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The effect of spillovers and congestion on the segregative properties of endogenous jurisdictions formation*

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Abstract

This paper analyzes the effect of spillovers and congestion of local public services on the segregative properties of endogenous formation of jurisdictions. Households choosing to live at the same place form a jurisdiction whose aim is to produce congested local public services, that can create positive spillovers to other jurisdictions. In every jurisdiction, the production of the local public services is financed through a local tax based on households' wealth. Local wealth tax rates are democratically determined in all jurisdictions. Households also consume housing in their jurisdiction. Any household is free to leave its jurisdiction for another one that would increase its utility. A necessary and sufficient condition to have every stable jurisdictions structure segregated by wealth, for a large class of congestion measures and any spillovers coefficients structure, is identified: the public services must be either a gross substitute or a gross complement to the private good and the housing.

JEL Classification: C78; D02; H73; R13

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1 Introduction

In most countries, local public spending accounts for a large share of the public spending (almost 50% in the USA), and this share has been increasing since the end of the Second World War. As a consequence of the growing role played by local jurisdictions, another phenomenon appeared: jurisdictions belonging to the same urban area seem to be more differentiated in terms of their inhabitants' wealth (see for instance [1]). A possible explanation is provided by Tiebout's 1956 article [2]. According to his intuitions, individuals choose their place of residence according to a trade-off between local tax rates and amounts of public services provided, which leads every jurisdiction to be homogeneous. The formation of jurisdictions structure is endogenous, due to the free mobility of households, that can "vote with their feet", that is to say leave their jurisdiction to another one, if they are unsatisfied with their jurisdiction's tax rate and amount of public services. An important literature dealing with the endogenous jurisdictions formation *à la* Tiebout exists. A widely spread belief is the self-sorting mechanism of the endogenous formation process: agents will live in homogeneous jurisdictions. This homogeneity can be expressed in terms of wealth, preferences on public services, on housing, on economic activity...

Westhoff [3] was among the first economists to provide a formal model based on Tiebout's intuitions. In this model, households can enjoy 2 goods, a local public good, financed through a local tax on wealth, which is a pure club good (only households living in the jurisdiction that produced it can enjoy the local public good, that does not suffer from congestion effect), and a composite private good, whose amount is equal to the after-tax wealth. He found a condition that ensures the existence of an equilibrium. This condition is for the slopes of individuals' indifference curves in the tax rate-amount of public good space to be ordered by their private wealth. If this condition is respected, not only an equilibrium will exist, but, at equilibrium, the jurisdictions structure will be segregated.

Gravel and Thoron [4] identified a necessary and sufficient condition that ensures the segregation, within Westhoff's meaning, of every stable jurisdictions structure: the public good must be, for all level of prices and wealth, either always a complement or always a substitute to the private good. This condition is called the Gross Substitutability/Complementarity (GSC) condition. This condition is equivalent to have the preferred tax rate being a monotonous function of the private wealth, for any level of prices and wealth. Biswas, Gravel and Oddou [5] integrated a welfarist central government to the model, whose purpose is to maximize a generalized utilitarian social welfare function by implementing an equalization payment policy. Equalization payment can be either vertical (the government taxes households and redistributes the revenues to jurisdictions), horizontal (the government redistributes local tax revenues between the jurisdictions), or mixed. They found that the GSC condition remains necessary and sufficient.

Greenberg [6] proposed a model of local public goods with spillovers among jurisdictions, but did not consider the possibility for households to leave their jurisdiction for another that offer a better "tax rate - amount of public good" package. He proved the existence of an equilibrium under the d-majority voting rule.

Nechyba [7] developed a model with spillovers and housing, but, contrary to Rose-Ackerman [8], housing is modelled as a discreet good, which differs on type, and households own their house instead of renting it, so wealth is not exogenous anymore, since housing price may vary. In his model, spillovers between jurisdictions were allowed, because households' utility depends not only on the amounts of local public good provided by its jurisdiction and of the national public good, but also on the amounts of public good provided by all other jurisdictions. After having ensured the existence of an equilibrium under certain conditions, he identified sufficient conditions for a stable jurisdictions structure to be segregated. Unfortunately, one of these sufficient conditions was the absence of spillovers between jurisdictions, which is a pretty strong assumption, that might not be necessary.

In a second article[**?**], Nechyba introduced a computable general equilibrium model of local public good economy based on [7] to provide intuitions about what policy should be implemented by the central government to avoid a sub-optimal provision of local public good in each jurisdiction. As the author pointed it out, the model may be improved by the introduction of interjurisdictional spillovers and peer effects.

The effect of spillovers on the provision of public goods and on the equilibrium have been analyzed by Bloch and Zenginobuz [9] and [10]. However, the authors do not examine the consequences in terms of segregation the existence of spillovers may generate.

This paper generalizes Gravel & Thoron's model by assuming that local public goods may suffer from congestion and create spillovers. Households choose a location, each set of households living in the same place forms a jurisdiction. In each jurisdiction, absentee landlords use the land available in the jurisdiction to produce housing. Housing price is competitive, so, at the equilibrium, the housing supply is equal to the housing demand for that price. Then every jurisdiction democratically determines its tax rate (which is applied to households' wealth), and the revenues generated by this tax are fully used to financed local public services, that may suffer of congestion.

Furthermore, households may benefit from other jurisdictions' local public services. This assumption differs from the main part of the literature on local public goods. However, considering that small towns belonging to a metropolitan area benefit from the public services provided by the main city is not outlandish.

Households are assumed to be freely mobile, so, once all jurisdictions have determined their tax rate, households can leave their jurisdiction for another one that would increase their utility. Equilibrium is reached when no household has incentive to leave unilaterally its jurisdiction or to modify its consumption bundle, the housing price clears the market in every jurisdiction and the local tax rates are democratic. We assume that preferences are homothetically separable between local public services on one hand and private spending and housing on the other hand within the meaning of Blackorby and alii[11]. This assumption, though restrictive, seems to be relevant with respect to recent data [12].

This paper aims at examining the segregative properties of the endogenous jurisdiction structure formation in such a framework. The article is organized as follows. The next section introduces the formal model. Section 3 provides an example of how congestion and spillovers can modify a jurisdictions structure. Section 4 states and proves the results. Finally, section 5 concludes.

2 The formal model

We consider a model *à la* Gravel & Thoron, improved by the presence of a competitive land market, and by the existence of spillovers and congestion effects. There is a continuum of households on the interval $[0; A]$ with Lebesgue measure λ, where, for any subset $I \subset [0; A]$, the mass of household in I is given by $\lambda(I)$. Households' wealth distribution is modelled as a Lebesgue measurable function $\omega : [0; A] \to \mathbb{R}_+^*$ - household i is endowed with a wealth $\omega_i \in \mathbb{R}_+^*$ - with ω being an increasing and bounded from above function.

Households have identical preferences, represented by a twice differentiable, increasing and concave utility function

$$U : \left| \begin{array}{ccc} \mathbb{R}^3_{++} & \longrightarrow & \mathbb{R}_+ \\ (Z, h, x) & \longmapsto & U(Z, h, x) \end{array} \right.$$

where

1. Z is the available amount of public services households can enjoy,
2. h is their amount of housing,
3. x is the amount of the households' expenditures for other things than housing.

We assume that public services are a non-Giffen good.

The utility function is assumed to be homothetically separable in the sense of [11] between the available public services on one hand, and the housing and other expenditures on the other hand. This property implies that the share of their after tax wealth households will spend on housing and on other expenditure does not depend on their after tax wealth, nor on the amount of the available amount of public services. Consequently, the indirect utility function, conditional to the amount of public services, is given by

$$V^C : \left| \begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathbb{R}_+ \\ \psi(p_h, p_x)(\omega_i - p_Z \bar{Z}) & \longmapsto & V^C(\bar{Z}; \psi(p_h, p_x)(\omega_i - p_Z \bar{Z})) \end{array} \right.$$

where

$$V^C(\bar{Z}; \psi(p_h, p_x)(\omega_i - p_Z \bar{Z}) = \max_{h,x} U(\bar{Z}, h, x)$$

$$\text{s.t. } p_h h + p_x x \leq \omega_i - p_Z \bar{Z}$$

and $\psi : \left| \begin{array}{ccc} \mathbb{R}^2_+ & \longrightarrow & \mathbb{R}_+ \\ (p_h, p_x) & \longmapsto & \psi(p_h, p_x) \end{array} \right.$ being a differentiable, increasing and quasi-convex[1] function.

We denote $\mathbb{U}$ as the set of all functions satisfying the properties defined above.

Each household has to choose a place of residence among all the conceivable location. $\mathbb{L} \subset \mathbb{N}$ is the finite set of locations.

Every location l has an amount $H_l \in \mathbb{R}_+^*$ of housing, belonging to absentee landlords that rent it at the unit price p_l. Since housing is costly, and that households only enjoy the housing that is located in their own location, then, obviously, no household will consume housing in more than one location.

[1]A function f is quasi-convex if $\forall x, y \in \mathbb{R}_+$ with $f(x) \geq f(y)$ and $\forall \lambda \in [0; 1], f(\lambda x + (1 - \lambda)y) \leq f(x)$

We do not rule out the possibility for some locations to be empty. If a location is empty, then we assume that $p_l = 0$ and that no tax will be collected. Households living at the same location form a jurisdiction. We denote $J \subseteq \mathbb{L}$ the set of jurisdictions, with $card(J) = M$.

We denote μ_i as the measure of households with private wealth ω_i.

Since local public services create spillovers in other jurisdictions, the total amount of public services a household in jurisdiction j can enjoy (Z_j) depends not only on the available amount of public services produced by jurisdiction j, but also on the amount of public services produced by the other jurisdictions. This amount is given by

$$Z_j = \pi(\zeta_j, S_j)$$

with :

- $\pi : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ being a non-decreasing (strictly increasing with respect to ζ_j), twice differentiable, concave function such that, $\forall S \in \mathbb{R}_+, \pi(0, S) = 0$,
- ζ_j is the available amount of public services produced by jurisdiction j,
- S_j is the amount of spillovers from other jurisdictions in jurisdiction j, given by

$$S_j = \sum_{k \in J - \{j\}} \beta_{jk} \zeta_k$$

  where $\beta_{jk} \in \mathbb{R}_+$ represents the spillovers coefficient of jurisdiction k's local public services in jurisdiction j.

Let us denote B as the square matrix of order M, that represents the spillovers coefficients, with $\forall (j, j') \in J^2, \beta_{jj'} \in [0; 1]$ and $\beta_{jj} = 1$. No assumption needs to be made on B. It can be exogenously determined[2], or depends positively or negatively on the amount of public services provided by the jurisdictions that generate and/or receive them. The matrix B is not necessarily symmetric. We denote $\mathbb{B}$ as the set of all square matrix of order M such that, $\forall (j, j') \in J^2, \beta_{jj'} \in [0; 1]$ and $\beta_{jj} = 1$

The amount of available local public services *produced* by jurisdiction j is given by

$$\zeta_j = \frac{t_j \varpi_j}{C_j}$$

with :

- t_j being the local tax rate,
- ϖ_j being the aggregated wealth in j,
- $C_j = C(\{\mu_k\}_{k \in J}, \{\beta_{kj}\}_{k \in J})$ is the congestion function, with $C : \mathbb{R}_+^{2M} \rightarrow [1; +\infty[$, continuous and non-decreasing (with respect to every argument).

Assuming that the intensity of the congestion faced by jurisdiction j's public services caused by the mass of households in jurisdiction j' depends on the spillovers coefficient j's public services creates in j' is quite reasonable, since public services will suffer more from an important mass of households in a jurisdiction that receives a lot

[2] For instance, spillovers coefficients can represent the distance between two jurisdictions (the closer jurisdiction j is to jurisdiction j', the closer to 1 will be $\beta_{jj'}$), or be affected by political agreements concluded between jurisdictions...

of spillovers from these public services than from a important mass of households that receives little spillovers. Moreover, it is assumed that if jurisdiction j's public services create no spillovers in jurisdiction j' (i.e. $\beta_{j'j} = 0$), then the congestion function is constant with respect to $\mu_{j'}$. Formally,

$$\beta_{j'j} = 0 \Rightarrow \frac{\partial C(\{\mu_k\}_{k \in J}, \{\beta_{kj}\}_{k \in J})}{\partial \mu_{j'}} = 0$$

However, one could specify the congestion measure in such a way that there would be no relation between congestion and the spillovers coefficients. The present definition of the congestion measure does not exclude such an assumption. The only properties assumed on the congestion function are that:

1. $\forall (j,k) \in J^2, \forall \{\mu_l\}_{l \in J} \in \mathbb{R}^M, \forall \{\beta_{lj}\}_{l \in J} \in [0;1]^M$:

$$\lim_{\mu_k \to +\infty} \frac{\partial C_j(\{\mu_l\}_{l \in J}, \{\beta_{lj}\}_{l \in J})}{\partial \mu_k} = 0$$

2. $\forall (j,k,k') \in J^3, \forall \{\mu_l\}_{l \in J} \in \mathbb{R}^M, \forall \{\beta_{lj}\}_{l \in J} \in [0;1]^M$:

$$\frac{\partial^2 C_j(\{\mu_l\}_{l \in J}, \{\beta_{lj}\}_{l \in J})}{\partial \mu_k \partial \mu_{k'}} \leq 0$$

The first property is certainly restrictive, but not unreasonable though, it requires that the marginal congestion in one jurisdiction generated by a infinitesimal increase of the mass of household, in this very jurisdiction or another one, tends to be null when the mass of households is infinite. Homogeneous functions of degree less than 1, for instance, respect this property.

The second property is more natural since, to a certain extent, the masses of households in different jurisdictions are substitutable concerning the congestion they provide in one jurisdiction.

We denote Γ as the set of all congestion function satisfying the properties defined above.

Definition A economy is composed of 5 elements:

- A wealth distribution ω
- Preferences represented by the utility function $U \in \mathbb{U}$
- A congestion measure $C \in \Gamma$
- A spillovers coefficient matrix $B \in \mathbb{B}$
- A set of location $\mathbb{L} \in \mathbb{N}$

We denote Δ as the set of all conceivable economies.

For simplicity, we denote $F_j = \frac{\varpi_j}{C_j}$ as jurisdiction j's fiscal potential, that is to say the maximal available amount of public services jurisdiction j can produce (if $t_j = 1$).

The demands for housing and private consumption of a household i depend on his after-tax wealth $(1 - t_j)\omega_i$ and on the housing price in jurisdiction j, p_j. In every jurisdiction, p_x is normalized to 1. We defined $h^M(p_j, (1-t_j)\omega_i)$ and $x^M(p_j, (1-t_j)\omega_i)$

as respectively the Marshallian demand for housing and for private consumption[3] of a household i in a jurisdiction j, e.g.

$$(h^M(p_j,(1-t_j)\omega_i), x^M(p_j,(1-t_j)\omega_i) \in \arg\max_{h,x} U(Z,h,x)$$

subject to

$$p_j h + x = (1-t_j)\omega_i$$

The local tax rate is determined according to a democratic rule. Hence, every household has to determine its favorite tax rate, denoted $t^* : \mathbb{R}^4_+ \to [0;1]$, which is a function of:

- the fiscal potential F,
- the spillovers from other jurisdictions' public services S (taken as given),
- the housing price p,
- the private wealth ω_i.

Formally,

$$t^*(F,S,p,\omega_i) \in \arg\max_t U(\pi(tF,S), h^M(p,(1-t)\omega_i), (1-t)\omega_i - ph^M(p,(1-t)\omega_i))$$

Lemma 1. *For all utility functions belonging to* $\mathbb{U}$, $\forall (F,S,p,\omega_i) \in \mathbb{R}^4_+$ *and for all functions* $\pi : \mathbb{R}^2_+ \to \mathbb{R}_+$ *satisfying the above properties, preferences are single-peaked with respect to* t, *so* $t^*(F,S,p,\omega_i)$ *exists* .

Proof. The proof of this lemma can be easily obtained by showing that the utility function is concave with respect to t, so there exists an unique

$$t^* \in \arg\max_{t\in[0;1]} U(\pi(tF,S), h^M(p,(1-t)\omega_i), (1-t)\omega_i - ph^M(p,(1-t)\omega_i))$$

Let us denote

$$g_{ij} :$$

$$The first derivative of this function with respect to t is : \frac{\partial g_{ij}(t)}{\partial t} = U_Z F \pi'(tF,S) - \omega_i[\frac{\partial h^M(p,R)}{\partial R} U_h - \frac{\partial x^M(p,R)}{\partial R} U_x]S$$

Let us define

$$\underline{t}^*_j = \min_{i\in I_j} t^*(F_j,S_j,p_j,(1-t_j)\omega_i)$$

and

$$\bar{t}^*_j = \max_{i\in I_j} t^*(F_j,S_j,p_j,(1-t_j)\omega_i)$$

as respectively the lowest and the highest tax rate preferred by a household living in jurisdiction j.

Definition A jurisdictions structure is a vector $\Omega = (J, \{I_j\}_{j\in J}; (\{t_j\}_{j\in J}); (\{p_j\}_{j\in J}), (\{S_j\}_{j\in J})$.

Definition A jurisdictions structure $\Omega = (J, (\{I_j\}_{j\in J}); (\{t_j\}_{j\in J}); (\{p_j\}_{j\in J}), (\{S_j\}_{j\in J})$ is **stable** in the economy $(\omega, U, C, B, \mathbb{L})$ if and only if

[3] Under the homothetic separability assumption, those Marshallian demands do not depend on the amount of public services, that is the reason why Z is withdrawn from the arguments of the Marshallian demands.

1. $\forall j, j' \in J, \forall i \in I_j, U(Z_j, h_{ij}, (1-t_j)\omega_i - p_j h_{ij}) \geq V(Z_{j'}, \psi(p_{j'})(1-t_{j'})\omega_i)$, with h_{ij} being the amount of housing in j consumed by household i,
2. $\forall j \in J, \int\limits_{I_j} h^M(p_j, (1-t_j)\omega_i)d\lambda = H_j$
3. $\forall j \in J, t_j \in [\underline{t}_j^*; \bar{t}_j^*]$

In words, a jurisdictions structure is stable if and only if :

1. No household can increase its utility by modifying its consumption bundle or by leaving its jurisdiction,
2. The housing prices are competitive in every jurisdiction (supply equals demand),
3. The tax rate is democratically chosen in every jurisdiction.

Let us now express formally the definition of the segregation, which is the same definition as in [3].

Definition A jurisdictions structure $\Omega = (J, (\{I_j\}_{j\in J}); (\{t_j\}_{j\in J}); (\{p_j\}_{j\in J}), (\{S_j\}_{j\in J})$ in the economy $(\omega, U, C, B, \mathbb{L})$ is **segregated** if and only if $\forall \omega_h, \omega_i, \omega_k \in \mathbb{R}_+$ such that $\omega_h < \omega_i < \omega_k$, $(h,k) \in I_j$ and $i \in I_{j'} \Rightarrow Z_j = Z_{j'}$ and $\forall \omega \in \mathbb{R}_+, V^C(Z_j, p_j, (1-t_j)\omega) = V^C(Z_{j'}, p_{j'}, (1-t_{j'})\omega)$

In words, a jurisdictions structure is wealth-segregated if, except for groups of jurisdictions offering the same amount of public services and in which every household would have the same utility, the poorest household of a jurisdiction with a high per capita wealth is (weakly) richer than the richest household in a jurisdiction with a lower per capita wealth.

Let us define $J_j = \{k \in J : Z_k = Z_j \text{ and } \psi(p_k)(1-t_k) = \psi(p_j)(1-t_j)\}$. In words, J_j is the set of all jurisdictions offering the same amount of available public services as j, and such that households have the same purchasing power within the meaning of Hicks[4]. Obviously, for all $j \in J$, households are indifferent between all jurisdictions belonging to J_j.

Formally, a jurisdictions structure is segregated if and only if, when, for all $j \in J$, the interval $\bigcup\limits_{k\in J_j} I_k$ is a connected subset of I.

3 Examples

This section presents 2 examples of economies, where congestion and spillovers are introduced in turn, so as to examine their impact on the jurisdictions structure at the equilibrium. In the first example, households' preferences are homothetically separable between the local public services on one hand, and the private spendings on the other hand, but violate the GCS condition (and so the monotonicity of the preferred tax rate function with respect to the private wealth).

In the second one, on the contrary, the GCS condition and the monotonicity of the preferred tax rate function hold, but not the homothetical separability. However, we construct a stable and yet non-segregated jurisdictions structure, which show that neither the GSC condition nor the monotonicity of the preferred tax rate function are

[4] 2 vectors of prices and wealth $(p_1, ...p_K, R)$ and $(p'_1, .., p'_K, R')$ provide the same purchasing power within the meaning of Hicks if and only if $V(p_1, ...p_K, R) = V(p'_1, .., p'_K, R')$

sufficient to ensure the segregation of any stable jurisdictions structure if preferences are not homothetically separable. We also show that the GSC condition, that implies the condition identified by Westhoff to ensure the existence of an equilibrium, is not anymore sufficient when congestion effects are allowed.

3.1 First example: the effect of congestion and spillovers on a stable jurisdictions structure

In this example, we start from a situation where the jurisdictions structure is stable and non segregated. We first assume that public services do not generate spillovers nor suffer from congestion. Then, we introduce congestion effects, which will lead to instability. The new jurisdictions structure will then be segregated. Finally, we consider that local public services generate spillovers, and again, the jurisdictions structure will not be stable anymore, and the first jurisdictions structure will arise. This example suggests that congestion effects increase the segregative properties of endogenous jurisdictions formation, while the presence of spillovers mitigates them.

Let us consider the example provided by Gravel & Thoron, improved by the presence of housing. Households' preferences are represented by

$$U(Z,h,x) = \begin{cases} \ln(Z) + 8\sqrt{hx} - 4hx & \text{if } \sqrt{hx} \leq \frac{7}{4} \\ \ln(Z) + (1 - \frac{14}{4(ln(1.75)-1)})\sqrt{hx} + \frac{49}{16(ln(1,75)-1)}\ln(2\sqrt{hx}) & \text{otherwise} \end{cases}$$

Such an utility function is continuous, twice differentiable, increasing and concave with respect to every argument. The indirect utility function conditional to the public services is given by

$$V^C(Z; p, (1-t)\omega_i)$$

$$=$$

$$\begin{cases} \ln(Z) + \frac{4(1-t)\omega_i}{\sqrt{p}} - \frac{(1-t)^2\omega_i^2)}{p} & \text{if } \frac{(1-t)\omega_i}{p} \leq \frac{7}{4} \\ \ln(Z) + (0,5 - \frac{7}{4(ln(1.75)-1)})\frac{(1-t)\omega_i}{\sqrt{p}} + \frac{49}{16(ln(1,75)-1)}\ln(\frac{(1-t)\omega_i)}{p}) & \text{otherwise} \end{cases}$$

Consider an economy with 2 jurisdictions j_1 and j_2 and 3 types of households a, b, c with private wealth $\omega_a = 2 - \sqrt{2}$, $\omega_b = 1.5$ and $\omega_c = 3$ and whose masses are $\mu_a = \frac{11.9}{2-\sqrt{2}}$, $\mu_b = 8$ and $\mu_c = \frac{1}{30}$. For simplicity, let us assume that the tax rate is determined through the majority voting rule, and that the housing supply is perfectly elastic with respect to its price, that will be considered as fixed to 1 in both jurisdictions.

For all $\omega_i \leq \frac{3+\sqrt{7}}{2}$, the preferred tax rate function is given by

$$t^*(F, S, \omega_i) = \frac{\omega_i - 2 + \sqrt{(\omega_i - 2)^2 + 2}}{2\omega_i}$$

Determining the preferred tax rate function of an households endowed with a private wealth greater than $\frac{3+\sqrt{7}}{2}$ will not be required, since households of type c will never be majoritary in their jurisdiction, so their preferred tax rate will never be applied. One can observe that the preferred tax rate does not depend on the fiscal potential, which will greatly facilitate the example.

Let us assume first that there is no congestion and no spillovers. Then, the available amount of public services in a jurisdiction j is simply the tax revenue: $Z_j = t_j \varpi_j$. Suppose that households of type a and c live in j_1, while households of type b live in j_2. Then, in both jurisdictions, the aggregate wealth will be equal to 12, the tax rate in j_1, denoted t_1, will be equal to $\frac{1}{2}$, while $t_2 = \frac{1}{3}$. Since, at first, it is assumed that $C_1 = C_2 = 1$ and $S_1 = S_2 = 0$, one has $Z_1 = 6$ and $Z_2 = 4$. Such a jurisdictions structure is stable, since the fiscal potential is the same in both jurisdictions, households of type a and b have their favorite tax rate in their respective jurisdiction, and households of type c are better-off in j_1, in which they enjoy an utility level equal to $ln(6) + \frac{15}{4} \approx 5.54$, against $ln(4) + 4 \approx 5.51$ if they would move to j_2.

Now, let us reconsider the example when the local public services suffer from congestion, with

$$C_k = \frac{30 + \sqrt{\mu_k}}{30}$$

Consequently,

$$Z_j = \frac{30(t_j \varpi_j)}{30 + \sqrt{\mu_k}}$$

Then, the amount of public services in j_1 will be equal to

$$Z_1 = \frac{180}{30 + \sqrt{\frac{11.9}{2-\sqrt{2}} + \frac{1}{30}}} \approx 5.216$$

and, in j_2,

$$Z_2 = \frac{120}{30 + \sqrt{8}} \approx 3.655$$

which will leads households of type c to move to j_2, in which they will enjoy an utility level of $ln(\frac{120}{30+\sqrt{8}}) + 2(0.5 - \frac{7}{4(ln(1.75)-1)}) + \frac{49ln2)}{16(ln(1,75)-1)} \approx 5.424$ while their utility level would have been $ln(\frac{180}{30+\sqrt{\frac{11.9}{2-\sqrt{2}}+\frac{1}{30}}}) + 15\frac{}{4\approx 5.402}$ if they had stayed in j_1. Households of type a and b would not have incentive to move, since households of type a would enjoy an utility level of approximatively 2.74 in j_1 against 2.71 in j_2, and households of type b, an utility level of 4.30 in j_2 and 4.09 in j_1.

Is the new jurisdictions structure stable after that households of type c had moved from j_1 to j_2? Since the preferred tax rate function is constant with respect to the fiscal potential, the tax rate will be the same in every jurisdiction. Once households of type c moved from j_1 to j_2, the new amount of public services in j_1 will be

$$Z_1 = \frac{178.5}{30 + \sqrt{\frac{11.9}{2-\sqrt{2}}}} \approx 5.173$$

and in j_2,

$$Z_2 = \frac{120 + \frac{1}{3}}{30 + \sqrt{8 + \frac{1}{30}}} \approx 3.665$$

so households of type a will get a higher utility level in j_1 than in j_2 (approximatively 2.73 against 2.71, while households of type b and of type c can enjoy a higher utility level by staying in j_2 than if they moved to j_1, respectively with 4.30 against 4.08 for households of type b and 5.43 against 5.39 for households of type c, so this new structure is stable and segregated, while the previous one was stable and non-segregated as long as no congestion effects were assumed. In this very specific example, the congestion seems to increase the segregative properties of the endogenous jurisdictions structure formation.

Let us now introduce spillovers in the example to observe what impact they can have. Suppose that jurisdiction j_1's local public services generates spillovers in jurisdiction j_2, and vice-versa. For instance, suppose that the available amount in a jurisdiction j is given by

$$Z_j = \frac{t_j \varpi_j (1 + S_j)}{C_j}$$

with $S_j = \sum\limits_{k \in J \setminus \{j\}} \beta_{jk} \frac{t_k \varpi_k}{C_k}$. Since local public services generate spillovers in other jurisdictions, it is not unreasonable to assume that the congestion function also depends on the mass of households in other jurisdictions and on the spillovers coefficients. Let us then redefine the congestion function as follows:

$$C_j = 1 + \frac{\sqrt{\mu_j + \sum\limits_{k \in J \setminus \{j\}} \beta_{kj} \mu_k}}{30}$$

Although j_1 produces more public services than j_2, suppose that jurisdiction j_1 receives more spillovers from j_2's public services than vice-versa [5]:

$$S_1 = \frac{40 + \frac{1}{9}}{30 + \sqrt{8 + \frac{1}{30}}} \approx 1.222$$

and

$$S_2 = \frac{36.3}{30 + \sqrt{\frac{11.9}{2 - \sqrt{2}}}} \approx 1.034$$

hence $\beta_{12} = \frac{1}{3}$ and $\beta_{21} = \frac{1}{5}$.

With such spillovers coefficients, the available amounts of public services households can enjoy are now

$$Z_1 = \zeta_1 (1 + \frac{1}{3} \zeta_2) \approx 11.29$$

and

$$Z_2 = \zeta_2 (1 + \frac{1}{5} \zeta_1) \approx 7.18$$

As a consequence, households of type c will move back to j_1, in which they will be able to enjoy an utility level of approximatively 6.17, against 6.10 if they stayed in j_2. Households of type a and b can not increase their utility by voting with their feet, since households of type a's utility is about 3.51 in j_1 while it would be about 3.38 in j_2, and households of type b have an utility level of approximatively 4.97 in j_2 against 4.86 if they moved to j_1.

Finally, the jurisdictions structure in which jurisdiction j_1 is composed of households of type a and c, and j_2 is composed of households of type b is stable: with households of type c in j_1 instead of j_2,

$$Z_1 \approx 11.513$$

[5] for instance, j_1 may have implemented a restrictive policy in order to prevent households living in j_2 from congesting its public services.

and

$$Z_2 \approx 7.18$$

One can observe that, due to the existence of spillovers between jurisdictions, households of type c's change of jurisdiction has almost no impact on the available amount of public services in each jurisdiction, then the utility levels will remains almost the same for all type of households.

As a conclusion, this example suggests that congestion favors the segregative properties of endogenous jurisdictions formation, whereas the existence of spillovers tends to decrease the number of stable segregated jurisdictions structures.

However, in the next section, the validity of the GCS condition, that was necessary and sufficient to ensure the segregation of every stable jurisdictions structure, is established within the existence of congestion effect and spillovers.

3.2 Second example: what if preferences are not homothetically separable?

The example follows 2 aims: showing that the GSC condition is not sufficient anymore if preferences are not homothetically separable between public services on one hand and the composite private good and the housing on the other, and then emphasizing the fact that more restrictive conditions must be found to ensure the existence of an equilibrium when public services suffer from congestion.

In this example, preferences are represented by the following function:

$$U : \left| \begin{array}{ccc} \mathbb{R}^3_+ & \longrightarrow & \mathbb{R}_+ \\ (Z,x,h) & \longmapsto & ln(Z)+ln(1+x)+ln(h) \end{array} \right.$$

Then, the Marshallian demands are given by:

$$Z^m(p_Z,p_x,p_h,R) = \frac{R+p_x}{3p_Z}$$
$$x^m(p_Z,p_x,p_h,R) = \frac{R}{3p_x} - \frac{2}{3}$$
$$h^m(p_Z,p_x,p_h,R) = \frac{R+p_x}{3p_h}$$

if $R \geq 2p_x$, and by:

$$Z^m(p_Z,p_x,p_h,R) = \frac{R}{2p_Z}$$
$$x^m(p_Z,p_x,p_h,R) = 0$$
$$h^m(p_Z,p_x,p_h,R) = \frac{R}{2p_h}$$

otherwise.

Clearly, local public services are a gross substitute to both the private good, and preferences are not homothetic between the composite private good and the housing. One can notice that it is equivalent to the condition identified by Westhoff to ensure the existence of an equilibrium (see lemma 4 below).

For a price of public service equal to $\frac{\omega_i}{\varpi}$ and a price of the composite private good normalized to 1, the preferred tax rate function is given by :

$$t^*(\varpi, p_h, R) = \begin{cases} \frac{1}{2} & \text{if } R < 2 \\ \frac{R+1}{3R} & \text{otherwise} \end{cases}$$

One can see that the preferred tax rate function is constant and then strictly decreasing with respect to the private wealth.

Let us suppose that there are 2 jurisdictions j_1 and j_2 and three types of households a, b, c respectively with private wealth $0,5$, 2 and 5 with $\mu_a = 2$, $\mu_b = 5$ and $\mu_c = 4.2$. Let assume first that there is no congestion effect nor spillovers, and that the technology is linear, so $Z = t\varpi$.

Suppose that households of type a and c live in j_1, while households of type b live in j_2. Consequently, if the tax rate is determined through the majority voting rule, then, one has $\varpi_1 = 22$, $t_1 = 0,4$ so $Z_1 = 8,8$, and $\varpi_2 = 10$, $t_2 = 0,5$ so $Z_1 = 5$. If we assume that housing prices are respectively 2 and 1 in j_1 and j_2 and the available amount of land, 4.5 and 5, then the jurisdictions structure is stable: households of type a have an utility of 0.28 while it would be only 0.22 in j_2, households of type b, an utility of 0.92 (against 0.88 in j_1) and households of type c, an utility of 2.17 (against 1.88 in j_2).

If congestion are introduced, with $Z = \frac{t\varpi}{C}$, with $C = 1+\sqrt{\mu}$, then the jurisdictions structure is not stable anymore, because households of type a would be better of by moving to j_2, since their utility would -0.86 against -0.97 if they stayed in j_1. The new jurisdictions structure will have households of type a moving in j_2, while other households will stay in their jurisdiction. But this jurisdictions structure will not be stable neither, because, due to the evolution of the total demand in each jurisdiction, housing prices will increase in j_2 to 1.1, and decrease in j_2 to $\frac{28}{15}$. Consequently, households of type a will have incentive to move back to j_1, because their utility would be -0.9 against -0.95 if they stayed in j_2, while other households could not increase their utility by moving to the other jurisdiction. This will lead to a cycle, so no equilibrium will arise.

Such a result proves that, contrary to Westhoff's model, the single crossing of the indifference curves in the (t, Z) space is not sufficient to ensure the existence of an equilibrium when there is a competitive housing market and when public services suffer from congestion effect.

4 Results

The main result of this paper is the robustness of the GSC condition to the existence of spillovers and congestion effect on the local public services to have all stable jurisdictions structures segregated. As in Gravel and Thoron (2007), this condition is equivalent to the monotonicity of the preferred tax rate function with respect to the private wealth, for any given amount of the other arguments. To prove this equivalence, let us first establish the following lemma.

Definition Let us define $\pi^{-1}: \left| \begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathbb{R}_+ \\ Z & \longmapsto & \pi^{-1}(Z;\bar{S}) \end{array} \right.$ as the amount of available public services produced in a jurisdiction needed to have a total available amount of public services Z if the amount of spillovers is $\bar{S}$. Formally, $\pi(\pi^{-1}(Z;\bar{S}),\bar{S}) = Z$

Since $\forall \bar{S} \in \mathbb{R}_+, \pi(\zeta,\bar{S})$ is continuous and strictly increasing with respect to ζ, $\pi^{-1}(Z;S)$ always exists.

Lemma 2. *$\forall U \in \mathbb{U}$, $\forall (F,S,p,\omega_i) \in \mathbb{R}^4_+$, the preferred tax rate function is a monotonic function of the Marshallian demand for the public good:*

$$t^*(F,S,p,\omega_i) \equiv \frac{1}{F}\pi^{-1}[Z^M(\frac{1}{F\pi_\zeta(t^*(F,S,p,\omega_i)F,S)},\frac{p}{\omega_i},\frac{1}{\omega_i},1);S] \tag{1}$$

Proof. At the optimum, the Marginal Rate of Substitution (MRS) is equal to the price ratio. Then:

$$\frac{U_Z(Z,h,x)}{U_x(Z,h,x)} = \frac{p_Z}{p_x} \tag{2}$$

The FOC of the utility maximization program with respect to t implies that:

$$\frac{U_Z(\pi(t^*F,S),h^M(p,(1-t^*)\omega_i),(1-t^*)\omega_i - ph^M(p,(1-t^*)\omega_i))}{U_x(\pi(t^*F,S),h^M(p,(1-t^*)\omega_i),(1-t^*)\omega_i - ph^M(p,(1-t^*)\omega_i))} = \frac{\omega_i}{F\pi_\zeta(t^*(F,S,p,\omega_i)F,S)} \tag{3}$$

because, using the envelop theorem, we know that $pU_x = U_h$. Then, using (2) and (3), we know that:

$$\pi(t^*(F,S,p,\omega_i)F,S) \equiv Z^M(\frac{\omega_i}{F\pi_\zeta(t^*(F,S,p,\omega_i)F,S)},p,1,\omega_i) \tag{4}$$

Since Marshallian demands are homogeneous of degree 0, we can divide all the arguments of the Marshallian demand for the public services by ω_i.

$$\pi(t^*(F,S,p,\omega_i)F,S) \equiv Z^M(\frac{1}{F\pi_\zeta(t^*(F,S,p,\omega_i)F,S)},\frac{p}{\omega_i},\frac{1}{\omega_i},1) \tag{5}$$

Then, using the definition of $\pi^{-1}(Z;S)$, we know that:

$$t^*(F,S,p,\omega_i)F,S) \equiv \pi^{-1}(Z^M(\frac{1}{F\pi_\zeta(t^*(F,S,p,\omega_i)F,S)},\frac{p}{\omega_i},\frac{1}{\omega_i},1);S) \tag{6}$$

This lemma states that the favorite tax rate function is equivalent to an increasing function of the Marshallian demand for public services. This lemma is used to prove that the favorite tax rate function is monotonic with respect to the private wealth if and only if the public services are either always a substitute or always a complement to the housing and the available wealth. This condition is called the Gross Substitutability Complementarity (GSC) condition.

If the GCS condition holds, then, one has either $Z^M(p_Z,p_x,p_h,R)p_x \leq 0 \forall (p_Z,p_x,p_h,R)$ (if Z is a gross complement to x) or $Z^M(p_Z,p_x,p_h,R)p_x \geq 0 \forall (p_Z,p_x,p_h,R)$ (if Z is a gross substitute to x)

For all utility function that are homothetically separable between the public services on one hand and the housing and private consumption on the other, the public services is a complement (resp. a substitute) to the private consumption if and only if it is also a complement (resp. a substitute) to the housing.

Lemma 3. *If public services are a non-Giffen good, then, for all utility functions belonging to* $\mathbb{U}$ *and all production function* π *that are increasing and concave with respect to* ζ*, the favorite tax rate function is always monotonic with respect to private wealth if and only if the public services are a gross substitute or a gross complement to the 2 other goods.*

Proof. To prove this lemma, we will show that the derivative of the preferred tax rate function with respect to the private wealth can be expressed as a negative function of the derivative of the Marshallian demand for the public services with respect to the housing price. Consequently, the preferred tax rate function will be monotonic with respect to the private wealth if and only if the Marshallian demand for public services are monotonic with respect to the housing price (or the composite private good price).

By deriving (1) with respect to the private wealth, one gets :

$$\frac{\partial t^*(F,S,p,\omega_i)}{\partial \omega_i}$$

$$=$$

$$\frac{-1}{\pi_\zeta(\zeta,S)}[\frac{\pi_{\zeta\zeta}(\zeta,S)}{F\pi_\zeta^2(\zeta,S)}\frac{\partial Z^M(\frac{1}{F\pi_\zeta(\zeta,S)},\frac{p}{\omega_i},\frac{1}{\omega_i},1)}{\partial p_Z}\frac{\partial t^*(F,S,p,\omega_i)}{\partial \omega_i}+\frac{p+k}{\omega_i^2}\frac{\partial Z^M(\frac{1}{F\pi_\zeta(\zeta,S)},\frac{p}{\omega_i},\frac{1}{\omega_i},1)}{\partial p_h}]$$

with

$$\frac{\partial Z^M(\frac{1}{F\pi_\zeta(\zeta,S)},\frac{p}{\omega_i},\frac{1}{\omega_i},1)}{\partial p_x}=k\frac{\partial Z^M(\frac{1}{F\pi_\zeta(\zeta,S)},\frac{p}{\omega_i},\frac{1}{\omega_i},1)}{\partial p_h}$$

with $k>0$, then one has :

$$(1+\frac{\pi_{\zeta\zeta}(\zeta,S)}{F\pi_\zeta^2(\zeta,S)}\frac{\partial Z^M(\frac{1}{F\pi_\zeta(\zeta,S)},\frac{p}{\omega_i},\frac{1}{\omega_i},1)}{\partial p_Z})\frac{\partial t^*(F,S,p,\omega_i)}{\partial \omega_i}=\frac{-(p+k)}{\omega_i^2}\frac{\partial Z^M(\frac{1}{F\pi_\zeta(\zeta,S)},\frac{p}{\omega_i},\frac{1}{\omega_i},1)}{\partial p_h}$$

Since the public services is assumed to be a non-Giffen good, and since $\pi(\zeta,S)$ is concave with respect to ζ, then $(1+\frac{\pi_{\zeta\zeta}(\zeta,S)}{F\pi_\zeta^2(\zeta,S)}\frac{\partial Z^M(\frac{1}{F\pi_\zeta(\zeta,S)},\frac{p}{\omega_i},\frac{1}{\omega_i},1)}{\partial p_Z})>0$, so

$$sign(\frac{\partial t^*(F,S,p,\omega_i)}{\partial \omega_i})=-sign(\frac{\partial Z^M(\frac{1}{F\pi_\zeta(\zeta,S)},\frac{p}{\omega_i},\frac{1}{\omega_i},1)}{\partial p_h})$$

The GSC condition is restrictive enough to be discussed, but nevertheless it is not outlandish. For instance, suppose that the only competence the central government has transferred to the studied jurisdictions level is social aid. Then, one may assume that the public services will be a substitute to the two other goods. On the contrary, if the jurisdiction is competent only in cultural activities, then the public services will probably be a complement. Suppose now that the jurisdiction is in charge of primary schools. In this case, the relation between the public services and the other goods is not trivial, and may vary with respect to the jurisdiction's parameters and the private wealth.

To prove the sufficiency of the GSC condition, the notion of indifference curve has to be introduced.

Definition $\forall(t,\bar{u},p,S,\omega_i)\in[0;1]\times\mathbb{R}_+^4$, let us define $F^{\bar{u}}(\bar{u},t,p,S,\omega_i)$:

$$U(\pi(tF^{\bar{u}}(\bar{u},p,t,S,\omega_i),S),h^M(p,(1-t)\omega_i),x^M(p,(1-t)\omega_i))\equiv\bar{u}$$

as the indifference curve of a household with private wealth ω_i, that is to say the amount of fiscal potential the household needs to reach utility $\bar{u}$ in a jurisdiction with housing price p, tax rate t and spillovers S.

The assumptions imposed on the utility function and on $\pi(.)$ ensure the existence and the derivability of F^u.The slope of the indifference curve in the plane (t,F) is given by

$$F_t^{\bar{u}}(\bar{u}, t, p, S), \omega_i) = \frac{1}{t}[\frac{\omega_i}{\pi_\zeta(tF, S) MRS^{\bar{u}}(\pi(t\bar{F}, S), (1-t)\omega_i)} - \bar{F}] \quad (7)$$

The next lemma, that will be used to prove the sufficiency of the GSC condition to have every stable jurisdictions structure segregated, states that the GSC condition implies the ordering of the indifference curves slopes with respect to wealth.

Lemma 4. *For any preferences belonging to $\mathbb{U}$ and any production function that is increasing and concave with respect to each argument, one has* $\frac{\partial F^{\bar{u}}(\bar{u},t,p,S,\omega_i)}{\partial t} \leq$ *(resp.* $\geq$*)* $\frac{\partial F^{\bar{u}}(\bar{u},t,p,S,\omega_k)}{\partial t} \forall (t,p,S) \in [0;1] \times \mathbb{R}_+^2, \omega_i < \omega_k$ *and* $\forall U \in \mathbb{U}$ *if the public services is a gross substitute for (resp. a gross complement to) the two other goods*[6]*.*

Proof. This lemma states that if the GSC condition holds, then, for any given housing price, any given amount of spillovers and any tax rate, the slope of the indifference curves in the (t, F) space is monotonic with respect to the private wealth. We prove this lemma by using the definition of $F^{\bar{u}}(\bar{u}, t, p, S, \omega_i)$ introduced above. The proof is provided for the gross complementary case, the gross substitutability case being symmetric. Assume that the public services is a gross complement to the two other goods. Then, by definition, $\frac{\partial Z^M(p_Z,p_h,p_x,\omega_i}{\partial p_h} < 0$ and $\frac{\partial Z^M(p_Z,p_h,p_x,\omega_i}{\partial p_x} < 0$. Let $(t, F, S, p) \in [0; 1] \times \mathbb{R}_+^4$ be a certain combination of tax rate, fiscal potential, spillovers and housing price and $(a, b) \in \mathbb{R}_+^2$ two amount of private wealth ($a < b$). Let us define (a) and $\omega(a)$ such that

$$Z^M(\frac{1}{\pi_\zeta(tF,S)F(a)}, \frac{p}{\omega(a)}, \frac{1}{\omega(a)}, 1) = \pi(tF, S)$$

and

$$ph^M(\frac{1}{\pi_\zeta(tF,S)F(a)}, \frac{p}{\omega(a)}, \frac{1}{\omega(a)}, 1) + m^M(\frac{1}{\pi_\zeta(tF,S)F(a)}, \frac{p}{\omega(a)}, \frac{1}{\omega(a)}, 1) = (1-t)a$$

Hence, the Marginal Rate of Substitution between the public services on one hand, and the housing and the other expenditure on the other hand, which is a function $MRS^u(Z, h+m)$ is equal, at the optimum, to the price ratio, and, by definition, the chosen bundle respects the budget constraint :

$$MRS^u(\pi(tF, S), (1-t)a) = \frac{\omega(a)}{\pi_\zeta(tF,S)F(a)} \quad (8)$$

$$\frac{\pi(tF,S)}{\pi_\zeta(tF,S)F(a)} + \frac{(1-t)a}{\omega(a)} = 1 \quad (9)$$

[6]Actually, the ordering of the indifference curve slopes with respect to the private wealth is equivalent to the GSC condition, but the implication is sufficient to prove our theorem. However, it shows that the GSC condition implies the condition identified by Westhoff to ensure the existence of an equilibrium, when households have identical preferences.

Combining (8) and (9) yields :

$$\frac{1-t}{\pi_\zeta(tF,S)F(a)+\pi(tF,S)} = \frac{MRS^u(\pi(tF,S),(1-t)a)}{a}$$

Let us now define $\omega(b)$ such that $\frac{p}{\omega(b)}$ and $\frac{1}{\omega(b)}$ are respectively the highest housing price and available money price that would allow a household with private wealth 1 to afford the bundle (not necessarily the optimal one) $(\pi(tF,S),h,x)$, with $\frac{ph+x}{\omega(b)} = (1-t)b$, if the public services price is still $\frac{1}{\pi_\zeta(tF,S)F(a)}$. Given the budget constraint, one has :

$$\omega(b) = \frac{\pi_\zeta(tF,S)F(a)(1-t)b}{\pi_\zeta(tF,S)F(a)-\pi(tF,S)} > \omega(a) \tag{10}$$

Since the public services is a complement, then one must have :

$$Z^M\left(\frac{1}{\pi_\zeta(tF,S)F(a)},\frac{p}{\omega(a)},\frac{1}{\omega(a)},1\right) \leq Z^M\left(\frac{1}{\pi_\zeta(tF,S)F(a)},\frac{p}{\omega(b)},\frac{1}{\omega(b)},1\right)$$

Moreover, the slope of the indifference curve must be, in absolute value, more than the price ratio $\frac{\omega(k)}{\pi_\zeta(tF,S)F(a)}$:

$$MRS^{\bar{u}}(\pi(tF,S),(1-t)b) \geq \frac{\omega(k)}{\pi_\zeta(tF,S)F(a)}$$

which is equivalent to

$$\frac{MRS^{\bar{u}}(\pi(tF,S),(1-t)b)}{b} \geq \frac{(1-t)}{\pi_\zeta(tF,S)F(a)-\pi(tF,S)}$$

Using (10), one obtains:

$$\frac{MRS^{\bar{u}}(\pi(tF,S),(1-t)b)}{b} \geq \frac{MRS^{\bar{u}}(\pi(tF,S),(1-t)a)}{a}$$

$$\Leftrightarrow$$

$$\frac{b}{MRS^{\bar{u}}(\pi(tF,S),(1-t)b)} \leq \frac{a}{MRS^{\bar{u}}(\pi(tF,S),(1-t)a)}$$

Using the definition of $F_t^{\bar{u}}$ given by (7), the implication is established.

This lemma is particularly important to prove the sufficiency of this article's main result, which is the following theorem.

Theorem 1. *For any possible economy $(\omega,U,C,B,\mathbb{L}) \in \Delta$, every stable jurisdictions structure will be segregated if and only if U is such that the GSC condition is satisfied.*

Let us begin the proof of this theorem by the sufficiency of the condition.

Proposition 1. *For all economies $(\omega,U,C,B,\mathbb{L}) \in \Delta$, **if** the GSC condition holds, then every stable jurisdictions structure is segregated.*

Proof. To prove this proposition, we use the lemma 4 to demonstrate that if a non-segregated jurisdictions structure arise at the equilibrium, then the GSC condition is not respected. This proof needs no assumption on how the spillovers coefficients are determined. Suppose that there exist 2 jurisdictions j_1 and j_2 with respective parameters (F_1, S_1, t_1, p_1) and (F_2, S_2, t_2, p_2) and 3 households with private wealth a, b, c, $a < b < c$, such that :

$$V^C(\pi(t_1F_1), S_1), \psi(p_1)(1-t_1)a) > V^C(\pi(t_2F_2, S_2), \psi(p_2)(1-t_2)a)$$
$$V^C(\pi(t_1F_1), S_1), \psi(p_1)(1-t_1)b) < V^C(\pi(t_2F_2, S_2), \psi(p_2)(1-t_2)b)$$
$$V^C(\pi(t_1F_1), S_1), \psi(p_1)(1-t_1)c) > V^C(\pi(t_2F_2, S_2), \psi(p_2)(1-t_2)c)$$

Suppose, with no loss of generality, that $p_1 > p_2$.
Consider the hypothetical jurisdiction j_0 with parameters (F_0, S_2, t_0, p_2) with $t_0 = 1 - (1-t_1)\frac{\phi(p_1)}{\phi(p_2)}$ and $F_0 = \frac{\pi^{-1}(\pi(t_1F_1, S_1); S_2)}{t_0}$ Hence, every household is indifferent between j_1 and j_0, because in both jurisdictions, the amount of available public services is the same and their purchasing power is the same. Then,

$$V^C(\pi(t_0F_0), S_2), \psi(p_2)(1-t_0)a) > V^C(Z_2, \psi(p_2)(1-t_2)a)$$
$$V^C(\pi(t_0F_0), S_2), \psi(p_2)(1-t_0)b) < V^C(Z_2, \psi(p_2)(1-t_2)b)$$
$$V^C(\pi(t_0F_0), S_2), \psi(p_2)(1-t_0)c) > V^C(Z_2, \psi(p_2)(1-t_2)c)$$

which, according to the lemma 4, is impossible if the GSC condition holds.

Now that the sufficiency of the GCS condition to have all stable jurisdictions structure segregated has been proved, the following proposition states that it is also necessary, by showing that any violation of the GCS condition allows to construct a non-segregated but yet stable jurisdictions structure.

Proposition 2. *For all economies belonging to Δ, every stable jurisdictions structure will be segregated **only if** the GSC condition holds.*

Proof. The proof of this proposition consists in constructing a stable and yet non-segregated jurisdictions structure. We are free to determine the number of jurisdictions, their available amount of housing, the mass of each type of households in every jurisdiction, and the spillovers coefficients matrix, in order to generate the housing price, the fiscal potential and the amount of spillovers for which the violation of the GSC condition arise.

Consider an utility function violating the GCS condition for some $(\bar{F}, \bar{S}, \bar{p}) \in \mathbb{R}^3_+$ and some non-degenerated interval $W \subset \mathbb{R}_+$. Using lemma 3, the monotonicity of the favorite tax rate function is known to be equivalent to the GSC condition, so we know for sure that there exist $(a, b, c) \in W^3$, with $a < b < c$, such that $t^*(\bar{F}, \bar{S}, \bar{p}, a) = t^*(\bar{F}, \bar{S}, \bar{p}, c) > t^*(\bar{F}, \bar{S}, \bar{p}, b)$ (the proof is the same if the favorite tax rate is increasing and then decreasing with respect to the private wealth). Then one can always construct a stable and non-segregated jurisdictions structure. Let us create 2 subsets of jurisdictions, both jurisdictions having a fiscal potential $\bar{F}$, a housing price p and receiving an amount of spillovers $\bar{S}$:

- jurisdictions belonging to J_1 are composed of certain measures μ_a and μ_c of households endowed with private wealth a and c, and apply a tax rate $t_1 = t^*(\bar{F}, \bar{S}, \bar{p}, a) = t^*(\bar{F}, \bar{S}, \bar{p}, c)$

- jurisdictions belonging to J_2 are composed of a certain measure μ_b of households endowed with private wealth b, and apply a tax rate $t_2 = t^*(\bar{F}, \bar{S}, \bar{p}, b)$.

Such a jurisdictions structure is clearly non-segregated, because the 2 different types of jurisdiction provide different amounts of public services. Moreover, no household has incentive to leave its jurisdiction, since its favorite tax rate is applied, and other parameters are the same in all the other jurisdictions. Let us now prove that there always exist positive measures μ_a, μ_b, μ_c, integers M_1 and M_2 of jurisdictions of respectively type 1 and type 2, a matrix of spillovers coefficients and available amount of housing H_1 and H_2 such that, in each jurisdiction :

- Spillovers are equal to $\bar{S}$,
- Fiscal potential is equal to $\bar{F}$,
- Housing price $\bar{p}$ is competitive.

We define respectively $\zeta_1 = t_1\bar{F}$ and $\zeta_2 = t_2\bar{F}$ as the amount of public services produced by jurisdictions of type 1 and of type 2. For simplicity, jurisdictions in J_1 will not create any spillovers for jurisdictions belonging to J_2, and vice-versa. Since the function $S(.)$ is non-bounded from above, we know that

$$\forall S \in \mathbb{R}_+, \forall(\tilde{\beta}, \tilde{\zeta}) \in \mathbb{R}_+^{*2}, \exists M \in \mathbb{N} : (M-1)\tilde{\beta}\tilde{\zeta} < S \leq M\tilde{\beta}\tilde{\zeta}$$

In words, for any strictly positive amount of produced public services and spillovers coefficient, by duplicating a jurisdiction a certain number of times, any amount of spillovers can be bounded from below and from above, even if spillovers coefficients depend on the amount of public services produced by the jurisdiction can generate or receive spillovers. Since $S = 0$ when the spillovers coefficient are null, one can deduce, using the Theorem of Intermediate Value, that $\exists \beta^* \in [0; \tilde{\beta}] : M\beta^*\tilde{\zeta} = S$

As a consequence, we can always find (M_1, β_1) and (M_2, β_2) such that

$$M_1\beta_1\zeta_1 = M_2\beta_2\zeta_2 = \bar{S}$$

Now, let us prove that we can always find a positive measures μ_a, μ_b, μ_c such that, in each jurisdiction, the fiscal potential is $\bar{F}$. Let us consider a jurisdiction j belonging to J_2. This jurisdiction is only composed of households endowed with a private wealth b. Since all jurisdictions in J_2 have the same measure of households and the same spillovers coefficients, the congestion function can be re-written as a function of the measure of households, the spillovers coefficient and the number of jurisdictions in J_2. Let us define $C^{J_2}(\beta_2, \mu_b, M_2) = C(\{\mu_b\}_{j\in J_2}, \{\beta_2\}_{l\in J_2})$. Hence, the fiscal potential of this jurisdiction j is $F_j = \frac{\mu_b b}{C(\beta_2, \mu_b, M_2)}$.

Using the properties assumed on the congestion function, we can prove that

$$\lim_{\mu_b \to +\infty} \frac{\mu_b b}{C(\beta_2, \mu_b, M_2)} \to +\infty$$

Indeed, since

$$\forall(j, k, k') \in J^3, \forall\{\mu_l\}_{l\in J} \in \mathbb{R}^M, \forall\{\beta_{lj}\}_{l\in J} \in [0; 1]^M$$

one has

$$\frac{\partial^2 C_j(\{\mu_l\}_{l\in J}, \{\beta_{lj}\}_{k\in J})}{\partial\mu_k\partial\mu_{k'}} \leq 0$$

Then $\forall (j,k,k') \in J^3, \forall (\mu_k, \mu_{k'}) \in \mathbb{R}^2_{++}$, and $\forall \mu'_{k'} > \bar{\mu}_{k'}$, one has

$$0 < \frac{\partial C_j(\mu_1, ..., \mu_k, ..., \mu'_{k'}, ..., \mu_{M_2}, \{\beta_{lj}\}_{l\in J})}{\partial \mu_k} \leq \frac{\partial C_j(\mu_1, ..., \mu_k, ..., \bar{\mu}_{k'}, ..., \mu_{M_2}, \{\beta_{lj}\}_{l\in J})}{\partial \mu_k}$$

As a consequence,

$$0 < \lim_{\mu'_{k'} \to +\infty} \frac{\partial C_j(\mu_1, ..., \mu_k, ..., \mu'_{k'}, ..., \mu_{M_2}, \{\beta_{lj}\}_{l\in J})}{\partial \mu_k} \leq \frac{\partial C_j(\mu_1, ..., \mu_k, ..., \bar{\mu}_{k'}, ..., \mu_{M_2}), \{\beta_{lj}\}_{l\in J}}{\partial \mu_k}$$

By definition, one has $\lim_{\mu_k \to +\infty} \frac{\partial C_j(\mu_1, ..., \mu_k, ..., \bar{\mu}_{k'}, ..., \mu_{M_2}, \{\beta_{lj}\}_{l\in J})}{\partial \mu_k} = 0$. Then, using the version of the Squeeze theorem[7] provided in Matousek[13][8], one can deduce that

$$\lim_{\mu_k \to +\infty} \lim_{\mu'_{k'} \to +\infty} \frac{\partial C_j(\mu_1, ..., \mu_k, ..., \mu'_{k'}, ..., \mu_{M_2}, \{\beta_{lj}\}_{l\in J})}{\partial \mu_k} = 0$$

This proof can be re-iterated to show that, $\forall (j,k) \in J^2, \forall \{\mu_l\}_{l\in J} \in \mathbb{R}^M_{++}, \forall \{\beta_{lj}\}_{l\in J} \in \mathbb{R}^M_{++}$,

$$\lim_{\mu_1 \to +\infty} \lim_{\mu_2 \to +\infty} ... \lim_{\mu_M \to +\infty} \frac{\partial C(\{\mu_l\}_{l\in J}, \{\beta_l j\}_{l\in J})}{\partial \mu_k} = 0$$

So, $\forall \beta_b \in [0;1]$ and $\forall M_2 \in \mathbb{N}$, one has

$$\lim_{\mu_b \to +\infty} \frac{\partial C^{J_2}(\mu_b, \beta_2, M_2)}{\partial \mu_b} = 0$$

Using L'hopital's rule[9] [14], one can show that:

$$\lim_{\mu_b \to +\infty} \frac{\mu_b b}{C^{J_2}(\mu_b, \beta_2, M_2)} = \lim_{\mu_b \to +\infty} \frac{b}{\frac{\partial C^{J_2}(\mu_b, \beta_2, M_2)}{\partial \mu_b}} = +\infty$$

Moreover, if the mass of households in a jurisdiction is null, then so is its fiscal potential. Hence, by the Intermediate Value Theorem, we know that there exists μ^*_b such that $\frac{\mu^*_b b}{C(\beta_2, \mu^*_b, M_2)} = F$. The same reasoning can be applied for jurisdictions in J_1, taking a constant mass of households with private wealth a over mass of households with private wealth c ratio. We now choose the available amount of housing H_1 and H_2 respectively in jurisdictions in J_1 and J_2 such that the housing price $\bar{p}$ is competitive, i.e.

$$H_1 = \mu_a h^M(\bar{p}, (1-t_1)a) + \mu_c h^M(\bar{p}, (1-t_1)c)$$

$$H_2 = \mu_b h^M(\bar{p}, (1-t_2)b)$$

Then, for any violation of the monotonicity of the preferred tax rate function with respect to the private wealth, one can always construct a stable and yet non-segregated jurisdictions structure.

[7]The Squeeze theorem (also called the Sandwich rule) states that if $\forall x \in E, u(x) \leq f(x) \leq v(x)$ then, $\forall \bar{x} \in E$ such that $\lim_{x\to\bar{x}} u(x) = l$ and $\lim_{x\to\bar{x}} v(x) = l$, one has $\lim_{x\to\bar{x}} f(x) = l$.

[8]2003, Chap. 3, Section 1, the Ham Sandwich Theorem

[9]L'Hopital's rule states that $\lim_{x\to\bar{x}} \frac{u(x)}{v(x)} = \lim_{x\to\bar{x}} \frac{u'(x)}{v'(x)}$ if u and v are differentiable.

5 Conclusion

The conclusion of this paper is that neither the congestion nor the existence of spillovers across jurisdictions modify the necessity or the sufficiency of the GSC condition to ensure the segregation of every stable jurisdictions structure in a model *a la* Westhoff.

However, this result does not imply that introducing congestion and spillovers into a model *a la* Westhoff would have no impact on the stability or on the segregative properties of endogenous jurisdictions structures formation, as the example provided in section 3 proved it.

The presence of congestion effects obviously mitigates the existence of an equilibrium. Consequently, stronger conditions must be found in order to ensure that a stable jurisdictions structure will arise.

This condition is robust to several generalizations of the model. Searching for a generalization that would make the condition either too weak or too strong to have all stable jurisdictions structures would be a interesting objective for further researches.

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