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Hybrid Procedures.

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Abstract

We consider hybrid procedures: a first step of reducing the game by iterated elimination of weakly dominated strategies ($IEWDS$) followed by a second step of applying an equilibrium refinement. We show that the set of perfect/proper outcomes of a reduced normal-form game might be larger than the set of the perfect/proper outcomes of the whole game by applying $IEWDS$. Even in dominance solvable games in which all the orders of $IEWDS$ select a unique singleton in the game, the surviving outcome need not be a proper equilibrium of the whole game. However, in dominance solvable games that satisfy the transference of decision maker indifference condition (TDI^* of Marx and Swinkels, 1997), the surviving outcome coincides with the unique stable one and hence is proper.

KEYWORDS: Weak dominance, Iterated elimination, Proper equilibrium.

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1 Introduction

Whereas the iterated elimination of strictly dominated strategies seems to be commonly accepted as an appealing procedure to simplify a game,[1] the procedure of iterated elimination of weakly dominated strategies ($IEWDS$) seems to be more controversial. Such a result is present, for instance, in the strategic voting literature (see De Sinopoli [1] among others). Indeed, $IEWDS$ is an order-dependent procedure that removes at each step some set of weakly dominated strategies; this order-dependency is among its least attractive features. In our paper, we ask a simple question: what can be inferred about the set of perfect/proper equilibria of the whole normal-form game from just focusing on the same set of equilibria of the fully reduced game(s) obtained through this procedure? In other words, does applying $IEWDS$ and then using a perfect/proper equilibrium in a normal-form game refine the set of perfect/proper equilibria? The answer is negative even for dominance solvable games. Nonetheless, we provide some sufficient conditions for a positive answer.

Why would one want to infer some information about the set of equilibria of the whole game by just focusing on the set of equilibria of the reduced game? From the point of view of computational complexity[2], one interesting venue of research could be to understand the properties of first applying $IEWDS$ and then solving the game. Kohlberg and Mertens [6] consider such a procedure[3] and then prove that such a method does not uniquely reach stability in a game in which a dominated strategy of a player is replaced with a constant-sum game that has a value equal to the initial payoff matrix. In a sense, they prove that such a method is too weak. Samuelson [14] also considers such a procedure[4] even though the focus of the paper is the interaction between the common knowledge of admissibility and iterated dominance. Our results imply that applying (any order of) $IEWDS$ and then applying properness might simply lead to different results than properness (both in the strategy profiles and in the payoffs) so that the "hybrid" procedure ensures neither perfection nor properness.

[1] The previous observation holds in finite games. When agents can choose among an infinite number of strategies, this need not even be the case (see Duwfenberg and Stegeman [2]).

[2] See the recent advances in computation of equilibria in finite games (for instance von Stengel et al. [17]).

[3] Kohlberg and Mertens [6] (p.1015) argue "that one might therefore conclude that strategic stability could be obtained by first reducing the normal form to some submatrix by iterative eliminations of dominated strategies, and then applying the relevant backwards induction solution (i.e. proper equilibrium)".

[4] Indeed, Samuelson [14] (p.287) states that "concepts such as properness perform well in all respects except admissibility calculations. In particular, the set of proper equilibria can be affected by the deletion of a dominated strategy from a game. One possible response is to construct a two-stage procedure. In the first step, the common knowledge of admissibility is applied to possibly eliminate some strategies. The second step then consists of the application of a solution concept such as properness to the resulting strategy sets".

A well-known property in the literature can be considered as a benchmark to our work. First, the set of Nash equilibria of a game G contains the set of NE of any game G' obtained from G by deletion of a (weakly) dominated strategy. Note that Mertens' stable sets (connected components of perfect equilibria) satisfy a weaker version of the property. The surviving profile in a dominance solvable game is hence a Nash equilibrium and is part of the unique stable set of the game. Therefore, it is perfect as any point in a stable set is perfect. The results get more icy when one scrutinizes the relation between perfect, proper equilibrium and $IEWDS$.

The problem for ensuring perfect and proper inclusion seems to be related to the existence of connected components of equilibria with a continuum of outcomes. Examples of such components can be found in Govindan and McLennan [4] and Kukushkin, Litan and Marhuenda [7].[5] We slightly modify the previously mentioned examples, in order to prove that removing weakly dominated strategies might enlarge the set of perfect and proper outcomes.

We provide a positive result concerning dominance solvable games, in which at least one order of $IEWDS$ selects a unique singleton from the game. Our question can be rephrased in dominance solvable games in the following terms: does the surviving outcome coincide with the outcome of a proper equilibrium? Indeed, as argued by Marx and Swinkels [8], "at an intuitive level, there seems to be an intimate relationship between backward induction and weak dominance." They prove that, in perfect information games, all orders of $IEWDS$ leave only strategy profiles that give rise to the unique backward induction payoff vector.[6] This result holds provided that when some player is indifferent between two strategy profiles that differ only in that player's choice of strategy, all other players are indifferent as well: this condition is denoted transference of decision-maker's indifference (TDI). Of course, as we deal with normal-form games, the precise definition of backward induction is elusive in contrast with perfect information games. The concept of proper equilibrium is often associated with backward induction since van Damme [15] and Kohlberg and Mertens [6] established that a proper equilibrium of a normal form game induces a quasi-perfect/sequential equilibrium in every extensive form game with that normal form.

We first provide an example of a dominance solvable game in which all the orders of deletion lead to the same strategy profile; this profile does not lead to a payoff associated with any proper equilibrium of the whole game, as it violates TDI. We then prove that the surviving outcome coincides with the unique stable one and hence is proper in dominance solvable games satisfying TDI^*. Note that TDI^* implies TDI, while both notions are generically equivalent. More precisely,

[5]See also Pimienta [13] which proves that such components do not exist in three-outcome bimatrix games.

[6]A related work (Hummel [5]) explores the relation of $IEWDS$ and backward induction in binary voting sequential games.

let Γ be a normal form game with associated strategy space S. Iteratively applying $IEWDS$ transforms S into a sequence of restrictions. Note that if the game is solvable then there is a unique stable set in the game. We prove that, if the solvable game satisfies TDI, this stable set is included within a connected component with a unique associated payoff. Hence, the singleton that survives $IEWDS$ leads to the stable outcome and hence its outcome is proper. Our contribution is related to Glazer and Rubistein [3], which underlines an interesting relationship between $IEWDS$ and backward induction. For dominance solvable games, it is proved that the elimination procedure is equivalent to backward induction in some appropriately chosen extensive game.[7] Their result holds provided that the agents are indifferent among the different outcomes, which is stronger than assuming TDI^*.

The work is structured as follows. Section 2 introduces the canonical framework in which we work. Section 3 presents the results dealing with perfection, and Section 4 is focused on the relationship between properness and $IEWDS$.

2 The setting

Let Γ be an n-person normal-form game $\Gamma = (S_1, \ldots, S_n; U_1, \ldots, U_n; N)$, where $N = \{1, 2, \ldots, n\}$ is the set of players, each S_i is a non-empty finite set of pure strategies, and each U_i is a real-valued utility function defined on the domain $S = S_1 \times S_2 \times \ldots \times S_n$. W.l.o.g we assume $S_i \cap S_j = \emptyset$ for any i and j.

For any finite set M, let $\Delta(M)$ be the set of all probability distributions over M. Thus, $\Delta(S_i)$ is the set of mixed strategies for player i in Γ with $\Delta(S_i)$. Similarly, $\Delta^0(S_i)$ stands for the set of *completely* mixed strategies in S_i and for player i. Furthermore, for any mixed strategy σ_i, its support is denoted by $\mathrm{Supp}(\sigma_i) = \{s_i \in S_i \mid \sigma_i(s_i) > 0\}$.

The utility functions are extended to mixed strategies in the usual way:

$$U_j(\sigma_1, \ldots, \sigma_n) = \sum_{(s_1,\ldots,s_n)\in S_1\times\ldots\times S_n} \left(\prod_{i=1}^{n} \sigma_i(s_i) \right) U_j(s_1, \ldots, s_n).$$

The pure strategy s_j^* is a best response to σ_{-j} for player j iff

$$U_j(s_j^*, \sigma_{-j}) = \max_{s_j' \in S_j} U_j(s_j', \sigma_{-j}).$$

An ε-perfect equilibrium of a normal-form game is a completely mixed strategy profile, such that whenever some pure strategy s_i is a worse reply than some other pure strategy t_i, the weight on s_i is smaller than ε. A perfect equilibrium of a normal form game is a limit of ε-perfect equilibria as $\varepsilon \to 0$.

[7] See Perea [12] for a summary of this literature.

An ε-proper equilibrium of a normal-form game is a completely mixed strategy profile, such that whenever some pure strategy s_i is a worse reply than some other pure strategy t_i, the weight on s_i is smaller than ε times the weight on t_i. A proper equilibrium of a normal form game is a limit of ε-proper equilibria as $\varepsilon \to 0$.

Iterated Dominance.

For $W \subseteq S$, let the strategies in W that belong to i be denoted $W_i = W \cap S_i$. Say that $W \subseteq S$ is a *restriction* of S if $\forall i$, $W_i \neq \emptyset$. Note that any restriction W of S generates a unique game given by strategy spaces W_i and the restriction of U_i to $\prod_{i=1}^{n} W_i$.

Let Γ^k denote the reduced game after k rounds of successive restrictions, and let $S_i^k \subseteq S_i^{k-1}$, $S^k \subseteq S^{k-1}$ be the corresponding strategy spaces. We write $S^0 = S$ and $\lim_{k\to\infty} S^k = \cap_{k=0}^{\infty} S^k = S^\infty$. Γ^∞ denotes the reduced game with strategy space S^∞ and the restriction of U_i to S^∞.

For all $i \in N$, let V_i be a nonempty finite subset of $\Delta(S_i) \cup S_i$, and let $V = \cup_{i\in N} V_i$.

Definition 1. [Weak Dominance] *Let $\sigma_i, \tau_i \in \Delta(S_i) \cup S_i$. Then,*
(i) σ_i very weakly dominates τ_i on V if $U_i(\sigma_i, \gamma_{-i}) \geq U_i(\tau_i, \gamma_{-i}) \forall \gamma_{-i} \in V_{-i} = \prod_{j\neq i} V_j$, and
(ii) σ_i weakly dominates τ_i on V if σ_i very weakly dominates τ_i on V, and, in addition, $U_i(\sigma_i, \gamma'_{-i}) > U_i(\tau_i, \gamma'_{-i})$ for some $\gamma'_{-i} \in V_{-i}$.

Redundancy on Mixed Strategies.

Definition 2. [Redundancy] *Let $\sigma_i, \tau_i \in \Delta(S_i) \cup S_i$. Then σ_i is redundant to τ_i on V if for all $\gamma_i \in V_{-i}$, $U_i(\sigma_i, \gamma_{-i}) = U_i(\tau_i, \gamma_{-i})$ implies $U(\sigma_i, \gamma_{-i}) = U(\tau_i, \gamma_{-i})$. A strategy τ_i is redundant on V if there is $\sigma_i \in V$ redundant to τ_i.*

Following Marx and Swinkels [8], we define nice weak dominance and the TDI^* condition.

Definition 3. [Nice Weak Dominance]. *Let $\sigma_i, \tau_i \in \Delta(S_i) \cup S_i$. σ_i nicely weakly dominates τ_i on V if σ_i weakly dominates τ_i on V and for all $\gamma_{-i} \in V_{-i}$, $U_i(\sigma_i, \gamma_{-i}) = U_i(\tau_i, \gamma_{-i})$ implies $U(\sigma_i, \gamma_{-i}) = U(\tau_i, \gamma_{-i})$.*

Definition 4. *Game Γ satisfies TDI^* if for all restrictions W, $\forall i \in N$, and $\forall s_i \in S_i$, if s_i is very weakly dominated on W by $\sigma_i \in \Delta(S_i \setminus s_i)$, then $\exists \sigma'_i \in \Delta(S_i \setminus s_i)$ such that either s_i is weakly dominated on W by σ'_i or s_i is redundant on W to σ'_i.*

If a game satisfies TDI^*, then whenever player i is indifferent between strategies s_i and σ_i, fixing the profile of opponents' strategies s_{-i}, either all players are indifferent between profiles (s_i, s_{-i}) and (σ_i, s_{-i}), or there is some strategy σ'_i such that i strictly prefers σ'_i over s_i and σ_i given s_{-i}.

Remark: For games satisfying TDI^*, weak dominance is equivalent to nice weak dominance.

Marx and Swinkels [8] show that if a game satisfies the following condition on pure strategies, then it generically satisfies TDI^*: $\forall i \in N, \forall s_i, r_i \in S_i, U_i(s_i, s_{-i}) = U_i(r_i, s_{-i}) \Longrightarrow U_j(s_i, s_{-i}) = U_j(r_i, s_{-i})$ (TDI).

3 Perfect equilibria

For any game $\Gamma = (S, U)$, let $Pe(\Gamma)$ denote its set of perfect equilibria and $Pro(\Gamma)$ denote its set of proper equilibria. The sets of (Nash) equilibria and undominated equilibria of Γ are respectively denoted $Ne(\Gamma)$ and $UNe(\Gamma)$.

By iterated weak dominance, there exists a finite number of orders (as there is a finite number of strategies, and we assume that at least one strategy is deleted at each stage until the game is fully reduced). Each order belongs to Θ. Hence the successive reductions of a game Γ due to order o are as follows:

$$\Gamma_o^0 = \Gamma = (S, U), \Gamma_o^1 = (S_o^1, U), \Gamma_o^2 = (S_o^2, U), \ldots, \Gamma_o^\infty = (S_o^\infty, U),$$

with $S_o^i \supseteq S_o^{i+1}$.

Γ_o^∞ stands for the fully reduced game obtained through iterated weak dominance by the order of reduction o.

It is simple to understand that the set of perfect equilibria of a reduced game is not nested in the whole set of perfect equilibria. The next well-known example proves that removing either M, C or both M and C leads to different sets of perfect equilibria on the reduced games, whereas the unique perfect equilibrium of the whole game is (T, L).

	L	C
T	2,1	1,1
M	2,1	0,0

However, despite this path-dependent procedure, we can state the following result.

Proposition 1. *For any order of deletion $o \in \Theta$, $Pe(\Gamma_o^k) \cap Pe(\Gamma) \neq \emptyset$ $\forall k \geq 1$.*

Proof. We omit the definition of Mertens' stable sets and refer to Mertens (1989) [9] for a complete definition. We simply use three of its properties. First, the existence property states that stable sets always exist. Second, stable sets are connected sets of normal-form perfect equilibria (connectedness). Third, stable sets of a game contain stable sets of any game obtained by deleting a pure strategy which is at its minimum probability in any normal form ε-perfect equilibrium in the neighborhood of the stable set (iterated dominance and forward induction). Hence, the last property applies in particular to any weakly dominated strategy. Therefore, there exists at least one stable set of Γ_o^k which is included in a stable set of Γ_o^{k-1}. As any point in a stable set is a perfect, we can directly conclude. □

We can therefore state the next corollary without proof.

Corollary 1. *For any order of deletion $o \in \Theta$, $Pe(\Gamma_o^\infty) \cap Pe(\Gamma) \neq \emptyset$.*

3.1 Bimatrix games

Within the set Θ, m stands for the maximal simultaneous reduction by weak dominance in which all mixed and pure strategies that are weakly dominated by some (mixed) strategy are removed at each step.

Proposition 2. *Let Γ be a bimatrix game. By maximal simultaneous deletion, $Pe(\Gamma_m^1) \subseteq Pe(\Gamma)$. Moreover, $Pe(\Gamma_m^\infty) \subseteq Pe(\Gamma)$.*

The converse of Proposition 2 does not hold. To see this, let us consider the example in Myerson (1978) [10]. There are two players 1,2 with three strategies each. There are two perfect equilibria (T,L) and (M,C); however the only equilibrium that survives maximal simultaneous deletion is (T,L).

	L	C	R
T	1,1	0,0	-9,-9
M	0,0	0,0	-7,-7
B	-9,-9	-7,-7	-7,-7

To see why, it suffices to understand that $M \succ_S B$ and that $C \succ_S R$ in Γ. Furthermore, in the game Γ^1 in which both B and R have been deleted, both $T \succ_{S^1} M$ and $L \succ_{S^1} C$, hence only (T,L) is perfect in the fully reduced game, and it is the unique proper equilibrium of the game.

We now state the proof of Proposition 2.

Proof. Let σ be a perfect equilibrium in the game Γ_m^1. In bimatrix games, an equilibrium is perfect if and only it is undominated. An equilibrium σ is undominated if each of its components σ_i of σ is undominated. Suppose that σ is not a perfect equilibrium in $\Gamma = \Gamma_m^0$.

Either σ is not an equilibrium in Γ or σ is an equilibrium in such a game, but some of the strategies in σ are dominated in Γ. In the former case, this is a contradiction with the definition of iterated dominance as an equilibrium σ of a reduced game is an equilibrium of the whole game. In the latter case, some of the strategies in σ are dominated in Γ so that by maximal simultaneous deletion, the strategy σ is not present in Γ_m^1, a contradiction. □

Proposition 3. *Let Γ be a bimatrix game satisfying TDI^*. For any order of deletion, the set of perfect outcomes of any fully reduced game is a subset of the set of perfect outcomes of Γ.*

Proof. By Proposition 2, the set of perfect equilibria of the fully reduced game Γ_m^∞ is a subset of the set of perfect equilibria of Γ. As stated by Marx and Swinkels [8], in any game satisfying TDI^*, any two full reductions by weak dominance are the same up to the addition or removal of redundant strategies. Moreover, the set of perfect equilibria is invariant to the addition of redundant strategies (see for instance Kohlberg and Mertens [6]). It hence follows that the set of outcomes of any fully reduced game is a subset of the set of outcomes of the whole game. □

3.2 Finite Games

To see why Proposition 2 does not hold with more than two players, let us consider the next example (p.29 Van Damme (1996) [16]).

A:

	L	C
T	1,1,1	1,0,1
M	1,1,1	0,0,1

B:

	L	C
T	1,1,0	0,0,0
M	0,1,0	1,0,0

In such a game, both $L \geq_S C$ and $A \geq_S B$. There is just one perfect equilibrium in Γ: (T,L,A). Nevertheless, applying maximal simultaneous deletion removes C and B from S, so that (T,L,A) and (M,L,A) are both perfect equilibria in the fully reduced game. In other words, removing weakly dominated strategies may enlarge the set of perfect equilibria.

Yet, the outcome is not enlarged in this example. One might wonder whether inclusion holds in terms of outcomes. The answer is again negative, as the following example shows.

This example is a modified version of the one present in Govindan and McLennan [4] with the addition of a weakly dominated strategy X for player 3 (as long as the payoff for player 3 in each of the outcomes is strictly positive). This is an outcome game that satisfies TDI and TDI^*.

U:

	L	R
T	a	a
M	b	b
B	a	b
D	e	f

D:

	L	R
T	c	c
M	d	d
B	c	d
D	e	f

X:

	L	R
T	c	c
M	d	d
B	0,0,0	0,0,0
D	e	f

There is a connected component of equilibria with a continuum of outcomes with support $\{T,M,B,D\}\times\{L,R\}\times\{U,D\}$. Hence, in the fully reduced game without X, this game has a continuum of perfect equilibria.

However, in the whole game, in any sequence of ε-perfect equilibria, $U_1(T,\sigma_{-1}^{\varepsilon}) > U_1(B,\sigma_{-1}^{\varepsilon})$ so that there is not a perfect equilibrium with both T and M in the support. There is not a continuum of outcomes anymore in the set of perfect equilibria. Hence, the perfect outcomes of the reduced game are a superset of the set of perfect outcomes of the whole game. Therefore, it is not even the case that $IEWDS$ restricts the set of perfect outcomes.

4 Proper Equilibria

4.1 A non-solvable game

This section presents an example that proves that the proper outcomes of the whole game and of the reduced game differ. This example is a modification of the one provided by Kukushkin, Litan and Marhuenda [7]: more precisely, two strictly dominated strategies (X and Y) have been added. Moreover, the game satisfies TDI and TDI^*. There are four outcomes: a, b, c and d. We let s_i stand for the payoff for player i associated to outcome s.

	L	C	R	S
T	c	a	b	b
M	d	a	a	b
B	c	d	b	c
X	0,0	1,1	1,1	0,0
Y	1,1	0,0	0,0	1,1

Note that X and Y are strictly dominated by T, B and M as long as $a_1, b_1, c_1, d_1 > 1$ (a). We assume that this inequality holds. If we remove this pair of strategies, the reduced game $\Gamma^\infty = \Gamma \setminus \{X, Y\}$ has no dominated strategies. Moreover, there is a connected component $\mathcal{C}$ with a continuum of outcomes as proved by Kukushkin, Litan and Marhuenda [7] provided that

$$d_1, b_1 < a_1, c_1 \text{ and } d_2 < b_2 < a_2, c_2, \ (b)$$

and that

$$b_2(d_1 - c_1) + b_1(c_2 - d_2) + c_1 d_2 - c_2 d_1 \neq 0 \ (c).$$

This component is defined by the following strategies

$$\sigma_1(u_2) = \frac{1}{a_2 - b_2 + c_2 - d_2}(b_2 - d_2, c_2 - b_2, a_2 - d_2),$$

and

$$\sigma_2(u_1;t) = (\frac{a_1-b_1}{a_1-b_1+c_1-d_1} - \frac{(a_1-b_1)t}{a_1-d_1}, \frac{(c_1-b_1)t}{a_1-d_1}$$
$$\frac{c_1-d_1}{a_1-b_1+c_1-d_1} - \frac{(c_1-d_1)t}{a_1-d_1}, t).$$

We assume that (a), (b) and (c) hold so that it is easy to check that the pair (σ_1, σ_2) defines a completely mixed strategy equilibrium in Γ^∞, provided t is positive and small enough.

We now prove that every equilibrium in $\mathcal{C}$ is not a proper equilibrium in Γ, proving that the set of proper equilibria of both games differ. Note that every equilibrium in $\mathcal{C}$ is an equilibrium in Γ and is also perfect as every undominated equilibrium is perfect in bimatrix games.

We consider the sequences $\sigma^\varepsilon = (\sigma_1^\varepsilon, \sigma_2^\varepsilon)$ of ε-proper equilibria converging towards the strategy profiles in $\mathcal{C}$.

By the definition of properness, $U_2(L, \sigma_1^\varepsilon) = U_2(S, \sigma_1^\varepsilon)$ as both are in the support of player 2's strategy. As the utility payoffs of L and S only differ when player 1 plays strategies T and M, it follows that in any ε-proper equilibrium, $\sigma_1^\varepsilon(M) = \frac{c_2-b_2}{b_2-d_2}\sigma_1^\varepsilon(T)$. Moreover, we must have that $U_2(C, \sigma_1^\varepsilon) = U_2(R, \sigma_1^\varepsilon)$ so that $\sigma_1^\varepsilon(B) = \frac{a_2-b_2}{b_2-d_2}\sigma_1^\varepsilon(T)$.

Hence, it follows that $\sigma_1^\varepsilon(B) = \frac{a_2-b_2}{c_2-b_2}\sigma_1^\varepsilon(M)$ (*).

Finally, in any equilibrium with full support for player 2, it must be the case that $U_2(R, \sigma_1^\varepsilon) = U_2(S, \sigma_1^\varepsilon)$. This implies that:

$$a_2\sigma_1^\varepsilon(M) + b_2\sigma_1^\varepsilon(B) + \sigma_1^\varepsilon(X) = b_2\sigma_1^\varepsilon(M) + c_2\sigma_1^\varepsilon(B) + \sigma_1^\varepsilon(Y).$$

Due to (*), one can check that the previous equality implies that $\sigma_1^\varepsilon(X) = \sigma_1^\varepsilon(Y)$. Hence, $U_1(X, \sigma_2^\varepsilon) = U_1(Y, \sigma_2^\varepsilon)$ as otherwise there is a contradiction with the definition of ε-properness. However, this implies that

$$\sigma_2^\varepsilon(C) + \sigma_2^\varepsilon(R) = \sigma_2^\varepsilon(L) + \sigma_2^\varepsilon(S).$$

It is clear that not every equilibrium in $\mathcal{C}$ satisfies this constraint, proving the claim.

4.2 Dominance Solvable Games

Dominance Solvability need not imply Properness

In this example, the unique strategy profile that survives all orders of deletion of $IEWDS$ need not be proper. Note that the game does not satisfy TDI. Furthermore, the outcomes by dominance solvability and properness need not coincide.

We focus on a bimatrix game in which each player has three strategies. Let us remark that L strictly dominates C.

	L	C	R
T	2,3	1,0	0,4
M	2,2	0,0	1,-1
B	2,3	1/2,-1	1/2,4

The set of Nash equilibria equals player 1 randomizing between his three strategies with the probability of M being higher or equal than 1/4 and player 2 playing L. Within this set, the unique pure strategy equilibrium is (M,L). Such an equilibrium is not proper since whenever the probability of player 1 playing M becomes sufficiently close to 1, player 2 strictly prefers to play C than R. Therefore, due to the definition of ε-properness, player 1 strictly prefers to play T than to play M for any $\varepsilon > 0$.

Furthermore, any order of deletion of $IEWDS$ singles out the singleton (M,L). To see this, it suffices to understand that it will first remove C then T and B (simultaneously or sequentially) and finally strategy R.

Hence, the strategy profile (M,L) satisfies three interesting features: (i) it is the unique strategy profile that survives all orders of deletion of $IEWDS$, (ii), it is not a proper equilibrium of the whole game and (iii) it does not lead to the same payoff outcome as any proper equilibrium of the whole game.

This happens because $IEWDS$ and properness choose different profiles in the Nash component. When the Nash component includes a continuum of outcomes as in this example, $IEWDS$ and properness need not induce the same outcome. One may think that this phenomenon is not surprising when we only consider dominance by pure strategies; the $IEWDS$ is a purely ordinal concept whereas the properness depends on the expected payoffs from the deviations, hence on the cardinality of the payoffs. However, the above example does not hinge on the exact cardinality of the payoffs, in the sense that we can find a class of games with the same structure; the fact that the solution (M,L) is not proper depends on the dominance relations between the pure strategies.

Therefore, one way to ensure that both concepts lead to the same prediction is to impose a condition on the payoff structure which provides restrictions on the other players' payoffs given a deviation of a player by a mixed strategy. One such condition is TDI^*, not just TDI, since the dominance by mixed strategy matters, as in the example of Myerson ([11], Table 5.2). The following Theorem shows that when TDI^* is combined with the dominance solvability, the outcome of the Nash component is singled out, and thus the predictions by the $IEWDS$ and by the properness coincide.

A Positive Result

Before stating our main positive result, we list four properties of stable sets (see Mertens [9] for a complete definition.).

1. Stable sets always exist (*Existence*).

2. Stable sets are connected sets of normal-form perfect equilibria (*Connectedness*).

3. Stable sets of a game contain stable sets of any game obtained by deleting a pure strategy which is at its minimum probability in any normal form ε-perfect equilibrium in the neighborhood of the stable set (*Iterated dominance and Forward Induction*).

4. Every stable set contains a proper (hence sequential) equilibrium (*Backwards induction.*).

Let us recall that the set of Nash equilibria consists of finitely many connected components (Kohlberg and Mertens [6]).

Observation 1: Let Γ be a normal-form game that is dominance-solvable while satisfying TDI^*. We let X and Y be two full reductions by weak dominance. X and Y are the same up to the addition or removal of redundant strategies (Marx and Swinkels [8]). Moreover, since Γ is dominance-solvable, there is some order of deletion that isolates some singleton $s = \{s_1, \ldots, s_n\}$. Therefore, any pure strategy profile t in both X and Y satisfies $U_i(t) = U_i(s)$ for any $i \in N$.

Theorem 1. *Let Γ be a dominance-solvable game satisfying TDI^* and let s be a surviving profile. Any equilibrium with s present in its support is payoff-equivalent to s.*

Proof. Let $\sigma = (\sigma_i, \sigma_i)$ be an equilibrium of Γ with $\sigma_i(s_i) > 0$, $\forall i \in N$. We have three cases: (**case 1**) σ is a pure strategy equilibrium, (**case** 2) σ is a mixed strategy equilibrium with exactly one player playing a mixed strategy, or (**case 3**) at least two players play a mixed strategy in σ.

Case 1. If σ is a pure strategy equilibrium, then $\sigma = s$ so that $U(\sigma) = U(s)$ holds by definition.

Case 2. If σ is a mixed strategy equilibrium in which just one player plays a mixed strategy, we let j be such a player and hence let $\#Supp(\sigma_j) \geq 2$. It follows that $\sigma_{-j} = s_{-j}$. Therefore, $U_j(s_j, s_{-j}) = U_j(t_j, s_{-j})$ for any $s_j, t_j \in Supp(\sigma_j)$. Since TDI holds, it follows that $U(s_j, s_{-j}) = U(t_j, s_{-j})$ and hence $U(\sigma) = U(s)$, as wanted.

Case 3. Assume finally that σ is a mixed strategy equilibrium in which at least two players play a mixed strategy ($\#Supp(\sigma_i) \geq 2$ for at least two players in N).

Since the game is dominance-solvable and satisfies TDI^*, we know that every order of deletion o leads to a fully reduced game G_o^∞ in which all pure strategy combinations t satisfy $U(t) = U(s)$ (Observation 1). Since nice weak dominance is equivalent to weak dominance in TDI^* games, without loss of generality we can consider the order of maximal elimination e that removes at each step every nicely weakly dominated strategy. We let D_e^k denote the set of pure nicely weakly dominated strategies after k steps of elimination according to e.

3.a: If there is no nicely weakly dominated strategy in S (which is equivalent to $D_e^0 = \emptyset$), then G is a fully reduced game so that every pure strategy profile t in S satisfies $U(t) = U(s)$. Hence $U(\sigma) = U(s)$, as wanted.

3.b: If, on the contrary, $D_e^0 \neq \emptyset$, then we let m_i in D_e^0. If m_i is in the support of σ, there are two possibilities: either $\sigma_i(m_i) = 1$ or $\sigma_i(m_i) < 1$.

If $\sigma_i(m_i) = 1$, then since $\sigma_i(s_i) > 0$ for all $i \in N$, we must have that $m_i = s_i$. Since now s_i is nicely weakly dominated, there must exist some t_i that nicely weakly dominates it in S. If $U_i(t_i, \sigma_{-i}) > U_i(s_i, \sigma_{-i})$ then s_i is not a best response, proving that σ is not an equilibrium. Hence, it must be the case that $U_i(t_i, \sigma_{-i}) = U_i(s_i, \sigma_{-i})$. However, the definition of nice weak dominance implies that if $U_i(t_i, \sigma_{-i}) = U_i(s_i, \sigma_{-i})$ then $U(t_i, \sigma_{-i}) = U(s_i, \sigma_{-i})$.

If $\sigma_i(m_i) < 1$, the equilibrium conditions imply that for every $i \in N$, $U_i(s_i, \sigma_{-i}) = U_i(m_i, \sigma_{-i})$ for any $m_i \in Supp(\sigma_i)$. However, nice weak dominance implies that if $U_i(s_i, \sigma_{-i}) = U_i(m_i, \sigma_{-i})$ then $U(s_i, \sigma_{-i}) = U(m_i, \sigma_{-i})$.

In both cases, the equilibrium payoff can be reached without the nicely weakly dominated strategy in the support.

3.c: Given that the payoff of σ does not depend on any nicely weakly dominated strategy in D_e^0, it must depend on the strategies in $S \setminus D_e^0$.

We let S^1 be the restriction $S \setminus D_e^0$. Note that the game $G^1 = (S^1, u)$ is the one obtained after one step of removing all nicely weakly dominated strategies.

3.d: If there are no nicely weakly dominated strategies in this restriction (i.e.$D_e^1 = \emptyset$), then the game is fully reduced so that every pure strategy combination t satisfies $U(t) = U(s)$ and hence $U(\sigma) = U(s)$, as wanted.

3.e: If, on the contrary, $D_e^1 \neq \emptyset$, the equilibrium payoff can be attained without the nicely weakly dominated strategies. Since the game is dominance-solvable, iterating this procedure until no nicely weakly dominated strategy is left leads to a game in which any pure strategy combination has the same payoff as s proving that $U(\sigma) = U(s)$, as wanted.

□

Theorem 2. *Let Γ be a dominance-solvable game that satisfies TDI*. Then:*

(i) *Under any order of iterative elimination of weakly dominated strategies, the outcome is the unique stable one.*

(ii) *For any equilibrium of the fully reduced game, there is a proper equilibrium of Γ which induces the same outcome.*

Proof. Since the game satisfies TDI^*, all fully reduced games lead to the payoff associated with s, the surviving singleton. Moreover the inclusion property of Mertens sets ensures that s is stable. Hence, there must exist some proper equilibrium in G with payoff identical to s (Backwards Induction property of stable sets). Moreover, since some order isolates s, then there is at most one stable set. Finally, all equilibria in the component of s lead to the same payoff (Theorem 1). Hence, the outcome of s is the unique stable one. □

Why do we need TDI^* rather than TDI ?

The main logic behind Theorem 1 is that all orders of deletion are equivalent under TDI^*. More specifically, nice weak dominance and weak dominance coincide whenever the game satisfies TDI^*. The proof of the theorem relies on the fact that (iteratively) applying nice weak dominance does not enlarge the set of Nash payoffs. Does the same result hold if we only apply TDI?

Suppose that a singleton is selected by some order of $IEWDS$. The outcome of this singleton must coincide with that of a proper equilibrium of the whole game if this precise order satisfies nice weak dominance (i.e. all removed strategies are nicely weakly dominated). Yet, the set of proper outcomes might be enlarged by applying $IEWDS$ in a game satisfying TDI but not TDI^*, as shown by the next example, related to the one provided by Marx and Swinkels [8] (p.233).

	L	C	R
T	2,1	4,3	0,2
M	0,3	3,1	4,2
B	1,4	1,4	1,4
D	1,4	0,3	0,2

This game satisfies TDI but not TDI^*. Indeed, the strategy R is very weakly dominated by $1/2L+1/2C$ in $S \setminus \{D,B\}$ but is neither weakly dominated nor redundant on $S \setminus \{D,B\}$. Moreover, this game is dominance solvable. After eliminating R, the strategies M,B,D are strictly dominated by T. Then eliminating L leads to (T,C) as the surviving profile.

On the contrary, if we only eliminate D and B from S, then we are left with the fully reduced game $\{T,M\} \times \{L,C,R\}$. In this game, there is a set of completely mixed strategy equilibria (hence proper) of the following type:

$$(1/2T + 1/2M, (pL + qC + (1-p-q)R)) \text{ as long as } 6p + 5q = 4.$$

However, some equilibria of this set are not proper equilibria of the whole game. Indeed, note that as far as R is in the support of an equilibrium (take for instance the equilibrium in which $p = \varepsilon$ and $q = (4-6\varepsilon)/5$), this equilibrium cannot be proper in the whole game since R is weakly dominated by $1/2L + 1/2R$ in S. Therefore, the set of proper equilibria might be enlarged by $IEWDS$ in a dominance solvable game that satisfies TDI but fails to satisfy TDI^*.

5 Conclusion

In this paper we explore the conditions under which simplification of the game by $IEWDS$ can be applied to analyze strategic stability of the equilibria.

We show that neither the TDI^* condition of Marx and Swinkels [8] nor the dominance solvability alone is sufficient to guarantee that the set of proper outcomes of the reduced game is included in the set of proper outcomes of the whole game (proper inclusion). We show by example that the TDI^* condition alone is not sufficient; indeed $IEWDS$ may *enlarge* the set of proper outcomes. Dominance solvability alone is not sufficient either: we give an example in which the outcome singled out by the dominance solvability does not coincide with any proper outcome of the whole game.

If the game satisfies both TDI^* and dominance solvability, we show that proper inclusion holds. Moreover, the uniqueness of the stable outcome is guaranteed.

There is a large class of games for which our sufficient conditions are satisfied. For example, in many strategic interactions in political competition, such as voting, players' payoff depends solely on the outcome, which is determined by the social choice, such as the winner of the election. TDI^* condition is relevant in many situations (see Marx and Swinkels [8]). Even in the games in which the DS condition is not satisfied, if the outcome is isolated, the proper inclusion is guaranteed. We can safely apply $IEWDS$ to simplify the game and analyze the strategic stability of the whole game by focusing on the reduced game.

This paper provides a set of sufficient conditions under which we can take advantage of both the simplicity of $IEWDS$ and the robustness of strategic stability. This is what we call hybrid procedures.

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