpha v(I)\tilde{w}_i + (1-\alpha)\lambda_A) = 0$ and, hence, $w_i^\alpha = 0$ for all $w^\alpha \in \Psi^{SFX}((\alpha v(I)\tilde{w} + (1-\alpha)\lambda))$. Furthermore, since $\alpha v(I)\tilde{w}_i + (1-\alpha)\lambda_i > 0$ for all non-null sources i, Part (ii) of Proposition 4.2 implies that $\Psi^{SFX}(\alpha v(I)\tilde{w} + (1-\alpha)\lambda)$ is a singleton and

$$\Psi^{SFX}(\alpha v(I)\tilde{w} + (1-\alpha)\lambda) = \Psi^{DIV}(\alpha v(I)\tilde{w} + (1-\alpha)\lambda)\,.$$

For a given α and $\{w^\alpha\} = \Psi^{DIV}(\alpha v(I)\tilde{w} + (1-\alpha)\lambda)$, define the sets $\lambda^A(\alpha)$ and the coefficients β_A^α as above. Note that expression (A-16) is satisfied for all i such that $w_i^\alpha > 0$, but also for all i such that $w_i^\alpha = 0$, since then $\alpha v(I)\tilde{w}_i + (1-\alpha)\lambda_i = 0$. Hence, $\alpha v(I)\tilde{w}_i + (1-\alpha)\lambda_i$ can be represented as a linear combination as in (A-17) and

$$\alpha v(I)\tilde{w} + (1-\alpha)\lambda = \sum_{\{A|\lambda_A>0,\, A\notin\{\{i\}_{i\in I}\}\}} \beta_A^\alpha \lambda^A(\alpha) + \left(1 - \sum_{\{A|\lambda_A>0,\, A\notin\{\{i\}_{i\in I}\}\}} \beta_A^\alpha\right) v(I)\, w^\alpha$$

If $\sum_{\left\{A|\lambda_A>0,\, A\notin\left\{\{j\}_{j\in I}\right\}\right\}}\beta_A^\alpha<1$, then, by Lemma A.3, $(w_i^\alpha)_{i\in I}\in\Psi\left(\lambda^A(\alpha)\right)$ for all A and by Axiom 1A, $\Psi\left(v\left(I\right)w^\alpha\right)=\{w^\alpha\}$. Then applying iteratively Axiom 4A, $\{w^\alpha\}=\Psi\left(\alpha v\left(I\right)\tilde{w}+(1-\alpha\lambda)\right)=\Psi^{SFX}\left(\alpha v\left(I\right)\tilde{w}+(1-\alpha\lambda)\right)$.

If $\sum_{\left\{A|\lambda_A>0,\, A\notin\left\{\{j\}_{j\in I}\right\}\right\}}\beta_A^\alpha\geq 1$, then for any α,

$$\beta_A^\alpha=\frac{(1-\alpha)\frac{\lambda_A}{v(I)}}{\sum_{j\in A}w_j^\alpha}\leq\frac{(1-\alpha)\frac{\lambda_A}{v(I)}}{\sum_{j\in A}\alpha\tilde{w}_j}.$$

Note that

$$\begin{aligned}\sum_{\left\{A|\lambda_A>0,\, A\notin\left\{\{j\}_{j\in I}\right\}\right\}}\beta_A^\alpha &\leq \sum_{\left\{A|\lambda_A>0,\, A\notin\left\{\{j\}_{j\in I}\right\}\right\}}\frac{(1-\alpha)\frac{\lambda_A}{v(I)}}{\sum_{j\in A}\alpha\tilde{w}_j}< \\ &< \frac{(1-\alpha)\sum_{\left\{A|\lambda_A>0,\, A\notin\left\{\{j\}_{j\in I}\right\}\right\}}\frac{\lambda_A}{v(I)}}{\alpha\min_{i\in supp(\tilde{w})}\tilde{w}_i}\end{aligned}$$

Hence, if

$$\begin{aligned}\frac{(1-\alpha)\sum_{\left\{A|\lambda_A>0,\, A\notin\left\{\{j\}_{j\in I}\right\}\right\}}\frac{\lambda_A}{v(I)}}{\alpha\min_{i\in supp(\tilde{w})}\tilde{w}_i} &< 1\text{, or}\\ \frac{\sum_{\left\{A|\lambda_A>0,\, A\notin\left\{\{j\}_{j\in I}\right\}\right\}}\frac{\lambda_A}{v(I)}}{\left[\min_{i\in supp(\tilde{w})}\tilde{w}_i+\sum_{\left\{A|\lambda_A>0,\, A\notin\left\{\{j\}_{j\in I}\right\}\right\}}\frac{\lambda_A}{v(I)}\right]} &= :\bar{\alpha}<\alpha\end{aligned}$$

then $\sum_{\left\{A|\lambda_A>0,\, A\notin\left\{\{j\}_{j\in I}\right\}\right\}}\beta_A^\alpha<1$ will obtain whenever $\alpha>\bar{\alpha}$. Again applying iteratively Axiom 4A, $\Psi\left(\alpha v\left(I\right)w+(1-\alpha)\lambda\right)=\Psi^{SFX}\left(\alpha v\left(I\right)w+(1-\alpha)\lambda\right)$ holds for all $\alpha\geq\bar{\alpha}$.□

Lemma A.6 *If $w\notin\Psi^{SFX}(v)$, then $w\notin\Psi^{SFX}\left(\alpha v+(1-\alpha)v\left(I\right)w\right)$.*

Proof of Lemma A.6:

Suppose that $w\in\Psi^{SFX}\left(\alpha\lambda+(1-\alpha)v\left(I\right)w\right)$, then

$$\left(\sum_{\{A|i\in A\}}\frac{\alpha\frac{\lambda_A}{v(I)}}{\sum_{i\in A}w_i}+\frac{(1-\alpha)w_i}{w_i}=1\right)_{i\in supp(w)}\text{, or, equivalently,}$$

$$\left(\sum_{\{A|i\in A\}}\frac{\frac{\lambda_A}{v(I)}}{\sum_{i\in A}w_i}=1\right)_{i\in supp(w)},$$

hence, $w\in\Psi^{SFX}(\lambda)$.□

Lemma A.7 *For all λ such that $\lambda_i>0$ for all $i\in I$, $\Psi(\lambda)\subseteq\Psi^{SFX}(\lambda)$.*

Proof of Lemma A.7:

Let w be an element of $\Psi(\lambda)$ with maximal support with respect to set inclusion. Such an element exists by Axiom 5A. Since $\tilde{w}_i=0$ for all $i\notin supp\left(w\right)$ and all $\tilde{w}\in\Psi(\lambda)$, it follows by Axiom 3A that $\Psi(\lambda)=\Psi\left(\lambda^{[supp(w)]}\right)$. We also know that by Axioms 1A and 4A,

$$\Psi\left(\alpha v\left(I\right)w+(1-\alpha)\lambda\right)=\Psi\left(\alpha v\left(I\right)w+(1-\alpha)\lambda^{[supp(w)]}\right)=\{w\}.$$

Note that $supp\,(w) = \left\{ i \in I, i \text{ is not null in } \lambda^{[supp(w)]} \right\}$. Hence, by Lemma A.5, there exists an $\alpha \in (0;1)$ such that:

$$\Psi\left(\alpha v\,(I)\,w + (1-\alpha)\,\lambda^{[supp(w)]}\right) = \Psi^{SFX}\left(\alpha v\,(I)\,w + (1-\alpha)\,\lambda^{[supp(w)]}\right) = \{w\}\,.$$

By Lemma A.6, this implies $w \in \Psi^{SFX}\left(\lambda^{[supp(w)]}\right) \subseteq \Psi^{SFX}(\lambda)$. Hence, $w \in \Psi^{SFX}(\lambda)$.□

To complete the proof of Proposition A.4, note that for any strictly monotone λ and any $w \in \Psi(\lambda)$, the fact that $\Psi(\lambda) \subseteq \Psi^{SFX}(\lambda) = \Psi^{DIV}(\lambda)$ implies $\Psi(\lambda) = \Psi^{DIV}(\lambda) = \{w\}$.□

Our next lemma shows that even for λ's not satisfying the conditions of Proposition A.4, $w \in \Psi(\lambda)$ only if $w \in \Psi^{SFX}(\lambda)$.

Lemma A.8 *If Ψ satisfies Axioms 1A – 6A, then for every λ, $w \in \Psi(\lambda)$ only if $w \in \Psi^{SFX}(\lambda)$.*

<u>Proof of Lemma A.8</u>:

For a given λ, let $w \in \Psi(\lambda)$, and assume that $w \notin \Psi^{SFX}(\lambda)$. Take the convex combination $\alpha\lambda + (1-\alpha)\,v\,(I)\,w$ for some $\alpha \in (0;1)$. By Axiom 4A, $\Psi(\alpha\lambda + (1-\alpha)\,v\,(I)\,w) = \{w\}$. Consider λ^{ϵ} s.t. $\lambda_i^{\epsilon} > 0$ for each $i \in I$ and each $\epsilon > 0$ and such that $\lim_{\epsilon\to 0}\lambda^{\epsilon} = \lambda$. Then, by Proposition A.4, for each $\epsilon > 0$, $\Psi(\alpha\lambda^{\epsilon} + (1-\alpha)\,v\,(I)\,w) = \Psi^{DIV}(\alpha\lambda^{\epsilon} + (1-\alpha)\,v\,(I)\,w)$. Furthermore, since Ψ satisfies Axiom 6A,

$$\lim_{\epsilon\to 0}\Psi^{DIV}(\alpha\lambda^{\epsilon} + (1-\alpha)\,v\,(I)\,w) = \lim_{\epsilon\to 0}\Psi(\alpha\lambda^{\epsilon} + (1-\alpha)\,v\,(I)\,w) \subseteq \Psi(\alpha\lambda + (1-\alpha)\,v\,(I)\,w) = \{w\}\,.$$

Since Ψ^{DIV} is upper hemicontinuous, we have that:

$$\lim_{\epsilon\to 0}\Psi(\alpha\lambda^{\epsilon} + (1-\alpha)\,v\,(I)\,w) = \lim_{\epsilon\to 0}\Psi^{DIV}(\alpha\lambda^{\epsilon} + (1-\alpha)\,v\,(I)\,w) \subseteq \Psi^{DIV}(\alpha\lambda + (1-\alpha)\,v\,(I)\,w)\,.$$

Hence, by Part (i) of Proposition 4.2, $w \in \Psi^{DIV}(\alpha\lambda + (1-\alpha)\,v\,(I)\,w) \subseteq \Psi^{SFX}(\alpha\lambda + (1-\alpha)\,v\,(I)\,w)$, and by Lemma A.6, $w \in \Psi^{SFX}(\lambda)$, in contradiction to our initial assumptions.□

Our next Lemma demonstrates that if Ψ satisfies Axioms 1A–6A, then $\Psi(\lambda) \supseteq \Psi^{DIV}(\lambda)$.

Lemma A.9 *If Ψ satisfies Axioms 1A–6A, then $\Psi(\lambda) \supseteq \Psi^{DIV}(\lambda)$.*

<u>Proof of Lemma A.9</u>:

Suppose to the contrary of the statement of the Lemma that there is a λ and a $w \in \Psi^{DIV}(\lambda)$ such that $w \notin \Psi(\lambda)$. Consider a sequence λ^{ϵ} s.t. $\lambda_i^{\epsilon} > 0$ for each $i \in I$ and each $\epsilon > 0$ and such that $\lim_{\epsilon\to 0}\lambda^{\epsilon} = \lambda$. Then, by Proposition A.4, for each $\epsilon > 0$, $\Psi(\lambda^{\epsilon}) = \Psi^{DIV}(\lambda^{\epsilon})$. Since Ψ^{DIV} satisfies Robustness, there exists a sequence λ^{ϵ} such that $\lim_{\epsilon\to 0}\Psi^{DIV}(\lambda^{\epsilon}) = \{w\}$ and, hence, $\lim_{\epsilon\to 0}\Psi(\lambda^{\epsilon}) = \{w\}$. Since by Axiom 6A, Ψ is upper hemicontinuous, $w \in \Psi(\lambda)$, a contradiction.□

By Lemmata A.8 and A.9, we know that for all λ's, $\Psi^{DIV}(\lambda) \subseteq \Psi(\lambda) \subseteq \Psi^{SFX}(\lambda)$. Since Ψ satisfies Axiom 5A, the proof of Part (iii) of Proposition 4.2 implies $\Psi(\lambda) = \Psi^{DIV}(\lambda)$ for all λ such that $\sum_{A\in\Lambda}\lambda_A = v\,(I)$ for some $v\,(I)$. Since $v\,(I)$ was chosen arbitrarily, we have that $\Psi(\lambda) = \Psi^{DIV}(\lambda)$

for all λ, thus concluding the proof of Theorem 4.1.□

Proof of Proposition 5.1:

The equivalence between (i), (ii) and (iii) is straightforward.

To prove the equivalence of (iii) and (iv), consider first the case of a diversity function, for which $w_1 > 0$ and $w_2 > 0$ obtain. The f.o.c. conditions for sources 1 and 2 are then given by:

$$\begin{aligned} 1 &= \frac{\lambda_1}{w_1} + \frac{\lambda_{12}}{w_1 + w_2} + \frac{\lambda_{13}}{w_1 + w_3}, \text{ and} \\ 1 &= \frac{\lambda_2}{w_2} + \frac{\lambda_{12}}{w_1 + w_2} + \frac{\lambda_{23}}{w_2 + w_3}. \end{aligned}$$

Hence

$$\frac{\lambda_1}{w_1} + \frac{\lambda_{13}}{w_1 + w_3} = \frac{\lambda_2}{w_2} + \frac{\lambda_{23}}{w_2 + w_3}.$$

Further, by uniformity

$$\lambda_1 + \lambda_{12} + \lambda_{13} = \lambda_2 + \lambda_{12} + \lambda_{23},$$

hence

$$\lambda_1 + \lambda_{13} = \lambda_2 + \lambda_{23} =: \ell.$$

Substituting, we have

$$\frac{\lambda_1}{w_1} + \frac{\ell - \lambda_1}{w_1 + w_3} = \frac{\lambda_2}{w_2} + \frac{\ell - \lambda_2}{w_2 + w_3},$$

i.e.

$$\left(\frac{1}{w_1} - \frac{1}{w_1 + w_3}\right)\lambda_1 + \frac{\ell}{w_1 + w_3} = \left(\frac{1}{w_2} - \frac{1}{w_2 + w_3}\right)\lambda_2 + \frac{\ell}{w_2 + w_3}. \tag{A-18}$$

Hence if $\lambda_1 \geq \lambda_2$ and $w_1 < w_2$, we get

$$\begin{aligned} \left(\frac{1}{w_1} - \frac{1}{w_1 + w_3}\right)\lambda_1 + \frac{\ell}{w_1 + w_3} &> \left(\frac{1}{w_2} - \frac{1}{w_2 + w_3}\right)\lambda_1 + \frac{\ell}{w_2 + w_3} \geq \\ &\geq \left(\frac{1}{w_2} - \frac{1}{w_2 + w_3}\right)\lambda_2 + \frac{\ell}{w_2 + w_3}, \end{aligned}$$

contradicting (A-18).

As we know $w_i = 0$ can only obtain if $\lambda_i = 0$. If $\lambda_1 > \lambda_2 = 0$, and $w_2 = 0$, then $w_1 > 0 = w_2$. If $\lambda_1 = \lambda_2 = 0$, we must have $\lambda_{13} = \lambda_{23}$. If $w_1 = 0$, then,

$$\frac{\lambda_{12}}{w_2} + \frac{\lambda_{13}}{w_3} - 1 < 0$$

and, hence, as long as $\lambda_{13} = \lambda_{23} > 0$, i.e., 1 and 2 are not clones,

$$\frac{\lambda_{12}}{w_2} + \frac{\lambda_{23}}{w_2 + w_3} - 1 < \frac{\lambda_{12}}{w_2} + \frac{\lambda_{13}}{w_3} - 1 < 0$$

for any positive w_2. It follows that $w_2 = w_1 = 0$.□

Proof of Proposition 5.2:

Suppose first that Ψ satisfies Fixed Weights at v. Then, for a given $w \in \Psi(v)$,

$$\left\{\sum_{j\in J} \frac{w_j x_j}{\sum_{k\in J} w_k}\right\} = \hat{\Psi}\left((x_i)_{i\in I}, v^{[J]}\right)$$

,

$$\left\{\sum_{i\in I\setminus J} \frac{w_i x_i}{\sum_{k\in I\setminus J} w_k}\right\} = \hat{\Psi}\left((x_i)_{i\in I}, v^{[I\setminus J]}\right)$$

and if the two are equal, then

$$\begin{aligned}\left\{\sum_{i\in I} w_i x_i\right\} &= \left\{\sum_{k\in J} w_k \sum_{j\in J} \frac{w_j x_j}{\sum_{k\in J} w_k} + \sum_{k\in I\setminus J} w_k \sum_{i\in I\setminus J} \frac{w_i x_i}{\sum_{k\in I\setminus J} w_k}\right\} = \\ &= \left\{\sum_{j\in J} \frac{w_j x_j}{\sum_{k\in J} w_k}\right\} = \left\{\sum_{i\in I\setminus J} \frac{w_i x_i}{\sum_{k\in I\setminus J} w_k}\right\} = \hat{\Psi}\left((x_i)_{i\in I}, v\right).\end{aligned}$$

Suppose now that Ψ satisfies Reinforcement at v and consider a $J \subset I$ and $j, k \in J$ such that $\{w\} = \Psi(v)$ and $\{\tilde{w}\} = \Psi\left(v^{[J]}\right)$ are such that $\frac{w_j}{w_k} < \frac{\tilde{w}_j}{\tilde{w}_k}$. Let x satisfy $x_j = \frac{1}{3}$, $x_k = \frac{2}{3}$ and $x_i = \gamma$ for all $i \notin \{j, k\}$, where

$$\gamma := \frac{2}{3}\frac{\tilde{w}_j}{\tilde{w}_j + \tilde{w}_k} + \frac{1}{3}\frac{\tilde{w}_k}{\tilde{w}_j + \tilde{w}_k}.$$

Then, by construction $\gamma \in \hat{\Psi}\left(x, v^{[J]}\right)$, $\{\gamma\} = \hat{\Psi}\left(x, v^{[I\setminus J]}\right)$ and hence, by Reinforcement, $\hat{\Psi}(x, v) = \{\gamma\}$. However, since

$$\frac{2}{3}\frac{w_j}{w_j + w_k} + \frac{1}{3}\frac{w_k}{w_j + w_k} < \gamma,$$

$\hat{\Psi}(x, v) = \left\{\sum_{i\in I} w_i x_i\right\}$ with $\sum_{i\in I} w_i x_i < \gamma$, a contradiction.□

Proof of Proposition 5.3:

To see that Reinforcement is violated on an open subset of $\mathbf{V}_M$, note that on $\mathbf{V}_M$, Reinforcement is defined by a finite set of equations that are continuous in v, respectively λ. These equations can be compactly written as

$$\Psi^{WSV}_{\Psi^{DIV}(v)}\left(v^{[J]}\right) = \Psi^{DIV}\left(v^{[J]}\right).$$

The set of v satisfying Reinforcement is thus closed, hence its complement is open.

We now show that the subset of $\mathbf{V}_M$ on which Reinforcement is violated is dense.

Suppose Ψ^{DIV} satisfies Reinforcement at $v \in \mathbf{V}_M$, with $\Psi^{DIV} = \{w\}$. Let A be any non-singleton strict subset of I. If v is not quasi-additive, take $A \in \Lambda$.

Define $v'' \in \mathbf{V}_M$ via λ'' as follows:

$$\begin{array}{ll} \lambda''_{\{i\}} = w_i & \text{if } i \notin A \\ \lambda''_{\{i\}} = \frac{1}{2} w_i & \text{if } i \in A \\ \lambda''_A = \frac{1}{2}\left(\sum_{i\in A} w_i\right) & \\ \lambda''_S = 0 & \text{otherwise.} \end{array}.$$

For any $\epsilon \in (0,1)$, let $v' = \epsilon v'' + (1-\epsilon)v$ and note that, by construction, $\Psi^{DIV}(v'') = \{w\}$. Hence, by Betweenness, also $\Psi^{DIV}(v') = \{w\}$.
Take any $J : A^c \subsetneq J \subsetneq I$. For any $i \in A$, $j \notin A$, $\frac{\Psi^{DIV}(v''^{[J]})_i}{\Psi^{DIV}(v''^{[J]})_j} > \frac{w_i}{w_j}$. Since v, v' and v'' are monotone, for each of them, Ψ^{DIV} and Ψ^{FSV} coincide and by Lemma A.6, we obtain

$$\Psi^{DIV}(v'^{[J]}) \neq \left\{ \left(\left(\frac{w_j}{\sum_{k\in J} w_k} \right)_{j\in J}, \underbrace{(0...0)}_{i\in I\setminus J} \right) \right\}.$$

Ψ^{DIV} thus fails to satisfy Fixed Weigths at v', and, hence also Reinforcement by Proposition 5.2. $\square$

Proof of Proposition 5.4:

We will prove the proposition using two Lemmata:

Lemma A.10 *v is exchangeable if and only if, either for all $J \subseteq I$,*

$$\Psi^{DIV}\left(v^{[J]}\right) = \Psi^{SV}\left(v^{[J]}\right) = \left\{ \left(\left(w_j = \frac{1}{|J|} \right)_{j\in J}, (w_j = 0)_{j\notin J} \right) \right\} \tag{A-19}$$

or for all $J \subseteq I$,

$$\Psi^{DIV}\left(v^{[J]}\right) = \left((w_j)_{j\in J} \in \Delta^{|J|-1}, (w_j = 0)_{j\notin J} \right). \tag{A-20}$$

Proof of Lemma A.10:

Suppose first that v is exchangeable, we have that for every $J \subseteq I$ and any J', $J'' \subseteq J$ such that $|J'| = |J''|$, $v^{[J]}(J') = v^{[J]}(J'')$.
Note that for any exchangeable v, we have that either $\lambda_I = v(I)$, and hence, for every $J \subseteq I$, (A-20) obtains, or there exists a natural K, $0 < K < I$ such that $\lambda_J > 0$ for all J with $|J| = K$. In the latter case, we have that $\{\mathbf{1}_A \mid A \in \Lambda \cup \{I\}\}$ has full rank and hence, by Proposition 3.1, the Diversity value of v is unique and given by

$$\Psi^{DIV}(v) = \Psi^{SV}(v) = \left\{ \left(\frac{1}{|I|} \cdots \frac{1}{|I|} \right) \right\}.$$

Furthermore, for a $J \subset I$ with $|J| > K$, we know that for some (and then all), $J' \subset J$, $|J'| = K$, $\lambda^{[J]}_{J'} \geq \lambda_{J'} > 0$. Hence, for such J, (A-19) obtains. For each $J \subset I$ with $|J| \leq K$, there is a $J' \subset J$, and a $J'' \subset I$ with $|J''| = K$ such that $J' = J'' \cap J$ and $|J| > |J'| > 0$. It then follows that $\lambda^{[J]}_{J''} \geq \lambda_{J''} > 0$ and hence, $\Psi^{DIV}\left(v^{[J]}\right)$ also satisfies (A-19).
Suppose, now that (A-19) holds for all $J \subseteq I$. We want to show that v is exchangeable. We will prove the claim by induction. The statement holds trivially for $|I| = 1$. Assume that it holds for all $I \leq K-1$ and let $I = K$. By the induction hypothesis, for any $i, j \in I$, $v^{[I\setminus\{i\}]}$ and $v^{[I\setminus\{j\}]}$ are both exchangeable, i.e.,

$$v(J) = v(J')$$

whenever $|J| = |J'|$ and either $J,\ J' \subset I\backslash\{i\}$ or $J,\ J' \subset I\backslash\{j\}$. Hence, $v(J) = v(J')$ whenever $|J| = |J'| < K-2$. From the symmetry of the Shapley value and the induction hypothesis, we obtain:

$$\begin{aligned}\Psi_i^{SV}(v) - \Psi_j^{SV}(v) &= \frac{1}{|I|}\left[v(I) - v(I\backslash\{i\})\right] - \frac{1}{|I|}\left[v(I) - v(I\backslash\{j\})\right] \\ &= \frac{1}{|I|}\left[v(I\backslash\{j\}) - v(I\backslash\{i\})\right] = 0\end{aligned}$$

Hence, $v(I\backslash\{i\}) = v(I\backslash\{j\})$ for all i and $j \in I$ and hence, v is exchangeable.

Finally, if (A-20) holds, we have that $\lambda_I = v(I)$, and, hence v is exchangeable.□

Lemma A.11 *A uniform v satisfies Reinforcement iff either for all $J \subseteq I$, (A-19) holds, or for all $J \subseteq I$, (A-20) holds.*

Proof of Lemma A.11:

It is clear that each of the conditions (A-19) and (A-20) implies Reinforcement. Assume thus that v is uniform and satisfies Reinforcement. Note that this implies that $v^{|J|}$ is uniform for all $J \subseteq I$. We will prove the claim by induction. If $|I| = 2$, uniformity implies either

$$\Psi^{DIV}(v) = \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\} = \Psi^{SV}(v),$$

or $\Psi^{DIV}(v) = \Delta^1$. Suppose that there is a uniform v on $|I| = 3$ such that $\Psi^{DIV}\left(v^{[i,j]}\right) = \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\}$ and $\Psi^{DIV}\left(v^{[i,k]}\right) = \Delta^1$. It is easy to see that this implies $\lambda_i = \lambda_k = \lambda_{ij} = \lambda_{jk} = 0$, $\lambda_j = \lambda_{ik}$. Hence, $\Psi^{DIV}(v) = \left\{\left(w_i \in \left[0, \frac{1}{2}\right]; w_j = \frac{1}{2}; w_k = \frac{1}{2} - w_i\right)\right\}$. Hence, (just as in Example 5.2), whenever $x_i \neq x_k$, we obtain a violation of Reinforcement. It follows that for $|I| = 3$, Reinforcement implies either $\Psi^{DIV}\left(v^{[i,j]}\right) = \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\}$ for all $i, j \in I$, and hence, $\Psi^{DIV}(v) = \left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$, or $\Psi^{DIV}\left(v^{[i,j]}\right) = \Delta^1$ for all $i, j \in I$, and hence, $\Psi^{DIV}(v) = \Delta^2$

Suppose thus that for all $I \leq K-1$ and all uniform v, for which Reinforcement is satisfied, either (A-19) holds for all $J \subseteq I$, or (A-20) holds for all $J \subseteq I$.

Consider a v on $I = K \geq 4$. If all $v^{[J]}$ with $|J| = K-1$ satisfy (A-19), then, by Reinforcement, we have that $\Psi^{DIV}(v) = \left\{\left(\frac{1}{|I|} \cdots \frac{1}{|I|}\right)\right\}$ and hence, v satisfies (A-19) as well. If all $v^{[J]}$ with $|J| = K-1$ satisfy (A-20), then, by Reinforcement, we have that $\Psi^{DIV}(v) = \Delta^{|I|-1}$ and hence, v satisfies (A-20) as well.

Finally, suppose that there are $J, J' \subset I$, $|J| = |J'| = K-1$ such that $v^{[J]}$ satisfies (A-19), whereas $v^{[J']}$ satisfies (A-20). Since $K-1 \geq 3$, any two such sets J and J' have at least two common elements, i and j. If $w_i = w_j$ for all $w \in \Psi(v)$, then setting $x_i \neq x_j$ and $\frac{1}{3}x_i + \frac{2}{3}x_j = x_k$ for all $k \notin \{i,j\}$ implies that

$$\begin{aligned}&\hat{\Psi}\left(x_1 ... x_{|I|}; v^{[I\backslash J']}\right) \cap \hat{\Psi}\left(x_1 ... x_{|I|}; v^{[J']}\right) = x_k \neq \\ &\neq \hat{\Psi}\left(x_1 ... x_{|I|}; v\right) = (w_i + w_j)\left(\frac{1}{2}x_i + \frac{1}{2}x_j\right) + (1 - w_i - w_j)\,x_k\end{aligned}$$

in violation of Reinforcement. Alternatively, if for some $w \in \Psi(v)$, $w_i \neq w_j$, then setting $x_i \neq x_j$ and $\frac{1}{2}x_i + \frac{1}{2}x_j = x_k$ for all $k \notin \{i, j\}$ implies that

$$\hat{\Psi}\left(x_1...x_{|I|}; v^{[I \setminus J]}\right) \cap \hat{\Psi}\left(x_1...x_{|I|}; v^{[J]}\right) = x_k \neq$$
$$\neq w_i x_i + w_j x_j + (1 - w_i - w_j) x_k \in \hat{\Psi}\left(x_1...x_{|I|}; v\right),$$

in violation of Reinforcement. This concludes the proof of the Lemma.□

Combining the results of Lemmata A.10 and A.11 implies the statement of the Proposition.□

7 References

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