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Fair Apportionment in the Italian Senate: Which Reform Should Be Implemented?

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Summary: In this paper we analyze the fairness of the 2007 reform proposal concerning the apportionment of the seats between the regions for the Italian Senate. Theory of power indices is used to compare the actual case with the proposed one. Two scenarios are proposed, senators belonging to the same region voting in blocks and senators voting according party lines, using both the Impartial Culture and the Impartial Anonymous Culture models. Our objective is to determine which apportionment is closer to the equal distribution of power among the citizens. In addition, we will seek for apportionments that are closer to the ideal representation than the ones proposed by politicians. We will also derive the probability that different apportionments produce a referendum paradox, i.e. exhibit a majority in the Senate different from the national popular majority.

JEL classification: **C7, D7**

Keywords: Power index, Banzhaf, Italian Senate, apportionment, voting paradox, Monte Carlo simulation.

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1 Introduction

Since the collapse of the Christian Democratic party in the 90s due to political scandals, Italian governments have always tried to face the need for wide constitutional reforms. In between the revision of several parts of the most important law of the country and some attempts in order to give a federal shape to the nation, a reduction in the number of deputies and senators was planned many times, as well as a deep change of the Senate's role. *In this paper we analyze the fairness of the 2007 reform proposal concerning its apportionment of the seats between the regions.* Esposito (2008) tells the history of the original creation of this chamber, firstly thought as the place in which the interests of the new Italian regions would have been represented. These proposed modifications to the Senate never saw the light, as the equilibrium formed by the several political colors in the assembly forbade it. The final shape of the Senate was just a copy of the Chamber of Deputies, with a smaller number of members[1].

The functions of this second chamber are identical to the ones of the first: elaborate, discuss and approve new laws at the national level. Article 57, that explains the constitution of the Senate, gives one senator every 200.000 people or a fraction larger than 100.000 to every region[2]. Some minima were guaranteed to every region, in particular one senator fixed for the Aosta Valley and a minimum of six senators for all other regions.

The introduction of Molise in 1963 gave two senators to this region (and raised the minimum of the other regions to seven). 1963 is also the year of a reform concerning the apportionment of the seats in both the Chamber of Deputies and the Senate. The size of the two assemblies were adjusted respectively to 630 and 315 members (plus honorific senators). The method chosen for the apportionment of the seats between the regions was the method of integers combined with the highest remainders: in other words, apportion to each region a number of seats equal to the whole number contained in each region's fair share with the remaining seats allocated to the regions with the highest remainders [3]. In January 17th 2000 a new constitutional law introduced the abroad constituency, modifying the articles

[1] Plus a maximum of 5 honorific senators named by the President of the Republic for a life time as well as all former Presidents of the Republic (who are senators by right).

[2] This method is equivalent to the Webster method (see Balinski and Young (2001)) with a fixed divisor and a moving chamber dimension (see Esposito (2008)).

[3] This method, called Hamilton method, was used for the U.S. House of representatives between 1850 and 1890.

48, 56 and 57 of the Constitution, ensuring twelve seats at the Chamber of Deputies and six seats at the Senate to this new region. The actual populations and number of senators for each region are displayed in Table 1.

In October 17th 2007 the commission for constitutional affairs of the Chamber of Deputies approved a new proposal implying a wide change of the Senate functions. Regional competencies should be in some way resurrected, and the apportionment rule deeply changed. Schematically, article 3 of the text, replacing the article 57 of the Constitution, gives a fixed number of seats to every region, and this number increases whenever the population rises above a certain threshold. A total of six seats is allocated to regions with less than one million of inhabitants, nine seats to the ones smaller than three million people, eleven to regions not bigger than five million citizens, twelve to the ones smaller than seven million and fourteen to the others. Most senators (from five to twelve according to the population of each region) would be elected by the Regional Council and a few (one or two, still depending on population) by the Council of the local autonomies, elected by the regional populations. Two senators are allocated both to Aosta Valley and Molise, and six both to Trentino-Alto Adige (three for each Province) and to the Italians resident abroad. With the data of the last census, the number of senators would be 186, as shown in table 1. The Italian Senate would then look like the German's federal upper House, the Bundesrat, where each land is represented by 3 to 6 members, designated by local authorities.

Then we face two situations for the Senate, the actual one, which nearly corresponds to the situation of 1963, except for the Abroad citizens, and the proposal, which would modify not only the number of seats but as well the relative apportionment between the regions.

The purpose of this paper is to analyze the fairness of this reform proposal compared to the *statu quo*. Our goal is not, like Balinski and Young (2001), to take into account the ideal situation "one man-one vote" that directly calls for the proportional principle. We want compare the actual apportionment and the proposal by using more sophisticated tools borrowed from game theory and social choice theory. First, following the power index literature, we wish to equalize the influence one Italian citizen gets, whatever the region he/she lives. This last situation is the ideal situation in terms of *power* or *influence*. Secondly, we wish to minimizes the situations where a majority in the Senate is different from the national popular majority. Such a situation is called a *referendum* paradox in Social Choice literature (see Nurmi (1999)) and any two tiers voting system can always

Table 1: Number of senators per region

Region	Population (2001)	Actual seats	Proposal seats
Lombardy	9 032 554	47	14
Campania	5 701 931	30	12
Lazio	5 112 413	27	12
Sicily	4 968 991	26	11
Veneto	4 527 694	24	11
Piedmont	4 214 677	22	11
Apulia	4 020 707	21	11
Emilia-Romagna	3 983 346	21	11
Tuscany	3 497 806	18	11
Calabria	2 011 466	10	9
Sardinia	1 631 880	9	9
Liguria	1 571 783	8	9
Marche	1 470 581	8	9
Abruzzo	1 262 392	7	9
Friuli-Venezia Giulia	1 184 764	7	9
Trentino-Alto Adige	940 016	7	6
Umbria	825 826	7	6
Basilicata	597 768	7	6
Molise	320 601	2	2
Aosta Valley	119 548	1	2
Abroad	3 649 377	6	6
Total	60 646 121	315	186

show this problem. We think that this phenomenon happened during the 2006 election, when the left coalition, *L'unione*, won a one seat majority at the Senate, while the right camp, *Casa delle libertà*, secured a thin majority in terms of votes[4].

For both criteria, a two parties/coalitions system (A and B) will be considered, and

[4]Our analysis is a reconstruction of the electoral data, by roughly aggregating the votes and seats of the minor parties competing in Valle d'Aosta, Trentino Alto-Adige and Abroad to the major camps. These parties protect the interests of these specific sensible areas, and in their own territory they are wide majority. We can anyway aggregate these votes to one of the two camps considering the specific history of a party and its general support to governments of the left or of the right. Even if one comes with a different conclusion with a different reconstruction, the situation was at least close to be a referendum paradox.

that is reasonable concerning the Italian framework. Smaller parties exist, but their role is irrelevant, while medium and big parties use to present themselves at the elections united in a bigger list.

The tasks of equalizing the influence and minimizing the probability of the referendum paradox both need the definition of an *a priori* probabilistic assumption to model the behavior of the voters. We will use two different assumptions to model the behavior of the voters, Impartial Culture (IC) and Impartial Anonymous Culture (IAC)[5]: In the first case electors cast their vote by flipping a fair coin, to decide whether to vote A or B, leading to Straffin's Independence assumption (1977) and Banzhaf measure of power (Banzhaf, 1965). In turns, each region will favor A or B with probability $\frac{1}{2}$. In the second one, every voter in region i has the same probability p_i to vote for A drawn from a uniform distribution on $[0, 1]$, giving a degree of homogeneity to the votes of that region (see Straffin, 1977). The Shapley-Shubik index is now the correct measure of the power of the citizens within each region (Shapley and Shubik, 1954). But in this paper, we further assume that each regions selects its common probability p_i independently, meaning that they can exhibit a different behavior in the same issue. However, on everage, each party can call a region with the same probability $\frac{1}{2}$.

Thus, when we add the results of all the regions, both the models will have, over a series of many elections, a distribution of the "A" votes at the federal level depicted by a normal law with 0.5 as mean. For the IC case, both the local and federal results will be close. In the IAC case, a region can exhibit a strong bias for a candidate if p_i is far from $\frac{1}{2}$, but at the federal level, the election will still be close, while the variance will be bigger. So, for both the models, we will use Banzhaf index to compute the power among the regions.

To compare the *statu quo* with the reform, we will present two scenarios. In the first one, all the senators belonging to a same region have the same preferences, that is to say we suppose that every region has only one senator who has several votes. This assumption seems to be strong, but the new proposal intends to build a Senate with regional competencies: the assumption that senators belonging to different parties but to the same region can vote in the same way now makes sense. And if the senators are elected by population mandate from the Regional Council how the proposal suggests, then we can suppose that there

[5]We borrowed the names of the probabilistic assumptions from the Social Choice literature and Fishburn and Gehrlein (1976) rather than from power index litteraure and Straffin (1977).

could be a “winner takes all” case like in the United States or in the German’s Bundesrat. Anyway we still do not have a precise mechanism saying how the cake will be split. The new system provides for a new institution, the “Council of the local autonomies”, electing parts of the senators, but it is not clear yet how and when the election of this assembly will work: if it is at the same time of the Regional Council election it is very probable that majorities inside the two houses will be the same, and by consequence we can consider the voting in blocks scenario as plausible.

The second scenario assumes that senators vote according to party lines even if they belong to the same region. The Law #270, approved in December 21st 2005, gives a rounded up 55% of the seats to the winning coalition in every region, in the case in which that coalition does not pass itself this threshold (see Esposito(2008)). We will see in section 3 how this threshold will alter dramatically the behavior of the electoral rule under IC.

To summarize, we intend to mobilize tolls from social choice theory and game theory to compare two possible apportionment of the seats among the Italian regions, using 1) two different criteria, 2) two different probabilistic assumptions 3) two different scenarios about selection of the senators in the regions.

Section 2 presents the concept of power that is widely used in the literature and the main power indices, due to Banzhaf (1965) and Shapley and Shubik (1953) as well as their applications to two tiers voting systems. In section 3 we provide the analysis for the two voting scenarios (regions voting as blocks, or along political party lines with a prize for the winner), and the two apportionments of the seats, following the structure of Felsenthal and Machover (2004)’s analysis. Our objective is to determine which apportionment is closer to the equal distribution of power among the citizens. But as we will see, none of them is satisfying. In section 4 we estimate the probability that different apportionments produce a referendum paradox. In section 5 we seek for apportionments of the seats that are closer to the ideal representation than the ones proposed by politicians. Section 6 concludes.

2 Equalizing power and influence

2.1 The concept of power

The theory of power indices shows that there exists an important difference between the number of mandates of a player (here an Italian region) and his/her influence in a voting situation. The classical example is the majority election where a party gets 50 seats, another one gets 49 seats and the third one gets 2 seats. If a winning coalition becomes a loosing coalition whenever a party leaves it, then we say that he/she has power. In such situation, the party is said to be a swing player. Thus, in our example, the party with 2 votes has as much influence as the big parties, as two parties, whatever their size, are necessary to build up a minimal winning coalition. Based on this notion, several power indices have been proposed in the literature (see for example Felsenthal and Machover (1998), or Laruelle (1998) for a clear presentation).

To present more formally the concept of power, it is useful to recall the idea of a weighted game, which is a particular voting game. Let $N = (1, 2...n)$ be the set of players and let a_i be the number of seats of player i, with $a = \sum_{i=1}^{n} a_i$. A weighted game is $G = [q; a_1, ..., a_n]$, where q is called the quota. As we will apply the theory to real voting bodies, we will assume throughout the paper that the a_i's and the q are integers. A winning coalition S (written $S \in W$, with W the set of winning coalitions) is a group of players such that

$$S \in W \iff \sum_{i \in S} a_i \geqslant q$$

The most famous voting game is the majority game, which perfectly corresponds to the method used in Italian Senate, where $q = \frac{a}{2} + 1$ if a is even and $q = \frac{a+1}{2}$ if a is odd.

Once the weighted game is defined, we can present the Banzhaf indices. Firstly, one have to determine all $2^n - 1$ possible coalitions (non empty) and the number of times for which the player i is a swing player. If this number is divided by 2^{n-1} (that is the number of coalitions containing the player i), we obtain the non-normalized Banzhaf power index, that is, the probability of being decisive. If it is divided by the total numbers of swing players, we obtain the normalized Banzhaf power index.

The non-normalized Banzhaf power index, or Penrose measure, is

$$\beta'_i = \frac{\sum_{S \subseteq N} |v(S) - v(S \backslash \{i\})|}{2^{n-1}}, \tag{1}$$

where $v(S) = 1$ if $S \in W$ and $v(S) = 0$ otherwise. Note that $[v(S) - v(S \backslash \{i\})]$ is different from 0 if and only if the player i is a swing player in coalition S.

Equivalently, β'_i gives the probability that a player i is decisive, given that he/she decides with equal probability and independently from the others whether joining or not a coalition. This assumption is known in the literature as the Independence assumption (see Straffin (1977)).

The formula of the normalized Banzhaf power index for the player i is

$$\beta_i = \frac{\sum_{S \subseteq N} |v(S) - v(S \backslash \{i\})|}{\sum_{j \in N} \sum_{S \subseteq N} |v(S) - v(S \backslash \{j\})|}. \tag{2}$$

The definition of the Shapley-Shubik index (1954) of a player i follows a different reasoning. Suppose voters join a coalition in a random order, there will always be a unique subject that will let the coalition win. This subject is the pivot. The index will be calculated as

$$\phi_i = \frac{number\ of\ orderings\ in\ which\ player\ i\ is\ pivotal}{n!}, \tag{3}$$

where $n!$ is the total number of orderings. Shapley-Shubik index is normalized by definition, the sum of ϕ_i for every i gives always 1, as there is always one (and only one) pivot in every ordering. Its formula in terms of swings is

$$\phi_i = \sum_{|v(S) - v(S \backslash \{i\}| \neq 0} \frac{(s-1)!(n-s)!}{n!}, \tag{4}$$

where s is the size of a coalition S. $(s-1)!$ and $(n-s)!$ are the permutations of the elements before and after the pivot i.

Consider the following example: Let a 3-player majority game $G = [3; 2, 1, 1]$. The different coalitions are $\{1\}$,$\{2\}$, $\{3\}$, $\{\underline{1}, \underline{2}\}$,$\{\underline{1}, \underline{3}\}$, $\{2, 3\}$ and $\{\underline{1}, 2, 3\}$, where the swing players are underlined. For example, the coalition $\{\underline{1}, \underline{2}\}$ becomes a losing one if one player leaves it. Thus the two players are both swing players. We obtain $\beta'_1 = 3/4$, $\beta'_2 = 1/4$ and $\beta'_3 = 1/4$ while $\beta_1 = 3/5$, $\beta_2 = 1/5$ and $\beta_3 = 1/5$ for $i = 1, 2, 3$.

All possible orderings of the three players are $\{1, \underline{2}, 3\}$, $\{1, \underline{3}, 2\}$, $\{2, \underline{1}, 3\}$, $\{2, 3, \underline{1}\}$, $\{3, \underline{1}, 2\}$ and $\{3, 2, \underline{1}\}$, where pivotal players are underlined. So, $\phi_1 = 2/3$, $\phi_2 = 1/6$ and $\phi_3 = 1/6$.

These two different indices answer the question that Straffin posed himself in 1977: "What is the the probability that player i's vote will make a difference in the outcome"? The solution depends from the probabilistic model we use to handle the behavior of voters. With the Independence assumption every player votes for a party with probability $\frac{1}{2}$, so the probability of each coalition S of size s is $p_S = (\frac{1}{2})^s(\frac{1}{2})^{n-s} = (\frac{1}{2})^n$. This value multiplied by 2 considers both the coalitions S and $S - i$, and multiplying by number of swings of player i, we have the probability of being pivotal for i, $p_i = \frac{\sum_{S \subseteq N} |v(S) - v(S \backslash \{i\})|}{2^{n-1}} = \beta'_i$.

With the Homogenous assumption, people inside a same region have common values, and the probability to vote for a party is drawn by a uniform variable between 0 and 1. A *priori*, not knowing this probability inside each region, each a coalition S of size s will have probability

$$P(S) = \int_0^1 p^s(1-p)^{n-s}dp, \tag{5}$$

the continuous sum for all the possible drawable probabilities. This integral is equivalent to a combinatoric expression passing through the use of Gamma and Beta functions (see Sebah and Gourdon (2002)). Knowing that $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $\Gamma(x+1) = x\Gamma(x)$, and that for natural values becomes $\Gamma(n+1) = n!$, we can state that, for $x = s+1$ and $y = n-s+1$,

$$P(S) = \frac{s!(n-s)!}{(n+1)!}. \tag{6}$$

Following Straffin (1977), taking the sums in the swing cases for player i, $|v(S) - v(S\backslash\{i\}| \neq 0$,

$$\sum(P(S)+P(S-\{i\})) = \sum(\frac{s!(n-s)!}{(n+1)!} + \frac{(s-1)!(n-s+1)!}{(n+1)!}) = \sum \frac{(s-1)!(n-s)!}{n!} = \phi_i. \tag{7}$$

If in a multi-region country we know that citizen in states vote independently, we know that we should calculate the power of each citizen inside the single regions using the non-normalized Banzhaf index. If citizens vote homogeneously, i.e. different regions have different common values in valuating a party, the power of each voter inside a single state is measured with the Shapley-Shubik index.

2.2 Evaluating power for two-tiers voting rules

A direct democracy, where the population can vote its rules by itself, is hardly implemented as form of government of a country. Small nations or municipalities can widely use popular *referendum*, as it is the case of Switzerland, where voters are called several times per year. But bigger societies may require a two tiers voting rule to handle an efficient law making process: the population elects the representatives and these ones produce the laws.

Actually, the electoral rule int the Italian Senate is a two-tiers vote: Every citizen of a region votes for senators and, in the second step, every senator votes the laws. And a subject has power when his/her presence in a coalition is determining for the victory of the latter. Hence, to be a swing player, an individual has to be a swing player in his/her region and the region has to be a swing region from the national point of view.

Since the works of Penrose (1946) and Banzhaf (1965), we can find in the literature many references to the Square Root Rule: It asserts that under the assumption of Independence, or IC (see Gehrlein (2006)), the voting power or influence of an individual in the union is proportional to the inverse of the square root of the population of his home state or region.

To state it, first assume there are n_i voters in state i and the same parties compete in each state of the Union. We assume that abstention is not allowed, and that each vote is casted flipping a fair coin, according to the IC assumption. Next, notice that the probability that a voter is decisive in the union is equal to the probability that the state the voter belongs to is decisive in voting a bill multiplied the probability that the voter is decisive in the election of the representatives inside his/her own state.

Penrose's limit theorem (Penrose (1952)) states that, if the union is wide enough and there's no state controlling a big amount of mandates, the Banzhaf power β'_i of state i with a_i mandates can be approximated by the number of its mandates:

$$\frac{\beta'_i}{\beta'_{i'}} \approx \frac{a_i}{a_{i'}}. \tag{8}$$

The validity of the approximation has been tested via simulations by Chang *et al.* (2006), while Feix *et al.* (2007) and Slomiczyński and Zyczkowski (2008) have found that this proportionality is even better when the state game is played with a super majority rule.

In a two-tiers voting system, we need to consider also the power of each voter j inside his/her state i. If n_i is odd[6], voter j will be decisive in changing the result of the election in

$$\binom{n_i - 1}{(n_i - 1)/2}$$

cases over the $2^{n_i - 1}$ configurations of votes. This probability can be written, using Stirling's approximation for $n!$ when n is large, as $\sqrt{\frac{2}{\pi n_i}}$. As immediate consequence, using the IC assumption, we have that

$$\frac{Probability\ that\ voter\ j\ from\ state\ i\ is\ decisive\ in\ the\ union}{Probability\ that\ voter\ j'\ from\ state\ i'\ is\ decisive\ in\ the\ union} \approx \frac{a_i}{a_{i'}} \frac{\sqrt{n_{i'}}}{\sqrt{n_i}}, \tag{9}$$

and so a fair allocation of power is achieved when the number of mandates a_i is proportional to the square root of the population of the state, $\sqrt{n_i}$.

Anyway, Gelman *et al.* (2004) have empirically shown by studying electoral data that IC assumption has to be rejected: there is correlation between the population sizes and

[6] A similar reasonning holds for n_i even.

the margin of victory for one party. To bypass this argument, we can say that computing power using IC gives us an *a priori* result. To project a system we need to be blind and we cannot consider the present situation that could change in the long run. Alternatively, the assumption of Homogeneity, or IAC, can be used as well to model the behavior of the voters inside the states. It enables us to circumvent Gelman's critics against IC, while keeping at the macro level the *a priori* position that no party is favored in the long run.

Thus, according to IAC, a random probability p_i to vote for party A is drawn from a uniform in each region i, and voters will be distributed around this value following a normal law. But each region is independent from each other, so the final distribution of the sum of the vote of the regions is a normal law centered on $\frac{1}{2}$. Notice that Homogeneity inside regions allow parties to overpass the 55% threshold of the regional majority prize, while the probability of this event is negligible under IC for large n_i.

We know that by assumption the Shapley Value is normalized to 1, and each voter is symmetric. Thus, $\phi_j = \frac{1}{n_i}$ in each region i. As regions vote independently from each other, Banzhaf power is still the right measure to consider at the second level, and by consequence Penrose's limit theorem holds:

$$\frac{Probability\ that\ voter\ j\ from\ state\ i\ is\ decisive\ in\ the\ union}{Probability\ that\ voter\ j'\ from\ state\ i'\ is\ decisive\ in\ the\ union} \approx \frac{a_i}{a_{i'}}\frac{n_{i'}}{n_i}, \tag{10}$$

and equal treatment is obtained apportioning seats proportionally to the population of each state.

2.3 Being swing in Trentino-Alto Adige

The theory we presented in the previous sections explains how to measure the power of a voter (using non-normalized Banzhaf and Shapley-Shubik indices) when the winner takes all principle applies. We will see in section 3.2 that the current voting rule (with a 55 % majority prize) can also be analyzed with the same tool. The only problem comes from the particular methods in use in Trentino-Alto Adige/Sud Tirol (TAA). The law # 422, approved in 1991, divides TAA in 6 electoral districts in which 6 different elections with different candidates are run. Only the most voted participant is elected in each district. The problem is in the attribution of the seventh seat that the actual apportionment gives to this region (the proposal gives 6 seats to TAA, so the problem disappears). To attribute the extra seat, first consider the votes obtained by the losers in each district and regroup these wasted votes according to party lines. The party which gathers the maximal number of wasted votes obtain the seventh seats, which goes to its candidate who obtained the most vote but was not elected directly.

The existence of an extra seat given to the best loser complicates the analysis. The situations where a voter is swing can be very peculiar. Using the IC, each district encounters a close election and vote for A with probability $\frac{1}{2}$. The situation is summarized in table 2. The probability of each possible allocation of the x seats to one of the two parties is given by $\frac{\binom{6}{x}}{2^6}$. In the case of 3 seats each party, the extra seat will go to one of the two rivals with probability 50%, forming two different cases.

Table 2: Trentino-Alto Adige attribution of the extra seat

Case	District seats	Extra seat	Final apportionment	Probability
I	6A, 0B	B	6A, 1B	$\frac{1}{64}$
II	5A, 1B	B	5A, 2B	$\frac{6}{64}$
III	4A, 2B	B	4A, 3B	$\frac{15}{64}$
IV.1	3A, 3B	A	4A, 3B	$\frac{10}{64}$
IV.2	3A, 3B	B	3A, 4B	$\frac{10}{64}$
V	2A, 4B	A	3A, 4B	$\frac{15}{64}$
VI	1A, 5B	A	2A, 5B	$\frac{6}{64}$
VII	0A, 6B	A	1A, 6B	$\frac{1}{64}$

The key point is that being pivotal in a district, due to the extra seat, does not mean that you are pivotal in the system, as passing from case IV.2 to case V (or from III to IV.1) does not change the final result. And you can be swing without being swing in a district: this is the case of a switch between case IV.1 and IV.2. To be pivotal between IV.1 and IV.2, we need that the sum of the votes of the losers from each party is equal. Then, choosing one side or the other, without being pivotal in my district (and losing), will give the extra seat to one of the two parties.

The situation is even worse under the IAC assumption, as a situation where the party gets more seats and the extra seat is not as unlikely as under IC (consider the case where A wins four districts with almost 100% of the votes and barely loses the other two). Finding the exact probability of being pivotal for a voter in TAA proves to be tricky and tedious, though not impossible technically.

Thus, to simplify the analysis, we will assume that each district have the chance to get the second seat with probability $\frac{1}{6}$. We will see that this will transform that district in a dummy player with zero power. This is only an approximation of the reality but should not change qualitatively the results.

3 Analysis

In this section we compare the actual seats apportionment and the proposed reform. This comparison is made on the basis of the relative power of each region. The ideal situation corresponds to the case where each citizen has the same amount of power. The power of the regions is computed, in both IC and IAC models, using the non-normalized Banzhaf power index developed in section 2. The next subsections will analyze, as seen in the work of Felsenthal and Machover (2004), the fairness of the different rules in the two scenarios described in the introduction of this paper.

The case of TAA is different in the two rules we are examining. In the actual situation the seats are given as described in section 2.3, with TAA divided in six districts (Trento-Valle di Non, Rovereto-Riva del Garda, Pergine-Fiemme-Fassa, belonging to the Autonomous Province of Trento, and Bolzano-Bassa Atesina, Merano and Bressanone-Brunico belonging to the Autonomous Province of Bolzano) each one having one mandate, with an extra mandate for one of them with equal probability (this is the sense of 1+ in the tables). The official populations of the districts are very difficult to find and even to calculate them using populations of every single cities: to remove this obstacle, we use the relative number of votes of each district multiplied by the total population of the belonging Province (known data) as a proxy. In the proposal we just have 3 seats for each Province, Trento and Bolzano.

The law concerning the vote abroad gives one senator to each of the four districts (1. Europe; 2. South America; 3. North and Central America; and 4. Asia, Africa, Oceania and Antartide, ranked from the most to the least populated with the actual census) and apportions the remaining seats using Hamilton method (see Balinski and Young, 2001). The extra seats are 2, so the two smallest districts will always have only one seat. Using the official populations, at the moment we have two seats the biggest districts, Europe and South America.

3.1 The first scenario: regions voting as blocks

In the following calculations, mostly obtained thanks to Leech computer algorithms (http://www.warwick.ac.uk/ ecaae/), we show the non-normalized (β') and the normalized (β) Banzhaf indices in the case in which the regions vote in blocks, i.e. the winning party in each district gets all the mandates. This situation is unrealistic in the actual situation, but very realistic in the proposal, as the Senate would be elected from regional assemblies in which there is a clear governing majority.

Table 3 displays the results of the IC assumptions for the apportionment rule. The objective here is to check the proportionality between the vector of power and the square root of the populations: the Quotient is the division between β and the relative square root population (IC). This variable should be as close as possible to 1 for each region; a value superior to 1 (inferior to 1) indicates that it is over weighted (under weighted). The Standard Deviation σ between the Quotient and the vector of ones will tell us how far from optimality we are[7]. We can observe in the Quotient column a typical pattern: the bigger the region, the higher the quotient is. This leads to high inequality: citizens from Lombardia are pivotal 12 times more than people from the Europe region. Even if we do not consider the abroad constituencies, the ratio is 6 to 1 between Lombardia and Trento-Valle di Non. The importance of the inequality is confirmed by the high value of $\sigma = 0.494$. None of these results are surprising, as the current apportionment in the Senate is more or less proportional to the populations while the IC scenario points towards the square root rule weights.

The proposal does not use the square root rule, but clearly levels the weights of all the regions. We may expect less inequalities from the results displayed on table 4. Indeed, the ratio between the most favored region (Friuli-Venezia Giulia) and the least favored (Europe) has been reduced to 6 to 1. Considering Italian regions only, the ratio even drops to 2.4 to 1, and the value of the Quotient of the major region is close to 1. Overall σ drops to 0.241. In an IC world, and if the senators vote as blocks, the proposal is clearly better than the actual situation in terms of equal power.

Let us turn now to the IAC assumption. The picture drawn by table 5 is much more favorable to the actual situation. First of all the vectors of power are now compared to the

[7] To calculate the standard deviation, we take the square root of sum of the distances between 1 and the quotients weighted by their ratio $\frac{p_i}{\sum p_i}$, where p_i is the population of each region i.

vectors of population as the square root rule do not apply under this assumption. Thus, all the Italian regions except the small ones have a Quotient close to 1. The maximal inequality is uncounted between a small Italian region, Basilicata (Quotient=2.188) and a large Foreign region, Europe (Quotient=0.180). σ equals now 0.237.

Moving to table 6, the situation becomes less favorable to the proposal, as proportionality is the right normative assessment under IAC. Large regions like Lombardia are clearly under weighted (Quotient=0.514) while the Quotient of the citizens of Valle d'Aosta jumps to 5.354! The ratio between Valle d'Aosta and Europe is now 17.3. σ here reaches its higher value, equal to 0.599.

To conclude, when regions vote as blocks, the comparison is completely driven by the underlying probability model. IC assumption points toward the proposal while the more realistic IAC model clearly declares as winner the actual situation.

3.2 The second scenario: senators voting following party lines

In this section we will assume that senators can vote according party lines in order to study the impact of the 55% prize in the actual Italian electoral system. Whenever a party gets more than 55% of the votes in a region the system works as a Hamilton apportionment rule (see Balinski Young, 2005). However the proportional principle is distorted in the range 45% - 55%; there is a premium for the party which gets in between 50% and 55%, and an equivalent harm for the party in between 45% and 50%. So the electoral rule becomes a mixture between a winner takes all system and a proportional system. This peculiarity will impact differently the analysis between the IC assumption and the IAC assumption.

With IC, elections results are always very tight elections: the bigger the populations, the smaller the deviations of each party from the 50%. This allow us to say that passing the threshold of 55% of the votes that will stop giving a majority prize to the winner is impossible, and by consequence one party will always enjoy the majority prize. By consequence, we can already give part of the seats to each party and consider only the prize seats as pivotal seats. For example, Lombardia has 47 seats in the actual system (see table 7). The majority prize will give a rounded up 55% of the seats to the winning party, that means 26. This means that each party will have 21 seats for sure, and only 5 seats can swing in block. Thus, under IC, the real weight of Lombardia is 5! The same logic applies to all the other regions.

Table 3: Actual situation, regions voting as blocks, IC

District	**Square Root Population**	**Relative S.R.P.**	**Weight**	β'	100β	**Quotient**
Lombardia	3005.42	8.40%	47	0.4788	16.40	**1.952**
Campania	2387.87	6.68%	30	0.2782	9.53	1.427
Lazio	2261.06	6.32%	27	0.2486	8.52	1.347
Sicily	2229.12	6.23%	26	0.2387	8.18	1.312
Veneto	2127.84	5.95%	24	0.2197	7.53	1.265
Piedmont	2052.97	5.74%	22	0.2001	6.86	1.194
Apulia	2005.17	5.61%	21	0.1907	6.53	1.165
Emilia-Romagna	1995.83	5.58%	21	0.1907	6.53	1.171
Tuscany	1870.24	5.23%	18	0.1626	5.57	1.065
Calabria	1418.26	3.97%	10	0.0904	3.10	0.781
Sardinia	1277.45	3.57%	9	0.0812	2.78	0.778
Liguria	1253.71	3.51%	8	0.0720	2.47	0.704
Marche	1212.68	3.39%	8	0.0720	2.47	0.728
Abruzzi	1123.56	3.14%	7	0.0630	2.16	0.686
Friuli-Venezia Giulia	1088.47	3.04%	7	0.0630	2.16	0.709
TAA - *Trento-Valle di Non*	457.03	1.28%	1+	0.0105	0.36	**0.280**
TAA - *Rovereto-Riva del Garda*	400.99	1.12%	1+	0.0105	0.36	0.320
TAA - *Pergine-Fiemme-Fassa*	327.64	0.92%	1+	0.0105	0.36	0.391
TAA - *Bolzano-Bassa Atesina*	402.13	1.12%	1+	0.0105	0.36	0.319
TAA - *Merano*	367.65	1.03%	1+	0.0105	0.36	0.349
TAA - *Bressanone-Brunico*	407.59	1.14%	1+	0.0105	0.36	0.314
Umbria	908.75	2.54%	7	0.0630	2.16	0.849
Basilicata	773.15	2.16%	7	0.0630	2.16	0.998
Molise	566.22	1.58%	2	0.0179	0.61	0.388
Valle d'Aosta	345.76	0.97%	1	0.0090	0.31	0.318
Europe	1439.59	4.03%	2	0.0179	0.61	**0.153**
South America	1008.85	2.82%	2	0.0179	0.61	0.218
North and Central America	599.88	1.68%	1	0.0090	0.31	0.183
Asia Africa Oceania Antartide	446.47	1.25%	1	0.0090	0.31	0.246

Another peculiarity of the Italian electoral law is that we can have dummy players: If a region have two seats (like Molise, Europe, South America) the premium does not apply and each party will have one seat for sure under IC[8]. The power of the citizens in these regions is therefore zero. Similarly, for TAA, each district has one seat with probability $\frac{5}{6}$ but it has two seats with probability $\frac{1}{6}$ due to extra seat awarded to a loser. Then, according to our simplifying assumption, it is dummy with probability $\frac{1}{6}$.

With IAC, we cannot use the same reasoning, as parties can overpass the 55% of the votes. A voter is pivotal when he/she can alter the number of seats of party A in each region and modify the global balance of power. At the 50% vote threshold he/she can move 10% of the seats but there are many other thresholds where he/she can move 1 seat. For example, due to the Hamilton apportionment method, in Lombardia party A will receive 0 seats if it gets less than $\frac{1}{94}$ of the votes and 1 seat if it gets exactly $\frac{1}{94}$. The next threshold will be $\frac{3}{94}$ where a pivotal voter can decide when party A gets 1 or 2 seats, etc. But to be pivotal in the federation the number of seats that A already received from Lombardia and all the other regions must be 157. More generally if region i has a_i seats, a voter will be pivotal at all the thresholds $\frac{2t+1}{2a_i}, t = 0, \ldots, a_i - 1$ which does not belong to the interval 45% - 55% plus the 50% threshold. As all these points are equally likely and all the voters are symmetric, the probability of being pivotal under IAC is $\frac{1}{n_i}$ in a specific region i.

Next, consider the seats gathered by A in the regions other than i. Define $L_i = \frac{a}{2} - (a_i - 1)$ and $U_i = \frac{a}{2}$, with a_i the number of seats of the considered region and $a = \sum_{t=1}^{N} a_t$ even, (respectively $L_i = \frac{a-1}{2} - (a_i - 1)$ and $U_i = \frac{a-1}{2}$ when $a = \sum_{t=1}^{N} a_t$ is odd). For each of the repartition of the other mandates for A in between L_i and U_i, there is a repartition of the vote in region i where voter j is pivotal. For example, each time the other regions will collect all together in between 111 and 157 seats for A with majority equal to 158, there exist situations in Lombardia (47 seats) where a voter will be pivotal. So for each region i, we need to estimate the probability P_{-i} that the sum of the mandates for A gathered by the other regions lies in between L_i and U_i.

To estimate the probability P_{-i} we use a Monte Carlo simulation. With IAC, for each region other than i we draw independently the result between 0 and 1 and add the seats obtained by A taking into account the majority prize. Then, for each region i, the distri-

[8] Giving to each party its integer number of seats and attributing the extra seats to the party with higher remainders will apportion both the seats to a single party if it gets more than 75% of the votes, that with IC is impossible for large population.

Figure 1: Voting power of Lombardia, actual situation, party lines, IAC

bution of the seats for a party obtained in the other regions follows a normal law (partially corrupted by the majority prize) and we can compute how many times the considered region is pivotal to reach a majority. The estimation of P_{-i} is obtained after 1 million iterations. Figure 1 shows the distribution of the votes of all regions but Lombardia. The black area, which is the power of this region, is equal to 0.71: Lombardia, with 47 seats, will be pivotal if the other regions will collect all together between 111 and 157 seats for A, with a majority equal to 158.

The index calculated with this simulation models the situation in which each senator can follow his/her party line. It is the probability for the region to be pivotal, and by consequence a power index. *In fact, if we gives all the seats of a region to the winning party inside this simulation, and not only the ones that a party gets considering the majority prize and that the latter applies only until having reached the 55% of the votes, the final indices are equal to the Banzhaf measure of power. In the following tables representing the IAC model's results, same size regions have very small differences in power, this is obviously a result of the simulations and these power levels should be considered as equal: for example Apulia and Emilia-Romagna, both having 21 seats in the actual situation, have a discrepancy of 0.0005 in their power that should be neglected.*

Table 8 shows the results for the IC assumption in the actual situation. We can immediately notice something bizarre: the weights are not monotone, for example Sicily, Veneto and Piedmont have more weight, and by consequence power, than Lazio. Molise, Europe and South America are now dummy players, with zero power! The ratio between the Quotients of Valle d'Aosta and Sardinia, respectively the most and the least advantaged regions (dummies excluded), is equal to 3.7, and $\sigma = 0.340$.

The proposal, as we can see in table 9, let the three biggest regions having less power than the block of regions going from Sicily to Tuscany. This is because the pivotal weights are no longer monotone if we apply the 55% rule! Umbria and Basilicata are better represented than all the group of bigger regions having weight equal to 1. Valle d'Aosta here becomes a dummy, while the situation for Molise, Europe and South America does not change in respect to table 8. Thus, Valle d'Aosta passes from being the top over weighted region in the actual situation to be a dummy with the proposal. The ratio between the most and least advantaged regions, Basilicata and Lombardia, is 3.9, and $\sigma = 0.442$.

Turning to the IAC model, we notice immediately from table 10, that presents the results in the actual situation, that dummies disappear. As we draw results from each region from $[0, 1]$ the peculiarities of the electoral system that the IC assumption magnified do not bite anymore. Basilicata is now the most advantaged region, and its ratio measured over the Quotient of Europe is 12, and over the least advantaged Italian region, Calabria, is 2.3. σ is equal to 0.239.

In the proposal, table 11 shows Valle d'Aosta as a winner and Europe and Lombardia as the losers: ratios between Quotients are respectively 17.3 and 10.6, with a large σ: 0.605.

Finally, IC model states as best option the proposal when regions vote in blocks. Voting in blocks is a reasonable assumption here as the members of the Senate in a region could all belong to the same party, being elected by the Regional Council. If senators can follow party lines, at the contrary, the actual situation is the best choice.

IAC model gives a different result: the actual situation is the best option in both the cases, the standard deviations being very close in the two assumptions. This means that,when parties can break the threshold of 55%, the peculiarities of the majority prize have no longer a big influence on the estmiation of power.

4 The probability of having a *referendum* paradox

Equalizing the influence of the citizens is not the only criteria that we can use in order to compare two-tiers voting systems. Using exactly the same probability assumptions we can also computing the probability of the *referendum* paradox and search for the apportionment method that minimizes it. A *referendum* paradox happens when the national outcome of an election as a whole is different from the majority in the assembly (Nurmi 1999). A typical example is the U.S. presidential election in 2000 when George W. Bush won with a minority in the national vote. Feix *et al.* (2004, 2009) have shown with computer simulations that the probability of the *referendum* paradox tends towards 20.5% under IC (respectively 16.5% under IAC) when the different regions have the same populations. We want to determine this probability in the Italian case, and see if this is consistent with the theoretical results.

In order to do so, we ran some simulations for several scenarios. Following Feix *et al.*

(2009, the distribution of the votes follows a normal law under the IC assumption when the population of a region n_i is large. By consequence, the excess of the votes towards one of the two parties will be $\varepsilon_i\sqrt{n_i}$, with ε_i drawn from a Gaussian. Without loss of generality we can say that A is the popular winner if

$$sgn\left(\sum_i \varepsilon_i\sqrt{n_i}\right) > 0 \tag{11}$$

and it is also supported by the representatives if

$$sgn\left(\sum_i a_i \cdot sgn\left(\varepsilon_i\right)\right) > 0, \tag{12}$$

where the function $sgn(x)$ gives 1 or -1 as output depending if x is more or less than 0. A referendum paradox occurs when expression (11) and (12) differ in sign. The same process applies for the IAC simulation, except that the ε_i are now drawn from a uniform distribution $[-1; 1]$ independently.

On the top of evaluating the probability of the referendum paradox of the eight scenarios studied in section 3 we will consider the whole family of α rules. An α rule give a_i seats to a region proportionally to $a_i = n_i^\alpha$. If α is equal to 0, the assembly is federal, and every region has the same representation. If $\alpha = 0.5$, it gives seats proportionally to the square root of the population. $\alpha = 1$ is the perfect proportionality while $\alpha \to \infty$ means dictatorship of the biggest region.

In our simulations, we let α going between 0 and 2, in order to find the best α rule that minimizes the possibility of a paradox. At this stage we do not take in consideration a fixed size for the Senate, here we care only of proportionality rules. Then, we will compute the exact probabilities for the real apportionments we consider. For every α we run 100,000 elections. When we consider the majority prize and regions voting following party lines, we have to determine the new pivotal seats. It is easy to see that the number of pivotal seats at the 50% threshold is $2R^+(R(n_i^\alpha \cdot 0.55)) - R(n_i^\alpha)$, where R^+ is a rounding up to the smaller higher integer and R a normal rounding. With IAC we will run elections giving to each party the seats considering the majority prize and the possibility for a party to overpass it itself.

The results for the IC case displayer in Figures 2 and 3, are perfectly in line with the ones of Feix *et al.* (2009), with a minimum probability of having a paradox using an apportionment method proportional with the square root of the population ($\alpha = 0.5$).

Table 4: Proposal, regions voting as blocks, IC

District	**Square Root Population**	**Relative S.R.P.**	**Weight**	β'	100β	**Quotient**
Lombardia	3005.42	8.64%	14	0.2684	7.66	0.886
Campania	2387.87	6.87%	12	0.2277	6.50	0.946
Lazio	2261.06	6.50%	12	0.2277	6.50	1.000
Sicily	2229.12	6.41%	11	0.2079	5.93	0.926
Veneto	2127.84	6.12%	11	0.2079	5.93	0.970
Piedmont	2052.97	5.90%	11	0.2079	5.93	1.005
Apulia	2005.17	5.77%	11	0.2079	5.93	1.029
Emilia-Romagna	1995.83	5.74%	11	0.2079	5.93	1.034
Tuscany	1870.24	5.38%	11	0.2079	5.93	1.103
Calabria	1418.26	4.08%	9	0.1688	4.82	1.181
Sardinia	1277.45	3.67%	9	0.1688	4.82	1.311
Liguria	1253.71	3.61%	9	0.1688	4.82	1.336
Marche	1212.68	3.49%	9	0.1688	4.82	1.381
Abruzzi	1123.56	3.23%	9	0.1688	4.82	1.491
Friuli-Venezia Giulia	1088.47	3.13%	9	0.1688	4.82	**1.539**
Trento	690.66	1.99%	3	0.0554	1.58	0.795
Bolzano	680.44	1.96%	3	0.0554	1.58	0.807
Umbria	908.75	2.61%	6	0.1121	3.20	1.224
Basilicata	773.15	2.22%	6	0.1121	3.20	1.439
Molise	566.22	1.63%	2	0.0370	1.06	**0.648**
Valle d'Aosta	345.76	0.99%	2	0.0370	1.06	1.061
Europe	1439.59	4.14%	2	0.0370	1.06	**0.255**
South America	1008.85	2.90%	2	0.0370	1.06	0.364
North and Central America	599.88	1.73%	1	0.0183	0.52	0.303
Asia Africa Oceania Antartide	446.47	1.28%	1	0.0183	0.52	0.408

Figure 2: *Referendum* paradox probabilities if regions vote in blocks, IC

Figure 3: *Referendum* paradox probabilities if senators vote following party lines, IC

Figure 4: *Referendum* paradox probabilities if regions vote in blocks, IAC

Figure 5: *Referendum* paradox probabilities if senators vote following party lines, IAC

Table 5: Actual situation, regions voting as blocks, IAC

District	**Population**	**Relative Pop**	**Weight**	β'	100β	**Quotient**
Lombardia	9 032 554	14.89%	47	0.4788	16.40	1.101
Campania	5 701 931	9.40%	30	0.2782	9.53	1.014
Lazio	5 112 413	8.43%	27	0.2486	8.52	1.010
Sicily	4 968 991	8.19%	26	0.2387	8.18	0.998
Veneto	4 527 694	7.47%	24	0.2197	7.53	1.008
Piedmont	4 214 677	6.95%	22	0.2001	6.86	0.987
Apulia	4 020 707	6.63%	21	0.1907	6.53	0.985
Emilia-Romagna	3 983 346	6.57%	21	0.1907	6.53	0.995
Tuscany	3 497 806	5.77%	18	0.1626	5.57	0.966
Calabria	2 011 466	3.32%	10	0.0904	3.10	**0.934**
Sardinia	1 631 880	2.69%	9	0.0812	2.78	1.033
Liguria	1 571 783	2.59%	8	0.0720	2.47	0.952
Marche	1 470 581	2.42%	8	0.0720	2.47	1.018
Abruzzi	1 262 392	2.08%	7	0.0630	2.16	1.036
Friuli-Venezia Giulia	1 184 764	1.95%	7	0.0630	2.16	1.104
Trento-Valle di Non	208 875	0.34%	1+	0.0105	0.36	1.040
Rovereto - Riva del Garda	160 792	0.27%	1+	0.0105	0.36	1.352
Pergine - Fiemme - Fassa	107 351	0.18%	1+	0.0105	0.36	2.024
Bolzano - Bassa Atesina	161 709	0.27%	1+	0.0105	0.36	1.344
Merano	135 163	0.22%	1+	0.0105	0.36	1.608
Bressanone - Brunico	166 127	0.27%	1+	0.0105	0.36	1.308
Umbria	825 826	1.36%	7	0.0630	2.16	1.584
Basilicata	597 768	0.99%	7	0.0630	2.16	**2.188**
Molise	320 601	0.53%	2	0.0179	0.61	1.162
Valle d'Aosta	119 548	0.20%	1	0.0090	0.31	1.558
Europe	2 072 410	3.42%	2	0.0179	0.61	**0.180**
South America	1 017 776	1.68%	2	0.0179	0.61	0.366
North and Central America	359 852	0.59%	1	0.0090	0.31	0.518
Asia Africa Oceania Antartide	199 339	0.33%	1	0.0090	0.31	0.934

Table 6: Proposal, regions voting as blocks, IAC

District	Population	Relative Pop	Weight	Beta'	100Beta	Quotient
Lombardia	9 032 554	14.89%	14	0.2684	7.66	**0.514**
Campania	5 701 931	9.40%	12	0.2277	6.50	0.691
Lazio	5 112 413	8.43%	12	0.2277	6.50	0.771
Sicily	4 968 991	8.19%	11	0.2079	5.93	0.724
Veneto	4 527 694	7.47%	11	0.2079	5.93	0.795
Piedmont	4 214 677	6.95%	11	0.2079	5.93	0.854
Apulia	4 020 707	6.63%	11	0.2079	5.93	0.895
Emilia-Romagna	3 983 346	6.57%	11	0.2079	5.93	0.903
Tuscany	3 497 806	5.77%	11	0.2079	5.93	1.029
Calabria	2 011 466	3.32%	9	0.1688	4.82	1.453
Sardinia	1 631 880	2.69%	9	0.1688	4.82	1.791
Liguria	1 571 783	2.59%	9	0.1688	4.82	1.859
Marche	1 470 581	2.42%	9	0.1688	4.82	1.987
Abruzzi	1 262 392	2.08%	9	0.1688	4.82	2.315
Friuli-Venezia Giulia	1 184 764	1.95%	9	0.1688	4.82	2.466
Trento	477 017	0.79%	3	0.0554	1.58	2.009
Bolzano	462 999	0.76%	3	0.0554	1.58	2.070
Umbria	825 826	1.36%	6	0.1121	3.20	2.350
Basilicata	597 768	0.99%	6	0.1121	3.20	3.246
Molise	320 601	0.53%	2	0.0370	1.06	1.996
Valle d'Aosta	119 548	0.20%	2	0.0370	1.06	**5.354**
Europe	2 072 410	3.42%	2	0.0370	1.06	**0.309**
South America	1 017 776	1.68%	2	0.0370	1.06	0.629
North and Central America	359 852	0.59%	1	0.0183	0.52	0.882
Asia Africa Oceania Antartide	199 339	0.33%	1	0.0183	0.52	1.593

Table 7: Example of pivotal seats used in the second scenario, with IC

Region	Seats	55%	Seats winner	Seats loser	Pivotal seats
Lombardia	47	25.85	26	21	5
Campania	30	16.5	17	13	4
Lazio	27	14.85	15	12	3

Table 8: Actual situation, senators voting following party lines, IC

District	Square Root Population	Relative S.R.P.	Weight	β'	100β	Quotient
Lombardia	3005.42	8.40%	5	0.3450	10.21	1.214
Campania	2387.87	6.68%	4	0.2688	7.95	1.191
Lazio	2261.06	6.32%	3	0.1981	5.86	0.927
Sicily	2229.12	6.23%	4	0.2688	7.95	1.276
Veneto	2127.84	5.95%	4	0.2688	7.95	1.336
Piedmont	2052.97	5.74%	4	0.2688	7.95	1.385
Apulia	2005.17	5.61%	3	0.1981	5.86	1.045
Emilia-Romagna	1995.83	5.58%	3	0.1981	5.86	1.050
Tuscany	1870.24	5.23%	2	0.1306	3.86	0.739
Calabria	1418.26	3.97%	2	0.1306	3.86	0.974
Sardinia	1277.45	3.57%	1	0.0649	1.92	**0.537**
Liguria	1253.71	3.51%	2	0.1306	3.86	1.102
Marche	1212.68	3.39%	2	0.1306	3.86	1.139
Abruzzi	1123.56	3.14%	1	0.0649	1.92	0.611
Friuli-Venezia Giulia	1088.47	3.04%	1	0.0649	1.92	0.631
TAA - *Trento-Valle di Non*	457.03	1.28%	1/0	0.0541	1.60	1.251
TAA - *Rovereto-Riva del Garda*	400.99	1.12%	1/0	0.0541	1.60	1.426
TAA - *Pergine-Fiemme-Fassa*	327.64	0.92%	1/0	0.0541	1.60	1.746
TAA - *Bolzano-Bassa Atesina*	402.13	1.12%	1/0	0.0541	1.60	1.422
TAA - *Merano*	367.65	1.03%	1/0	0.0541	1.60	1.556
TAA - *Bressanone-Brunico*	407.59	1.14%	1/0	0.0541	1.60	1.403
Umbria	908.75	2.54%	1	0.0649	1.92	0.755
Basilicata	773.15	2.16%	1	0.0649	1.92	0.888
Molise	566.22	1.58%	0	0.0000	0.00	**0.000**
Valle d'Aosta	345.76	0.97%	1	0.0649	1.92	**1.985**
Europe	1439.59	4.03%	0	0.0000	0.00	**0.000**
South America	1008.85	2.82%	0	0.0000	0.00	**0.000**
North and Central America	599.88	1.68%	1	0.0649	1.92	1.144
Asia Africa Oceania Antartide	446.47	1.25%	1	0.0649	1.92	1.537

The probability of the referendum paradox then reaches a minimum of about 20%. The turbulence that can be seen in Figure 3 for α between 0 and 0.2, where the probability of having a paradox jumps till almost 30% and then decreases very fast, can be explained as follows: When α is equal to 0 all the regions have the same amount of senators, one each one. Next, when α increases a few, the 10% seats prize that are pivotal not only take the values of 1,2, and 3 but also behave in a non monotonic way. In fact, at $\alpha = 0.15$, the reader can check that the number of pivotal seats is 2 for the most populous region, Lombardia and Campania, while the next six regions (Lazio, Sicily, Veneto, Piedmont, Apulia, Emilia-Romagna) enjoy 3 pivotal seats! As α increases, non monotonic behavior can still occur, but this will touch smaller regions, with less impact on the probability of the referendum paradox.

With IC the probabilities of having a paradox using the four real apportionments, founded with a similar simulation (the only difference is that here we use the real values of the seats and not $a_i = n_i^\alpha$), are:

- 22.73% for the actual system with regions voting in blocks,
- 21.56% for the proposal and regions voting in blocks,
- 22.29% for the actual system when the senators follow party lines and
- 23.85% for the proposal with senators following party lines.

If we put these values in order from the the best situation to the worst, the proposal is the best but only if regions vote in blocks, as seen with the power indices analysis. With the second scenario that would be the worst case, and the actual system is rather good if senators can vote following the instructions of their parties.

With the IAC model things are different. Figure 4 is consistent with the value found in Feix *et al.* (2009), with a minimum probability of about 16% to have a paradox, when $\alpha = 1$. Figure 5 shows a case when the weight of a region is not fixed. Parties get, even if with a majority prize, a share of seats in each regions more proportional to the share of votes they have. By consequence, when $\alpha = 1$, the probability of having a paradox is almost zero. The majority prize has almost no effect in obtaining a paradox, compared to an attribution of a fixed number of seats to the winner in each region. In the four real cases, probabilities are

- 16.78% for the actual system, when regions vote in blocks,
- 20.97% for the proposal and regions voting in blocks,
- 6.63% for the actual system as the senators follow party lines and
- 16.83% for the proposal with senators following party lines.

Here the *statu quo* is definitely the best situation, with a probability close to the ideal. As the apportionment is not perfectly proportional, the value doesn't approach the zero. The proposal with the assumption of voting in blocks has the highest value, as it is the least proportional case.

5 Seeking for a better apportionment: new proposals

Even if the previous results may already give hints about the best way to apportion seats among regions and select senators in a region, they may not be optimal, as actual rule and the proposal were not designed to match normative criteria. In this section we will seek for better apportionment methods and we will formulate our proposal. As the new reform proposal has shown no interest to keep a fixed size of the Senate, so we will not care about that either. We will just try to keep that size closer or slightly smaller than the actual one.

5.1 Voting as block

Finding a possible solution in the case of regions voting as blocks is rather immediate: according to the previous sections of this paper we will just need to find an apportionment list of seats proportional to the square roots of the populations of every region for the IC model, and proportional to he populations for the IAC model. Moreover, we will round the remainders using the method of Webster, as suggested by Balinski and Young (2001). Table 12 displays as a possibility a Senate with 231 members, and a standard deviation equal to just 0.034. Its probability to have a *referendum* paradox is just 20.21%, close to the minimum possible in the Italian system with IC.

In the case of IAC, we apportion the seats proportionally to the populations. Table 13 is an example of a Senate with 303 members, a standard deviation of 0.061 and a *referendum* paradox probability of 16.34%, in line with its minimal value in Figure 6.

5.2 Partisan votes

In the case in which senators vote according party lines the ideal apportionment depends from the electoral law: we need to take in account the majority prize of a rounded up 55% of the seats for the winning party.

Under IC, the seats to be proportional to the square roots of the populations are now the pivotal ones. But while proposing an apportionment, we need to consider also the other seats, the ones already automatically given to the parties using the IC model. To build the real apportionment starting from the one of the pivotal seats, we need to use the equation:

$$ps_i = 2R^+(0.55a_i) - a_i$$

where ps are the pivotal seats of region i and a_i the total seats we are seeking. This equation is impossible to solve analytically as there is a rounding up function, and there are even several possible results for every value of ps_i. With an easy loop procedure that increases the seats starting from a value of zero until it doesn't find the first solution of the equation, we can anyway find the values, and use the smallest possible to keep the size of the Senate small. Notice that we cannot give two seats to a region as in these cases the majority prize is not used and normally the apportionment of the seats between the parties is one and one. So in that case we will directly give the region four seats, the second smallest possible solution if we have two pivotal seats. Table 14 shows a possible apportionment, both the complete scheme and the pivotal seats only. This Senate has 310 members and a small standard deviation, 0.094. The probability to have a *referendum* paradox is 20.36%.

Under IAC, when senators vote following party lines, it is impossible to determine easily an ideal apportionment, as the seats the winning party will get in each region here are not fixed. But if we compare tables 5 and 10, and tables 6 and 11, we can notice that power is almost proportional region by region, as an effect of the Penrose's limit theorem. Hence, we can propose the same apportionment weights of the voting in blocks case and recompute the power with the Monte Carlo simulation in the second scenario. In fact, the same weights work very well: the standard deviation is 0.044 and the probability to have a *referendum* paradox is 6.33%. This value is still bigger than zero, because proportionality is still slightly imperfect, as we need to keep compact the size of the Senate and still to deal with the distortion effects of the majority prize.

6 Conclusions

Which reform should be implemented if we seek for fair apportionment in the Italian Senate? If "fair apportionment" means giving to every citizen the same amount of power, and if an *a priori* framework is enough, we can use the IC and the IAC models to analyze the situation. If we think that each voter decides his/her favorite candidate flipping a fair coin, the actual situation is not very good, but it works better if the senators can vote according to the preferences of their parties: this happens at the moment. The proposal reform is not that bad, but only if, at the opposite, regions voting in blocks, like having one senator with some mandates equal to the weight of his/her region. This is realistic too, as the reform purpose is the creation of a federal Senate in which the members are elected from the Regional Councils, which have a majority. The probabilities to have a *referendum* paradox are in both cases close to the minimum possible, but higher than that of 1-2 percentage points.

If we think that there is homogeneity, i.e. each Italian region has its own common values and opinions and voters don't flip a fair coin to cast their ballots, and that independence in voting will be reached only at a national level, the actual situation is the best option in both cases, considering both power and paradox propensity.

Lacking a detailed analysis of Italian electoral data, we can anyway say that the latter assumption should be preferred to model the behavior of the electors of this country: several regions traditionally vote consistently for one political color. Possible examples are Tuscany and Emilia-Romagna for the Left or Lombardia and Veneto for the Right. This makes the IC assumption hardly defendable, voters do not behave independently. Much more reasonable is the assumption of homogeneity, or IAC. This assumption forecasts a close election at macro level, but this is still realistic, as the final results are always very close to a tie. So, we suggest that the analysis done with this second method is more realistic: by consequence the proposal should be avoided.

Anyway, we can do much better. Ideal apportionments are presented for the two models and the two scenarios, all of them quite performing in terms of fairness. Moreover, if we consider the IAC case as the most realistic one, our proposal is unique in both the scenarios, and performing in both of them.

In this work, we provided a wide range of tools to deeply analyze the Italian situation concerning the representation of the regions at the Senate, and we think that politicians have now all the instruments to take the best decision when the constitutional reforming process will start again during the current legislature.

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Table 9: Proposal, senators voting following party lines, IC

District	**Square Root Population**	**Relative S.R.P.**	**Weight**	β'	100β	**Quotient**
Lombardia	3005.42	8.64%	2	0.1725	5.23	**0.605**
Campania	2387.87	6.87%	2	0.1725	5.23	0.761
Lazio	2261.06	6.50%	2	0.1725	5.23	0.804
Sicily	2229.12	6.41%	3	0.2643	8.01	1.249
Veneto	2127.84	6.12%	3	0.2643	8.01	1.308
Piedmont	2052.97	5.90%	3	0.2643	8.01	1.356
Apulia	2005.17	5.77%	3	0.2643	8.01	1.388
Emilia-Romagna	1995.83	5.74%	3	0.2643	8.01	1.395
Tuscany	1870.24	5.38%	3	0.2643	8.01	1.489
Calabria	1418.26	4.08%	1	0.0853	2.58	0.633
Sardinia	1277.45	3.67%	1	0.0853	2.58	0.703
Liguria	1253.71	3.61%	1	0.0853	2.58	0.716
Marche	1212.68	3.49%	1	0.0853	2.58	0.741
Abruzzi	1123.56	3.23%	1	0.0853	2.58	0.799
Friuli-Venezia Giulia	1088.47	3.13%	1	0.0853	2.58	0.825
Trento	690.66	1.99%	1	0.0853	2.58	1.300
Bolzano	680.44	1.96%	1	0.0853	2.58	1.320
Umbria	908.75	2.61%	2	0.1725	5.23	1.999
Basilicata	773.15	2.22%	2	0.1725	5.23	**2.350**
Molise	566.22	1.63%	0	0.0000	0.00	**0.000**
Valle d'Aosta	345.76	0.99%	0	0.0000	0.00	**0.000**
Europe	1439.59	4.14%	0	0.0000	0.00	**0.000**
South America	1008.85	2.90%	0	0.0000	0.00	**0.000**
North and Central America	599.88	1.73%	1	0.0853	2.58	1.497
Asia Africa Oceania Antartide	446.47	1.28%	1	0.0853	2.58	2.011

Table 10: Actual situation, senators voting following party lines, IAC

District	Population	Relative Pop	IAC Power	Norm. IAC Power	Quotient
Lombardia	9 032 554	14.89%	0.7148	15.02	1.008
Campania	5 701 931	9.40%	0.4549	9.56	1.017
Lazio	5 112 413	8.43%	0.4100	8.62	1.022
Sicily	4 968 991	8.19%	0.3946	8.29	1.012
Veneto	4 527 694	7.47%	0.3645	7.66	1.026
Piedmont	4 214 677	6.95%	0.3336	7.01	1.009
Apulia	4 020 707	6.63%	0.3182	6.69	1.009
Emilia-Romagna	3 983 346	6.57%	0.3177	6.68	1.016
Tuscany	3 497 806	5.77%	0.2725	5.73	0.993
Calabria	2 011 466	3.32%	0.1514	3.18	**0.959**
Sardinia	1 631 880	2.69%	0.1359	2.86	1.062
Liguria	1 571 783	2.59%	0.1207	2.54	0.979
Marche	1 470 581	2.42%	0.1210	2.54	1.048
Abruzzi	1 262 392	2.08%	0.1055	2.22	1.065
Friuli-Venezia Giulia	1 184 764	1.95%	0.1053	2.21	1.133
TAA - *Trento-Valle di Non*	208 875	0.34%	0.0177	0.37	1.079
TAA - *Rovereto-Riva del Garda*	160 792	0.27%	0.0177	0.37	1.401
TAA - *Pergine-Fiemme-Fassa*	107 351	0.18%	0.0177	0.37	2.099
TAA - *Bolzano-Bassa Atesina*	161 709	0.27%	0.0177	0.37	1.393
TAA - *Merano*	135 163	0.22%	0.0177	0.37	1.667
TAA - *Bressanone-Brunico*	166 127	0.27%	0.0177	0.37	1.356
Umbria	825 826	1.36%	0.1057	2.22	1.631
Basilicata	597 768	0.99%	0.1055	2.22	**2.250**
Molise	320 601	0.53%	0.0303	0.64	1.203
Valle d'Aosta	119 548	0.20%	0.0150	0.31	1.597
Europe	2 072 410	3.42%	0.0302	0.63	**0.186**
South America	1 017 776	1.68%	0.0300	0.63	0.376
North and Central America	359 852	0.59%	0.0153	0.32	0.541
Asia Africa Oceania Antartide	199 339	0.33%	0.0152	0.32	0.971

Table 11: Proposal, senators voting following party lines, IAC

District	Population	Relative Pop	IAC Power	Norm. IAC Power	Quotient
Lombardia	9 032 554	14.89%	0.4052	7.54	**0.506**
Campania	5 701 931	9.40%	0.3468	6.45	0.686
Lazio	5 112 413	8.43%	0.3473	6.46	0.767
Sicily	4 968 991	8.19%	0.3185	5.93	0.723
Veneto	4 527 694	7.47%	0.3186	5.93	0.794
Piedmont	4 214 677	6.95%	0.3181	5.92	0.852
Apulia	4 020 707	6.63%	0.3183	5.92	0.893
Emilia-Romagna	3 983 346	6.57%	0.3190	5.94	0.904
Tuscany	3 497 806	5.77%	0.3178	5.91	1.025
Calabria	2 011 466	3.32%	0.2606	4.85	1.462
Sardinia	1 631 880	2.69%	0.2605	4.85	1.801
Liguria	1 571 783	2.59%	0.2606	4.85	1.871
Marche	1 470 581	2.42%	0.2597	4.83	1.992
Abruzzi	1 262 392	2.08%	0.2601	4.84	2.324
Friuli-Venezia Giulia	1 184 764	1.95%	0.2600	4.84	2.476
Trento	477 017	0.79%	0.0863	1.61	2.042
Bolzano	462 999	0.76%	0.0864	1.61	2.105
Umbria	825 826	1.36%	0.1734	3.23	2.369
Basilicata	597 768	0.99%	0.1730	3.22	3.265
Molise	320 601	0.53%	0.0571	1.06	2.010
Valle d'Aosta	119 548	0.20%	0.0569	1.06	**5.372**
Europe	2 072 410	3.42%	0.0570	1.06	**0.310**
South America	1 017 776	1.68%	0.0571	1.06	0.633
North and Central America	359 852	0.59%	0.0284	0.53	0.890
Asia Africa Oceania Antartide	199 339	0.33%	0.0285	0.53	1.611

Table 12: An ideal apportionment, regions voting as blocks, IC

District	Square Root Population	Relative S.R.P.	Weight	β'	100β	Quotient
Lombardia	3005.42	8.64%	20	0.3200	8.97	1.038
Campania	2387.87	6.87%	16	0.2503	7.02	1.022
Lazio	2261.06	6.50%	15	0.2337	6.55	1.007
Sicily	2229.12	6.41%	15	0.2337	6.55	1.022
Veneto	2127.84	6.12%	14	0.2172	6.09	0.995
Piedmont	2052.97	5.90%	14	0.2172	6.09	1.031
Apulia	2005.17	5.77%	13	0.2010	5.63	0.977
Emilia-Romagna	1995.83	5.74%	13	0.2010	5.63	0.981
Tuscany	1870.24	5.38%	12	0.1849	5.18	0.964
Calabria	1418.26	4.08%	9	0.1376	3.86	0.946
Sardinia	1277.45	3.67%	9	0.1376	3.86	1.050
Liguria	1253.71	3.61%	8	0.1221	3.42	0.949
Marche	1212.68	3.49%	8	0.1221	3.42	0.981
Abruzzi	1123.56	3.23%	7	0.1066	2.99	0.925
Friuli-Venezia Giulia	1088.47	3.13%	7	0.1066	2.99	0.955
Trento	690.66	1.99%	5	0.0759	2.13	**1.072**
Bolzano	680.44	1.96%	5	0.0759	2.13	1.088
Umbria	908.75	2.61%	6	0.0912	2.56	0.978
Basilicata	773.15	2.22%	5	0.0759	2.13	0.957
Molise	566.22	1.63%	4	0.0607	1.70	1.045
Valle d'Aosta	345.76	0.99%	2	0.0303	0.85	**0.854**
Europe	1439.59	4.14%	10	0.1533	4.30	1.037
South America	1008.85	2.90%	7	0.1066	2.99	1.030
North and Central America	599.88	1.73%	4	0.0607	1.70	0.986
Asia Africa Oceania Antartide	446.47	1.28%	3	0.0455	1.27	0.993

Table 13: An ideal apportionment, regions voting as blocks, IAC

District	Population	Relative Pop	Weight	IAC Power	Norm. IAC Power	Quotient
Lombardia	9 032 554	14.89%	45	0.7099	14.89	1.000
Campania	5 701 931	9.40%	29	0.4564	9.57	1.018
Lazio	5 112 413	8.43%	26	0.4089	8.58	1.017
Sicily	4 968 991	8.19%	25	0.3932	8.25	1.007
Veneto	4 527 694	7.47%	23	0.3613	7.58	1.015
Piedmont	4 214 677	6.95%	21	0.3299	6.92	0.996
Apulia	4 020 707	6.63%	20	0.3144	6.59	0.995
Emilia-Romagna	3 983 346	6.57%	20	0.3144	6.59	1.004
Tuscany	3 497 806	5.77%	17	0.2670	5.60	0.971
Calabria	2 011 466	3.32%	10	0.1574	3.30	0.996
Sardinia	1 631 880	2.69%	8	0.1259	2.64	0.981
Liguria	1 571 783	2.59%	8	0.1259	2.64	1.019
Marche	1 470 581	2.42%	7	0.1104	2.32	0.955
Abruzzi	1 262 392	2.08%	6	0.0946	1.98	0.953
Friuli-Venezia Giulia	1 184 764	1.95%	6	0.0948	1.99	1.017
Trento	477 017	0.79%	2	0.0315	0.66	**0.839**
Bolzano	462 999	0.76%	2	0.0313	0.66	0.859
Umbria	825 826	1.36%	4	0.0630	1.32	0.970
Basilicata	597 768	0.99%	3	0.0474	0.99	1.008
Molise	320 601	0.53%	2	0.0315	0.66	1.248
Valle d'Aosta	119 548	0.20%	1	0.0157	0.33	**1.668**
Europe	2 072 410	3.42%	10	0.1575	3.30	0.967
South America	1 017 776	1.68%	5	0.0784	1.65	0.980
North and Central America	359 852	0.59%	2	0.0315	0.66	1.115
Asia Africa Oceania Antartide	199 339	0.33%	1	0.0156	0.33	0.994

Table 14: An ideal apportionment, senators voting following party lines, IC

District	Relative S.R.P.	Total Weight	Pivotal Weight	β'	100β	Quotient
Lombardia	8.64%	42	6	0.3141	8.86	1.025
Campania	6.87%	31	5	0.2570	7.25	1.055
Lazio	6.50%	31	5	0.2570	7.25	1.115
Sicily	6.41%	22	4	0.2029	5.72	0.893
Veneto	6.12%	22	4	0.2029	5.72	0.935
Piedmont	5.90%	22	4	0.2029	5.72	0.969
Apulia	5.77%	22	4	0.2029	5.72	0.992
Emilia-Romagna	5.74%	22	4	0.2029	5.72	0.997
Tuscany	5.38%	22	4	0.2029	5.72	1.064
Calabria	4.08%	11	3	0.1507	4.25	1.042
Sardinia	3.67%	11	3	0.1507	4.25	1.157
Liguria	3.61%	11	3	0.1507	4.25	1.179
Marche	3.49%	4	2	0.0998	2.82	0.807
Abruzzi	3.23%	4	2	0.0998	2.82	0.871
Friuli-Venezia Giulia	3.13%	4	2	0.0998	2.82	0.899
Trento	1.99%	1	1	0.0497	1.40	**0.706**
Bolzano	1.96%	1	1	0.0497	1.40	0.717
Umbria	2.61%	4	2	0.0998	2.82	1.077
Basilicata	2.22%	4	2	0.0998	2.82	1.266
Molise	1.63%	1	1	0.0497	1.40	0.861
Valle d'Aosta	0.99%	1	1	0.0497	1.40	**1.410**
Europe	4.14%	11	3	0.1507	4.25	1.027
South America	2.90%	4	2	0.0998	2.82	0.970
North and Central America	1.73%	1	1	0.0497	1.40	0.813
Asia Africa Oceania Antartide	1.28%	1	1	0.0497	1.40	1.092

Table 15: An ideal apportionment, senators voting following party lines, IAC

District	**Population**	**Relative Pop**	**Weight**	**Beta'**	**100Beta**	**Quotient**
Lombardia	9 032 554	14.89%	45	0.4771	16.32	1.096
Campania	5 701 931	9.40%	29	0.2803	9.59	1.020
Lazio	5 112 413	8.43%	26	0.2493	8.53	1.012
Sicily	4 968 991	8.19%	25	0.2392	8.18	0.999
Veneto	4 527 694	7.47%	23	0.2190	7.49	1.004
Piedmont	4 214 677	6.95%	21	0.1991	6.81	0.980
Apulia	4 020 707	6.63%	20	0.1893	6.48	0.977
Emilia-Romagna	3 983 346	6.57%	20	0.1893	6.48	0.986
Tuscany	3 497 806	5.77%	17	0.1600	5.47	0.949
Calabria	2 011 466	3.32%	10	0.0939	3.21	0.968
Sardinia	1 631 880	2.69%	8	0.0749	2.56	0.952
Liguria	1 571 783	2.59%	8	0.0749	2.56	0.988
Marche	1 470 581	2.42%	7	0.0654	2.24	0.923
Abruzzi	1 262 392	2.08%	6	0.0561	1.92	0.921
Friuli-Venezia Giulia	1 184 764	1.95%	6	0.0561	1.92	0.982
Trento	477 017	0.79%	2	0.0187	0.64	**0.811**
Bolzano	462 999	0.76%	2	0.0187	0.64	0.836
Umbria	825 826	1.36%	4	0.0373	1.28	0.938
Basilicata	597 768	0.99%	3	0.0280	0.96	0.971
Molise	320 601	0.53%	2	0.0187	0.64	1.207
Valle d'Aosta	119 548	0.20%	1	0.0093	0.32	**1.618**
Europe	2 072 410	3.42%	10	0.0939	3.21	0.940
South America	1 017 776	1.68%	5	0.0467	1.60	0.952
North and Central America	359 852	0.59%	2	0.0187	0.64	1.075
Asia Africa Oceania Antartide	199 339	0.33%	1	0.0093	0.32	0.971