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The q-Condorcet efficiency of positional rules

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Abstract

According to a given quota q, a candidate a is beaten by another candidate b if at least a proportion of q individuals prefer b to a. The q-Condorcet efficiency of a voting rule is the probability that the rule selects a q-Condorcet winner (q-CW), that is any candidate who is never beaten under the q-majority. Closed form representations are obtained for the q-Condorcet efficiency of positional rules (simple and sequential) in three-candidate elections. This efficiency is significantly greater for sequential rules than for simple positional rules.

Keywords Positional rules (Simple and Sequential) • Condorcet efficiency • q-majority
JEL Classification D71,D72

1 Introduction

In voting theory, the Condorcet winner is any candidate that would be able to defeat each of the other candidates in a series of pairwise majority comparisons. A Condorcet voting rule selects a Condorcet winner (CW), whenever one exists. While it is very appealing for a voting rule to select a CW when one exists, another famous class of voting rules, the positional rules, lacks to satisfy this requirement (Condorcet, 1785). This is certaintly a failure of the positional approach.

However, this negative point of view can be contrasted. By computing the probability that a voting rule selects the Condorcet winner, one can check whether such a phenomenon is sufficiently rare to be ignored or not. In the context of the majority rule, this approach has been particularly studied by Gehrlein and Lepelley (2006)[1]. Under the majority rule a candidate a is preferred to a candidate b if the total number of voters who prefer a to b is greater than the total number of voters who prefer b to a. In this paper, we aim to extend Gehrlein and Lepelley analysis to qualified majorities. Although a simple majority is the rule most often used, qualified majorities are also common in actual parliaments on important constitutional issues.

According to a given quota q, a candidate a is beaten by a candidate b if at least a proportion of q individuals prefer b to a. Following the terminology of Baharad and Nitzan (2003) and Courtin

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[1] Their book "voting paradoxes and group coherence: The Condorcet efficiency of voting rules" summarizes many of the existing papers on that topics.

et al. (2012), the q-Condorcet winner (q-CW) is the candidate who is never beaten under the q-majority. For a set of positional rules, Courtin *et al.* (2012) provide a necessary and sufficient condition on the quota for the winner of the election not to be beaten under the q-majority. For each voting rule considered[2], they provide the lower bound of the quota (q^*) which guarantees that a q-CW is always selected, whenever one exists. Indeed, the threshold q^* is such that: *a)* for any quota q less than or equal to q^*, there always exists a profile of individual preferences for which the winner of the positional rule is beaten under the q-majority; and *b)* for any quota q greater than q^*, the winner of the positional rule is never beaten under the q-majority.

In this paper, along the line of Gehrlein and Lepelley (2006), we evaluate the propensity of positional rules to select the q-CW when the quota is not achieved. This entails computing the probability (hereafter called q-Condorcet efficiency) that a voting rule selects the q-CW. We know from Courtin *et al.* (2012) that given a positional rule, the q-Condorcet efficiency is equal to one for any quota q greater than q^*, whereas the pioneering work of Gehrlein and Lepelley (2006) summarizes the result for simple majority. Our purpose is then to fill the gap for $\frac{1}{2} < q < q^*$. For three-candidate elections, the q-Condorcet efficiency is evaluated for a wide class of positional rules (simple and sequential). We observe that the q-Condorcet efficiency is sometimes very low for some simple positional rules. This efficiency is significantly greater for sequential rules than for simple positional rules.

The remainder of the paper is organized as follows: Section 2 is a presentation of the general framework with notation and definitions. Section 3 and 4 provide computations for the simple positional rules (*PR*) and the sequential positional rules (*SPR*) respectively. Section 5 concludes the paper.

2 Notation and definitions

We consider an election with three candidates (a, b, c) and n voters who have preferences represented by linear orders (indifference is not allowed). The six possible preference rankings are given by

n_1	n_2	n_3	n_4	n_5	n_6
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

where for instance, n_1 individuals prefer the candidate a to the candidate b and the candidate b to the candidate c. Obviously $n = \sum_{i=1}^{6} n_i$. A specific combination of n_i's that sum to n is referred to as an anonymous profile or a voting situation.

In the present paper, we focus our attention on the simple positional rules (*PR*) and the sequential positional rules (*SPR*). We assume that each individual gives 1 point to the candidate ranked first in his preferences, a weight of α points ($0 \leq \alpha \leq 1$) to the candidate ranked second and 0 points to the last candidate.

In a *PR*, a winner is any candidate who receives the largest number of points[3]. For instance, the candidate a is a winner if $n_1+n_2+\alpha(n_3+n_5) \geq n_3+n_4+\alpha(n_1+n_6)$ and $n_1+n_2+\alpha(n_3+n_5) \geq$

[2] A complete solution is given for the classical positional rules (plurality rule, negative plurality rule, Borda rule, Hare's rule, Coombs rule and Nanson's rule).

[3] If more than one candidate have the same score, then they all belong to the "winning set".

$n_5+n_6+\alpha(n_2+n_4)$. Three particular *PR*s are well-known in the literature: plurality rule ($\alpha = 0$), negative plurality rule ($\alpha = 1$) and Borda rule ($\alpha = \frac{1}{2}$). The *PR* that corresponds to the weight α is denoted F_α.

In a *SPR*, there are two steps. The candidate who obtains the smallest number of points is first eliminated[4]. The preferences concerning the two other candidates are not changed. A majority comparison gives the winner at the second step. For instance, if $n_1 + n_2 + \alpha(n_3 + n_5) < n_3 + n_4 + \alpha(n_1 + n_6)$ and $n_1 + n_2 + \alpha(n_3 + n_5) < n_5 + n_6 + \alpha(n_2 + n_4)$, then a is eliminated at the first step. If $n_1 + n_3 + n_4 > n_2 + n_5 + n_6$, then b is the winner. Once again, three usual *SPR*s are well-known in the literature: Hare's rule ($\alpha = 0$), Coombs rule ($\alpha = 1$) and Nanson's rule ($\alpha = \frac{1}{2}$). The *SPR* that corresponds to the weight α is denoted $\overline{F}_\alpha$.

A candidate x is beaten under a q-majority by a candidate y if the number of individuals who prefer y to x is greater than or equal to qn, with $\frac{1}{2} < q \leq 1$[5]. Given a *PR* or a *SPR*, Courtin *et al.* (2012) determine the smallest values q^* for which no candidate in the winning set is beaten under the q-majority. They show that there always exists a profile of individual preferences at which a candidate in the winning set is q-majority beaten whenever $1/2 < q < q^*$.

In order to compute the q-Condorcet efficiency, some probabilistic assumptions must be made regarding the likelihood that various profiles are observed. The impartial anonymous culture (IAC) is one of the most used probabilistic models in the literature[6]. Under the IAC assumption, all anonymous profiles are equally likely to be observed. The q-Condorcet efficiency under IAC is then the ratio

$$\frac{\text{number of all anonymous profiles at which no candidate in the winning set is beaten under the } q\text{−majority}}{\text{total number of all possible anonymous profiles}}$$

It is worth noting that the q-Condorcet efficiency considered in this paper is slightly different from the one used by Gehrlein and Lepelley (2006). In fact, the (conditional) Condorcet efficiency by Gehrlein and Lepelley is the conditional probability that a positional rule will elect the CW, when one exists. We compute the probability that a positional rule selects a q-CW. By so doing we aim to highlight the concordance of the two main social choice approaches, namely the positional approach and the Condorcet approach. Note that to obtain the conditional version of our q-Condorcet efficiency, one only needs to divide our results by the probability that a q-CW exists.

To evaluate the q-Condorcet efficiency of each positional rule under study, we first have a typology of various voting situations for which no candidate in the winning set is beaten under the q-majority. These voting situations are completely described by a set of linear inequalities (technicalities are available from the authors upon simple request). From this typology, by the use of computerized evaluation processes, we obtain various results measuring the q-Condorcet efficiency of each voting rule under consideration. The computer evaluation process is based on the same technique whose origins are in Gehrlein and Fishburn (1976) but are more recently based on Ehrhart polynomials (see Lepelley *et al.* 2008).

[4]Note that if two candidates have the same score, they are eliminated at the first step and then there is no second step. If all the three candidates have the same score, then they all belong to the winning set.

[5]Note that for $q \leq \frac{1}{2}$, one may find some configurations of individual preferences where for some candidates a and b, a is beaten by b and b is beaten by a. Therefore those quotas are omitted to avoid ambiguous situations.

[6]For a detailed discussion of this hypothese and some others, see Regenwetter et al. (2006).

Also note that in order to derive exact representations for a fixed size n of the electorate, computations will be performed only on integer variables. Thus we have to rewrite all the initial constraints that involve the term qn which may not be integer. Then given the size n of the electorate and the parameter q of the majority, a new parameter p has been introduced to allow integer computerized evaluation; that is $p = \lceil qn \rceil - 1$ where $\lceil y \rceil$ is the smallest integer greater than or equal to y.

The q-Condorcet efficiency for a given simple positional rule α, a given n and a given q, will be denoted $CE(F_\alpha, n, q)$, while the q-Condorcet efficiency for a given sequential positional rule α, n and q, will be denoted $CE(\overline{F}_\alpha, n, q)$.

3 Simple Positional rules

Three usual *PRs* are under consideration here: plurality ($\alpha = 0$), negative plurality ($\alpha = 1$) and Borda ($\alpha = \frac{1}{2}$). For each of the three rules, two series of results are provided: (i) the q-Condorcet efficiency in terms of n and q; and (ii) the q-Condorcet efficiency in terms of q as the size of the electorate tends to infinity[7]. More precisely, results for a finite number of voters will be given in terms of p instead of q; recall that $p = \lceil qn \rceil - 1$ and that $\frac{n-1}{2} \leq p \leq n-1$ for $\frac{1}{2} < q \leq 1$.

3.1 Plurality

Proposition 1 below provides exact representation of the q-Condorcet efficiency for the plurality rule.

Proposition 1. *Let F_0 be the PR. If n is a multiple of 6,*[8] *and $p \geq \frac{1}{2}n$ with $p = \lceil qn \rceil - 1$, then*

$$CE(F_0, n, p) = \begin{cases} \frac{9n^5+(-320p-505)n^4+\left(1680p^2+2960p-915\right)n^3+\left(-3240p^3-5040p^2+7560p+3105\right)n^2}{9(n+1)(n+2)(n+3)(n+4)(n+5)} \\ +\frac{\left(2700p^4+3780p^3-11\,340p^2-2340p+4266\right)n-810p^5-1080p^4+5670p^3+1080p^2-2700p+1080}{9(n+1)(n+2)(n+3)(n+4)(n+5)} & \text{if } \frac{1}{2}n \leq p \leq \frac{2}{3}n \\ 1 & \text{if } \frac{2}{3}n < p \leq n-1 \end{cases}$$

For large electorates, that is when the total number of voters tends to infinity, $\frac{p}{n}$ tends to q. We then deduce the following result.

Proposition 2. *Let F_0 be the PR. Then for large electorates,*

$$CE(F_0, \infty, q) = \begin{cases} -90q^5 + 300q^4 - 360q^3 + \frac{560}{3}q^2 - \frac{320}{9}q + 1 & \text{if } \frac{1}{2} < q \leq \frac{2}{3} \\ 1 & \text{if } \frac{2}{3} \leq q \leq 1 \end{cases}$$

We present in Table 1 some numerical values of the q-Condorcet efficiency of the plurality rule[9] with illustrations in Figure 1.

[7] The same technique can be applied to derive more results in terms of the quota q given some other value of the weight α; or conversely other results in terms of the weight α given some value of the quota q.

[8] Data for other values of n are available from the authors upon simple request.

[9] Note that for each table provided in this paper, 1^- means that frequencies are almost 1, but not 1; and $\epsilon = 0.001$.

Table 1 q-Condorcet efficiency of the *plurality rule*

n \ q	$\frac{1}{2}+\epsilon$	$0,54$	$0,57$	$0,6$	$0,63$	$\frac{2}{3}$	$> \frac{2}{3}$
12	0.8932	0.8932	0.8932	0.9842	0.9842	0.9842	1
24	0.8623	0.8623	0.9376	0.9794	0.9967	0.9967	1
30	0.8555	0.9221	0.9648	0.9648	0.9884	0.9981	1
42	0.8476	0.9009	0.9405	0.9856	0.9952	0.9992	1
60	0.8414	0.9154	0.9625	0.9776	0.9946	0.9997	1
66	0.8401	0.9092	0.9555	0.9828	0.9959	0.9998	1
78	0.8381	0.9230	0.9593	0.9820	0.9974	0.9999	1
96	0.8359	0.9090	0.9571	0.9846	0.9968	0.9999	1
102	0.8354	0.9233	0.9639	0.9870	0.9973	0.9999	1
114	0.8344	0.9156	0.9556	0.9862	0.9964	1^-	1
126	0.8337	0.9235	0.9580	0.9855	0.9973	1^-	1
144	0.8328	0.9143	0.9645	0.9866	0.9969	1^-	1
156	0.8323	0.9209	0.9585	0.9860	0.9976	1^-	1
∞	0.8294	0.9216	0.9646	0.9883	0.9980	1	1

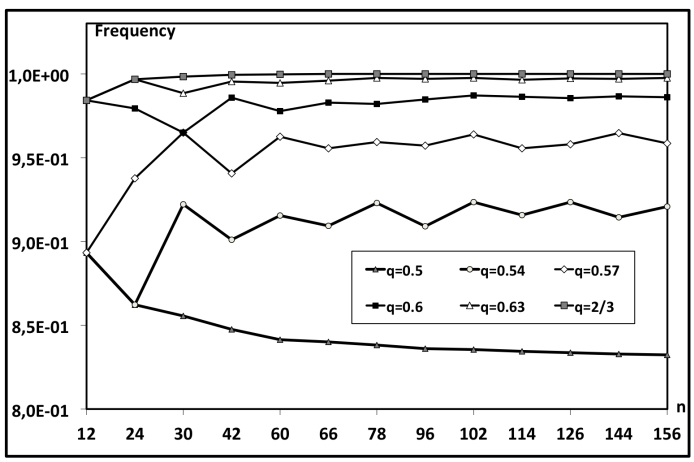

Fig. 1 *q-Condorcet efficiency of the plurality rule*

These results illustrate the following facts:

(i) For large electorates, with only a probability of 83% the q-Condorcet efficiency of the plurality rule is small as q tends to $\frac{1}{2}$. In other words, for 17% of voting situations, a majority of voters prefer another candidate to the plurality winner.

(ii) The q-Condorcet efficiency increases as q increases and tends quickly to 1. The computed values in Table 1 indicate that the plurality rule always leads to the selection of a q-CW as soon as $q > 2/3$. This result is consistent with the general bound proposed by Courtin *et al.* (2012) which is $\frac{k-1}{k}$, when the total number of candidates is k.

(iii) The variations of frequencies are relatively small as n increases.

3.2 Negative Plurality

The answer of our main question is given below for the negative plurality rule.

Proposition 3. *Let F_1 be the PR. If n is a multiple of* 6, *and $p \geq \frac{1}{2}n$ with $p = \lceil qn \rceil - 1$, then*
$CE(F_1, n, p) =$

$$\begin{cases} \frac{5\left[-20n^5+(200p+60)n^4+\left(-816p^2-448p-500\right)n^3+\left(1728p^3+1440p^2+2808p+612\right)n^2\right]}{36(n+1)(n+2)(n+3)(n+4)(n+5)} + \frac{5\left[(-1080p^3-1728p^4-2592p^2+144p+72)n+\left(648p^5+81p^4+1080p^3+1620p^2+2592p+864\right)\right]}{36(n+1)(n+2)(n+3)(n+4)(n+5)} & \text{if } \frac{1}{2}n \leq p \leq \frac{2}{3}n \text{ and } p \text{ even} \\ \frac{-100n^5+(1000p+300)n^4\left(-4080p^2-2240p-2500\right)n^3+\left(8640p^3+7200p^2+14\,040p+3060\right)n^2}{36(n+1)(n+2)(n+3)(n+4)(n+5)} + \frac{\left(-8640p^4-5400p^3-12\,960p^2+3960p+6840\right)n+3240p^5+405p^4+5400p^3+2430p^2+1485}{36(n+1)(n+2)(n+3)(n+4)(n+5)} & \text{if } \frac{1}{2}n \leq p \leq \frac{2}{3}n \text{ and } p \text{ odd} \\ \frac{-17n^5+(90p-105)n^4+\left(-180p^2+480p-185\right)n^3+\left(180p^3-720p^2+810p-15\right)n^2}{(n+1)(n+2)(n+3)(n+4)(n+5)} + \frac{\left(480p^3-90p^4-810p^2+480p+202\right)n+18p^5-120p^4+270p^3-240p^2+72p+120}{(n+1)(n+2)(n+3)(n+4)(n+5)} & \text{if } \frac{2}{3}n < p \leq n-1 \end{cases}$$

For a large electorates, we have the following results

Proposition 4. *Let F_1 be the PR. Then for large electorates,*

$$CE(F_1, \infty, q) = \begin{cases} 90q^5 - 240q^4 + 240q^3 - \frac{340}{3}q^2 + \frac{250}{9}q - \frac{25}{9} & \text{if } \frac{1}{2} < q \leq \frac{2}{3} \\ 18q^5 - 90q^4 + 180q^3 - 180q^2 + 90q - 17 & \text{if } \frac{2}{3} \leq q \leq 1 \end{cases}$$

Table 2 q-Condorcet efficiency of the *negative-plurality rule*

n \ q	$\frac{1}{2}+\epsilon$	0,54	0,57	0,6	0,63	$\frac{2}{3}$	0,69	0,75	0,84	0,96	1
12	0.6130	0.6734	0.6734	0.7776	0.7776	0.7776	0.9050	0.9050	0.9922	0.9990	0.9990
24	0.6026	0.6370	0.6965	0.8091	0.8558	0.8558	0.9133	0.9512	0.9950	0.9999	0.9999
30	0.6004	0.7038	0.7498	0.7498	0.8352	0.8709	0.9154	0.9683	0.9956	0.9999	0.9999
42	0.5976	0.6743	0.7090	0.8083	0.8639	0.8876	0.9181	0.9726	0.9962	1^-	1^-
60	0.5954	0.6899	0.7639	0.7862	0.8488	0.8997	0.9373	0.9718	0.9967	1^-	1^-
66	0.5950	0.6685	0.7382	0.8017	0.8570	0.9022	0.9363	0.9763	0.9968	1^-	1^-
78	0.5943	0.6985	0.7560	0.8087	0.8691	0.9060	0.9348	0.9773	0.9969	1^-	1^-
96	0.5936	0.6703	0.7510	0.8084	0.8704	0.9099	0.9430	0.9761	0.9972	1^-	1^-
102	0.5934	0.6892	0.7639	0.8168	0.8741	0.9108	0.9421	0.9786	0.9971	1^-	1^-
114	0.5930	0.6793	0.7475	0.8200	0.8616	0.9125	0.9406	0.9790	0.9972	1^-	1^-
126	0.5928	0.6971	0.7452	0.8119	0.8687	0.9138	0.9393	0.9793	0.9972	1^-	1^-
144	0.5925	0.6783	0.7639	0.8207	0.8697	0.9154	0.9441	0.9783	0.9973	1^-	1^-
156	0.5923	0.6873	0.7467	0.8140	0.8712	0.9162	0.9428	0.9787	0.9978	1^-	1^-
∞	0.5928	0.6907	0.7607	0.8233	0.8763	0.9259	0.9485	0.9824	0.9981	1^-	1

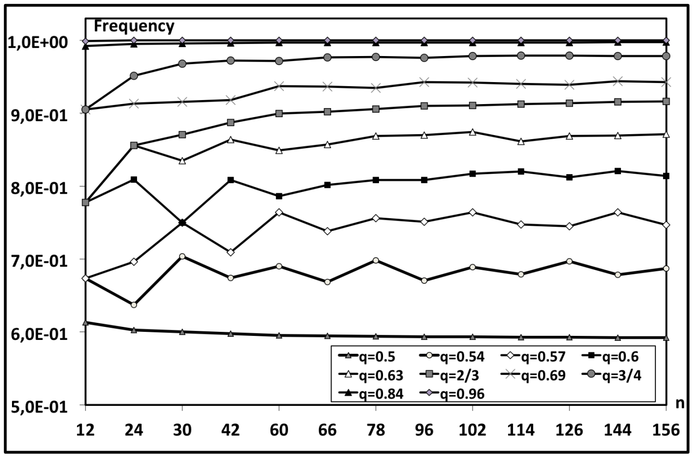

Fig. 2 *q-Condorcet efficiency of the negative plurality rule*

Once again, frequencies are relatively constant when n increases and increase as q rises. Table 2 and Figure 2 show that no quota garantees the selection of a q-CW under the negative plurality rule. Indeed, for large electorate and when q tends to $\frac{1}{2}$, the q-CW is chosen for only 60% of the voting situations. A quota of at least $\frac{3}{4}$ is needed for a q-Condorcet efficiency greater than 95%.

3.3 Borda rule

The following results deal with the Borda rule.

Proposition 5. *Let* F_2 *be the PR. If* n *is a multiple of* 1008 *and* p *multiple of* 336, *with* $p = \lceil qn \rceil - 1$, *then* $CE\left(F_2, n, p\right) =$

$$\begin{cases} \dfrac{105n^5+(-1440p-825)n^4+\left(6480p^2+5040p-1115\right)n^3+\left(-12\,960p^3-9720p^2+9360p+3465\right)n^2}{9(n+1)(n+2)(n+3)(n+4)(n+5)} \\ +\dfrac{\left(12\,150p^4+8100p^3-15\,390p^2-3780p+3690\right)n-4374p^5-2430p^4+8370p^3+2430p^2-1836p+1080}{9(n+1)(n+2)(n+3)(n+4)(n+5)} & \text{if } \frac{1}{2}n \leq p \leq \frac{2}{3}n \\ 1 & \text{if } \frac{2}{3}n < p \leq n-1 \end{cases}$$

When the total number of voters tends to infinity, the following results hold.

Proposition 6. *Let* F_2 *be the PR. Then for large electorates,*

$$CE\left(F_2, \infty, q\right) = \begin{cases} -486q^5 + 1350q^4 - 1440q^3 + 720q^2 - 160q + \dfrac{35}{3} & \text{if } \frac{1}{2} < q \leq \frac{2}{3} \\ 1 & \text{if } \frac{2}{3} \leq q \leq 1 \end{cases}$$

Table 3 q-Condorcet efficiency of the *Borda rule*

$n \backslash q$	$\frac{1}{2}+\epsilon$	$0,54$	$0,57$	$0,6$	$0,63$	$\frac{2}{3}$	$> \frac{2}{3}$
∞	0.8573	0.9463	0.9805	0.9953	0.9995	1	1

With the Borda rule, computations reveal hight periodicity both on n and p. This phenomenon is frequent with the Borda rule (see Lepelley *et al.* 2008). In Table 3, we then provide only results when n tends to infinity. It appears that the Borda rule particularly performs well. The q-Condorcet efficiency of the Borda rule converges quickly to one as q tends to the threshold $\frac{2}{3}$.

4 Sequential Positional Rules

We now consider the following three *SPR*s*:* Hare's rule ($\alpha = 0$), Coombs rule ($\alpha = 1$) and Nanson's rule ($\alpha = \frac{1}{2}$).

4.1 Hare's rule

The following results solve the Hare's rule.

Proposition 7. *Let* $\overline{F}_0$ *be the SPR. If* n *is a multiple of* 24 *and* $p \geq \frac{1}{2}n$ *with* $p = \lceil qn \rceil - 1$*, then* $CE\left(\overline{F}_0, n, p\right) =$

$$
\begin{cases}
\frac{(12\,850p+10\,785)n^4-1933n^5+\left(-25\,920p^2-53\,160p-93\,640\right)n^3+\left(25\,920p^3+184\,320p^2+682\,560p+293\,040\right)n^2}{576(n+1)(n+2)(n+3)(n+4)(n+5)} \\ +\frac{\left(-12\,960p^4-224\,640p^3-1123\,200p^2-587\,520p+119\,808\right)n+\left(2592p^5+90\,720p^4+630\,720p^3+518\,400p^2+62\,208p+69\,120\right)}{576(n+1)(n+2)(n+3)(n+4)(n+5)} & \text{if } \frac{1}{2}n \leq p \leq \frac{2}{3}n \text{ and } p \text{ even} \\[2ex]
\frac{(12\,850p+10\,785)n^4-1933n^5+\left(-25\,920p^2-53\,160p-93\,640\right)n^3+\left(25\,920p^3+184\,320p^2+682\,560p+318\,960\right)n^2}{576(n+1)(n+2)(n+3)(n+4)(n+5)} \\ +\frac{\left(-12\,960p^4-224\,640p^3-1123\,200p^2-691\,200p+29\,088\right)n+\left(2592p^5+90\,720p^4+630\,720p^3+596\,160p^2+127\,008p+4320\right)}{576(n+1)(n+2)(n+3)(n+4)(n+5)} & \text{if } \frac{1}{2}n \leq p \leq \frac{2}{3}n \text{ and } p \text{ odd} \\[2ex]
\frac{-1637n^5+(10\,530p-3495)n^4+\left(22\,680p-25\,920p^2+6520\right)n^3+\left(31\,680p^3-43\,200p^2-2880p+22\,320\right)n^2}{64(n+1)(n+2)(n+3)(n+4)(n+5)} \\ +\frac{\left(-19\,200p^4+36\,480p^3+1920p^2-19\,200p+20\,992\right)n+\left(4608p^5-11\,520p^4+11\,520p^2-4608p+7680\right)}{64(n+1)(n+2)(n+3)(n+4)(n+5)} & \text{if } \frac{2}{3}n < p \leq \frac{3}{4}n \\
1 & \text{if } \frac{3}{4}n < p \leq n-1
\end{cases}
$$

For large electorates, we have the following proposition.

Proposition 8. *Let* $\overline{F}_0$ *be the SPR. Then for large electorates,*

$$
CE\left(\overline{F}_0, \infty, q\right) = \begin{cases}
\frac{9}{2}q^5 - \frac{45}{2}q^4 + 45q^3 - 45q^2 + \frac{6425}{288}q - \frac{1933}{576} & \text{if } \frac{1}{2} < q \leq \frac{2}{3} \\
72q^5 - 300q^4 + 495q^3 - 405q^2 + \frac{5265}{32}q - \frac{1637}{64} & \text{if } \frac{2}{3} \leq q \leq \frac{3}{4} \\
1 & \text{if } \frac{3}{4} \leq q \leq 1
\end{cases}
$$

Table 4 and Figure 3 below present some exact frequencies.

Table 4 q-Condorcet efficiency of the *Hare's rule*

n \ q	$\frac{1}{2}+\epsilon$	0,54	0,57	0,6	0,63	$\frac{2}{3}$	0,7	$\geq \frac{3}{4}$
24	0.9121	0.9121	0.9486	0.9747	0.9898	0.9898	0.9988	1
48	0.9096	0.9391	0.9636	0.9754	0.9910	0.9955	0.9955	1
72	0.9090	0.9375	0.9681	0.9818	0.9911	0.9967	0.9992	1
96	0.9087	0.9406	0.9641	0.9805	0.9912	0.9972	0.9995	1
120	0.9085	0.9424	0.9667	0.9797	0.9912	0.9974	0.9993	1
144	0.9084	0.9436	0.9684	0.9821	0.9913	0.9976	0.9995	1
168	0.9083	0.9445	0.9661	0.9813	0.9913	0.9977	0.9994	1
∞	0.9092	0.9483	0.9691	0.9834	0.9926	0.9983	0.9996	1

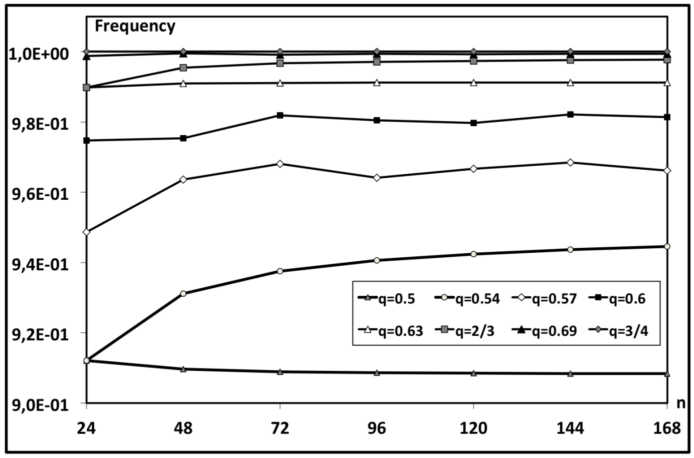

Fig. 3 *q-Condorcet efficiency of the Hare's rule*

Once again, it can be observed that the q-Condorcet efficiency of the Hare's rule increases as q increases. Moreover, frequencies converge quickly to 1 as q tends to one. For $q \geq \frac{3}{4}$, the Hare's rule always selects the q-CW in accordance with Courtin *et al.* (2012). Besides it appears in terms of q-Condorcet efficiency that the Hare's rule always performs better than the plurality rule.

4.2 Coombs rule

The following results concern Coombs' procedure.

Proposition 9. *Let $\overline{F}_1$ be the SPR. If n is a multiple of 6 and and $p \geq \frac{1}{2}n$ with $p = \lceil qn \rceil - 1$, then*

$$CE\left(\overline{F}_1, n, p\right) =$$

$$\begin{cases} \frac{57n^5+(-560p-185)n^4+\left(2160p^2+1320p-1495\right)n^3+\left(11\,700p-1800p^2-3780p^3+6165\right)n^2}{9(n+1)(n+2)(n+3)(n+4)(n+5)} \\ +\frac{\left(3105p^4+810p^3-20\,385p^2-14\,850p-342\right)n+\left(11\,880p^3-972p^5+12\,960p^2+4212p+1080\right)}{9(n+1)(n+2)(n+3)(n+4)(n+5)} & \text{if } \frac{1}{2}n \leq p \leq \frac{2}{3}n \\ 1 & \text{if } \frac{2}{3}n < p \leq n-1 \end{cases}$$

In a large society, we have:

Proposition 10. *Let $\overline{F}_1$ be the SPR. Then for large electorates,*

$$CE\left(\overline{F}_1, \infty, q\right) = \begin{cases} -108q^5 + 345q^4 - 420q^3 + 240q^2 - \frac{560}{9}q + \frac{19}{3} & \text{if } \frac{1}{2} < q \leq \frac{2}{3} \\ 1 & \text{if } \frac{2}{3} \leq q \leq 1 \end{cases}$$

The corresponding table and figure are:

Table 5 q-Condorcet efficiency of the *Coombs rule*

n \ q	$\frac{1}{2}+\epsilon$	$0,54$	$0,57$	$0,6$	$0,63$	$\frac{2}{3}$	$>\frac{2}{3}$
12	0.9349	0.9349	0.9349	0.9842	0.9842	0.9842	1
24	0.9243	0.9243	0.9644	0.9869	0.9967	0.9967	1
30	0.9217	0.9569	0.9798	0.9798	0.9925	0.9981	1
42	0.9185	0.9465	0.9676	0.9917	0.9969	0.9992	1
60	0.9159	0.9548	0.9798	0.9878	0.9969	0.9997	1
66	0.9154	0.9516	0.9761	0.9907	0.9976	0.9998	1
78	0.9145	0.9591	0.9783	0.9903	0.9985	0.9999	1
96	0.9136	0.9518	0.9772	0.9918	0.9983	0.9999	1
102	0.9134	0.9594	0.9808	0.9931	0.9985	0.9999	1
114	0.9130	0.9554	0.9765	0.9927	0.9980	1^-	1
126	0.9127	0.9596	0.9778	0.9923	0.9985	1^-	1
144	0.9124	0.9548	0.9813	0.9929	0.9984	1^-	1
156	0.9122	0.9583	0.9781	0.9926	0.9987	1^-	1
∞	0.9112	0.9590	0.9815	0.9939	0.9990	1	1

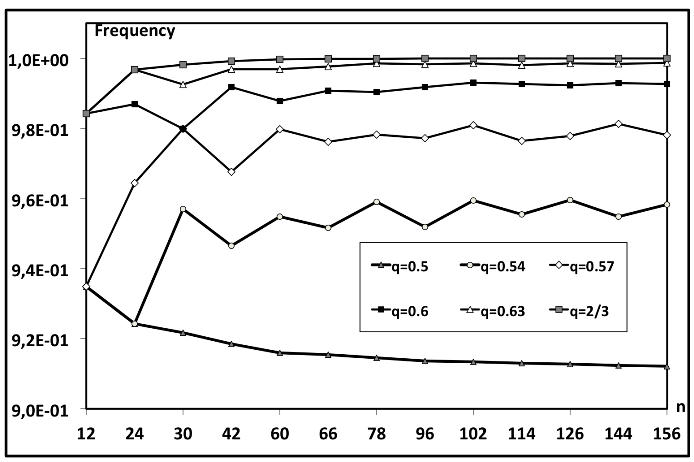

Fig. 4 *q-Condorcet efficiency of the Coombs rule*

With contrast to the poor performance of the negative plurality rule, its sequential version, the Coombs rule, performs very well in selecting the q-CW. With the Coombs rule, the q-Condorcet efficiency is above 91% as soon as q is greater than $\frac{1}{2}$. Moreover, the Coombs rule admits a threshold quota of $\frac{2}{3}$ above which it always selects the q-CW, whereas the negative plurality does not admit any threshold.

4.3 Nanson's rule

The following result deals with the Nanson's rule.

Proposition 11. *Let $\overline{F}_2$ be the SPR. If n is a multiple of 72 and p multiple of 36 with $p = \lceil qn \rceil - 1$, then*

$CE\left(\overline{F}_2, n, p\right) =$

$$\begin{cases} \frac{17n^5+(-160p-225)n^4+\left(600p^2+1480p+225\right)n^3+\left(485-3420p^2-580p-1080p^3\right)n^2}{(n+1)(n+2)(n+3)(n+4)(n+5)} \\ +\frac{\left(945p^4+3510p^3+795p^2-810p+138\right)n+\left(630p^2-1350p^4-360p^3-324p^5+204p+120\right)}{(n+1)(n+2)(n+3)(n+4)(n+5)} & \text{if } \frac{1}{2}n \leq p \leq \frac{2}{3}n \\ 1 & \text{if } \frac{2}{3}n < p \leq n-1 \end{cases}$$

In a large society, we have:

Proposition 12. *Let $\overline{F}_2$ be the SPR. Then for large electorates,*

$$CE\left(\overline{F}_2, \infty, q\right) = \begin{cases} -324q^5 + 945q^4 - 1080q^3 + 600q^2 - 160q + 17 & \text{if } \frac{1}{2} < q \leq \frac{2}{3} \\ 1 & \text{if } \frac{2}{3} \leq q \leq 1 \end{cases}$$

Table 6 illustrates these results for a large set of voters.

Table 6 q-Condorcet efficiency of the *Nanson's rule*

n \ q	$\frac{1}{2}+\epsilon$	$0,54$	$0,57$	$0,6$	$0,63$	$\frac{2}{3}$	$> \frac{2}{3}$
∞	0.9387	0.9758	0.9909	0.9978	0.9998	1	1

Yet again, the Nanson's rule performs better than the Borda rule in the selection of a q-CW. Indeed, the q-Condorcet efficiency of the Nanson's rule is above 93% as soon as q is greater than $\frac{1}{2}$.

5 Concluding discussion

It seems desirable that the winner of an election should not be beaten under a qualified majority rule. Unfortunately for three-candidate elections, positional rules (simple or sequential) fail to satisfy this requirement for any quota of the qualified majority between $\frac{1}{2}$ and q^*. Our contribution in the present paper underlines the respective behaviors of six positional rules with respect to their ability to select the q-CW. Our overall comments of the results obtained are as follows:

(i) for each rule studied, the q-Condorcet efficiency increases as the quota increases up to a threshold above which the rule always selects a q-CW;

(ii) among the usual simple positional rules, the Borda rule consistently has significantly greater q-Condorcet efficiency than both the plurality and the negative plurality rules. The negative plurality appears to be the worst simple positional rule;

(iii) the Nanson's rule is the top-performing rule as well among the sequential positional rules as among the simple positional rules under consideration here;

(iv) the improvement in q-Condorcet efficiency seems greater in moving from a simple positional rule to its corresponding sequential form.

In a different but similar investigation, our results are in accordance with earlier observations by Gehrlein and Lepelley (2006). With three candidates, the use of a sequential positionnal rule in combination with a larger quota will significantly improve the stability of the winner.

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