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A Map of Approval Voting Equilibria Outcomes*

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Abstract

It is commonly accepted that the multiplicity of equilibria is ubiquitous in preference aggregation games with any voting method. We prove that this multiplicity is greatly reduced under some mild restrictions over social preferences when each voter can vote for as many candidates as she wishes (the Approval voting method). For scenarios with three candidates, we can hence build a map that associates any preference profile to its set of equilibria outcomes; this map is very close to the most well-known Tournament solutions.

Keywords Approval voting; Condorcet winner; Voting equilibria; Asymmetric Societies

JEL Classification D70, D72

1 Introduction

Approval Voting (AV) is the method of election according to which a voter can vote for as many candidates as she wishes, the elected candidate(s) being the one(s) who

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receives the most votes. This simple voting rule has attracted interest from scholars in political science and economics[1] due to its flexibility: voters approve of each candidate independently of the rest of the candidates. This rule plays a distinct role in information aggregation settings (see [1], [2], [9] among others). As far as preference aggregation is concerned (in which we focus), one main result emerges: this rule (as many others) tends to generate a multiplicity of outcomes, independently of whether one assumes sincere[2] or strategic voting. Concerning strategic voting, theorists unambiguously consider that the multiplicity of voting outcomes is an unattractive feature as argued by Myerson and Weber [16]. Nonetheless, there are two important aspects of the multiplicity of equilibria under AV that need to be highlighted. First, the literature suggests that the multiplicity of equilibria is less severe under AV than under other voting rules such as Plurality. Indeed, under Plurality, the set of possible winners in any electoral situation, includes any candidate who is not a Condorcet loser and may also include the Condorcet Loser in some situations. The second feature of AV equilibria is the one presented in [12] which provides in a mass elections model[3] a strong argument for the use of AV: in the absence of a tie in the expected scores of the candidates, AV uniquely selects the Condorcet Winner (CW), the candidate that beats every other candidate in pairwise contests [4].

However, the previous papers do not provide a full description of AV equilibria since the multiplicity of equilibria seems unavoidable. This paper proves that for a wide family of preference distributions, *the asymmetric societies*, this multiplicity of equilibria vanishes. This property of asymmetric societies allows us to draw a map of approval voting equilibria. A preference distribution is asymmetric if two conditions hold: the Simple Asymmetry (*SA*) and the Inverse Asymmetry (*IA*). According to *SA*, for any pair of candidates x, y the number of voters who prefer x to y must be different from the number of voters who prefer y to x. *IA* states that for any triple of

[1] The reader can refer to [3], [19] and the recent handbook of approval voting ([13]).

[2] The notion of sincerity under AV is not completely obvious since there is not a one-to-one correspondence between the set of ballots and the set of preference profiles. According to the main used one, an AV ballot is sincere if whenever a voter approves of a given candidate c, she also approves of any candidate preferred to c.

[3] We do not attempt here to give a full review of the literature. There is also a literature in small elections which reaches the same conclusion concerning the multiplicity of equilibria under AV, while using very different methods. For instance, see [6] and [7] and [8].

[4] Note that these results hinge on the ordering condition. Indeed, [14, 15] develop the Poisson voting games, a formal game-theory model to analyze large elections. However, [18] proves that in these games the ordering condition is not satisfied which leads to the victory of a candidate who is not the Condorcet Winner under AV.

candidates x, y, z, the number of voters who prefer x to y and y to z must be different from the number of voters who prefer z to y and y to x. Note that both conditions are very mild.

One main result emerges. On asymmetric large elections, AV produces two type of equilibria. In the former one, the Condorcet winner is the unique front-runner (*i.e.* he is the candidate with the highest expected score), whereas in the latter one, there are at least three candidates tied for victory. In other words, if there exists an equilibrium with a unique front-runner, then the front-runner must be the CW. If the preference profile does not admit a Condorcet Winner, then there must be at least three candidates tied for victory. Moreover, we prove that if there exists a Condorcet winner then the game has an equilibrium in which he is elected.

Together, these results allows us to build a map (see Figure 1) that associates any preference profile to its set of voting equilibria outcomes in three candidate elections. This map is deeply related to the Tournament majority solutions such as the Top-Cycle set or the Bipartisan set. To build it, we combine equilibrium behavior and the approval relation (A) introduced by [5]. We say that a candidate x is $A-$preferred to a candidate y if the number of voters who rank x first and y last in their individual preference is higher than the number of voters who strictly prefer y to x minus the number of voters who rank y above x, none of them being either first or last. Note that with just three candidates, the definition of the relation is simpler since the number of voters who rank y above x, none of them being either first or last equals zero. A candidate y belongs to the *Approval domain* if there exists a candidate who is A-preferred to him.

Therefore there are three regions in this map. In region (i), the unique equilibrium of the game selects the CW as the front-runner of the election. In this region, the *Approval domain* is non-empty and hence a CW exists. In region (ii), while there exists a CW, the *Approval domain* is empty. In this region, there is always an equilibrium in which the CW wins but this equilibrium need not be unique (see Example 1). Finally, region (iii) simply stands for the preference distributions in which there is no CW. In this region, the unique equilibrium outcome distribution is a tie among all the candidates.

In other words, in three candidates settings, the set of approval voting equilibria is close, even though, not equivalent to the Top-Cycle set (and hence to any other major tournament solution since they are equivalent with just three candidates). Indeed, the Top-Cycle set stands for the set of candidates that are preferred to any

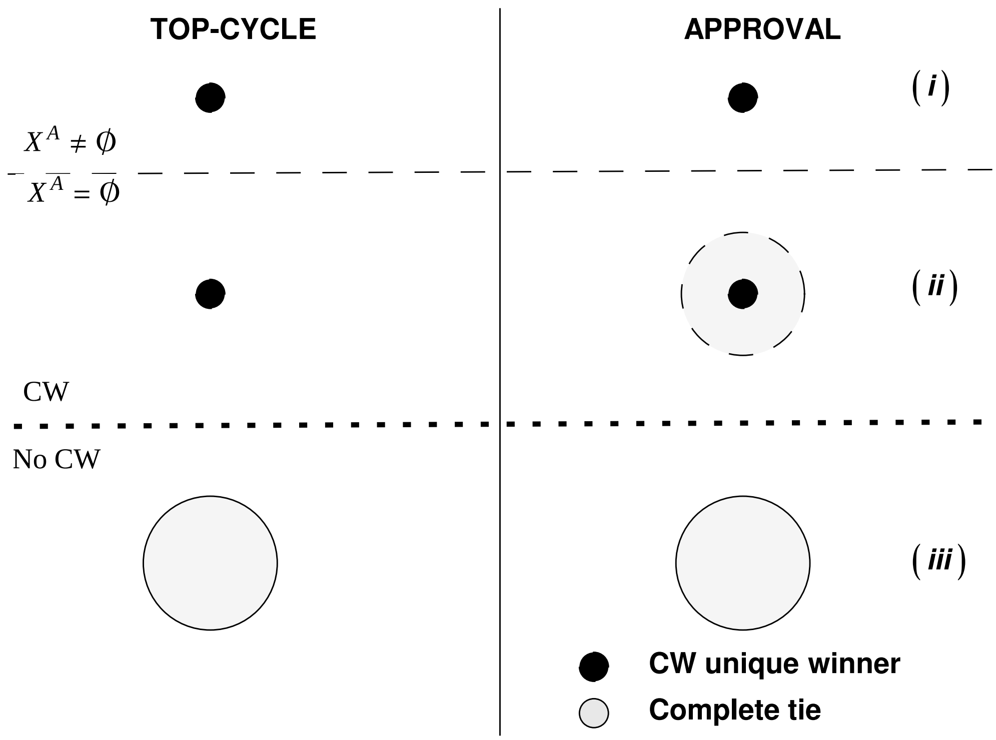

Figure 1: Top-cycle and the AV Equilibria Outcomes.

other candidate via a chain of majority relations. In our setting, the Top-Cycle is equal either to the CW or the whole set of candidates in the absence of a CW.

This work is structured as follows. Section 2 introduces the general framework and Section 3 analyze AV outcomes on asymmetric societies. As far as the scenario with three candidates is concerned, the full description of AV equilibrium outcomes is included in Section 4 and Section 5 concludes the paper.

2 The electoral setting

The finite set of voters and candidates are respectively denoted by $\mathcal{N} = \{1, \ldots, n\}$ and $\mathcal{X} = \{a, b, \ldots, k\}$. Note that n is supposed to be large. The strict preferences of a voter are defined by a von Neumann–Morgenstern (*vN-M*) utility function $u : \mathcal{X} \to \mathbb{R}$, in which $u(x)$ denotes the utility a voter gets if candidate x wins the election. In other words, for each $i \in \mathcal{N}$ and for any pair of candidates $x, y \in \mathcal{X}$, $x \succ_i y \Longleftrightarrow u_i(x) > u_i(y)$.

Voting

All voters vote simultaneously. Each voter can approve as many candidates as she wishes by choosing a ballot $v = (v_a, \dots, v_k)$ where $v_x \in \{0,1\}$ denotes the number of points given to candidate x. With three candidates, the set of all possible ballots that a voter could submit under AV is $V = \Big\{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),$ $(0,0,0),(1,1,1)\Big\}$.

An AV strategy is *undominated* if the voter approves her most preferred candidate and never approves of her worst preferred one. An AV strategy is *sincere* if, given the lowest-ranked candidate that a voter approves of, she also approves of all candidates ranked higher (see [4] and [13]).

Society

Given the individual preferences, one can derive social preferences over pairs of candidates. For any pair of candidates x, $y \in \mathcal{X}$, the majority relation M is defined as follows. We say that x is M-preferred to y, denoted xMy, if and only if $N(x,y) > N(y,x)$, with $N(x,y) = \#\{i \in N \mid x \succ_i y\}$.

Throughout the work, we make two slight assumptions that ensure that social preferences are asymmetric. Note that both conditions are mild since they will generically hold if one considers for instance an odd number of voters.

SA: Simple Asymmetry. For any $x, y \in \mathcal{X}$, $N(x,y) \neq N(y,x)$.

For any triple of candidates x, $y, z \in \mathcal{X}$, we let $N(x,y,z)$ denote the number of voters who prefer x to y and y to z; formally, $N(x,y,z) = \#\{i \in N \mid x \succ_i y \succ_i z\}$.

IA: Inverse Asymmetry. For any triple of candidates $x, y, z \in \mathcal{X}$, $N(x,y,z) \neq N(z,y,x)$.

The Condorcet Winner (CW) is the candidate who is M-preferred to any other candidate in the election: x is the Condorcet Winner if and only if xMy for any $y \in \mathcal{X} \setminus \{x\}$.

Consider a chain between candidates x and y which is a sequence of candidates $d_1, \dots, d_m$ such that $d_1 = x$, $d_m = y$, and not $d_{l+1} M d_l$ for each $l = 1, \dots, m-1$. Then the

Top-Cycle set, denoted $\mathcal{X}^{TC}$ is the set

$$\mathcal{X}^{TC} = \{x \in \mathcal{X} \mid \forall y \in \mathcal{X}, \exists \text{ a chain between } x \text{ and } y\}.$$

Note that with three candidates, if both SA and IA hold, we have that either $\mathcal{X}^{TC} = \mathcal{X}$ if there is no Condorcet Winner or $\mathcal{X}^{TC} = CW$. With three candidates, the Top-Cycle set coincides with several tournament solution concepts such as the Uncovered set or the Bipartisan set (see [11]). We just define the Top-Cycle as it is the simplest one of these concepts.

Approval relation

We now define the approval relation A. We say that candidate x is A-preferred to y if the number of voters who rank x first and y last (denoted $N(x,\ldots,y)$) is higher than the number of voters who strictly prefer y to x plus the number of voters who prefer x to y but do not rank x first or y last (denoted $N(\cdot,x,y,\cdot)$). Formally, we have

$$xAy \text{ if and only if } N(x,\ldots,y) > N(y,x) + N(\cdot,x,y,\cdot).$$

Since with three candidates $N(\cdot,x,y,\cdot) = 0$ given that at least one of the two candidates is ranked first or last in a voter's preference ordering, it follows that

$$xAy \text{ if and only if } N(x,\ldots,y) > N(y,x) \text{ when } k = 3.$$

Note that the A relation is asymmetric and that A need not be complete.

The set $\mathcal{X}^A$ (the Approval domain) stands for the candidates for which there exists a candidate who is A-preferred to them so that,

$$\mathcal{X}^A = \{x \in \mathcal{X} \mid \exists y \text{ s.t. } yAx\}.$$

[5] state among other things the following properties:

1. $\mathcal{X}^A$ might be empty with $\mathcal{X}^A \neq \mathcal{X}$ (α);

2. a candidate cannot win in any undominated strategy combination under AV if and only if it belongs to $\mathcal{X}^A$ (β);

3. if $\#\mathcal{X}^A \geq k-2$, then the game has a CW (γ).

We now introduce a property of the A-relation that proves the existence of a link between Approval voting and the Tournament solutions. Indeed, any A-dominated candidate does not belong to the Top-cycle and to neither of its refinements.

Proposition 1. *If $k \in \mathcal{X}^A$ then $k \notin \mathcal{X}^{TC}$.*

Proof. Let $\mathcal{X} = \{a, b, c, \ldots, k\}$ and $k \in \mathcal{X}^A$. We assume that aAk and that $k \in \mathcal{X}^{TC}$.
If aMx $\forall x \neq a$, then a is the CW so that $k \notin \mathcal{X}^{TC}$, a contradiction.
Similarly, if yMk $\forall y \neq k$, then $k \notin \mathcal{X}^{TC}$, a contradiction.
We hence assume that there exists some pair of candidates x and y with xMa and kMy.

Assume first that $x = y$ which implies that xMa and kMx. However, Lemma 9 in [5] proves that for any triple a, b, c if aMb and bAc then aMc. In our case, we have xMa and aAk so that xMk, a contradiction with kMx.

Assume now that $x \neq y$. W.l.o.g. we let $x = b$ and $y = c$ so that bMa and kMc.

1) bMa implies that $N(a, b) < \frac{1}{2}$. The previous inequality can be rewritten as :

$$N(a, ..., k) + N(a, ., k, .) \leq N(a, b) < \frac{1}{2}.$$

with $N(a, ., k, .)$ the number of voters who rank a first and k not last.

2) aAk is equivalent to $N(k, a) + N(., a, k, .) < N(a, ..., k)$.

3) If we combine (1) and (2), then

$$N(k, a) + N(., a, k, .) + N(a, ., k, .) < \frac{1}{2}.$$

4) kMc implies $N(\ldots, k) < \frac{1}{2}$ with $N(\ldots, k)$ the number of voters who rank k last.

5) Then if we add (3) and (4) we have

$$N(k, a) + N(., a, k, .) + N(a, ., k, .) + N(\ldots, k) < 1.$$

6) Finally note that $N(a, k) = N(., a, k, .) + N(a, ., k, .) + N(\ldots, k)$. But, by definition, $N(k, a) + N(a, k) = 1$ which is a contradiction with (5). □

Large Elections

Let H be the set of all unordered pairs of candidates; a pair $\{x, y\}$ in H is denoted by xy with $xy = yx$. The xy-pivot probability p_{xy} is the probability (perceived by the voters) that candidates x and y are tied for first place in the election. A vector listing the pivot probabilities for all pairs of candidates is denoted by $p = (p_{xy})_{xy \in H}$. This vector p is assumed to be identical and common knowledge for all voters in the election. W.l.o.g. we let p represent a probability distribution so that $\sum_{xy \in H} p_{xy} = 1$ with $p_{xy} \geq 0$.

A strategy is a probability distribution σ over the set V that describes the voting behavior of the voters. For any ballot v, $\sigma_i(v)$ equals the probability that voter i casts ballot v. The expected utility gain of a voter when she plays the strategy $\sigma_i(\cdot)$ equals $U_i(\sigma_i(\cdot); p)$. Slightly abusing notation, we let $U_i(v; p)$ denote the expected utility gain of voter i from casting ballot v with:

$$U_i(v; p) = \sum_{xy \in H} (v_x - v_y) \cdot p_{xy} \cdot [u_i(x) - u_i(y)]. \qquad (U)$$

Given the strategy combination σ, the share of the electorate who casts ballot v is denoted by $\tau(v) = \sum_{i \in \mathcal{N}} \sigma_i(v)$. Hence, the expected score of candidate x is $S(x) = \sum_{v \in V} v_x \tau(v)$.

The set of *front runners* of the election contains the candidates whose expected score $S(x)$ is maximal given the strategy σ. *MW* impose the following consistency requirement in equilibrium: $S(x) > S(y) \implies \varepsilon p_{xz} \geq p_{yz}$, $\forall \varepsilon \in (0,1)$, $\forall x, y, z$. This implies that pivot probabilities involving candidates with low vote shares are zero in a similar fashion to the definition of proper equilibrium.

We impose on beliefs a weaker version of *MW*'s ordering condition. For any candidate x, let l be the unique leading candidate without x (if it exists): i.e. $l = \arg\max_{y \in \mathcal{X} \setminus \{x\}} S(y)$. We simply assume that

$$\varepsilon p_{xl} \geq p_{xz}, \forall \varepsilon \in (0,1), \forall x, z.$$

Our condition simply requires that the voters anticipate that the most likely pivot in which a candidate is involved is almost surely against the leading candidate. With three candidates, both conditions are quite close even though ours is substantially weaker with more candidates[5]. In order to see the main difference, take four candi-

[5]The ordering condition plays a central role in these mass election models (see [17] for a review of this condition). Indeed, [12] proves that when this condition holds (for a large number of voters), there exists an equilibrium under AV that leads to the election of the CW. In Poisson games, this condition need not be verified ([18]) which leads to bad preference aggregation; so is the case if one focuses on classic equilibrium refinements such as perfection or Mertens' stability as proved by [7, 8].

dates with $S(x) > S(y) > S(z) > S(w)$ and consider the pivot probabilities in which x is involved. The *MW* condition implies that $\varepsilon p_{xy} \geq p_{xz}$ and $\varepsilon p_{xz} \geq p_{xw}$. Ours simply implies that $\varepsilon p_{xy} \geq p_{xz}, p_{xw}$ without any implication over the ratio between p_{xz} and p_{xw}.

The probability of three (or more) candidates being tied for first place is infinitesimal in comparison to the probability of a two-candidate tie.

Given a pivot probability vector p, the set of pure best replies of a voter equals $\mathrm{BR}_i(p) = \{v \in V \mid \ v \in \arg\max_{v' \in V} U_i(v'; p)\}$. Given the strategy σ_i of a voter i, its support denotes the set of pure strategies played with positive probability according to σ_i: $\mathrm{Supp}(\sigma_i) = \{v \in V \mid \sigma_i(v) > 0\}$.

The strategy σ is a voting *equilibrium* of the game if and only if, for every positive number ε, there exists a vector p^ε of positive pivot probabilities that satisfies the ordering condition for ε given σ, and such that, for each ballot v and for each voter $i \in \mathcal{N}$, if $v \in \mathrm{Supp}(\sigma_i)$, then $v \in \mathrm{BR}_i(p^\varepsilon)$. The set of equilibria is non-empty since our assumptions are weaker than those in *MW* which prove the existence of equilibrium.

3 Equilibria Outcomes in Asymmetric Societies

We now discuss the main implications of our work in scenarios with any number of candidates. The main characteristic of our results is that they do not depend *explicitly* on the voters' best responses. In other words, we do not need to completely define how the voters vote in order to predict how the equilibrium outcomes are. The main logic is driven by the voters' anticipations to the possible scores of the candidates, greatly simplifying the task at hand.

Proposition 2. *If SA holds, then there is no AV equilibrium with two front-runners.*

Proof. Assume, by contradiction, that there is an equilibrium with two front-runners. W.l.o.g. we let x and y be this pair of candidates. Due to the ordering condition, the most probable pivot outcome in which x (resp. y) is involved is against y (resp. x). Therefore, the voters who strictly prefer x over y vote for x and the ones who strictly prefer y over x vote for y. Hence, the score of x equals $N(x,y)$ whereas the one of y equals $N(y,x)$. However, since *SA* holds, the scores of such candidates must be different, contradicting the assumption that both x and y are tied. □

See the recent contribution by [10] for a different weakening of the ordering condition in a related framework.

Proposition 3. *If IA holds and there is an AV equilibrium with a unique front-runner, then this candidate must be the Condorcet Winner.*

Proof. Assume that there is a unique front-runner in equilibrium, denoted a. Due to the ordering condition, every voter knows that, when $\varepsilon \to 0$, the pivot outcome in which any candidate $x \neq a$ is involved against a becomes infinitely more likely than the rest of pivot events.

We have two cases: either there is a tie in the scores of two candidates (who are not the front-runners) or there is no tie.

Case 1: Assume first that, given σ, there is a tie in the expected score of two candidates who are not the front-runners. We denote them b and c w.l.o.g. As the most likely pivot outcome in which both are involved is against a, we know that the unique voters who vote for b (resp. c) are the ones who prefer b (resp. c) to a.

Therefore, the scores of both candidates are the following ones:

$$S(b) = N(b,a,c) + N(b,c,a) + N(c,b,a),$$

and

$$S(c) = N(c,a,b) + N(c,b,a) + N(b,c,a).$$

Since the condition IA holds, it follows that the scores of b and c cannot be equal, a contradiction.

In other words, when IA holds, there is not an equilibrium with a unique front-runner in which two candidates are tied (in expected scores). So that, if there is a unique front-runner in equilibrium, the only possible case is that there is no tie in the expected score, to be analyzed in the *Case* 2.

Case 2: Assume now that there are no ties in the scores. Note first that $N(x,a) \neq N(y,a)$ for any pair $x,y \in \mathcal{X}$. To prove this, it suffices to see that $N(x,a) = N(x,a,y) + N(x,y,a) + N(y,x,a)$ and $N(y,a) = N(y,a,x) + N(y,x,a) + N(x,y,a)$. The condition IA implies that $N(x,a,y) \neq N(y,a,x)$. Therefore, it must be the case that $N(x,a) \neq N(y,a)$ for any pair $x,y \in \mathcal{X}$.

W.l.o.g. we assume that $N(b,a) > N(c,a) > \ldots > N(k,a) \;\forall\; b,c,\ldots,k \in \mathcal{X}$.

Since every voter anticipates that the most likely pivot outcome involving any candidate $x \neq a$ is against a, it follows that the score of each candidate $x \neq a$ equals $N(x,a)$ the share of voters who strictly prefer x to a whereas the one of a equals $N(a,b)$. Hence, the scores of the candidates satisfy $S(a) > S(b) > \ldots > S(k)$.

Assume that a is not the CW so that there is some candidate y with yMa. If $y = b$, then $N(b,a) > N(a,b)$ so that the score of b is higher than the score of a, a contradiction with a being the front-runner. If $y \neq b$, then $N(y,a) > 1/2$ so that $S(y) = N(y,a) > 1/2 > N(b,a) = S(b)$. Therefore, y is ranked second. In this case, the score of a equals $N(a,y) < 1/2$, a contradiction with a being the front-runner. Hence, it can only be the case that a is M-preferred to the rest of the candidates: for any $x \in \mathcal{X} \setminus \{a\}$, aMx. In other words, it must be the case that a is the Condorcet winner. □

Therefore, we can establish without proof the following corollary.

Corollary 1. *If IA holds and there is no Condorcet winner, the set of front-runners contains at least three candidates in equilibrium.*

Proposition 4. *If IA holds and there is a Condorcet winner, then there must exist an AV equilibrium that uniquely selects this candidate.*

Proof. Take a society in which there is a CW (denoted a) and in which IA holds. Since IA holds, we can assume w.l.o.g. that $N(b,a) > N(c,a) > \ldots > N(k,a)$. Indeed, as shown in the proof of Proposition 3 (case 2), if IA holds, then $N(x,a) \neq N(y,a)$ $\forall x, y \in \mathcal{X}$.

Assume that the scores satisfy $S(a) > S(b) > \ldots > S(k)$. Due to the ordering condition, it follows that the most likely pivot in which a is involved is against b whereas the most likely pivot outcome in which any other candidate x is against a. Thus, the score of a equals $N(a,b)$ whereas the score of x ($x \neq a$) equals $N(x,a)$. As a is the CW, it follows that $N(a,b) > 1/2$ and that $N(x,a) < 1/2$ for any $x \neq a$. Finally, since $N(b,a) > N(c,a) > \ldots > N(k,a)$, the scores satisfy $S(a) > S(b) > \ldots > S(k)$ as wanted. Thus we have proved that there exists an equilibrium in which the CW is the unique front-runner, concluding the proof. □

4 The Map

We now describe the main implications of the conditions of asymmetry on the shape of voting equilibria by constructing a map that associates any society to its set of front-runners in equilibrium in electoral situations with three candidates. The following example proves that some equilibria might exist only for certain utility levels of the voters, making more difficult the construction of the map.

Example 1: Let $\mathcal{X} = \{a, b, c\}$ and consider a society with the following proportions : $\frac{1}{9}$ of the voters with $u_A = (10, \mu, 0)$; $\frac{2}{9}$ of the voters with $u_B = (10, 0, \mu)$; $\frac{4}{9}$ of the voters with $u_C = (10 - \mu, 10, 0)$ and $\frac{2}{9}$ of the voters with $u_D = (10 - \mu, 0, 10)$.

The candidate a is the CW and $\mathcal{X}^A = \emptyset$. Note that both SA and IA hold.

Since $\mathcal{X}^A = \emptyset$, property β implies that, for each candidate, there is a pure strategy combination under which the candidate wins with positive probability. However, due to SA and IA, some equilibria are removed.

Indeed, Proposition 2 implies that there is no equilibrium with two front-runners. Moreover, since there is a CW, Proposition 4 ensures that there exists an equilibrium in which a is the unique front-runner. Finally, there is no other equilibrium with a unique front-runner as ensured by Proposition 3. In other words, neither b or c can win alone.

One question remains to be answered: is there an equilibrium with the three candidates tied for victory? These equilibria might or not exist as a function of the voters' intensities of preferences. More formally, when $0 < \mu < 5$, there is no equilibrium with three front-runners (see Appendix A). Hence, in any equilibrium outcome, the CW is selected.

This example illustrates the fact that for a open set of utilities, when there is a CW and $\mathcal{X}^A = \emptyset$, the unique outcome AV might uniquely select the CW. However, for a different utility representation, we can find an equilibrium in which the three candidates get the same score. For example, if we set $\mu = 6$, there is an equilibrium in which the three candidates are tied for victory as long as $p^\varepsilon = (5/8\varepsilon, 1/8\varepsilon, 2/8\varepsilon)$.

Building on Example 1, the following result describes the equilibria outcomes with three candidates and allows us to build the map in Figure 1.

Theorem 1. *Let $k = 3$. If both SA and IA hold, the voting equilibria are as follows:*

i) If there is a Condorcet Winner with $\mathcal{X}^A$ non-empty, then the Condorcet winner is the unique winner.

ii)If there is a Condorcet Winner with $\mathcal{X}^A$ empty, there must exist an equilibrium in which he is the unique winner. The three candidates might be tied for victory.

iii) If there is no Condorcet Winner, the three candidates must be tied for victory.

Proof. i) Since $\mathcal{X}^A \neq \emptyset$ and $k = 3$, the property γ implies that there is a CW. Proposition 4 entails that if there is a CW in the profile, there must exist an equilibrium in which this candidate is the unique front-runner. As ensured by Proposition 2, there is no equilibrium with two front-runners. Moreover, there is no other equilibrium with a unique front-runner as ensured by Proposition 3. Therefore, the only type of equilibrium that might exist is the one in which the three candidates are tied. However, since $\mathcal{X}^A \neq \emptyset$, property β ensures that some candidate is not in the set of front-runners for any undominated strategy combination. Hence, there is no equilibrium in which the three candidates are tied, proving the claim.

ii) Proposition 4 ensures that there exists an equilibrium in which the CW is the unique front-runner. Example 1 proves that there exists elections in which both situations might arise: either the CW is the unique candidate in the set of front-runners or both equilibria are possible.

iii) This final point is a direct implication of Corollary 1. □

Theorem 1 proves the existence of a deep link between the equilibria outcomes under AV and the theory of Tournament solutions. In a sense, AV almost implements the different solutions with just three candidates.

5 Concluding Comments

One of the main findings of this work is that in asymmetric societies, the Approval voting method leads to outcomes very close to the Top-Cycle set with three candidates. In other words, when voters are strategic, this rule gives incentives to voters to coordinate in a particular way: the equilibria outcomes (almost) coincide with the recommendations of the literature in Tournament solutions.

How does the map extend to more candidates? While at first one might think it is not the case, the answer is not completely straightforward. As shown by Proposition 1, the Approval Voting rule has a remarkable property: any A-dominated candidate does not belong to the Top-Cycle. Since the rest of Tournament solutions are refinements of the Top-Cycle, it follows that any A-dominated candidate cannot belong to any of them. A reasonable conjecture seems to be that for any preference profile, there must exist an equilibrium in which all the elements in the Top-Cycle or the Bipartisan set are tied for victory[6]. This equilibrium should exist independently of the utility levels of the voters. If either $\#\mathcal{X}^A \geq k-3$ or there is a CW, the conjecture holds whereas it is not so clear when the previous restrictions are not met.

[6]We would like to thank Jean-François Laslier for the idea of the Bipartisan Set.

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A Appendix: Example 1

There is no equilibrium in this election in which the three candidates have the same expected score with no voter being indifferent between single and double voting. Indeed, when no voter is indifferent between single and double voting, it follows that all the voters with the same utility vector vote in the same way. One can check that in any possible combination of pure undominated strategies (each voter voting for her top candidate or for her two top candidates), there is no equality between the scores of the candidates. Hence, there is no such equilibrium with a three-way tie.

Thus, in order to have such an outcome, some type of voters must be indifferent between single and double voting. In equilibrium, voters always approve of their most preferred candidate and never approve of their worst preferred one.

If just one type of voters play a mixed strategy, then it is not possible to obtain a three-way tie. If at least two types play in mixed strategies, then either C or D voters vote also for their middle ranked candidate so that a has the highest expected score.

Indeed, assume first that a C voter plays a mixed strategy over her two undominated strategies so that $U_C(0,1,0) = U_C(1,1,0)$. Due to (U), the previous equality is equivalent to $U_C(1,0,0) = 0$ so that

$$p^{\varepsilon}_{13}(10-\mu) - p^{\varepsilon}_{12}\mu = 0. \qquad (*)$$

However, when $(*)$ holds, we have that $U_D(1,0,1) > U_D(0,0,1)$. To see why, note first that $U_D(1,0,1) > U_D(0,0,1) \iff U_D(1,0,0) > 0$. Moreover, remark that $U_D(1,0,0) = (10-\mu)p^{\varepsilon}_{12} - \mu p^{\varepsilon}_{13}$ so that, when $(*)$ holds,

$$U_D(1,0,0) = \frac{10(10-2\mu)}{\mu} p^{\varepsilon}_{13} > 0.$$

which holds since $\mu < 5$.

Therefore, if a C voter plays a mixed strategy, D voters must vote for their second ranked candidate a, leading to its victory. A symmetric argument applies when a D voter plays a mixed strategy. Therefore, in any mixed strategy profile in which either C or D voters play a mixed strategy between their two undominated ballots, a is the sole winner of the election.

Hence, the only possibility for the existence of an equilibrium in which the three candidates get the same outcome is to assume that A and B voters both play a mixed strategy. However, this implies that

$$U_A(0,1,0) = 0 \iff -p^{\varepsilon}_{12}(10-\mu) + p^{\varepsilon}_{23}\mu = 0,$$

and

$$U_B(0,0,1) = 0 \iff -p^{\varepsilon}_{13}(10-\mu) + p^{\varepsilon}_{23}\mu = 0.$$

The previous two equalities imply that the unique pivot probability vector justifying such best responses equals $p^{\varepsilon} = (\frac{\mu}{10+\mu}\varepsilon, \frac{\mu}{10+\mu}\varepsilon, \frac{10-\mu}{10+\mu}\varepsilon)$. However, as previously noted, $U_C(1,0,0) = p^{\varepsilon}_{13}(10-\mu) - p^{\varepsilon}_{12}\mu$ which is strictly positive given p^{ε} since $\mu < 5$. Hence, as in the previous case, if both A and B voters play a mixed strategy, C voters give one point to a, leading to its victory. Therefore, there is no equilibrium with three front-runners. Moreover, by Proposition 4, we know that there must exist an equilibrium in which a is the unique front-runner. Furthermore, Proposition 2 implies that there is no equilibrium with two front-runners. Hence, in any equilibrium outcome, a is the unique front-runner as long as $\mu < 5$.