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Positional rules and $q-$Condorcet consistency

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Abstract

A well-known result in Social Choice theory is the following: every scoring rule (positional rules) violates Condorcet consistency. A rule is Condorcet consistent when it selects the alternative that is preferred to every other alternative by a majority of individuals. In this paper, we investigate some limits of this negative result. We expose the relationship between a weaker version of the Condorcet consistency principle and the scoring rules. Our main objective is then to study the condition on the quota that ensure that positional rules (simple and sequential) satisfy this principle.

Keywords Positional rules (Simple and Sequential) . Condorcet Consistency . q-majority

JEL Classification D71,D72

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1 Introduction

A wide literature on voting theory is concerned with the theoretical debate between Condorcet social choice methods on one hand, and positional (or scoring) systems - voting *à la* Borda - on the other hand. The first ones consider, following Condorcet (1785), that the collective choice has to be based on majority duals between alternatives; while the second ones introduced by Borda (1781), suggest to deduce the collective preference from a numerical evaluation taking into account the positions of the alternatives in the orders of individual preferences.

There are several arguments in favor of either type of procedures, discussed in an abundant literature (see Nurmi 1987 and 1999, and Saari 2006 among others). The most significant contributions concerning positional rules are due to Smith (1973), Young (1974, 1975) and Saari (1994). The results presented by these three authors reveal some very natural properties that are satisfied only by positional mechanisms. They are certainlty powerful arguments for adopting a positional approach. Unfortunately we know from Condorcet (1785) that the Borda rule and more generally the positional rules do not necessarily choose the Condorcet winner when it exists which is not the case of Condorcet consistent rule. A Condorcet winner is an alternative that is preferred to every other alternative by a majority of individuals: such an alternative would beat every other in majority comparisons. A Condorcet consistent rule selects the Condorcet winner when it exists. Similarly, it is well known that positional rules fail also to satisfy some weaker version of this property.

There is no doubt about the importance of these results, since it provides a clear axiomatic boundary between Borda's and Condorcet's approaches of social choice mechanisms. It is worth noting, however, that the above-mentioned negative results have been obtained by assuming that the simple majority is used. An alternative is majority preferred to an other alternative if at least more than one half of the individuals prefer this alternative to the other one[1].

The question we propose to tackle in this paper is then the following: to what extent does a larger majority (supra-majority) modify these negative results? To assume a larger majority is not so common in social choice theory. An important result concerning a supra-majority have been established by Ferejohn and Grether (1974). According to a given supra-majority m, an alternative is socially preferred to an other alternative if at least m individuals prefer this alternative to the other one. They give a necessary and sufficient condition on the majority needed in order to ensure that the rule associated with that majority always selects an alternative[2]. In a voting game context, they give simple and elegant condition for which the Core is non empty[3].

The purpose of this paper is then to give conditions such that the alternatives chosen by a rule belongs to the Core. For that, we follow Baharad and Nitzan (2003) who study the relationship between the positional rule and what they call the "$q-$Condorcet consistency" principle. Given a supra-majority q, a rule satisfies this principle if the winner of this rule is not preferred by any other alternative by a $q-$majority of individuals. Unfortunately, they have results for only one special positional rule, the Borda rule. We want to extent these results by investigating more positional rules.

Note that the well known positional approach takes place in a single stage process, where the winner(s) is (are) the alternative(s) with the highest score. But it can also be used in a multi-stage process of sequential eliminations, in each stage of which the alternative(s) with the least votes is (are) eliminated. We will then focus also on this kind of rules in this paper.

[1]We assume in this paper that individual preferences are strict.

[2]See section 2 for the recall of this result.

[3]This well known concept is largely study in the literature with a remarkable result due to Nakamura (1979).

Our main objective is then to study the conditions on the majority that ensure that positional rules (simple and sequential) satisfy the $q-$Condorcet consistency principle.

The remainder of this work is organized as follows: Section 2 is a presentation of the general framework with notations and definitions. Section 3 provides a characterization of the most famous simple positional rules *vis-à-vis* the $q-$Condorcet Consitency principle. Then, Section 4 provides results for sequential positional rules. Finally, Section 5 concludes the paper.

2 Notations and definitions

Consider a finite set N of n individuals or *voters*. For any subset S of N, S^c denotes the complementary of S, that is $N \setminus S$. Consider a finite set $A = \{a_1, ..., a_k\}$ of k distinct alternatives (or candidates), $k \geq 3$. Let 2^A be the set of nonempty subsets of A. Individual preference relations are defined over A and are assumed to be strict (indifference is not allowed). Assume that the preference relation R^i of individual i, $i \in N$, is a complete, antisymmetric and transitive binary relation (or simply a linear order) on A and let L be the set of all linear orders on A. A profile is an n-tuple $R = (R^i)_{i\in N}$ of individual preference relations, one for each individual. The set of all profiles on N will be denoted by L^N.

A social choice correspondence (SCC) is a mapping F from L^N to the set of nonempty subsets of A. This rule specifies the collective choice for any preference profile, $F : L^N \rightarrow 2^A$. For all profile R, $F(R)$ is the set of winners according to R.

Note that we do not introduce in this paper a mechanism in order to break ties among alternatives. That is why we focus on social choice correspondences and not on social choice functions which assign a single alternative to each profile. It should however be emphasized that the results presented here remain valid when we take into account a mechanism in order to break ties.

Now, we need additional notations to present the simple positional rules (PR) and the sequential positional rules (SPR) under focus in the present paper.

Given $B \in 2^A$ such that $|B| \geq 2$ ($|B|$ is the cardinality of the set B) and $a_h \in B$, let $r(B, a_h, R^i)$ be the rank, according to R^i, of a_h among alternatives in B. A scoring vector is a $|B|$-tuple $\alpha = \left(\alpha_1, ..., \alpha_r, ..., \alpha_{|B|}\right)$ of real numbers such that for all $r = 1, ..., |B| - 1$, $\alpha_r \geq \alpha_{r+1}$, and $\alpha_1 = 1$, $\alpha_{|B|} = 0$. $\alpha_r(B, a_h, R^i)$ is the number of points given to the alternative $a_h \in B$ by the individual i.

Definition 1. Given $B \in 2^A$, $R \in L^N$, $a_h \in B$ and a scoring vector $\alpha \in \mathbb{R}^{|B|}$, the score of a_h denoted $Sc(B, a_h, R, \alpha)$ is defined as follows: $Sc(B, a_h, R, \alpha) = \sum\limits_{i=1}^{n} \alpha_{r(B,a_h,R^i)}$.

In social choice processes using PR, the winning alternatives are those with the highest score, as stated in the following definition.

Definition 2. Let $B \in 2^A$ and a scoring vector $\alpha \in \mathbb{R}^{|B|}$. A simple positional rule (PR) with scoring vector α denoted F_α is a social choice correspondence F such that $\forall R \in L^N$, $\forall a_h \in B$, $a_h \in F_\alpha(R)$ if $[Sc(B, a_h, R, \alpha) \geq Sc(B, a_j, R, \alpha)\ \forall a_j \in B]$.

From the definitions of PR, it appears that given some issue B, a PR is defined by a vector $\alpha \in \mathbb{R}^{|B|}$. We can then express the three usual procedures: plurality rule denoted F_{α_0} where $\alpha = (1, 0, ..., 0)$; negative plurality rule denoted F_{α_1} with $\alpha = (1, ..., 1, 0)$ and Borda rule denoted F_{α_2} corresponding to $\alpha = (1, \frac{k-2}{k-1}, ..., \frac{k-r}{k-1}, ..., \frac{1}{k-1}, 0)$.

We now introduce sequential positional rules (*SPR*).

Let $\alpha^A = \left(\alpha^k, ..., \alpha^{|B|}, ..., \alpha^2\right)$ a collection of scoring vectors, each vector $\alpha^{|B|}$ being associated with each possible cardinality $|B|$ of the subset $B \subseteq A$, $|B| \geq 2$. At the first step of the sequential process, scores are computed using the vector α^k and the losing alternatives defined below are eliminated. In the next step, the corresponding vector is used to compute the scores and again the losing alternatives are eliminated. The sequential process is repeated until a set of winners (possibly one) is obtained (see Lepelley 1996).

More formally,

Definition 3. Given $B \in 2^A$, $R \in L^N$ and a scoring vector $\alpha^{|B|} \in \mathbb{R}^{|B|}$, the set of losing alternatives denoted $L\left(B, R, \alpha^{|B|}\right)$ is such that:

$$L\left(B, R, \alpha^{|B|}\right) = \left\{ a_h \in B : \left\{ \begin{array}{l} Sc\left(B, a_h, R, \alpha^{|B|}\right) \leq Sc\left(B, a_j, R, \alpha^{|B|}\right) \text{ for all } a_j \in B \\ \text{and } \exists a_j \in B : Sc\left(B, a_h, R, \alpha^{|B|}\right) < Sc\left(B, a_j, R, \alpha^{|B|}\right) \end{array} \right\} \right..$$

Note that when there is only one element in B, $L\left(B, R, \alpha^1\right) = \emptyset$.

Then we have the following definition for the SPR.

Definition 4. A sequential positional rule (*SPR*) associated with a collection of vectors $\alpha^A = \left(\alpha^k, ..., \alpha^{|B|}, ..., \alpha^2\right)$, denoted F_{α^A} is a social choice correspondence F such that given $B \subseteq A$, $R \in L^N$ and $a_h \in A$, $a_h \in F_{\alpha^A}(R) \Longleftrightarrow a_h \in A_p$, with A_p sequentially defined in the following way:

$$\begin{array}{l} A_1 = A \\ A_2 = A - L\left(A_1, R, \alpha^{|A_1|}\right) \\ \quad \vdots \\ A_{|B|+1} = A_{|B|} - L\left(A_{|B|}, R, \alpha^{|B|}\right) \\ \quad \vdots \\ A_p = A_{p-1} - L\left(A_{p-1}, R, \alpha^{|A_{p-1}|}\right) \text{ and } L\left(A_p, R, \alpha^{|A_p|}\right) = \emptyset. \end{array}$$

In order to illustrate this definition, let us consider the three-alternative case, that is $A = \{a_1, a_2, a_3\}$. We then have $\alpha^A = (\alpha^3, \alpha^2)$, with $\alpha^3 = (1, \lambda, 0)$, $0 \leq \lambda \leq 1$, where 1, λ and 0 are the scores of the alternatives ranked first, second and third respectively, in individual preference relations in the first step, and $\alpha^2 = (1, 0)$. More generally, given a profile of individual preferences, the total score of an alternative is the sum of individual scores, over the whole set of individuals. A *PR* selects the set of alternatives with the highest score. As a difference, an *SPR* first eliminates the alternatives with the smallest score at the first step, and then selects the alternatives with the highest score at the last step.

Note that with this formulation of sequential positional rules, there is no fixed relation between two scoring vectors of different steps: for example, we may use plurality at step 1 and negative plurality at step 2, etc. However, the most well known sequential positional rules are the iterative positional rules, where at each step the scoring vector changes only with respect to the number of alternatives. More formally, if at each step, $\alpha^{|B|}$ is the $|B|$-tuple $(1, 0, ..., 0)$ then it is Hare's Procedure denoted $F_{\alpha_0^A}$. Likewise if at each step, $\alpha^{|B|}$ is the $|B|$-tuple $\alpha = (1, ..., 1, 0)$ or $\alpha = (1, \frac{|B|-2}{|B|-1}, ..., \frac{1}{|B|-1}, 0)$, then we obtain Coombs' Procedure denoted $F_{\alpha_1^A}$ and Nanson's Procedure denoted $F_{\alpha_2^A}$, respectively.

It should be also noted that alternatives are not removed one after the other if several of them have the worst score. They should be removed all together. Furthermore, if all alternatives have the same score at a given step, then there is no elimination: all the alternatives are selected.

We can now define the $q-$Condorcet consistency principle.

Definition 5. Given a profile R, a rational number q, an alternative $a_h \in A$ is q-majority preferred to an alternative $a_j \in A$ iff $|\{i \in N : a_h R^i a_j\}| \geq q.n$, with $q \in \left]\frac{1}{2}, 1\right]$. a_j is said to be q-majority beaten by a_h.

In this paper we assume that n is large enough such that for all rational number k, $k.n$ is an integer.[4]

Let $C_q(R)$, the $q-$Condorcet majority set, defined as the set of alternatives in A which are not $q-$majority beaten by any other alternative in A.

It is well known from Ferejohn and Grether (1974) that a necessary and sufficient condition under which $C_q(R) \neq \emptyset$ for all R is that $q > \frac{k-1}{k}$.

Definition 6. Given a rational number $q \in \left]\frac{1}{2}, 1\right]$, a *SCC* F satisfies the $q-$Condorcet consistency ($q-CC$) principle if $F(R) \subseteq C_q(R)$ for any profile R.

Our main objective is then to determine conditions on q that ensure that usual *PR* and *SPR* satisfy $q-CC$.

3 Simple Positional rules

3.1 Usual procedures

We will first present results for the three usual simple positional rules, *i.e.* plurality, negative plurality and Borda.

The following result completely solves the plurality case.

Theorem 1. *Let $A = \{a_1, ..., a_k\}$, the scoring vector $\alpha = (1, 0, ..., 0)$. Let $q \in \left]\frac{1}{2}, 1\right]$.*

$$F_{\alpha_0} \text{ satisfies } q\text{-}CC \text{ if and only if } q > \frac{k-1}{k}.$$

Proof. 1) We know by Ferejohn and Grether (1974) that if $q \leq \frac{k-1}{k}$ there exists a profile R such that $C_q(R)$ is empty. For any such profile R, $F_{\alpha_0}(R)$ is q-majority beaten (is not included in $C_q(R)$) and thus F_{α_0} does not satisfies q-CC.

2) Conversely, assume that $q > \frac{k-1}{k}$. In order to show that F_{α_0} satisfies q-CC, it is enough to show that if R is a profile for which a_1 is q-majority beaten by (say) a_2 then $a_1 \notin F_{\alpha_0}(R)$.

Since a_1 is q-majority beaten by a_2, we have $|S| \geq qn$ where $S = \{i \in N : a_2 R^i a_1\}$. Since the scoring vector is $\alpha = (1, 0, ..., 0) : |S| \geq qn$ implies $Sc(A, a_1, R, \alpha) \leq n - qn = (1 - q)n$.

Let a_{max} be a candidate such that $Sc(A, a_{max}, R, \alpha) = \max_{2 \leq j \leq k} Sc(A, a_j, R, \alpha)$.

Since $\sum\limits_{j=2}^{k} Sc(A, a_j, R, \alpha) \geq qn$ we have $Sc(A, a_{max}, R, \alpha) \geq \frac{qn}{k-1}$.

Thus, $Sc(A, a_{max}, R, \alpha) - Sc(A, a_1, R, \alpha) \geq n(\frac{q}{k-1} - (1-q)) = n(\frac{k}{k-1}q - 1) > 0$ because $q > \frac{k-1}{k}$, and hence $a_1 \notin F_{\alpha_0}(R)$. This ends the proof. □

The answer of our main question is given below for the negative plurality rule.

[4]This assumption aims at simplifying our proofs. One could also assume that there is a continuum of voters.

Theorem 2. *Let $A = \{a_1, ..., a_k\}$, the scoring vector $\alpha = (1, 1, ..., 1, 0)$. For all $q \in \left]\frac{1}{2}, 1\right]$, F_{α_1} does not satisfies q-CC.*

Proof. Let $q \in \left]\frac{1}{2}, 1\right]$: To prove that F_{α_1} does not satisfies q-CC, we shall determine a profile R for which a winner, according to F_{α_1} is q-majority beaten.

For that, consider the profile R in which each individual i has the preference R^i defined as follows :

$$a_2 R^i a_1 R^i a_3 R^i a_4 ... R^i a_{k-1} R^i a_k$$

It is obvious that $F_{\alpha_1}(R) = \{a_1, ..., a_{k-1}\}$ because $\begin{cases} \forall j = 1, ..., k-1, Sc(A, a_j, R, \alpha) = n \\ Sc(A, a_k, R, \alpha) = 0 \end{cases}$.

The candidate a_1 is q-majority beaten (everybody prefers a_2 to a_1) meanwhile $a_1 \in F_{\alpha_1}(R)$. □

As Baharad and Nitzan (2003), we now solve the Borda case (with a slightly different proof).

Theorem 3. *Let $A = \{a_1, ..., a_k\}$, the scoring vector $\alpha = (1, \frac{k-2}{k-1}, ..., \frac{1}{k-1}, 0)$. Let $q \in \left]\frac{1}{2}, 1\right]$.*

$$F_{\alpha_2} \textit{ satisfies q-CC if and only if } q > \frac{k-1}{k}.$$

Proof. 1) Again, it is obvious that if $q \leq \frac{k-1}{k}$ then F_{α_2} does not satisfies q-CC property.

2) Conversely, assume that $q > \frac{k-1}{k}$. In order to show that F_{α_2} satisfies q-CC, it is enough to show that if R is a profile for which a_1 is q-majority beaten by a_2 then $a_1 \notin F_{\alpha_2}(R)$.

Since a_1 is q-majority beaten by a_2, we have $|S| \geq qn$ where $S = \{i \in N : a_2 R^i a_1\}$.

Furthermore, $\begin{cases} \alpha_{r(A,a_2,R^i)} - \alpha_{r(A,a_1,R^i)} \geq \frac{1}{k-1} \text{ for all } i \in S \text{ and} \\ \alpha_{r(A,a_2,R^i)} - \alpha_{r(A,a_1,R^i)} \geq -1 \text{ for all } i \in S^c. \end{cases}$

$$\begin{aligned} Sc(A, a_2, R, \alpha) - Sc(A, a_1, R, \alpha) &= \sum_{i \in S} (\alpha_{r(A,a_2,R^i)} - \alpha_{r(A,a_1,R^i)}) + \sum_{i \in S^c} (\alpha_{r(A,a_2,R^i)} - \alpha_{r(A,a_1,R^i)}) \\ &\geq \tfrac{1}{k-1}|S| - |S^c| \\ &\geq qn\tfrac{1}{k-1} - (1-q)n = n((\tfrac{1}{k-1} + 1)q - 1) \\ &= n(\tfrac{k}{k-1}q - 1) \\ &> 0 \ \ (\text{because } q > \tfrac{k-1}{k}). \end{aligned}$$

Hence $a_1 \notin F_{\alpha_2}(R)$. This ends the proof. □

3.2 General Case

The aim of this section is to identify the conditions on the quota q under which a given PR satisfies $q-CC$.

3.2.1 A sufficient condition.

We will first introduce the following proposition giving a sufficient condition under which winners for a given simple positional rule are never q-majority beaten.

For any scoring vector $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k)$, we let $\overline{\alpha} = \min\{\alpha_r - \alpha_{r+1}; r = 1, 2, ..., k-1\}$.

Proposition 1. *Let $A = \{a_1, ..., a_k\}$ and F_α the simple positional rule with scoring vector $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k)$ with $\alpha_1 = 1, \alpha_k = 0$ and $q \in \left]\frac{1}{2}, 1\right]$.*

$$\textit{If } q > \frac{1}{1+\overline{\alpha}}, \textit{ then } F_\alpha \textit{ satisfies } q\textit{-CC.}$$

Proof. Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k)$ and $q \in \left]\frac{1}{2}, 1\right]$ such that $q > \frac{1}{1+\overline{\alpha}}$.

In order to prove that F_α satisfies q-CC it is enough to show that for any profile R for which a_1 is q-majority beaten by a_2, $a_1 \notin F_\alpha(R)$.

Let R be such a profile : then $|S| \geq qn$ where $S = \{i \in N : a_2 R^i a_1\}$.

$Sc(A, a_2, R, \alpha) = \sum\limits_{i\in N} \alpha_{r(A,a_2,R^i)} = \sum\limits_{i\in S} \alpha_{r(A,a_2,R^i)} + \sum\limits_{i\in S^c} \alpha_{r(A,a_2,R^i)}$ and

$Sc(A, a_1, R, \alpha) = \sum\limits_{i\in N} \alpha_{r(A,a_1,R^i)} = \sum\limits_{i\in S} \alpha_{r(A,a_1,R^i)} + \sum\limits_{i\in S^c} \alpha_{r(A,a_1,R^i)}$; thus

$Sc(A, a_2, R, \alpha) - Sc(A, a_1, R, \alpha) = \sum\limits_{i\in S}(\alpha_{r(A,a_2,R^i)} - \alpha_{r(A,a_1,R^i)}) + \sum\limits_{i\in S^c}(\alpha_{r(A,a_2,R^i)} - \alpha_{r(A,a_1,R^i)})$.

Since $a_2 R^i a_1$ for all $i \in S$, we have $\alpha_{r(A,a_2,R^i)} - \alpha_{r(A,a_1,R^i)} \geq \overline{\alpha}$ for all $i \in S$.

On the other hand, for $i \notin S$, $\alpha_{r(A,a_2,R^i)} - \alpha_{r(A,a_1,R^i)} \geq -1$.

Finally,

$$\begin{aligned} Sc(A, a_2, R, \alpha) - Sc(A, a_1, R, \alpha) &= \sum_{i\in S}(\alpha_{r(A,a_2,R^i)} - \alpha_{r(A,a_1,R^i)}) + \sum_{i\in S^c}(\alpha_{r(A,a_2,R^i)} - \alpha_{r(A,a_1,R^i)}) \\ &\geq \overline{\alpha}|S| - |S^c| \\ &\geq qn\overline{\alpha} - (1-q)n = n((1+\overline{\alpha})q - 1) \\ &> n((1+\overline{\alpha}) \times \tfrac{1}{1+\overline{\alpha}} - 1) \text{ (because } q > \tfrac{1}{1+\overline{\alpha}}) \\ &= 0. \end{aligned}$$

Since $Sc(A, a_2, R, \alpha) - Sc(A, a_1, R, \alpha) > 0$, it follows that $a_1 \notin F_\alpha(R)$ and this ends the proof. □

3.2.2 A necessary and sufficient condition: the particular three alternative case.

Let $A = \{a_1, a_2, a_3\}$ and a scoring vector $\alpha = (\alpha_1, \alpha_2, \alpha_3) = (1, \lambda, 0)$ with $\lambda \in [0, 1]$. The cases $\lambda = 0$, $\lambda = \frac{1}{2}$ and $\lambda = 1$ are already solved : next we consider the two following cases: $\lambda > \frac{1}{2}$ and $\lambda < \frac{1}{2}$.

The case $\lambda > \frac{1}{2}$ is handled in the following result.

Proposition 2. *Let $A = \{a_1, a_2, a_3\}$ and let $\alpha = (1, \lambda, 0)$ with $\lambda > \frac{1}{2}$, F_α the simple positional rule associated with α and $q \in \left]\frac{1}{2}, 1\right]$.*

$$F_\alpha \textit{ satisfies } q\textit{-CC if and only if } q > \frac{1}{2-\lambda}.$$

Proof. Let $\alpha = (1, \lambda, 0)$ with $\lambda > \frac{1}{2}$.

Then $\overline{\alpha} = 1 - \lambda$ and therefore $\frac{1}{1+\overline{\alpha}} = \frac{1}{2-\lambda}$. Thanks to proposition 1, if $q > \frac{1}{2-\lambda}$, then F_α satisfies q-CC.

Conversely, assume that $q \leq \frac{1}{2-\lambda}$. To show that F_α does not satisfies q-CC, consider the following profile R in which:

$$\begin{cases} qn \text{ individuals have preference } a_2 R^i a_1 R^i a_3 \\ (1-q)n \text{ have preference } a_1 R^i a_3 R^i a_2 \end{cases}$$

For the profile R so defined, a_1 is q-majority beaten by a_2. However let us compute the scores:

we have :

$$\begin{cases} Sc(A, a_1, R, \alpha) = \lambda qn + (1-q)n \\ Sc(A, a_2, R, \alpha) = qn \text{ and} \\ Sc(A, a_3, R, \alpha) = \lambda(1-q)n \end{cases}$$

$$\begin{array}{rcl} Sc(A, a_1, R, \alpha) - Sc(A, a_2, R, \alpha) & = & \lambda qn + (1-q)n - qn \\ & = & n((\lambda - 2)q + 1) \\ & = & -n((2-\lambda)q - 1) \geq 0 \text{ because } q \leq \frac{1}{2-\lambda} \\ Sc(A, a_1, R, \alpha) - Sc(A, a_3, R, \alpha) & = & \lambda qn + (1-q)n - \lambda(1-q)n \\ & = & n(\lambda q + 1 - q - \lambda + \lambda q) \\ & = & n((2\lambda - 1)q + 1 - \lambda) > 0 \text{ because } 2\lambda - 1 > 0 \text{ and } 1 - \lambda > 0 \end{array}$$

It then follows that $a_1 \in F_\alpha(R)$ and hence, F_α does not satisfy q-CC. □

The following result deals with the case $\lambda < \frac{1}{2}$.

Proposition 3. *Let $A = \{a_1, a_2, a_3\}$ and let $\alpha = (1, \lambda, 0)$ with $\lambda < \frac{1}{2}$, F_α the simple positional rule associated with α and $q \in \left]\frac{1}{2}, 1\right]$.*

$$F_\alpha \textit{ satisfies } q\text{-}CC \textit{ if and only if } q > \frac{2}{3}.$$

Proof. 1) By Ferejohn and Grether (1974), it is obvious that if $q \leq \frac{2}{3}$ then F_α does not satisfies q-CC.

2) Conversely, let $q > \frac{2}{3}$. In order to prove that F_α satisfies q-CC, we will show that for any profile R for which a_1 is $q-$majority beaten by a_2, $a_1 \notin F_\alpha(R)$. Let R be such a profile. Assume on the contrary that $a_1 \in F_\alpha(R)$.

There exists a subset S such that $|S| = qn$ and $a_2 R^i a_1$ for all $i \in S$.

Let θ be the proportion of individuals in S holding preference $a_2 R^i a_1 R^i a_3$.

With no loss of generality, we can then assume that in S:

$$\begin{cases} \theta n \text{ individuals have preference } a_2 R^i a_1 R^i a_3 \text{ and} \\ (q-\theta)n \text{ individuals have preference } a_3 R^i a_2 R^i a_1 \end{cases} \text{ with } 0 \leq \theta \leq q.$$

The score of a_1 is bounded above by $\lambda\theta n + (1-q)n$ and furthermore, the value $\lambda\theta n + (1-q)n$ is obtained if and only if every individual in S^c ranks a_1 at the first position.

Now let δ be the proportion of individuals in S^c with preference $a_1 R^i a_3 R^i a_2$.

With no loss of generality, we can then assume that in S^c:

$$\begin{cases} \delta n \text{ individuals have preference } a_1 R^i a_3 R^i a_2 \text{ and} \\ (1-q-\delta)n \text{ individuals have } a_1 R^i a_2 R^i a_3 \end{cases} \text{ with } 0 \leq \delta \leq 1-q.$$

At this level we will distinguish two cases : $\delta = 1-q$ and $\delta < 1-q$:

First case : $\delta = 1 - q$

In this case, $\begin{cases} Sc(A, a_1, R, \alpha) = \lambda\theta n + (1-q)n = n(\lambda\theta - q + 1) \\ Sc(A, a_2, R, \alpha) = \theta n + \lambda(q-\theta)n = n((1-\lambda)\theta + \lambda q) \\ Sc(A, a_3, R, \alpha) = (q-\theta)n + \lambda(1-q)n = n(-\theta + (1-\lambda)q + \lambda) \end{cases}$.

If $a_1 \in F_\alpha(R)$ then :

$$\begin{cases} Sc(A, a_1, R, \alpha) - Sc(A, a_2, R, \alpha) \geq 0 \text{ and} \\ Sc(A, a_1, R, \alpha) - Sc(A, a_3, R, \alpha) \geq 0 \end{cases} \Rightarrow \begin{cases} (2\lambda - 1)\theta - (1+\lambda)q + 1 \geq 0 & (1) \text{ and} \\ (1+\lambda)\theta + (-2+\lambda)q + (1-\lambda) \geq 0 & (2) \end{cases}$$

$$\begin{array}{rcl} (1) & \Rightarrow & (2\lambda - 1)\theta \geq (1+\lambda)q - 1 \\ & \Rightarrow & \theta \leq \frac{1}{1-2\lambda}[-(1+\lambda)q + 1] \end{array}$$

$$
\begin{array}{rll}
(2) & \Rightarrow & (2-\lambda)q \leq (1+\lambda)\theta + (1-\lambda) \\
 & \Rightarrow & (2-\lambda)q \leq (1+\lambda)\frac{1}{1-2\lambda}[-(1+\lambda)q+1] + (1-\lambda) \text{ with } (1) \\
 & \Rightarrow & (2-\lambda)q - (1-\lambda) \leq \frac{1+\lambda}{1-2\lambda}[-(1+\lambda)q+1] \\
 & \Rightarrow & (1-2\lambda)(2-\lambda)q - (1-\lambda)(1-2\lambda) \leq -(1+\lambda)^2 q + (1+\lambda) \\
 & \Rightarrow & (3\lambda^2 - 3\lambda + 3)q \leq 2\lambda^2 - 2\lambda + 2 \\
 & \Rightarrow & 3q \leq 2 \text{ because } 3\lambda^2 - 3\lambda + 3 > 0. \\
 & & \text{a contradiction since } q > \frac{2}{3}.
\end{array}
$$

Second case : $\delta < 1-q$

Assume that $a_1 \in F_\alpha(R)$:

then $Sc(A, a_1, R, \alpha) - Sc(A, a_2, R, \alpha) \geq 0$ and $Sc(A, a_1, R, \alpha) - Sc(A, a_3, R, \alpha) \geq 0$.

$$
\begin{cases}
Sc(A, a_1, R, \alpha) = \lambda\theta n + (1-q)n = n[\lambda\theta + (1-q)] \\
Sc(A, a_2, R, \alpha) = n[\theta + \lambda(q-\theta) + \lambda(1-q-\delta)] = n[(1-\lambda)\theta + \lambda - \lambda\delta] \\
Sc(A, a_3, R, \alpha) = (q-\theta)n + \lambda\delta n = n[-\theta + q + \lambda\delta]
\end{cases}.
$$

As $Sc(A, a_1, R, \alpha) - Sc(A, a_2, R, \alpha) \geq 0$ and $Sc(A, a_1, R, \alpha) - Sc(A, a_3, R, \alpha) \geq 0$, we have:

$$
\begin{array}{rll}
\begin{cases}
(2\lambda - 1)\theta - q + 1 - \lambda + \lambda\delta \geq 0 \\
(\lambda+1)\theta - 2q + 1 - \lambda\delta \geq 0
\end{cases}
& \Rightarrow & \begin{cases}
\lambda\delta \geq (1-2\lambda)\theta + q + \lambda - 1 \\
(\lambda+1)\theta - 2q + 1 \geq \lambda\delta
\end{cases} \quad (*) \\
 & \Rightarrow & (1-2\lambda)\theta + q + \lambda - 1 \leq \lambda\delta \leq (\lambda+1)\theta - 2q + 1 \\
 & \Rightarrow & (1-2\lambda)\theta + q + \lambda - 1 \leq (\lambda+1)\theta - 2q + 1 \\
 & \Rightarrow & (1-2\lambda-\lambda-1)\theta \leq -3q - \lambda + 2 \\
 & \Rightarrow & -3\lambda\theta \leq -(3q+\lambda-2) \text{ that is, } \theta \geq \frac{1}{3\lambda}(3q+\lambda-2)(**).
\end{array}
$$

On the other hand we have $0 \leq \delta < 1-q$ which implies $\lambda - \lambda q > \delta\lambda \geq (1-2\lambda)\theta + q + \lambda - 1$ (thanks to $(*)$).

$$
\begin{array}{rll}
\lambda - \lambda q > (1-2\lambda)\theta + q + \lambda - 1 & \Rightarrow & (-\lambda-1)q + 1 > (1-2\lambda)\frac{1}{3\lambda}(3q+\lambda-2) \text{ (thanks to } (**)) \\
 & \Rightarrow & 3\lambda(-\lambda-1)q + 3\lambda > 3q(1-2\lambda) + (1-2\lambda)(\lambda-2) \\
 & \Rightarrow & (-3\lambda^2 - 3\lambda - 3 + 6\lambda)q > \lambda - 2 - 2\lambda^2 + 4\lambda - 3\lambda \\
 & \Rightarrow & 3q(-\lambda^2 + \lambda - 1) > 2(-\lambda^2 + \lambda - 1) \\
 & \Rightarrow & 3q < 2 \text{ because } -\lambda^2 + \lambda - 1 < 0 \\
 & & \text{which is a contradiction since } q > \frac{2}{3}.
\end{array}
$$

□

To summarize, for $A = \{a_1, a_2, a_3\}$, $\alpha = (1, \lambda, 0)$, let $q_\alpha = \begin{cases} \frac{1}{1+\overline{\alpha}} \text{ if } \overline{\alpha} = 1-\lambda \\ \frac{2}{3} \text{ if } \overline{\alpha} = \lambda \end{cases}$ that is,

$q_\alpha = \begin{cases} \frac{1}{2-\lambda} \text{ if } \lambda > \frac{1}{2} \\ \frac{2}{3} \text{ if } \lambda < \frac{1}{2} \end{cases}$.

Thanks to propositions 2 and 3 above, we can state the following result.

Theorem 4. *Let $A = \{a_1, a_2, a_3\}$, and let $\alpha = (1, \lambda, 0)$, F_α the simple positional rule associated with α and $q \in \left]\frac{1}{2}, 1\right]$.*

$$F_\alpha \text{ satisfies } q\text{-}CC \text{ if and only if } q > q_\alpha.$$

We can remark that this result is consistent with the previous results for Borda, Plurality and Negative Plurality. Indeed when $\lambda = 0$ and $\lambda = \frac{1}{2}$, $q_\alpha = \frac{2}{3}$, and when $\lambda = 1$, $q_\alpha = 1$.

4 Sequential Positional Rules

We will present results in this section for the three usual *SPR*, *i.e.* Hare's Procedure ($F_{\alpha_0^A}$), Coombs' Procedure ($F_{\alpha_1^A}$) and Nanson's Procedure ($F_{\alpha_2^A}$). Before presenting the results of this section, we recall that we assumed that n, the number of voters is large enough so that kn is an integer whenever k is a rational number.

The following result completely solves Hare's case.

Theorem 5. *Let $A = \{a_1, ..., a_k\}$, and let $\alpha = (1, 0, ..., 0)$ the scoring vector at each step. Let $q \in \left]\frac{1}{2}, 1\right]$.*

$$\text{Then } F_{\alpha_0^A} \text{ satisfies } q\text{-CC if and only if } q > (1 - \frac{1}{2^{k-1}}) - \frac{1}{n}.$$

Proof. First we will show that if $q \leq (1 - \frac{1}{2^{k-1}}) - \frac{1}{n}$ then $F_{\alpha_0^A}$ does not satisfy q-CC. We then need to construct a profile R such that (say) $a_1 \in F_{\alpha_0^A}(R)$ meanwhile a_1 is q-majority beaten by a_2.

Let $\theta = \frac{n}{2^{k-1}}$ and consider the following profile :

$\theta + 1$ individuals	:	$a_1 R^i a_k R^i a_{k-1} R^i ... R^i a_2$
θ	:	$a_2 R^i a_1 R^i a_k R^i a_{k-1} R^i ... R^i a_3$
2θ	:	$a_3 R^i a_2 R^i a_1 R^i a_k R^i a_{k-1} R^i ... R^i a_4$
$2^2\theta$	:	$a_4 R^i a_3 R^i a_2 R^i a_1 R^i a_k R^i a_{k-1} R^i ... R^i a_5$
...	:	
$2^{k-4}\theta$	:	$a_{k-2} R^i a_{k-3} R^i ... R^i a_2 R^i a_1 R^i a_k R^i a_{k-1}$
$2^{k-3}\theta - 1$	:	$a_{k-1} R^i a_{k-2} R^i ... R^i a_2 R^i a_1 R^i a_k$
$2^{k-2}\theta$	:	$a_k R^i a_{k-1} R^i a_{k-2} R^i ... R^i a_2 R^i a_1$

With respect to R a_1 is q-majority beaten by a_2. Indeed, the number of voters who prefer a_2 to a_1 is

$$\begin{array}{rcl} \theta + 2\theta + 2^2\theta + ... + 2^{k-3}\theta - 1 + 2^{k-2}\theta & = & \theta(2^{k-1} - 1) - 1 \\ & = & n[(1 - \frac{1}{2^{k-1}}) - \frac{1}{n}] \\ & \geq & qn \text{ since } q \leq (1 - \frac{1}{2^{k-1}}) - \frac{1}{n} \end{array}.$$

Then a_1 is $q-$majority beaten by a_2.

Now let us show that $a_1 \in F_{\alpha_0^A}(R)$.

Let $L(A_t, R, \alpha^{|A_t|})$ be the set of losing candidates at step t, we have $F_{\alpha_0^A}(R) = A \backslash \bigcup_{t=1}^{p} L(A_t, R, \alpha^{|A_t|})$ where p is the number of steps. The scoring vector at any step t is the $|A_t|$-tuple $(1, 0, 0, ..., 0)$.

From the definition of R it is obvious that $Sc(A_1, a_2, R, \alpha^{|A_1|}) = \theta < Sc(A_1, a_i, R, \alpha^{|A_1|})$ for all $i \neq 2$; thus $L(A_1, R, \alpha^{|A_1|}) = \{a_2\}$.

Note that $Sc(A_2, a_1, R, \alpha^{|A_2|}) = \theta + \theta + 1 = 2\theta + 1$ whereas $Sc(A_2, a_3, R, \alpha^{|A_2|}) = 2\theta$.

It can be easily seen that :

$L(A_2, R, \alpha^{|A_2|}) = \{a_3\}$, $L(A_3, R, \alpha^{|A_3|}) = \{a_4\}$, and so on and $L(A_{k-2}, R, \alpha^{|A_{k-2}|}) = \{a_{k-1}\}$.

Finally, at the last step, a_1 is pitted against a_k.

$Sc(A_{k-1}, a_1, R, \alpha^{|A_{k-1}|}) = \theta + \theta + 2\theta + ... + 2^{k-3}\theta = (2 + 2^1 + ... + 2^{k-3})\theta = 2^{k-2}\theta$

and $Sc(A_{k-1}, a_k, R, \alpha^{|A_{k-1}|}) = 2^{k-2}\theta = Sc(A_{k-1}, a_1, R, \alpha^{|A_{k-1}|})$ thus, $F_{\alpha_0^A}(R) = \{a_1, a_k\}$.

Conversely, assume that $q > 1 - \frac{1}{2^{k-1}} - \frac{1}{n}$, that is, $qn > n(1 - \frac{1}{2^{k-1}}) - 1$ or $qn \geq n(1 - \frac{1}{2^{k-1}})(*)$

Let us show that $F_{\alpha_0^A}(R)$ satisfies q-CC. For that, let R be a profile for which a_1 is q-majority beaten by a_2 and $a_1 \in F_{\alpha_0^A}(R)$.

Assume that for the profile R, there are p steps and that the set of losing candidates are respectively $L(A_1, R, \alpha^{|A_1|})$, $L(A_2, R, \alpha^{|A_2|})$, ... and $L(A_p, R, \alpha^{|A_p|})$ and the set of winners is $F_{\alpha_0^A}(R) = A \setminus \bigcup_{j=1}^{p} L(A_j, R, \alpha^{|A_j|})$.

Let n_j be the cardinality of $L(A_j, R, \alpha^{|A_j|})$ and $r+1$ the cardinality of $F_{\alpha_0^A}(R)$. (If $r = 0$ then a_1 is the only winner).

Since a_1 is q-majority beaten by a_2 , $Sc(A_1, a_1, R, \alpha^{|A_1|}) \leq (1-q)n$.

As n_1 candidates are eliminated at the first step, we have

$Sc(A_1, a_h, R, \alpha^{|A_1|}) < Sc(A_1, a_1, R, \alpha^{|A_1|}) \leq (1-q)n \ \forall a_h \in L(A_1, R, \alpha^{|A_1|})$

and thus, $\sum\limits_{a_h \in L(A_1, R, \alpha^{|A_1|})} Sc(A_1, a_h, R, \alpha^{|A_1|}) < n_1(1-q)n$, therefore at the second step,

$Sc(A_2, a_1, R, \alpha^{|A_2|}) < (1-q)n + n_1(1-q)n = (1+n_1)(1-q)n$.

Likewise, since candidates in $L(A_2, R, \alpha^{|A_2|})$ are eliminated at step 2, we have

$Sc(A_2, a_r, R, \alpha^{|A_2|}) < (1+n_1)(1-q)n \ \forall a_r \in L(A_2, R, \alpha^{|A_2|})$ and therefore

$\sum\limits_{a_r \in L(A_2, R, \alpha^{|A_2|})} Sc(A_2, a, R, \alpha^{|A_2|}) < n_2(1+n_1)(1-q)n$.

We deduce that

$Sc(A_3, a_1, R, \alpha^{|A_3|}) < (1+n_1)(1-q)n + n_2(1+n_1)(1-q)n = [(1+n_2)(1+n_1)]\,(1-q)n$.

At the last step, after the elimination of $L(A_1, R, \alpha^{|A_1|})$, $L(A_2, R, \alpha^{|A_2|})$, ... and $L(A_p, R, \alpha^{|A_p|})$, we have:

$Sc(A_p, a_1, R, \alpha^{|A_p|}) < [(1+n_p)(1+n_{p-1})...(1+n_2)(1+n_1)]\,(1-q)n$.

At the end, all the candidates in $F_{\alpha_0^A}(R)$ have the same score say s^*, implying that $(r+1)s^* = n$.

However, since $a_1 \in F_{\alpha_0^A}(R)$, s^* should be equal to $Sc(A_p, a_1, R, \alpha^{|A_p|})$.

Finally,

$$\begin{array}{rcl} n = (r+1)s^* & = & (r+1)Sc(A_p, a_1, R, \alpha^{|A_p|}) \\ & < & (1+r)(1+n_p)(1+n_{p-1})...(1+n_2)(1+n_1)(1-q)n \\ & & \text{note that for all nonnegative integer } m,\ 1+m \leq 2^m \\ & \leq & 2^{r+n_p+n_{p-1}+...+n_1}(1-q)n \\ & = & 2^{k-1}(1-q)n \\ & & \text{because } (1+r)+(n_p)+....+(n_1) = k \\ & \leq & 2^{k-1}(\frac{1}{2^{k-1}})n = n \text{ (thanks to } (*)) \\ & & \text{This is a contradiction.} \end{array}$$

Conclusion : we cannot have a_1 $q-$majority beaten and $a_1 \in F_{\alpha_0^A}(R)$. □

The following result deals with Coombs' procedure.

Theorem 6. *Let $A = \{a_1, ..., a_k\}$, and let $\alpha = (1, 1, ...1, 0)$ the scoring vector at each step. Let $q \in \left]\frac{1}{2}, 1\right]$.*

$$F_{\alpha_1^A} \textit{ satisfies q-CC if and only if } q > \frac{k-1}{k}.$$

Proof. 1) By Ferejohn and Grether (1974) it is obvious that if $q \leq \frac{k-1}{k}$, $F_{\alpha_1^A}$ does not satisfies q-CC.

2) It then suffices to prove that if $q > \frac{k-1}{k}$ then for any R for which a_1 is q-majority beaten by a_2, $a_1 \notin F_{\alpha_1^A}(R)$. Let R be such a profile. Assume the contrary : $a_1 \in F_{\alpha_1^A}(R)$.

Since a_1 is q-majority beaten by a_2, we have $|S| \geq qn$ where $S = \{i \in N : a_2 R^i a_1\}$. Recall that at any step t in which the set of alternatives is A_t, the scoring vector is the $|A_t|$-tuple $(1, 1, ...1, 0)$.

Assuming that there are p steps and that $L(A_t, R, \alpha^{|A_t|})$ is the set of eliminated candidates at step t, we have $F_{\alpha_1^A}(R) = A \setminus \bigcup_{t=1}^{p} L(A_t, R, \alpha^{|A_t|})$.

We will prove that if $a_1 \in F_{\alpha_1^A}(R)$ then $a_2 \in F_{\alpha_1^A}(R)$. We will then show that at any step t, if $a_1 \notin L(A_t, R, \alpha^{|A_t|})$ then $a_2 \notin L(A_t, R, \alpha^{|A_t|})$.

Consider a step t and assume, with no loss of generality that v $(v > 0)$ candidates have so far been eliminated with $a_1 \notin L(A_t, R, \alpha^{|A_t|})$. v is the cardinality of $\bigcup_{h=1}^{t-1} L(A_h, R, \alpha^{|A_h|})$

Let a_0 be a candidate such that :

$Sc(A_t, a_0, R, \alpha^{|A_t|}) = min \left\{ Sc(A_t, a_j, R, \alpha^{|A_t|}) : a_j \in A_t = A \setminus \bigcup_{h=1}^{t-1} L(A_h, R, \alpha^{|A_h|}) \right\}$.

As A_t has exactly $k - v$ candidates and the scoring vector is $\alpha^{|A_t|} = (1, 1, ..., 1, 0)$, we have :

$\sum_{a_h \in A_t} Sc(A_t, a_h, R, \alpha^{|A_t|}) = n(k - v - 1)$.

By the definition of a_0, it holds $Sc(A_t, a_0, R, \alpha^{|A_t|}) \leq Sc(A_t, a_h, R, \alpha^{|A_t|})$ for all $a_h \in A_t$. This implies that

$(k - v)Sc(A_t, a_0, R, \alpha^{|A_t|}) \leq \sum_{a_h \in A_t} Sc(A_t, a_h, R, \alpha^{|A_t|})$, that is, $Sc(A_t, a_0, R, \alpha^{|A_t|}) \leq \frac{k-v-1}{k-v} n$

Since $a_2 R^i a_1$ for all $i \in S$, we have $Sc(A_t, a_2, R, \alpha^{|A_t|}) \geq qn > \frac{k-1}{k} n$

$Sc(A_t, a_2, R, \alpha^{|A_t|}) - Sc(A_t, a_0, R, \alpha^{|A_t|}) > \frac{k-1}{k} n - \frac{k-v-1}{k-v} n = \frac{v}{k(k-v)} \geq 0$ which means that $a_2 \notin L(A_t, R, \alpha^{|A_t|})$.

We have just proved that at each step t, if $a_1 \notin L(A_t, R, \alpha^{|A_t|})$ then $a_2 \notin L(A_t, R, \alpha^{|A_t|})$.

This means that if $a_1 \in F_{\alpha_1^A}(R)$ then $a_2 \in F_{\alpha_1^A}(R)$.

Let $k - \mu = |F_{\alpha_1^A}(R)|$ which means that at the end μ candidates have been eliminated. As all candidates in $F_{\alpha_1^A}(R)$ have the same score say s^*, it holds $(k - \mu)s^* = n(k - \mu - 1)$. Note that there is at least two alternatives in $F_{\alpha_1^A}(R)$, since $a_1 \in F_{\alpha_1^A}(R)$ and therefore $a_2 \in F_{\alpha_1^A}(R)$.

We should have $Sc(A_p, a_2, R, \alpha^{|A_p|}) = s^*$ and again $Sc(A_p, a_2, R, \alpha^{|A_p|}) \geq |S|$ since $\forall i \in S, a_2 R^i a_1$.

Therefore, $(k - \mu)s^* \geq (k - \mu)|S| > (k - \mu)\frac{k-1}{k} n$

$$\begin{aligned}
(k - \mu)s^* > (k - \mu)\tfrac{k-1}{k} n \quad &\Rightarrow \quad n(k - \mu - 1) > (k - \mu)\tfrac{k-1}{k} n \\
&\Rightarrow \quad k(k - \mu) - k > k(k - \mu) - (k - \mu) \\
&\Rightarrow \quad -k > -(k - \mu) \text{ which is a contradiction.}
\end{aligned}$$

We conclude that $a_1 \notin F_{\alpha_1^A}(R)$. □

The last result solves Nanson's rule.

Theorem 7. *Let $A = \{a_1, ..., a_k\}$, and let $\alpha = (1, \frac{|A_t|-2}{|A_t|-1}, ... \frac{1}{|A_t|-1}, 0)$ the scoring vector at each step t where the set of alternatives is A_t. Let $q \in]\frac{1}{2}, 1]$.*

$$\textit{Then } F_{\alpha_2^A} \textit{ satisfies } q\textit{-CC if and only if } q > \frac{k-1}{k}.$$

Proof. 1) Once again, it is obvious that if $q \leq \frac{k-1}{k}$, $F_{\alpha_2^A}$ does not satisfies q-CC.

2) It suffices to prove that if $q > \frac{k-1}{k}$ then for any R for which a_1 is q-majority beaten by a_2, $a_1 \notin F_{\alpha_2^A}(R)$.

Since a_1 is q-majority beaten by a_2, we have $|S| \geq qn$ where $S = \{i \in N : a_2 R^i a_1\}$. Recall that at any step t in which the set of alternatives is A_t, the scoring vector is the $|A_t|$-tuple $(1, \frac{|A_t|-2}{|A_t|-1}, \ldots \frac{1}{|A_t|-1}, 0)$.

Assuming that there are p steps and that $L(A_t, R, \alpha^{|A_t|})$ is the set of losing candidates at step t, we have $F_{\alpha_2^A}(R) = A \setminus \bigcup_{t=1}^{p} L(A_t, R, \alpha^{|A_t|})$.

In order to prove $a_1 \notin F_{\alpha_2^A}(R)$, it suffices to show that at any step t whenever a_1 and a_2 belong to A_t, we have $Sc(A_t, a_2, R, \alpha^{|A_t|}) > Sc(A_t, a_1, R, \alpha^{|A_t|})$.

Now consider a given step t such that $a_1, a_2 \in A_t$:

We have $\begin{cases} \alpha_{r(A_t,a_2,R^i)} - \alpha_{r(A_t,a_1,R^i)} \geq \frac{1}{k-1} \text{ for all } i \in S \text{ and} \\ \alpha_{r(A_t,a_2,R^i)} - \alpha_{r(A_t,a_1,R^i)} \geq -1 \text{ for all } i \in S^c. \end{cases}$

Thus,

$$\begin{array}{rcl} Sc(A_t, a_2, R, \alpha^{|A_t|}) - Sc(A_t, a_1, R, \alpha^{|A_t|}) & = & \sum\limits_{i \in S} (\alpha_{r(A_t,a_2,R^i)} - \alpha_{r(A_t,a_1,R^i)}) + \sum\limits_{i \in S^c} (\alpha_{r(A_t,a_2,R^i)} - \alpha_{r(A_t,a_1,R^i)}) \\ & \geq & |S| \frac{1}{k-1} - |S^c| \\ & = & |S| (\frac{1}{k-1} + 1) - n \\ & \geq & n(\frac{k}{k-1} q - 1) \text{ since } |S| \geq qn \\ & > & n(\frac{k}{k-1} \frac{k-1}{k} - 1) = 0 \text{ since } q > \frac{k-1}{k} \\ & & \text{which means that } Sc(A_t, a_2, R, \alpha^{|A_t|}) > Sc(A_t, a_1, R, \alpha^{|A_t|}). \end{array}$$

At a given step $t \in \{1, 2, ..., p-1\}$, if $a_1, a_2 \in A_t$ then $Sc(A_t, a_2, R, \alpha^{|A_t|}) > Sc(A_t, a_1, R_t, \alpha^{|A_t|})$.

At the last step p, we cannot have both a_1 and a_2 in $F_{\alpha_2^A}(R)$ since all candidates in $F_{\alpha_2^A}(R)$ have exactly the same score, hence $a_1 \notin F_{\alpha_2^A}(R)$.

□

5 Conclusion

The question motivating this study is whether a larger majority (supra-majority) can ensure that positional rules (simple and sequential) satisfy a weaker version of the Condorcet consistency principle. The affirmative answer to this question is the main message of our analysis. Indeed, all usual positional rules, that is Plurality, Borda, Hare, Nanson and Coombs satisfy this principle. The only one which does not verifies it, is Negative Plurality. This is an interesting result, since none of these rules except Nanson satisfy Condorcet Consitency when the simple majority is required. This means that Condorcet's approach and Borda's approach are not so mutually incompatible. In this respect, our results can be seen as positive results.

Moreover we give the precise condition on the majority (on the quota) needed to ensure that this rules satisfies the $q-$Condorcet consistency principle. Our results show that in general, the principle is verify as soon as the quota is greater than $\frac{k-1}{k}$. This is true for Plurality, Borda, Nanson and Coombs. This result is interesting since this is also the quota which guarantees that a rule associated with this quota always select an alternative, as established by Ferejohn and Grether

(1974). Regarding Hare, the quota of $(1-\frac{1}{2^{k-1}})-\frac{1}{n}$ is greater than the previous one, which suppose that this is more difficult with Hare to respect $q-$Condorcet Consistency.

We also consider the case of all simple positional rules. However our study is limited to the three alternative case. It appears indeed that the issue of characterizing quota under which a given rule satisfies our consistency property is a difficult task. We do not try to solve the problem for general sequential positional rules since it would require to define what is the scoring vectors at each step of the process.

Finally, it is worth noting that it may seems appropriate to extend our results by evaluating the propensitivity of positional rules to violate this majority condition when the quota is not achieved. This is a very common approach in the context of simple majority. That is why it can be interesting to apply it in the context of supra-majority.

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