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Abstract

This work provides necessary and sufficient conditions for the dominance solvability of approval voting games. Our conditions are very simple since they are based on the approval relation, a binary relation between the alternatives. We distinguish between two sorts of dominance solvability and prove that the most stringent one leads to the election of the set of Condorcet Winners whereas this need not be the case for the weak version.

KEYWORDS: Approval voting, Strategic voting, Dominance-solvability, Condorcet Winner

JEL Classification Numbers: C72 ;D71; D72.

1 Introduction

What would be the outcome of an election if each voter was allowed to vote for as many candidates as she wanted rather than just for one as it is the case now? This is one of the main questions that the recent literature on voting theory has been concerned with, both in political science and economics. This idea of switching from

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the “One man, One vote” principle (i.e. Plurality Voting) to the “One Man, Many Votes” one is indeed the focus of the theoretical debate over Approval Voting (AV)[1]. Our paper contributes this debate by considering the strategic analysis of the preference aggregation games. Rather than focusing on some normative criteria such as the Condorcet consistency[2], we evaluate the merits of this rule by considering the equilibria of the following game. The voters' preferences are common knowledge and the voters play according to the Approval voting method: they can approve of as many candidates as they want. Voters are instrumental so that they only get utility from the outcome of the election. Ties are broken by a fair lottery.

As this game typically has (too) many equilibria, we focus on the situations in which most of the game-theoretical tools give a sharp and unique prediction. Indeed, we just consider which consider equilibria that survive to the iterated elimination of weakly dominated strategies (i.e. sophisticated voting à la Farquharson [10]). The games in which sophisticated voting leads to a unique outcome are dominance-solvable. Note that, as we consider generic games, this outcome satisfies a list of desiderata: this outcome is perfect and proper and moreover is the unique stable one à la Mertens [13]. In a recent contribution, Buenrostro et al. [6] underline the importance of the conditions for dominance solvability of approval games. They state some sufficient conditions for the solvability of scoring rule games and hence prove that whenever the conditions are satisfied for a scoring rule, so are the ones for AV. This is important since, whenever the conditions they describe hold, the outcome coincides with the set of Condorcet Winners and so, in a sense, these conditions are some sort of normative criteria for voting rules.

We distinguish two sorts of dominance solvability: cardinal (CS) and ordinal solvability (OS). Typically, the cardinal utilities of the voters are defined as the primitives of the game. Hence, for us, a game is CS if it is dominance solvable (i.e. there is some order of deletion that isolates a unique outcome). However, following the Arrovian tradition, one might consider that interpersonal comparable utilities

[1]AV is the method of election according to which a voter can vote for as many candidates as she wishes and the candidate with the highest number of approval votes wins the election. The reader can refer to Brams [3], Weber [14] and the recent handbook of approval voting (Laslier and Sanver [12]) for an account of this literature. The literature of information aggregation under AV contains many interesting contributions such as the ones by Ahn and Oliveros [1], Bouton and Castanheira [2] and Goertz and Maniquet [11].

[2]A Condorcet Winner is a candidate that beats every other candidate in pair wises comparison. A rule is Condorcet consistent if it selects the Condorcet Winner whenever it exists.

should not be taken into account so that only the voters' ordinal preferences might have an impact on the outcome of the game. As will be shown, this need not be the case as the outcome by iterated removal of weakly dominated strategies might depend on the voters' cardinal preferences. In other words, given a ordinal preference ordering, different specifications of the utilities might have an impact on the prediction of dominance-solvability. Hence, we define a stronger notion of dominance solvability, OS, just based on the ordinal preferences. We say that a game is OS if every game with the same ordinal preference ordering is CS and the same outcome is reached. It follows that any OS game is CS whereas the contrary need not be true (see Example 1).

We answer two main questions: (i) when is a game OS and CS? and (ii) does the surviving outcome satisfy Condorcet Consistency?

As far as the first question is concerned, our answer is rather simple. Our conditions are based on the social preferences of the voters, and more precisely the approval relation (A) and the weak approval relation (ωA). We say that a candidate x is (weakly) A–preferred to a candidate y if the number of voters who rank x first and y last in their individual preference is (weakly) higher than the number of voters who strictly prefer y to x minus the number of voters who rank y above x, none of them being either first or last. Note that with just three candidates, the definition of the relation is much simpler since the number of voters who rank y above x, none of them being either first or last equals zero. A candidate y belongs to the *(weak) Approval domain* if there exists a candidate who is A-preferred to him. We first prove that the A-relation fully characterizes the set of winners under AV provided that the voters use weakly undominated strategies. In other words, a candidate belongs to the Approval domain if and only if he cannot win the election since its maximal score is strictly lower than the minimal score of some other candidate.

Building on this result, we provide a characterization of dominance-solvability.

With just three candidates, we first prove that if the Approval domain is non-empty, then the game is OS. Moreover, we show that if the Approval domain is empty but the weak Approval domain is non-empty, a game might be CS but not OS. Moreover, we prove that for such games, there is always a non-empty set of utilities for which the game is CS. In other words, the conditions for a game to be CS without being OS do not simply hinge on the number of the voters but also on their cardinal utilities. Finally, we prove that if the weak Approval domain is empty, then the game is not CS. Therefore, in a scenario with three alternatives, a game is

OS if and only the Approval Domain is non-empty. More generally, if k candidates are present in the election, the requirements for *OS* are still based on the Approval Domain. If at least $k-2$ candidates belong to the Approval domain, the game is *OS*. Again, if the weak Approval domain is empty then the game cannot be *CS*. Note that if less than $k-2$ candidates belong to the Approval domain, the game need not be *CS*. Finally, it seems important to remark that our condition for *OS* implies the previous sufficient condition recently stated by Buenrostro et al. [6]. Indeed, they prove that if the largest fraction of players that agree on the best or on the worst candidate is high enough, the game is *CS*.

Regarding the second question, we prove that the difference between *OS* and *CS* is key. Indeed, the literature was rather ambiguous on this precise point. On the one hand, Buenrostro et al. [6] prove that under their sufficient conditions for dominance solvability, the set of Condorcet Winners was elected. On the other hand, the nice examples in De Sinopoli et al. [7, 8] emphasize that this need not be the case. Our work gives a clear-cut answer to this controversy. We first prove that if a game is *OS*, then the outcome of dominance solvability equals the set of Condorcet Winners. On the contrary, if the game is *CS* but not *OS* it need not be the case that the outcome satisfies Condorcet Consistency. If one considers the whole set of *CS* games, this violation occurs under very precise conditions which can be stated with just three candidates as follows: it must be the case that the number of voters that rank some candidate x first and some candidate y must be *equal* to the number of voters who prefer y to x.

Intuitively, the main rationale behind the selection in Condorcet winners in *OS* games is as follows. In these games the procedure of dominance-solvability takes a very natural structure described by the following three steps:

1. Voters agree that every voter should only use weakly undominated strategies (voting for your top-candidate and never voting for your least-preferred one).

2. Given the previous agreement, at least $k-2$ candidates cannot be in the winning set since their maximal score is lower than the minimal score of some other candidate. Hence, it is irrelevant for the outcome of the election whether the voters vote for any of these candidates.

3. Finally, the majority winner among the (at most) two remaining candidates, wins the election.

Given that, when there is least $k-2$ candidates in the Approval domain there must exist a CW, the procedure of dominance-solvability correctly selects the CW. In games which are CS but not OS, the procedure does not follow the same steps since there are some candidates which maximal score *coincides* with the minimal score of some other candidates. This explains why a CS game fails to select the Condorcet winner.

The remainder of this paper is organized as follows. Section 2 introduces the model and defines concepts and notation that will be used in the rest of the article. As far as the scenario with three alternatives is concerned, Section 3 presents the sufficient conditions for an AV game to be OS and CS and Section 4 shows the necessary conditions. Finally, Section 5 discusses the Condorcet consistency, extends the results to the many alternatives case and concludes the paper.

2 Model

The voting game

There is a set $N = \{1, ..., n\}$ of voters with $n \geq 4$ and a set $X = \{a, b, ..., k\}$ of alternatives. With just three alternatives, we let $X = \{a, b, c\}$. The voting game is as follows. All voters vote simultaneously. A voter can approve as many candidates as she wishes by choosing a vector v_i from the set of pure strategy vectors V_i. With just three alternatives, we have:

$$V_i = \Big\{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(0,0,0),(1,1,1)\Big\}.$$

The profile of vote vectors is $v \in V = \prod_i V_i$. The profile $v_{-i} \in V_{-i}$ denotes the vote profile excluding voter i. Let $s_j(v)$ be the number of votes for candidate j if the profile is v and $s_j(v_{-i})$ the total score for candidate j when the vote of i is excluded. The vector $s(v) = (s_a(v), \ldots, s_k(v))$ stands for the total score vector. $s(v_{-i})$ records the total votes for each candidate in X given a profile v_{-i}; we write s_{-i} for short. The winning set of candidates, that is, the outcome corresponding to profile v, is denoted by $W(v)$ and consists of those candidates that get the maximum total score:

$$W(v) = \{x \in X \mid s_x(v) = \max_{y \in X} s_y(v)\}.$$

Voter $i \in N$ has strict preferences on the set of candidates X, which are strict orderings of X. $a \succ_i b$ means that a is strictly preferred to b by voter i. The preference orderings of the voters are summarized by $\succ = (\succ_i)_{i\in N}$.

We assume that preferences can be represented by a von Neumann–Morgenstern (*vN-M*) utility function $u_i : X \to \mathbb{R}$ over lotteries on the set of candidates. Given that ties are broken by a fair lottery, the expected utility of a voter i is a function of the profile v given by :

$$U_i(v) = \frac{1}{\#W(v)} \sum_{x\in W(v)} u_i(x).$$

We will impose the following regularity condition (Dhillon and Lockwood [9]) to ensure that the order of deletion of weakly dominated strategies is irrelevant[3]: $\forall i, v, v'$ with

$$W(v) \neq W(v') \Longrightarrow U_i(W(v)) \neq U_i(W(v')). \tag{1}$$

This condition implies that no voter is indifferent between any pair of candidates and moreover no voter is indifferent between any pair of winning sets.

Letting $u = (u_i)_{i\in N}$, the strategic form voting game Γ is then defined by $\Gamma = (u, V)$. Given a preference ordering $\succ$, we define the ordinal game as the class of games represented by the following set:

$$\Gamma_{\succ} = \{\Gamma = (u, V)\,|\, u \text{ is a vN-M representation of } \succ\}.$$

Iterated weak dominance

Let $\emptyset \neq V_i' \subseteq V_i$ and $V' = \prod_{i\in N} V_i'$ be a restriction of V. Γ' denotes the reduced game of V with strategy space $V' \subseteq V$ and winning set $W' = \{W(v)\,|\, v \in V'\}$ the restriction of W to V'. Γ^m, $m = 1, 2, \ldots$ denotes the sequence of reduced games of V with $V^m \subseteq V^{m-1}$ and W^m denotes the restriction of W to V^m. Let $V^0 = V$, $\lim_{m\to\infty} V^m = V^\infty$ and $\lim_{m\to\infty} W^m = W^\infty$. Γ^∞ denotes the fully reduced game with $\Gamma^\infty = (u, V^\infty)$.

Definition 1. *A pure strategy $v_i \in V_i$ very weakly dominates pure strategy $\omega_i \in V_i$ in Γ iff $\forall v_{-i} \in V_{-i}$, $u_i(v_i, v_{-i}) \geq u_i(\omega_i, v_{-i})$. v_i weakly dominates pure strategy ω_i in Γ iff v_i very weakly dominates ω_i in Γ and $\exists v'_{-i} \in V_{-i}$ such that $u_i(v_i, v'_{-i}) > u_i(\omega_i, v'_{-i})$.*

[3] It is irrelevant in the sense that two full reductions of a game through iterated weak dominance lead to the same winning set when condition 1 holds.

Fact 1: An AV strategy is not weakly dominated in V^0 if and only if the voter approves her most preferred candidate and never approves of her worst preferred one (see Brams and Fishburn [4]).

V^∞ stands for the full reduction of V through the sequence of restrictions V^m in which there is no weakly dominated strategy.

A full reduction by pure weak dominance is achieved by maximal simultaneous deletion (*msd*) if at each step all the weakly dominated strategies of all players are deleted.

Definition 2. *A pure strategy $v_i \in V$ is redundant in $\Gamma = (u, V)$ if there is some pure strategy $\omega_i \in V$ such that $u_i(v_i, v_{-i}) = u_i(\omega_i, v_{-i})$, $\forall v_{-i} \in V_{-i}$ and $\forall\, i \in N$.*

We define two notions of dominance solvability.

Definition 3. *The game $\Gamma = (u, V)$ is cardinally solvable (CS) if there is a full reduction by pure weak dominance such that W^∞ is a singleton.*

The outcome of a CS game $\Gamma = (u, V)$ is denoted W^∞_Γ.

The first notion is a standard one in the literature. It just requires that a unique winning set is reached through one order of deletion of weakly dominated strategies.

Definition 4. *Given a preference ordering $\succ$, the game $\Gamma_\succ$ is ordinally solvable (OS) if :*

1. *any game Γ in $\Gamma_\succ$ is CS and*
2. *$W^\infty_\Gamma = W^\infty_{\Gamma'}$ for any pair Γ, Γ' in $\Gamma_\succ$.*

The second notion of dominance solvability is stronger. Indeed, as will be shown by the next example, the usual definition CS is not fully satisfactory in voting games since it does not simply hinge on the voters' ordinal utilities: it might lead to a different outcome as a function of the preferences' intensities of the voters. This second notion precludes this property of dominance solvability by requiring that the same outcome is reached in all the games that represent the same preference ordering.

Example 1: A non OS ordinal game.

Let $n = 4$ and $X = \{a, b, c\}$ and consider the preference ordering $\succ$ given by

$$\begin{aligned} i = 1,2 &\ : \ a \succ_i b \succ_i c \\ i = 3,4 &\ : \ c \succ_i b \succ_i a. \end{aligned}$$

Due to Fact 1, the only weakly undominated strategies in V are $\{a\}$ and $\{a,b\}$ for $i = 1,2$ and $\{c\}$ and $\{c,b\}$ for $i = 3,4$. Take the game $\Gamma^1 = (u, V^1)$ after one step of *msd*. Note that for any profile in this game, $s_a(v) = s_c(v) = 2$. Hence, any voter i with $i = 1,2$ is indifferent between her two pure strategies when $s_{-i} \in \{(1,0,2),(1,3,2)\}$. However, her decision when $s_{-i} \in \{(1,1,2),(1,2,2)\}$ depends on her cardinal utility. If $u_i(b) < \frac{u_i(a)+u_i(c)}{2}$, she weakly prefers to vote $\{a\}$. If $u_i(b) > \frac{u_i(a)+u_i(c)}{2}$, she weakly prefers to vote $\{a,b\}$.

A similar claim applies to a voter i with $i = 3,4$. If $u_i(b) < \frac{u_i(a)+u_i(c)}{2}$, she weakly prefers to vote $\{c\}$. If $u_i(b) > \frac{u_i(a)+u_i(c)}{2}$, she weakly prefers to vote $\{b,c\}$. As we have assumed that no voter is indifferent between two winning sets, it follows that each voter has a unique weakly undominated strategy in V^1 that depends on her cardinal utilities. Therefore each game Γ in $\Gamma_\succ$ is CS.

However, the outcome W_Γ^∞ depends on the cardinal utilities. For instance, assume that $u_i(b) < \frac{u_i(a)+u_i(c)}{2}$ holds for every voter. Then $W_\Gamma^\infty = \{a,c\}$. If, on the contrary, $u_i(b) > \frac{u_i(a)+u_i(c)}{2}$ is satisfied for every voter, it follows that $W_\Gamma^\infty = \{b\}$. Therefore, the ordinal game $\Gamma_\succ$ is not OS since the outcome W_Γ^∞ might differ across the different games in $\Gamma_\succ$.

3 The Approval Domain: a Sufficient Condition

Prior to stating the sufficient conditions for dominance solvability, we define some binary relations that will be necessary throughout.

We let $N(x,y)$ denote the number of voters who prefer candidate x to candidate y: $N(x,y) = \#\{i \in N \mid x \succ_i y\}$.

x is M-preferred to y, denoted xMy, if and only if $N(x,y) > N(y,x)$. The weak version of the relation M denoted $x\,\omega My$, is such that $x\,\omega My$ if and only if $N(x,y) \geq N(y,x)$. The set of Condorcet Winners (CW) are the candidate(s) who are preferred by a majority of voters to any other candidate in the election.

$$X^{CW} = \{x \in X | x\,\omega My\ \forall y \neq x \in X\}.$$

We now define the approval relation A. We say that candidate x is A-preferred to y if the number of voters who rank x first and y last is higher than the number of voters who strictly prefer y to x plus the number of voters who prefer x to y but do not rank x first or y last (denoted $N(\cdot, x, y, \cdot)$). Formally, letting $N(x, \dots, y) = \#\{i \in N \mid x \succ_i \dots \succ_i y\}$,

$$xAy \text{ if and only if } N(x, \dots, y) > N(y, x) + N(\cdot, x, y, \cdot).$$

Since with three alternatives $N(\cdot, x, y, \cdot) = 0$ given that at least one of the two candidates is ranked first or last in a voter's preference ordering, it follows that

$$xAy \text{ if and only if } N(x, \dots, y) > N(y, x) \text{ when } k = 3.$$

Note that the A relation is asymmetric and that A need not be complete. Moreover, note that if xAy then xMy. Indeed, if xAy then $N(x, \dots, y) > N(y, x) + N(\cdot, x, y, \cdot)$. Given that $N(x, y) \geq N(x, \dots, y)$, it follows that

$$N(x, y) > N(y, x) + N(\cdot, x, y, \cdot) \geq N(y, x) \Rightarrow xMy.$$

The set X^A (the Approval domain) stands for the candidates for which there exists a candidate who is A-preferred to them with

$$X^A = \{x \in X \mid \exists y \text{ s.t. } yAx\}.$$

Note that this set might be empty. Similarly the weak version of the relation A, denoted $x\,\omega Ay$, is such that $x\,\omega Ay$ if and only if $N(x, \dots, y) \geq N(y, x) + N(\cdot, x, y, \cdot)$, and with associated set $X^{\omega A} = \{x \in X \mid \exists y \text{ s.t. } y\,\omega Ax\}$.

We can state some simple claims concerning the properties of relation A.

Lemma 1. *$X^A \neq X$.*

Proof. Assume, by contradiction, that $X^A = X$. Hence, for any candidate x, there exists a y for which yAx, so that $N(y, \dots, x) > N(x, y) + N(\cdot, y, x, \cdot)$ (1). Since $X^A = X$, there must exist a z with zAy which implies that $N(z, \dots, y) > N(y, z) + N(\cdot, z, y, \cdot)$ (2). However, note that $N(x, y) \geq N(z, \dots, y)$ as $N(z, \dots, y)$ stands for the share of voters who rank z first and y last so that x is preferred to y in these profiles. Moreover, remark that $N(y, z) \geq N(y, \dots, x)$ by an analogous reasoning. Combining these two inequalities with (1) leads to the following inequality,

$$N(y,z) \geq N(y,\ldots,x) > N(x,y) + N(\cdot,y,x,\cdot) \geq N(z,\ldots,y) + N(\cdot,y,x,\cdot),$$

a contradiction with (2). □

The following lemma will be useful throughout:

Lemma 2. *If the voters use weakly undominated strategies, then:*

$$xAy \iff s_x(v) > s_y(v) \text{ for any } v.$$

Proof. Take a pair of alternatives x and y. Assume that $s_x(v) > s_y(v)$ for any v when voters use weakly undominated strategies. This is equivalent to assume that the minimal score of x is higher than the maximal score of y (denoted $\underline{s_x} > \overline{s_y}$). Under AV, it is weakly undominated for a voter to approve of her top preferred candidate and never vote her worst preferred one.

Therefore, using weakly undominated strategies, the minimal score for a candidate x equals the number of voters who have x as their top-preferred candidate (denoted $N(x,\ldots)$). Note that these voters can be divided between those who rank y last ($N(x,\ldots,y)$) and those who do not ($N(x,\ldots,y^c)$, so that $N(x,\ldots) = N(x,\ldots,y) + N(x,\ldots,y^c)$.

Similarly, the maximal score of a candidate y equals the total number of voters minus the number of voters who rank this candidate last ($n - N(\ldots,y)$). Again, all the voters who rank y over x ($N(y,x)$) are in this group. The rest of the voters in this group so are the ones who rank x over y without the voters who rank y last ($N(x,y,\cdot)$) so that $n - N(\ldots,y)) = N(y,x) + N(x,y,\cdot)$.

Hence, we can write that:

$$\begin{aligned}\underline{s_x} > \overline{s_y} &\iff N(x,\cdots) > n - N(\cdots,y)) \\ &\iff N(x,\ldots,y) + N(x,\ldots,y^c) > N(y,x) + N(x,y,\cdot).\end{aligned}$$

Removing the voters who rank x first and y not last from both sides of the inequality (in other words, $N(x,\ldots,y^c) = N(x,y,\cdot)$) leads to

$$\underline{s_x} > \overline{s_y} \iff N(x,\ldots,y) > N(y,x) + N(\cdot,x,y,\cdot).$$

Since this is the precise definition of xAy, this concludes the proof. □

More formally, the previous lemma can be restated as follows:

$$y \in X^A \Longleftrightarrow X^{'} \notin W^1 \text{ for any } X' \subseteq X \text{with } y \in X'. \tag{2}$$

In other words, any set of candidates in which $y \in X^A$ is included can not be in the winning set W^1, which corresponds to the possible winning sets in the game Γ^1 obtained after one step of *msd*. Our condition coincides with the one given by Brams and Sanver [5] when $k = 3$. This is not surprising since they prove a characterization of a candidate being an AV winner, provided that the voters use sincere and undominated strategies (which is the case in our setting with just three alterrnatives).

3.1 The Approval Domain

Our analysis first focuses on a three candidates scenario which helps to present the main intuition for the many-alternatives case.

Voters' preferences are strict so that we divide the voters into six groups as follows:

N_1	N_2	N_3	N_4	N_5	N_6
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a
n_1	n_2	n_3	n_4	n_5	n_6

with for example N_1 be the set of voters i with preference ordering $a \succ_i b \succ_i c$, with $\#N_1 = n_1$.

As condition (1) holds, we apply first one step of *msd* and so *w.l.o.g.* we focus on $\Gamma^1 = (u, V^1)$ the game in which every voter chooses among her undominated strategies in Γ. As we focus on three candidates, each voter i has only two weakly undominated strategies under AV: either simple-voting for her top-ranked candidate (t_i) or double-voting for her first two top-ranked candidates (d_i). Note that W^1 is the winning set associated to Γ^1 with

$$W^1 \in \{\{a\}; \{b\}; \{c\}; \{a,b\}; \{a,c\}; \{b,c\}; \{a,b,c\}\}.$$

Hence, the scores of the candidates under undominated AV voting strategies (i.e.

any vote distribution in V^1) can be written for any vote distribution v as follows with

$$\begin{aligned} s_a(v) &= n_1 + n_2 + \eta_3(v) + \eta_5(v) \\ s_b(v) &= n_3 + n_4 + \eta_1(v) + \eta_6(v) \\ s_c(v) &= n_5 + n_6 + \eta_2(v) + \eta_4(v), \end{aligned}$$

with $\eta_j(v) \in \{0, \ldots, n_j\}$ with $j = 1, \ldots, 6$. Note that for a given candidate the sum of the n_j accounts for the votes he gets from simple-voting whereas the sum of η_j stands for the votes which stem from double-voting. For a given Γ^1 the n_j are constant whereas the η_j depend on the vote profile.

We can now state a simple sufficient condition for an ordinal game to be OS.

Theorem 1. *If $X^A \neq \emptyset$, then the ordinal game $\Gamma_{\succ}$ is ordinally solvable.*

Proof. Take a game $\Gamma = (u, V)$. After one step of *msd*, the set of winners in the game $\Gamma^1 = (u, V^1)$ equals the set of AV winners under sincere and undominated strategies. If $X^A \neq \emptyset$, then there is some candidate which is not present in any winning set W^1 by (2).

If two candidates are in X^A, then there is a unique winning set that coincides with $X \setminus X^A$. The claim is proved as then the game is CS, as wanted. Moreover, the argument does not hinge on the voters' cardinal utilities proving that the game is OS.

If there is just one candidate in X^A, then let $X^A = \{c\}$ *w.l.o.g*. Therefore $W^1 = \{\{a\}, \{b\}, \{a, b\}\}$. Any strategy that assigns one point to c is redundant to any strategy $v_i \in V^1$ that assigns no point to it. We remove all the strategies in which voters assign one point to c as their second-ranked candidate[4]. Note that this has no impact of iterated removal of weakly dominated strategies due to condition (1). The game after removal of redundant strategies is denoted $\Gamma^2 = (u, V^2)$.

Consider two vote profiles v and v' with $v = (t_i, v_{-i})$ and $v' = (d_i, v_{-i})$. Note that the only difference between both profiles is that voter i switches from simple to double voting. Hence, as we are in Γ^2, this change might only add one point to the voter's second ranked candidate, that is either a or b. The proof is now divided in two cases:

[4]If we remove all the strategies that assign a score of 1 to c, we might modify the game as some voters vote for c and for some other candidate. Indeed, voters of groups N_5 and N_6 always assign one point to c and maybe one point to either a or b in Γ^1. Hence, one cannot ensure that removing their strategies from the game does not affect the winning set since they might assign a positive score to both a and b.

Case I. Suppose first that for some voter i, there exists $v_{-i} \in V^2_{-i}$ for which $U_i(v) \neq U_i(v')$. Then it must be the case that $W(v) \neq W(v')$. Consider first that $a \succ_i b \succ_i c$. As $t_i = (1,0,0)$ and $d_i = (1,1,0)$, switching from t_i to d_i adds one point to b.
If $W(v) = \{a\}$, then it can only be the case that $W(v') = \{a,b\}$.
If $W(v) = \{b\}$, this is a contradiction since adding one point to b cannot change the winning set.
If $W(v) = \{a,b\}$, then it follows that $W(v') = \{b\}$.
As voter i prefers a over b, she weakly prefers to choose t_i.

The same argument applies to each possible preference ordering implying that no voter is indifferent between t_i and d_i so that the game is CS.

Case II. Suppose now that there is some voter i for which $U_i(v) = U_i(v')$ for any $v, v' \in V^2$. Then it must be the case that $W(v) = W(v')$.
II.1 If $W(v)$ is the same one for any $v \in V^2$, then W^2 is a singleton so that the game is CS by definition.

II.2 Suppose now that there exist v_{-i} and v'_{-i} with $W(t_i, v_{-i}) = W(d_i, v_{-i})$, $W(t_i, v'_{-i}) = W(d_i, v'_{-i})$ and $W(t_i, v_{-i}) \neq W(t_i, v'_{-i})$.
Note that in Γ^2, switching from t_i to d_i can only alter the score of a and b.

a. Assume first that $W(t_i, v_{-i}) = \{a,b\}$ for some v_{-i} in V^2_{-i}. If a is the second-ranked candidate then $W(t_i, v_{-i}) = \{a,b\} \Longrightarrow W(d_i, v_{-i}) = \{a\}$ entailing a contradiction. The same contradiction arises if b is the second-ranked candidate.

A similar claim applies when $W(t_i, v'_{-i}) = \{a,b\}$.

b. Suppose now that $W(t_i, v_{-i}) \neq \{a,b\}$ for any v_{-i} in V^2_{-i}.
If $W(t_i, v_{-i}) = \{a\}$ and $W(t_i, v'_{-i}) = \{b\}$, then $s_a(t_i, v_{-i}) > s_b(t_i, v_{-i})$ and $s_a(t_i, v'_{-i}) < s_b(t_i, v'_{-i})$. By Lemma 7 in the appendix, there must exist some distribution v''_{-i} with $W(t_i, v''_{-i}) = \{a,b\}$. However, as previously, one can prove that there is a contradiction since adding an extra-point for a or b breaks the tie.

The same logic applies if $W(t_i, v_{-i}) = \{b\}$ and $W(t_i, v'_{-i}) = \{a\}$.

In all the previous cases, either the game is DS or there is a contradiction, proving the claim. Finally, as the only driving force of the proof is the voters' ordinal preferences, we can conclude that the game $\Gamma_{\succ}$ is OS.

□

3.2 Comparison to the previous sufficient conditions.

Buenrostro et al. [6] provide sufficient conditions for the dominance solvability of AV games with just three candidates. Indeed, they focus on conditions in terms of one statistic of the game: sufficient agreement on the best candidate or on the worst candidate. We now prove that our condition (i.e. $X^A \neq \emptyset$) is strictly weaker than theirs. Indeed, any game satisfying conditions on the agreement on the best or on the worst candidate satisfy ours (Lemmata 3 and 4) whereas the contrary need not be true (Example 2).

We now summarize their two conditions and prove how they relate to ours.

Condition on the best candidate:

Their first condition states that if more than 2/3 of the voters have the same top-ranked candidate, the game is dominance solvable. So if a is this candidate, the condition can be written as follows:

$$\text{If } n_1 + n_2 > \frac{2}{3}n, \text{ then the game is dominance solvable } (CBest).$$

Lemma 3. *If CBest holds, then* $X^A \neq \emptyset$*.*

Proof. Take a game in which $CBest$ holds for a so that $n_1 + n_2 > \frac{2}{3}n$. It follows that $\sum_{i=3}^{6} n_i < \frac{1}{3}n$ as $\sum_{i=1}^{6} n_i = n$ by definition. Moreover, it must be the case that either $n_1 > \frac{1}{3}n$ or $n_2 > \frac{1}{3}n$ in order to ensure that $n_1 + n_2 > \frac{2}{3}n$. Assume first that $n_1 > \frac{1}{3}n$. Then, $n_1 > \frac{1}{3}n > \sum_{i=4}^{6} n_i$. However, $n_1 = N(a,b,c)$ and $\sum_{i=4}^{6} n_i = N(c,a)$ so that the previous inequality is equivalent to $N(a,b,c) > N(c,a)$. Therefore, aAc so that $X^A \neq \emptyset$ as wanted. A similar claim holds if $n_2 > \frac{1}{3}n$, finishing the proof. □

Condition on the worst candidate:

Their second condition states that if more than 2/3 of the voters have the same worst-ranked candidate, the game is dominance solvable. So if c is this candidate, the condition can be written as follows:

$$\text{If } n_1 + n_3 > \frac{2}{3}n, \text{ then the game is dominance solvable. } (CWorst)$$

Lemma 4. *If CWorst holds, then* $X^A \neq \emptyset$*.*

Proof. Take a game in which $CWorst$ holds for c so that $n_1 + n_3 > \frac{2}{3}n$. It follows that $n_2 + n_4 + n_5 + n_6 < \frac{1}{3}n$ as $\sum_{i=1}^{6} n_i = n$ by definition. Moreover, it must be the case that either $n_1 > \frac{1}{3}n$ or $n_3 > \frac{1}{3}n$ in order to ensure that $n_1 + n_3 > \frac{2}{3}n$. Assume first that $n_1 > \frac{1}{3}n$. Then, $n_1 > \frac{1}{3}n > n_4 + n_5 + n_6$, so that the previous inequality is equivalent to $N(a,b,c) > N(c,a)$. Therefore, aAc so that $X^A \neq \emptyset$ as wanted. A similar claim holds if $n_3 > \frac{1}{3}n$, concluding the proof. □

The next example proves that $X^A \neq \emptyset$ does not imply $CBest$ and $CWorst$.

Example 2: $X^A \neq \emptyset$ **need not imply neither** $CBest$ **nor** $CWorst$**.**

Let $n = 5$ and $X = \{a,b,c\}$ and consider the preference ordering $\succ$ given by

$$\begin{aligned} i = 1,2,3 &\quad : \quad a \succ_i b \succ_i c \\ i = 4,5 &\quad : \quad c \succ_i b \succ_i a \end{aligned}$$

This example is discussed by Buenrostro et al. [6] in order to prove that their sufficient conditions are not necessary. Indeed, only 3/5 of the voters agree on the best and on the worst candidate and, nevertheless, the game is dominance solvable. Note that in this game, $N(a,b,c) > N(c,a)$ so that $X^A \neq \emptyset$ proving that our sufficient condition for dominance solvability is weaker than theirs.

4 The weak Approval Domain: a Necessary Condition

In this section, we derive necessary condition for an ordinal game to be ordinally solvable with just three candidates. Indeed if the game is OS then the Approval domain can not be empty, as presented by the next result.

Theorem 2. *If an ordinal game $\Gamma_{\succ}$ is ordinally solvable, then $X^A \neq \emptyset$*

The proof of this theorem is the purpose of this section. The three main steps of the proof are the following ones:

Step 1: Proposition 1 shows that if $X^{\omega A} = \emptyset$, then the game is not CS. Therefore if $X^{\omega A} = \emptyset$ for some game Γ, then the ordinal game $\Gamma_{\succ}$ to which Γ belongs is not OS.

Step 2: Consider the games with $X^A = \emptyset$ and $X^{\omega A} \neq \emptyset$. These games might be CS (as shown in Example 1) but need not (see Example 3). Proposition 2 proves that none of these games is OS.

Step 3: Since the ordinal game is not OS when $X^{\omega A} = \emptyset$ or when $X^{\omega A} \neq \emptyset$ and $X^A = \emptyset$, it follows that if an ordinal game is OS then $X^A \neq \emptyset$.

Proposition 1. *If $X^{\omega A} = \emptyset$, then the game is not cardinally solvable.*

Proof. Take a game with $X^{\omega A} = \emptyset$ (and hence $X^A = \emptyset$). By definition, $x\omega Ay \iff s_x(v) \geq s_y(v)$ for any $v \in V^1$. As there is no candidate in the set $X^{\omega A}$, for any pair of candidates x and y, there exist two vote distributions v and v' in V^1 with $s_x(v) > s_y(v)$ and $s_x(v') < s_y(v')$. Therefore, Lemma 7, included in the appendix, implies that all the winning sets are possible in V^1.

We focus on the game Γ^1 after one step of *msd*. *W.l.o.g.* take a voter i with $a \succ_i b \succ_i c$ so that $t_i = (1,0,0)$ and $d_i = (1,1,0)$.

As all the winning sets are possible in V^1, there must exist some $v_{-i} \in V^1_{-i}$ for which either $W(t_i, v_{-i}) = \{a,b\}$ or $W(d_i, v_{-i}) = \{a,b\}$.
If $W(t_i, v_{-i}) = \{a,b\}$ then as switching from t_i to d_i adds one point to b, then $W(d_i, v_{-i}) = \{b\}$. Hence, given v_{-i}, the voter i strictly prefers t_i.
If $W(d_i, v_{-i}) = \{a,b\}$ then as switching from d_i to t_i removes one point to b, then $W(t_i, v_{-i}) = \{a\}$. Hence, given v_{-i}, the voter i strictly prefers t_i.

Similarly, there must exist some $v'_{-i} \in V^1_{-i}$ for which either $W(t_i, v'_{-i}) = \{b,c\}$ or $W(d_i, v'_{-i}) = \{b,c\}$. If $W(t_i, v'_{-i}) = \{b,c\}$ then $W(d_i, v'_{-i}) = \{b\}$. Hence, given v'_{-i}, the voter i strictly prefers d_i. Again, if $W(d_i, v'_{-i}) = \{b,c\}$, then $W(t_i, v'_{-i}) = \{c\}$ so that the voter prefers d_i.

Hence, the voter i has no weakly dominated strategy finishing the proof. □

Example 3: $X^A = \emptyset$ and $X^{\omega A} \neq \emptyset$ need not imply that the game is CS.

As shown by Example 1, a game with $X^A = \emptyset$ and $X^{\omega A} \neq \emptyset$ might be cardinally solvable. To see that this type of games need not be CS take the following game. Let $n = 5$ and $X = \{a,b,c\}$ and consider the preference ordering $\succ$ given by

$$\begin{aligned} i = 1,2: &\quad a \succ_i b \succ_i c \\ i = 3: &\quad a \succ_i c \succ_i b \\ i = 4,5: &\quad c \succ_i b \succ_i a. \end{aligned}$$

Note that $a\omega Ac$ as $N(a, \ldots, c) = N(c, a)$.

Take first $i = 1$ and assume that $u_i = (3,2,0)$. If $s_{-i} = (2,3,2)$, then the voter prefers t_i to d_i as $U_i(t_i, v_{-i}) = 5/2 > U_i(d_i, v_{-i}) = 2$. On the contrary, if $s_{-i} = (2,3,3)$, then the voter prefers d_i to t_i as $U_i(t_i, v_{-i}) = 5/3 < U_i(d_i, v_{-i}) = 2$. The same reasoning applies to $i = 2$ proving that neither of these voters has a weakly dominated strategy in Γ^1.

Take now $i = 3$ and assume that $u_i = (3,0,2)$. If $s_{-i} = (2,2,2)$, then the voter prefers t_i to d_i as $U_i(t_i, v_{-i}) = 3 > U_i(d_i, v_{-i}) = 5/2$. If we assume that $s_{-i} = (2,3,2)$, then the voter prefers d_i to t_i as $U_i(t_i, v_{-i}) = 3/2 < U_i(d_i, v_{-i}) = 5/3$. This proves that the voter 3 has no weakly undominated strategy.

Finally, take $i = 4$ with $u_i = (0,1,3)$. Note that when $s_{-i} = (3,3,1)$, then the voter prefers d_i to t_i as $U_i(t_i, v_{-i}) = 1/2 < U_i(d_i, v_{-i}) = 1$. However, when the score $s_{-i} = (3,3,2)$, then it follows that $U_i(t_i, v_{-i}) = 4/3 > U_i(d_i, v_{-i}) = 1$ proving that this voter has no weakly undominated strategy in this game. The same claim applies to $i = 5$ by symmetry provided that she has the same utility profile.

Therefore, no voter has weakly dominated strategies after one step of *msd* proving that the game is not CS.

Note that in the game Γ^1, the minimal score for a candidate x equals the number of voters who have x as their top-preferred candidate. Similarly, the maximal score of a candidate x equals the total number of voters minus the number of voters who rank this candidate last. Let us recall that $\underline{s_j}$ and $\overline{s_j}$ respectively denote $\min_{V^1} s_j(v)$ and $\max_{V^1} s_j(v)$.

Using the previous notation, the set of possible score vectors $S = \{s = (s_j(v))_{j \in X} | v \in V^1\}$ can be characterized as follows:

$$s \in S \Longleftrightarrow s_j(v) \in \{\underline{s_j}, \ldots, \overline{s_j}\}. \tag{3}$$

While the sufficient condition ($\Longrightarrow$) holds by definition, the necessary condition ($\Longleftarrow$) is slightly more technical. Indeed, note under AV whether a voter approves of a candidate does not imply any restriction on the scores of the rest of the candidates (unlike under plurality voting). Hence it follows that any score for each possible candidate (between $\underline{s_j}$ and $\overline{s_j}$) is possible, independently of the scores of the rest of the candidates. In other words, for any possible vector $s \in \mathbb{N}^3$ with $s_j \in \{\underline{s_j}, \ldots, \overline{s_j}\}$ for any $j \in X$, we can find a vote profile $v'' \in V^1$ with $s(v'') = s$.

Building on the previous characterization, the next result describes which games can be dominance-solvable without being embedded in an ordinally-solvable ordinal game. As will be shown, these conditions are quite stringent.

Proposition 2. *If $X^A = \emptyset$ and $X^{\omega A} \neq \emptyset$ for some game Γ, then the ordinal game $\Gamma_>$ to which Γ belongs is not OS.*

Proof. Take a game with $X^A = \emptyset$ and $X^{\omega A} \neq \emptyset$.

We focus on the game Γ^1 after one step of *msd*. Assume *w.l.o.g.* that $a\omega Ac$ so that $\underline{s_a} = \overline{s_c}$ as $X^A = \emptyset$. Therefore if c is in the winning set so is a which implies that the possible winning sets are as follows:

$$W^1 \subseteq \{\{a\},\{b\},\{a,b\},\{a,c\},\{a,b,c\}\}.$$

Note that Lemma 8 ensures that $\{a,b,c\} \in W^1$.

The proof is now structured in three cases. The first two cases analyze the problem when $\{a,b\} \notin W^1$ and when $\{a,c\} \notin W^1$. Finally, the third case focuses on the situation in which both $\{a,b\}$ and $\{a,c\}$ belong to W^1.

Case I.

We assume in this case that there is not a vote profile $v \in V^1$ for which $W(v) = \{a,b\}$. Given that $a\,\omega Ac$, there are three cases: $a\omega Ab$, $b\omega Aa$ or neither of both.

I.1 If $a\omega Ab$, then $\underline{s_a} = \overline{s_b}$ so that $\underline{s_a} = \overline{s_b} = \overline{s_c}$. As there is no v for which $W(v) = \{a,b\}$, then we must have $\underline{s_c} = \overline{s_c}$. Indeed, if $\underline{s_c} < \overline{s_c}$, by (3), there must exist a vector v' for which $s_a(v') = s_b(v') > s_c(v')$, a contradiction.

As $\underline{s_c} = \overline{s_c}$, the candidate c is ranked either first or last by all voters since its minimal and maximal scores coincide. However, we have assumed that $a\omega Ab$ so that $N(a,\ldots,b) = N(b,a)$. Since no voter ranks c second, then $N(a,\ldots,b) = 0$ so that $N(b,a) = 0$ in order to satisfy the equality. Hence, there are only two preference orderings $a \succ_i b \succ_i c$ and $c \succ_i a \succ_i b$ with the same number of voters (say m) for each ordering as $a\omega Ac$. So the game has $2m$ voters with the voters $i = 1,\ldots,m$ with $a \succ_i b \succ_i c$ and the voters $i = m+1,\ldots,2m$ with $c \succ_i a \succ_i b$.

Take a voter with preferences $a \succ_i b \succ_i c$. The only vote profile v_{-i} in which she is not indifferent between voting $t_i = \{a\}$ and $d_i = \{a,b\}$ is $v_{-i} = (m-1, m-1, m)$ since in the rest either a wins alone or both a and c win. If $v_{-i} = (m-1, m-1, m)$, then $U_i(t_i, v_{-i}) > U_i(d_i, v_{-i}) \iff \frac{u_i(a)+u_i(c)}{2} > u_i(b)$. By symmetry, the same reasoning applies to all the voters with these preferences.

Take now a voter with preferences $c \succ_i a \succ_i b$. For any utility representation, the voter strictly prefers to vote $t_i = \{c\}$ than $d_i = \{c,a\}$ when v_{-i} equals $(m, p, m-1)$ for any $0 \le p \le m-1$. Similarly, she prefers to vote t_i to d_i when $v_{-i} = (m, m, m-1)$ iff $\frac{u_i(c)+u_i(b)}{2} > u_i(a)$. For the rest of the vote profiles, she is indifferent. The same applies for the voters with identical ordinal preferences.

Assume first that $\frac{u_i(a)+u_i(c)}{2} > u_i(b)$ for any $i = 1,\ldots,m$ and that $\frac{u_i(c)+u_i(b)}{2} > u_i(a)$ for any $i = m+1,\ldots,2m$. Hence, the game is CS and the voters $i = 1,\ldots,m$ vote $\{a\}$

whereas the rest of the voters vote $\{c\}$, implying that $W^\infty = \{a, c\}$.

Assume now that $\frac{u_i(a)+u_i(c)}{2} < u_i(b)$ for any $i = 1, \ldots, m$ and that $\frac{u_i(c)+u_i(b)}{2} < u_i(a)$ for any $i = m+1, \ldots, 2m$. Hence, the voters $i = 1, \ldots, m$ vote $\{a, b\}$ as $\{a\}$ becomes a weakly dominated strategy whereas the rest of the voters do not have any weakly dominated strategy. After one new step of msd (i.e assuming that the voters $i = 1, \ldots, m$ vote $\{a, b\}$) then the voters $i = m+1, \ldots, 2m$ vote $\{c, a\}$ as it becomes weakly dominant. Hence $W^\infty = \{a\}$ since a gets a score of $2m$ whereas the rest of the candidates get a score of m.

This proves that the game cannot belong to an OS ordinal game since we have proved that W^∞ might depend on the utility levels of the voters.

I.2 If $b\omega Aa$, then $\underline{s_b} = \overline{s_a}$ so that $\underline{s_b} = \overline{s_a} \geq \underline{s_a} = \overline{s_c}$ as $a\omega Ac$. It follows that $\overline{s_a} = \underline{s_a}$. Indeed, if $\overline{s_a} > \underline{s_a}$, then $\underline{s_b} > \overline{s_c}$ which is equivalent bAc, a contradiction as $X^A = \emptyset$. Moreover, as there is no v for which $W(v) = \{a, b\}$, then we must have $\underline{s_c} = \overline{s_c}$. Indeed, if $\underline{s_c} < \overline{s_c}$, by (3), there must exist a vector v' for which $s_a(v') = s_b(v') > s_c(v')$, a contradiction.

Therefore, we have $\underline{s_b} = \overline{s_a} = \underline{s_a} = \overline{s_c} = \underline{s_c}$. As the minimal and maximal scores of both candidates a and c coincide, a and c can only be ranked either first or last. This implies that b is never ranked first so that $\underline{s_b} = 0$. However, in order to satisfy the inequality we must have that $\underline{s_a} = \underline{s_c} = 0$, a contradiction as by definition all voters always approve of their preferred candidate.

I.3 Suppose finally that neither $a\,\omega Ab$ nor $b\,\omega Aa$. Therefore, we have $\underline{s_a} < \overline{s_b}$ and $\underline{s_b} < \overline{s_a}$. Hence, due to (4), it follows that $\underline{s_b} \leq \underline{s_a} < \overline{s_b}$. Moreover, $\underline{s_a} = \overline{s_c}$ as $a\,\omega Ac$.

First suppose that $\underline{s_b} = \underline{s_a}$. Notice that we must have $\underline{s_a} = \overline{s_c}$ and $\underline{s_b} < \overline{s_a}$ by assumption. Hence, combining the previous expressions, it follows that

$$\underline{s_a} = \overline{s_c} = \underline{s_b} < \overline{s_a}.$$

Hence, due to (3), there must exist some v for which $W(v) = \{a, b\}$, a contradiction.

Now if $\underline{s_b} < \underline{s_a}$, there are two possibilities: either $\underline{s_a} = \overline{s_a}$ or $\underline{s_a} < \overline{s_a}$. If the latter holds, then again by (3), there must exist some v for which $W(v) = \{a, b\}$, a contradiction. On the contrary, assume that $\underline{s_a} = \overline{s_a}$. Then a cannot be ranked second so that $\underline{s_a} = \overline{s_a} = n_1 + n_2$ and $\overline{s_c} = n_2 + n_4 + n_6$. Since $\underline{s_a} = \overline{s_c}$ as $a\,\omega Ac$, then $n_1 + n_2 = n_2 + n_4 + n_6$ so that $n_1 = n_4 + n_6$. However, note that $n_1 = n_4 + n_6$ is equivalent to $a\omega Ab$ in this case, which does not hold by assumption.

This concludes the case I.

Case II.

We assume in this case that there is not a vote profile $v \in V^1$ for which $W(v) = \{a, c\}$. Given that $a\,\omega Ac$, there are three cases: $a\omega Ab$, $b\omega Aa$ or neither of both.

II.1 If $a\omega Ab$, then $\underline{s_a} = \overline{s_b}$ so that $\underline{s_a} = \overline{s_b} = \overline{s_c}$. As there is no v for which $W(v) = \{a, c\}$, then we must have $\underline{s_b} = \overline{s_b}$. Indeed, if $\underline{s_b} < \overline{s_b}$, by (3), there must exist a vector v' for which $s_a(v') = s_c(v') > s_b(v')$, a contradiction.

As $\underline{s_b} = \overline{s_b}$, the candidate b is ranked either first or last by all voters. However, we have assumed that $a\omega Ac$ so that $N(a,\ldots,c) = N(c,a)$. Since there is no voter without b ranked either first or last then $N(a,\ldots,c) = 0$ so that $N(c,a) = 0$ in order to satisfy the equality. Hence, there are only two preference orderings $a \succ_i c \succ_i b$ and $b \succ_i a \succ_i c$ with the same number of voters (say m) for each ordering as $a\omega Ab$.

The same argument as the one in case I.A proves that the game cannot belong to an OS ordinal game.

II.2 If $b\omega Aa$, then $\underline{s_b} = \overline{s_a}$ so that $\underline{s_b} = \overline{s_a} \geq \underline{s_a} = \overline{s_c}$ as $a\omega Ac$. It follows that $\overline{s_a} = \underline{s_a}$. Indeed, if $\overline{s_a} > \underline{s_a}$, then $\underline{s_b} > \overline{s_c}$ which is equivalent bAc, a contradiction as $X^A = \emptyset$.

Therefore, we have $\underline{s_b} = \overline{s_a} = \underline{s_a} = \overline{s_c} \geq \underline{s_c}$. As the minimal and maximal scores of candidate a coincides, a is ranked either first or last. Therefore, there only four possible preferences orderings $a \succ_i b \succ_i c$ (n_1 voters), $a \succ_i c \succ_i b$ (n_2 voters), $b \succ_i c \succ_i a$ (n_4 voters) and $c \succ_i b \succ_i a$ (n_6 voters). Since $N(a,\ldots,c) = N(c,a)$ it follows that $n_1 = n_4 + n_6$. Similarly, as $N(b,\ldots,a) = N(a,b)$, we must have $n_4 = n_1 + n_2$. However, both equalities together imply that $n_1 = n_1 + n_2 + n_6$ so that $n_2 + n_6 = 0$, which implies that both n_2 and n_6 equal zero. Hence there are only two preference orderings $a \succ_i b \succ_i c$ (n_1 voters) and $c \succ_i b \succ_i a$ (n_6 voters). Again, since $a\omega Ac$, $n_1 = n_6 = m$. However, this game does not belong to an OS ordinal game since the only pivotal event involves the three-way tie (as in Example 1).

II.3 Suppose finally that neither $a\,\omega Ab$ nor $b\,\omega Aa$. Therefore, we have $\underline{s_a} < \overline{s_b}$ and $\underline{s_b} < \overline{s_a}$. Hence, due to (4), it follows that $\underline{s_b} \leq \underline{s_a} < \overline{s_b}$. Moreover, $\underline{s_a} = \overline{s_c}$ as $a\,\omega Ac$.

If $\underline{s_b} < \underline{s_a}$, then due to (3), there must exist a vote profile v for which $W(v) = \{a, c\}$, a contradiction.

If, on the contrary, $\underline{s_b} = \underline{s_a}$, then $\overline{s_c} = \underline{s_b} = \underline{s_a}$. Note that it must be the case that $\underline{s_a} < \overline{s_a}$.

We must have $\{a, b\} \in W^1$ as otherwise there is a contradiction as proved by case I. As by assumption $\{a, c\} \notin W^1$, the voters in the groups N_2 and N_4 have always a weakly dominated strategy. Indeed, they are indifferent between t_i and d_i if the pivot event equals $\{a, b\}$ as the only difference between both ballots is adding one point to candidate c. Moreover, if the pivot event involves the three candidates, they vote for d_i if and only if $\frac{u_i(a)+u_i(b)}{2} < u_i(c)$. We assume that this is the case.

Hence, we can remove the strategies t_i for both N_2 and N_4 voters which leads to the game $\Gamma^2 = (u, V^2)$.

Hence it must be the case that $s_c(v) = \overline{s_c}$ for any $v \in V^2$. As $\underline{s_a} = \underline{s_b} = \overline{s_c}$, there must exist a $v \in V^2$ for which $s_a(v) = s_b(v) = s_c(v)$. Moreover, as $\overline{s_c} < \overline{s_a}, \overline{s_b}$, we must also have some $v' \in V^2$ for which $s_a(v') = s_b(v') > s_c(v')$.

Take a voter in the N_1 group with preferences $a \succ_i b \succ_i c$. She prefers t_i to d_i whenever $W(t_i, v_{-i}) = \{a, b\}$ or $W(d_i, v_{-i}) = \{a, b\}$. Note that one of these two vote profiles must exist in V^2. On the contrary, she prefers d_i to t_i when $W(t_i, v_{-i}) = \{a, b, c\}$ if and only if $\frac{u_i(a)+u_i(c)}{2} < u_i(b)$. Hence, this voter has no weakly dominated strategy. We assume that this is the case for all the voters in N_1.

Moreover, for any $i \in N_3$, we let $\frac{u_i(b)+u_i(c)}{2} < u_i(a)$.

For any $i \in N_5$, we let $\frac{u_i(b)+u_i(c)}{2} > u_i(a)$.

For any $i \in N_6$, we let $\frac{u_i(a)+u_i(c)}{2} > u_i(b)$.

The previous inequalities imply that none of these voters have a weakly dominated strategy. This proves that the game Γ is not CS and hence the ordinal game $\Gamma_\succ$ to which Γ belongs cannot be OS.

Case III.

We assume in this case that $\{\{a, b\}, \{a, c\}, \{a, b, c\}\} \subseteq W^1$. Hence, all voters face an unambiguous pivot event which is independent of their utility levels (the two way ties) and an ambiguous one which involves the three way tie. Hence, for some utility level, each voter does not have a weakly dominated strategy, proving that the game cannot belong to an OS ordinal game.

□

For any preference ordering $\succ$, let $\Gamma_\succ$ be an ordinal game. The set $U_\succ$ represents the set of cardinal utilities that represent $\succ$.

Corollary 1. *Let $\Gamma_\succ$ be an ordinal game with $X^A = \emptyset$ and $X^{\omega A} \neq \emptyset$. Then there is a non-empty set of utilities $\hat{U} \subseteq U_\succ$ for which any $\Gamma = (V, u)$ with $u \in \hat{U}$ is CS.*

Proof. The proof of Theorem 2 analyzes the different games for which $X^A = \emptyset$ and $X^{\omega A} \neq \emptyset$. In each of the different cases, either the game does not exist or the game is CS for some utility levels proving the claim. □

5 Concluding discussion

This section comments on the main remaining aspects of dominance solvability that have not been addressed so far: the extension of our results to a many-alternatives scenario and the Condorcet consistency of dominance solvability.

5.1 The general case

We now extend our results to the games with any number of candidates. The main difference is that it is not anymore the case that sincerity is weakly undominated. Hence, the voters might approve of some candidate without approving of a most preferred candidate.

Theorem 3. *If $\#X^A \geq k-2$, then the ordinal game $\Gamma_{\succ}$ is ordinally solvable.*

Proof. Take a game $\Gamma = (u, V)$ with $\#X^A \geq k-2$. After one step of *msd*, the set of winners in the game $\Gamma^1 = (u, V^1)$ equals the set of AV winners under undominated strategies. Note that due to (2), any set of candidates which includes some candidate in X^A does not belong to the winning set W^1.

Since $X^A \neq X$ (Lemma 1), there are just two cases.

If $k-1$ candidates are in X^A, then the unique winning set corresponds to the victory of the unique candidate not present in X^A proving that the game is CS.

If, on the contrary, there are $k-2$ candidates are in X^A, then there are only two candidates (denoted x and y) who can win the election. Due to Lemma 7 (see the appendix), there must exist some vote profile v in which both x and y are tied. Moreover, this the only possible pivot event. Hence, since each voter has strict preferences over x and y, each voter approves of the one she prefers among both (and her top preferred alternative). Therefore, there is a unique winning set proving that the game is CS, as wanted. Moreover, the argument does not hinge on the voters' cardinal utilities proving that the game is OS. □

Theorem 4. *If $X^{\omega A} = \emptyset$, then the game is not dominance solvable.*

Proof. The proof is analogous to the one of Proposition 1. Indeed, since $X^{\omega A} = \emptyset$, there is a vote profile in which each candidate can win alone. Hence, due to Lemma 7 (see the appendix), for each pair of candidates x and y, there is a vote profile in which both candidates are tied. Moreover, due to (3), there must exist a vote profile in which both candidates are tied and with a score strictly above of the one corresponding to the rest of the candidates. Hence, no voter has a weakly undominated strategy since approving of a candidate might be pivotal against a most preferred candidate or a less preferred one. This proves that the game is not dominance solvable and concludes the proof. □

We conjecture that if $X^A = \emptyset$ and $X^{\omega A} \neq \emptyset$, then the game cannot be OS but might be CS some non-empty set of utilities. In other words, Proposition 2 holds for any number of candidates. However, the proof seems to be technically challenging. Moreover, the boundary of the number of candidates in Theorem 3 is tight: that is, if $\#X^A < k-2$, the game need not be CS and hence the ordinal game in which it is embedded is not OS. To see why, consider the next example.

Example 4: If $\#X^A < k-2$, then the game need not be CS. Consider the election depicted in Example 3 in which one extra-candidate is added at the bottom of the voters' preferences; this candidate is unanimously disliked. Let $n = 5$ and $X = \{a, b, c, d\}$ and consider the preference ordering $\succ$ given by

$$\begin{aligned} i = 1,2: &\quad a \succ_i b \succ_i c \succ_i d \\ i = 3: &\quad a \succ_i c \succ_i b \succ_i d \\ i = 4,5: &\quad c \succ_i b \succ_i a \succ_i d. \end{aligned}$$

We set $u_1 = u_2 = (4,3,1,0)$, $u_3 = (4,1,3,0)$ and $u_4 = u_5 = (1,2,4,0)$.
In this game, $X^A = \{d\}$ so that $\#X^A = 1 = k-3 < k-2$. Moreover, after one step of msd, each voter can approve of each of their three preferred candidates. Note that all the possible strategy combinations depicted in the analysis of Example 3 are present in our game, whereas the contrary need not be true. Indeed, in our game the voters can approve of their three preferred candidates whereas in Example 3 they could only approve their two preferred candidates after one step of msd. Therefore, no voter has a weakly undominated strategy proving that the game is not CS as wanted.

5.2 Welfare Analysis

In Sections 3 and 4, a simple condition on the *(weak) Approval domain* is given in order to ensure OS. The next two lemmata (Lemma 5 and 6) prove that ordinally solvability satisfies Condorcet consistency. In other words, if a game is OS, then the outcome of dominance solvability equals the set of Condorcet Winners. To prove this claim, Lemma 5 proves that whenever the sufficient conditions for OS hold, then the voters' preference profile admits a CW. Building on this result, Lemma 6 proves it must be the Condorcet winner(s) that is to be selected by the procedure of dominance solvability.

We can also notice that (non-ordinal) solvability need not imply the selection of the CW as already proved by Example 1. Indeed, in this example, b might be the unique outcome whereas a and c are the CW.

Lemma 5. *If $\#X^A \geq k-2$, then the game has a CW.*

Proof. Take a game with $\#X^A \geq k-2$. Note that as $X^A \neq X$ (Lemma 1), there are only two possibilities: $\#X^A = k-1$ or $\#X^A = k-2$. Assume by contradiction that there is not a CW.

Assume first that $\#X^A = k-2$. W.l.o.g. we let $X = \{x_1, x_2, \ldots, x_k\}$ and $X^A = \{x_3, \ldots, x_k\}$. It follows that for every x_i with $i = 3, \ldots, k$, there exists some $x_j \in X$ with $x_j A x_i$. We denote the set of A-dominating candidates by $X_A = \{x \in X | xAy \text{for some} y \in X\}$. As previously stated, $x_j A x_i \iff \underline{s_j} > \overline{s_i}$. Take a candidate x_{j^*} with the maximal minimal score among the candidates in X_A: formally,

$$x_{j^*} = \arg\max_{x_j \in X_A} \underline{s_j}.$$

Note that this candidate exists since there is a finite number of candidates in X_A. Since $\underline{s_{j^*}} \geq \underline{s_j}$ and $\underline{s_j} > \overline{s_i}$ for any $x_j \in X_A$ and any $i = 3, \ldots, k$, it follows that

$$\underline{s_{j^*}} > \overline{s_i},$$

which is equivalent to $x_{j^*} A x_i$ for any $i = 3, \ldots, k$.

Assume that $x_{j^*} \in \{x_3, \ldots, x_k\}$. Then, $x_{j^*} A x_{j^*}$, a contradiction. Hence, $x_{j^*} \in \{x_1, x_2\}$.

W.l.o.g we let $x_{j^*} = x_1$. It follows that $x_1 A x_i$ for any $i = 3, \ldots, k$. If moreover $x_1 M x_2$, then x_1 is the CW, a contradiction. Assume, on the contrary, that $x_2 \omega M x_1$. Since $x_2 \omega M x_1$ and $x_1 A x_i$ for any $i = 3, \ldots, k$, Lemma 9 (included in the appendix) entails that $x_2 M x_i$ for any $i = 3, \ldots, k$. In other words, x_2 is the CW, a contradiction.

A similar claim applies when $\#X^A = k-1$, finishing the proof.

□

Lemma 6. *If $\#X^A \geq k-2$, then the outcome W^∞ is a CW.*

Proof. Take a game with $\#X^A \geq k-2$. Then the game is cardinally solvable as shown by Theorem 3. Moreover, the game admits a CW as proved by Lemma 5. The candidate which is in the outcome W^∞ is not in X^A. Note that the CW cannot be in X^A since every alternative in this set is M-dominated.

If $\#X^A = k-1$, then there is a unique winning set after one step of *msd*. Hence, this candidate coincides with the CW of the election.

If $\#X^A = k-2$, then assume *w.l.o.g.* $X \setminus X^A = \{a, b\}$ and aAc. By Theorem 1, the game is CS. Moreover, there are three possible winning sets $\{a\}$,$\{b\}$ and $\{a, b\}$ after one step of *msd*.

Hence, all the voters approve of either a or b (the one they prefer) and they approve their most preferred candidate. Hence the winning set coincides with the set of CW. Indeed, if aMb, then a wins and conversely if bMa then b wins alone. Finally, if $a\omega Mb$ and $b\omega Ma$ then both candidates are weak CW and are tied for victory. □

Note that with $k = 3$, this simply implies that if $\#X^A \neq \emptyset$, then dominance solvability implies Condorcet consistency.

As a conclusion, the outcome of a dominance solvable game need not be a CW. This is a consequence of the game being CS but the ordinal game $\Gamma_{\succ}$ in which the game is embedded not being OS. Therefore, if one considers the whole set of CS games, Condorcet consistency is violated when the weak Approval domain is non empty and the Approval domain is empty. However, when one focuses on the most stringent condition of dominance solvability, the surviving outcome must coincide with the set of CW.

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A Appendix: Proofs

Lemma 7. *Let $k = 3$. Let $\Gamma^1 = (u, V^1)$. If there exists two vote profiles v and v' for which $s_x(v) > s_y(v)$ and $s_x(v') < s_y(v')$, then there must exist a vote profile v'' for which $s_x(v'') = s_y(v'')$*

Proof. W.l.o.g. we prove the claim for $x = a$ and $y = b$. Denote by $\Delta(w) = s_a(w) - s_b(w) = (n_1 + n_2 - n_3 - n_4) - (\eta_3 + \eta_5 - \eta_1 - \eta_6)$ the difference between the scores of candidates a and b for any vote profile w. As $(n_1 + n_2 - n_3 - n_4)$ is constant for a given game, it follows that $\Delta(w)$ can be rewritten as $\Delta(w) = c + f(w)$ with $f(w) \in \{-\eta_1 - \eta_6, ..., \eta_3 + \eta_5\}$ and $c \in \mathbb{R}$. Hence $\Delta(w) \in \{c - \eta_1 - \eta_6, ..., c + \eta_3 + \eta_5\}$.

Take two vote profiles v and v' with $\Delta(v) > 0$ and $\Delta(v') < 0$. It must be the case that $c - \eta_1 - \eta_6 < 0$ and $c + \eta_3 + \eta_5 > 0$ in order to satisfy the definition of $\Delta(w)$. Therefore there must exist some vote distribution v'' for which $\Delta(v'') = 0$, which concludes the proof. □

Lemma 8. *If $X^A = \emptyset$ and $X^{\omega A} \neq \emptyset$, then there is some vote profile in V^1 for which the three candidates are tied, i.e. some $v \in V^1$ with $s_a(v) = s_b(v) = s_c(v)$.*

Proof. Take a game with $X^A = \emptyset$ and $X^{\omega A} \neq \emptyset$. Assume *w.l.o.g.* that $c \in X^{\omega A}$ with $a\,\omega A c$. Then, it follows that $\underline{s_a} \geq \overline{s_c}$. Since $X^A = \emptyset$, there does not exist a pair of candidates x, y with $\underline{s_x} > \overline{s_y}$. Hence, it must be the case that $\underline{s_a} = \overline{s_c}$.

Consider now the pair of candidates a and b. There are three possible cases: either $a\,\omega A b$ (**Case I.**), or $b\,\omega A a$ (**Case II.**) or finally that neither of the previous relations hold (**Case III.**).

Case I. Suppose first $a\,\omega A b$ which is equivalent to $\underline{s_a} \geq \overline{s_b}$. Once again since $X^A = \emptyset$, it must be the case $\underline{s_a} = \overline{s_b}$. Therefore, we have $\underline{s_a} = \overline{s_b} = \overline{s_c}$ so that there exists a vote profile v with $s_a(v) = s_b(v) = s_c(v)$ as wanted.

Case II. Suppose now that $b\,\omega A a$. This relation is equivalent to $\underline{s_b} = \overline{s_a}$ as $X^A \neq \emptyset$. As $\underline{s_a} = \overline{s_c}$ holds, we can write that $\underline{s_b} = \overline{s_a} \geq \underline{s_a} = \overline{s_c}$ since by definition $\overline{s_a} \geq \underline{s_a}$. If $\overline{s_a} = \underline{s_a}$, then $\underline{s_b} = \underline{s_a} = \overline{s_c}$, so that there exists a vote profile for which the three candidates are tied. If, on the contrary, $\overline{s_a} > \underline{s_a}$, it follows that $\underline{s_b} > \overline{s_c}$ so that aAc a contradiction with $X^A = \emptyset$.

Case III. Suppose finally that neither $a\omega Ab$ nor $b\omega Aa$. Then as $X^A = \emptyset$, we can write that

$$\underline{s_a} < \overline{s_b} \text{ and } \underline{s_b} < \overline{s_a}.$$

Given that $\underline{s_a} = \overline{s_c}$, there must be the case that $\underline{s_b} \leq \underline{s_a}$. Indeed, suppose that $\underline{s_b} > \underline{s_a}$. The previous inequality jointly with $\underline{s_a} = \overline{s_c}$ implies that $\underline{s_b} > \overline{s_c}$ implying that bAc, which contradicts $X^A = \emptyset$.

Hence we have that

$$\underline{s_b} \leq \underline{s_a} < \overline{s_b}. \tag{4}$$

Let $\hat{V}$ be the set of vote distributions for which $s_a(v) = s_c(v)$. Note that this set is non-empty as $\underline{s_a} = \overline{s_c}$. We assume that $s_a(v) = p$. Due to (4), we know that $p \in \{\underline{s_b}, \ldots, \overline{s_b} - 1\}$. Moreover, due to (3), for any possible score vector s with $s_j \in \{\underline{s_j}, \ldots, \overline{s_j}\}$, we can find a vote profile $v'' \in V^1$ with $s(v'') = s$. Therefore, there exists a vote profile $v'' \in \hat{V}$ for which $s_a(v'') = s_b(v'') = s_c(v'')$ as wanted, concluding the proof. □

Lemma 9. *For any triple of candidates $x, y, z \in X$:*

1. *if $x\omega My$ and yAz then xMz.*
2. *if xAy and $y\omega Mz$ then xMz.*

Proof. 1. Take a triple of candidates x, y, z with $x\omega My$ and yAz.

Assume by contradiction that zMx so that $N(z, x) > N(x, z)$. The previous inequality is equivalent to :

$$N(z,x,y)+N(z,y,x)+N(y,z,x) > N(x,z,y)+N(x,y,z)+N(y,x,z). \quad (1)$$

However, by assumption yAz, so that $N(y,x,z) \geq N(y,\cdot,z) > N(z,y) \geq N(z,x,y) + N(z,y,x)$ so that for (1) to hold, we need $N(y,z,x) > N(x,y,z) + N(x,z,y)$ (2).

Moreover, we have assumed that $x\omega My$ so that $N(x,y) \geq N(y,x)$, which is equivalent to:

$$N(x,y,z)+N(x,z,y)+N(z,x,y) \geq N(y,x,z)+N(y,z,x)+N(z,y,x). \quad (3)$$

Again, as yAz, it follows that $N(y,x,z) \geq N(y,\cdot,z) > N(z,y) \geq N(x,z,y) + N(z,x,y)$. Hence, in order to ensure that (3) holds, we must have $N(x,y,z) > N(y,z,x)+N(z,y,x)$ (4).

However, there is a contradiction between (2) and (4) proving the claim.

2. The proof of this claim is analogous to the one of 1. and hence is omitted. □