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U.S. Presidential Elections and the Referendum Paradox*

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Abstract

In the United States, the president is elected by the Electoral College (EC) and not directly by individual voters. This can give rise to a so-called "referendum paradox" in which one candidate receives more popular votes than any other, but this candidate is not elected. The 2000 election is an example of this phenomenon. Can the EC be reformed so that a referendum paradox never arises? We consider varying three natural parameters. First, we consider changing the method of apportioning seats in the House of Representatives to states. Second, we consider changing the total number of seats in the House. Intuition suggests that as the number of seats approaches the number

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of voters, the referendum paradox should disappear. Finally, we consider varying the fixed and proportional components of each state's EC vote. Using data from U.S. presidential elections we show that none of these reforms can prevent a referendum paradox from occurring. We conclude that susceptibility to a referendum paradox is an inescapable feature of the system for electing presidents. An interesting corollary of our analysis is that seemingly insignificant changes to the EC can cause different candidates to be elected president.

JEL classification: D72.

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1 Introduction

The U.S. presidential election in 2000 was close. George W. Bush obtained 47.9% of the popular vote against 48.4% for Al Gore. Despite receiving 543,895 more individual votes than Bush, Gore obtained only 266 electors in the Electoral College (EC) whereas Bush was supported by 271 electors. This situation is known as a "referendum paradox" (Nurmi, 1998). One candidate receives more popular votes than any other, but this candidate is not elected. The fact that such a situation can arise is an obvious weakness of the EC system.[1] A referendum paradox has occurred three times in U.S. history, in 1876, 1888 and 2000.[2] On each occasion it favoured Republicans over Democrats (see Table 1).[3]

[1]Merlin and Senne (2008) compute the probability of obtaining a referendum paradox under certain probabilistic assumptions.

[2]Strictly speaking, the paradox also occurred in 1824. Four candidates secured EC votes, but none received an electoral majority. Therefore, the election was determined by the House of Representatives. John Quincy Adams won the vote with the support of 13 states compared to 7 for Andrew Jackson and 4 for William H. Crawford. However, Jackson won the popular vote.

[3]Note that one so-called "faithless" elector from the District of Columbia, Barbara Lett-Simmons, abstained from voting in the 2000 election as a means of protesting against the

Table 1: The referendum paradox in U.S. presidential elections.

Year	**Popular vote (%)**		**Electoral vote**	
	Dem.	Rep.	Dem.	Rep.
1876	51.0	48.0	184	185
1888	48.6	47.8	168	233
2000	48.4	47.9	266	271

538 electors currently belong to the EC. This number corresponds to the size of Congress divided in two parts: the House of Representatives (435) plus two senators for each state (100). The last 3 members belong to the District of Columbia.[4] Each state's delegation to the EC (its electors) equals the size of the state's delegation in the House of Representatives plus two for its senators. For example, California currently has 55 electors in the EC. This number corresponds to its 53 representatives and 2 senators. Crucially, most states operate a "winner-takes-all" rule under which the candidate with the largest popular vote in the state takes *all* of the state's EC votes. As we will see, this turns out to be critical in what follows.

The two exceptions to this are Maine and Nebraska. In these states, there could be a split of EC votes among candidates. These states select one elector within each congressional district by popular vote, and additionally select their remaining two electors by the aggregate, statewide popular vote. For example, in the 2008 presidential election, Barack Obama won one of Nebraska's EC votes while John McCain won the remaining four. This is because Obama received more votes than any other candidate in Nebraska's second congressional district.

The 435 seats in the House are apportioned to the 50 states on the basis of their populations. A census to determine the U.S. population is carried out every 10 years. An important mathematical issue which arises here is that

District's lack of voting representation in Congress. She had been expected to vote for Gore/Lieberman.

[4]The 23rd Amendment to the U.S. Constitution specifies that the number of electors for the District of Columbia is equal to the number given to the least populous state. Currently this number is 3.

a state's natural allocation of seats, reflecting its share of the population, is rarely an integer value. A state's natural allocation of seats is called its quota. For example, the 1990 census apportioned 435 representatives to the 50 states and this apportionment was used for the 2000 presidential election. According to the census, California had a population of 29,839,250 out of a total U.S. population of 249,022,783.[5] Therefore its quota is 52.124. Since seats are indivisible, the problem is how to round this fraction to an integer. Different methods have been proposed throughout history, usually by famous American politicians. A comprehensive analysis of this problem can be found in Balinski and Young (2001) and we give a brief overview of the different methods of apportionment in section 2 of this paper. Although the difference between these methods appears small, as we demonstrate, changing the method of apportionment can change the outcome of a presidential election. For example, had Jefferson's method been used for determining the 1990 apportionment instead of Hill's method (which is the current method), then Al Gore would have been elected president in 2000 and not George W. Bush. The referendum paradox would have been avoided.

The inspiration for this work comes from an important paper by Neubauer and Zeitlin (2003). Rather than consider changing the method of apportionment, Neubauer and Zeitlin analyse the effects of changes in House size on the 2000 presidential election. The size of the House is determined by law and not by the Constitution. This means that, in principle, it is easy to change. Neubauer and Zeitlin's results are striking. The number of seats in the House was fixed at 435 in 1911 and has not changed since.[6] Using voting data from the 2000 presidential election, Neubauer and Zeitlin show that if the size of the House is less than 491, then Bush is always the winner, and if it is greater than 597 then Gore is always the winner (with, somewhat surprisingly, a tie at 655). Between these two numbers, sometimes Bush wins, sometimes Gore wins, and sometimes there is a tie. In other words, behaviour in this interval of integers is "non-monotonic" (i.e. initial increases in House size cause Gore to win, but further increases cause the winner to revert back to Bush, and

[5]This data comes from Balinski and Young (2001, pp. 178-179).

[6]There was a temporary increase to 437 at the time of admission of Alaska and Hawaii as states.

so on). This means that, *without changing anyone's vote*, simply increasing the size of the House can cause a different candidate to be elected president.

This suggests that there might be another way of avoiding a referendum paradox, one that does not involve changing the method of apportionment. As Neubauer and Zeitlin point out, due to population growth there was one representative per 301,000 citizens in 1941 and by 1990 this ratio had fallen to one per 572,000. If we consider that, for whatever reason, the "appropriate" ratio is the one that existed in 1941, then the size of the House after the 1990 census should have been 830. In this hypothetical situation, Gore would have obtained 471 votes in the EC compared to Bush's 463 votes.[7] Again, the referendum paradox would have been avoided. Rather disturbingly then, the outcome of the 2000 election was influenced in a critical way by an arbitrary decision taken in 1911.

We have hinted at two possible ways of avoiding the paradox, either of which would have worked in the 2000 presidential election: change the method of apportionment or increase sufficiently the size of the House. But are there reforms to the EC system that will *always* ensure that a referendum paradox never arises? This is the important question that we pose in this paper, and we show that the answer is no. Our hypothetical reforms involve varying several fundamental parameters, each of which is a part of the architecture of the EC system. First, we consider changing the method of apportioning seats to states. Second, we consider increasing the total number of seats in the House. Intuition suggests that as the number of seats approaches the number of voters, the referendum paradox should disappear. Surprisingly, we show that this is *not* the case. Finally, we consider varying the fixed component and the proportional component of each state's EC vote. In the language of our paper, a state's "fixed" component is simply its number of senators and its "proportional" component is its number of representatives in the House. A state's EC vote is simply the sum of these two components. We shall let k denote each state's fixed component (currently two), and let m

[7]In this hypothetical situation, the number of electors is 934. 830 is the size of the House, 100 is the number of senators, and 4 is the number of electors for the District of Columbia. As we mentioned earlier the District of Columbia receives a number of electors equal to the number received by the least populous state. When the House size is 830, this number is 4.

denote a floor (a lower bound) on each state's proportional component. Each state is entitled to at least one representative in the House, and so currently m is one. Varying these parameters allows us to consider a range of potential reforms, including a purely federal system (where the president is elected by the states with no weight given to their relative populations), and a system of perfect proportionality (where $k = 0$ and a state's proportional component equals its population).

Using voting data from actual U.S. presidential elections we show that *none of these reforms can prevent a referendum paradox from occurring.* We conclude that susceptibility to a referendum paradox is an inescapable feature of the system for electing presidents. Of course, whether you find this conclusion disturbing or not depends on your point of view. There may be benefits from the EC system that outweigh its susceptibility to a referendum paradox.[8] These arguments fall outside the scope of this paper, and so we do not attempt to address them here. The point of this paper is simply to prove the susceptibility of the EC system to a referendum paradox under a range of hypothetical reforms. It is worth pointing out, however, that in the literature on social choice theory, a mathematical condition called "anonymity" is often considered to be a property that a voting system ought to satisfy.[9] This condition says that the personal characteristics of the voters (their names, where they live, etc.) should not influence the outcome of an election; all that matters is how they vote. Clearly, a corollary of our analysis is that the system for electing presidents in the U.S. violates this requirement. We suspect that many social choice theorists would favour abolishing the EC for this reason, and replace it with a system in which the citizens directly elect the president. An alternative would be to introduce a Maine/Nebraska-type mechanism which would attempt to incorporate an element of proportionality into the allocation of a state's EC votes. This would replace the current "winner-takes-all" rule.

[8] See, for example, Best (1975). A good exploration of the logical foundations of the EC is Belenky (2009).

[9] A classic reference in this literature is Sen (1970).

2 Apportionment

As we noted above, the apportionment of seats in the House influences the number of votes each state has in the EC. In this section we briefly explain how this apportionment is currently done, and describe several alternatives. Our exposition of the underlying theory follows Balinski and Young (2001).

Imagine that the states are placed in alphabetical order. Next to each state's name we can write down its population. What we have done through this process is create an ordered list of numbers; this is called a vector. Let this vector of populations be denoted by $\mathbf{p} = (p_1, ..., p_n)$ where n is the total number of states (in the U.S. case, $n = 50$). A fixed number of seats a must be apportioned among these n states. A vector $\mathbf{a} = (a_1, ..., a_n)$ is an apportionment of a, with the requirement that $a_i > 0$ is a positive integer for each state i. Obviously, constraints can be imposed on the apportionment. As noted above, $a_i \geq m = 1$.

The quota for state i is its share of the population multiplied by the total number of seats. Let q_i denote the quota for state i. Therefore, $q_i = \frac{p_i}{\sum_{j=1}^{n} p_j} \times a$.

The easiest way to determine an apportionment is to use Hamilton's method: compute the quotas for each state, and then give each state the largest integer contained in its quota. For example, as previously noted, California's quota in 1990 was 52.124, and so the largest integer contained in this quota is 52. After this, give any seats as yet unapportioned to those states with the largest remainders. This method was used in the U.S. from 1850 to 1900.

An alternative methodology is provided by the so-called "divisor" methods. In this paper we consider five of these methods; the Jefferson, Adams, Webster, Dean and Hill methods. These are the most important methods from a historical point of view.

The vector $\mathbf{a}$ is a Jefferson apportionment if and only if

$$\text{for all } i = 1, \ldots, n, \quad a_i = \lfloor \frac{p_i}{x} \rfloor$$

with x being a "divisor" such that $\sum_{i=1}^{n} a_i = a$ and $\lfloor \frac{p_i}{x} \rfloor$ is the largest integer contained in $\frac{p_i}{x}$. In other words, once a is fixed, we have to find a divisor x

such that when we divide each state's population by x, and then sum the largest integers contained in these numbers, we obtain a number that is equal to a. For example, for a given divisor x if $p_i/x = 3.22$ then state i gets 3 seats under this method. The Jefferson method was used from 1790 to 1830 to apportion seats in the House of Representatives.

To explain more precisely how these divisor methods work, we reproduce the following table from Balinksi and Young (2001, p. 19). It shows the Jefferson method at work. Note that $\frac{p_i}{x}$ is called state i's "quotient". In this table the divisor is set at 28,500.

Table 2: Apportionment of 120 seats by Jefferson's Method, 1791 Census.

State	**Representative Population**	**Quotient (28,500)**	**Jefferson Apportionment**
Connecticut	236,841	8.310	8
Delaware	55,540	1.949	1
Georgia	70,835	2.485	2
Kentucky	68,705	2.411	2
Maryland	278,514	9.772	9
Massachusetts	475,327	16.678	16
New Hampshire	141,822	4.976	4
New Jersey	179,570	6.301	6
New York	331,589	11.635	11
North Carolina	353,523	12.404	12
Pennsylvania	432,879	15.189	15
Rhode Island	68,446	2.402	2
South Carolina	206,236	7.236	7
Vermont	85,533	3.001	3
Virginia	630,560	22.125	22
Total	**3,615,920**	**126.874**	**120**

As is clear from this table, more than one number can serve as the divisor. In fact, in this example, any divisor between 28,356 and 28,511 would produce the same Jefferson apportionment.[10]

[10] Balinski and Young (2001, p.19). As Balinski and Young note, a divisor of 30,000 only

We now briefly describe the other main methods of apportionment.

The vector **a** is an Adams apportionment if and only if

$$\text{for all } i = 1, \ldots, n, \quad a_i = \lceil \frac{p_i}{x} \rceil$$

where x is a divisor such that $\sum_{i=1}^{n} a_i = a$ and where $\lceil y \rceil$ is the smallest integer greater than or equal to y. The construction of Adams's method is identical to Jefferson's, the only difference being the way of rounding a number. For example, if $p_i/x = 4.28$ then state i gets 5 seats under the Adams method while it would only get 4 seats under the Jefferson method.

The vector **a** is a Webster apportionment if and only if

$$\text{for all } i = 1, \ldots, n, \quad a_i = [\frac{p_i}{x}]$$

with x a divisor such that $\sum_{i=1}^{n} a_i = a$ and where $[y]$ is the nearest integer to y. For example, if $y = 0.51$, then $[y] = 1$ and if $y = 3.45$, then $[y] = 3$. In the particular case where y is an integer plus 0.5, then there are two solutions. Therefore, if $y = 8.5$ then $[y] = 8$ or $[y] = 9$. This method (Webster's method) was used for apportioning seats in the House of Representatives in 1840, and also from 1910 to 1930.

We now describe our final two divisor methods, Dean's method and Hill's method.

Before we do so, a technical point. Note that between two successive integers, the value that changes the rounding (for Webster's method) is the arithmetic mean. So, if n is an integer we have

$$\text{for all } y \in [n, n+1], \begin{cases} [y] = n, & \text{if } y \leqslant \frac{(n+(n+1))}{2} \\ [y] = n+1, & \text{if } y \geqslant \frac{(n+(n+1))}{2} \end{cases}$$

The methods of Dean and Hill are similar to Webster's except in one important respect. Instead of using the arithmetic mean as the basis for rounding, Dean's method uses the harmonic mean and Hill's method uses the geometric mean. This means that for Dean's method the following is true. If n is an integer we have

apportions 112 seats using Jefferson's method, and is therefore too high.

$$\text{for all } y \in [n, n+1], \begin{cases} [y] = n, & \text{if } y \leqslant \frac{2}{\frac{1}{n}+\frac{1}{n+1}} \\ [y] = n+1, & \text{if } y \geqslant \frac{2}{\frac{1}{n}+\frac{1}{n+1}} \end{cases}$$

For Hill's method the following is true. If n is an integer we have

$$\text{for all } y \in [n, n+1], \begin{cases} [y] = n, & \text{if } y \leqslant \sqrt{(n \times (n+1))} \\ [y] = n+1, & \text{if } y \geqslant \sqrt{(n \times (n+1))} \end{cases}$$

Hill's method has been used for apportioning seats in the House of Representatives since 1940.

In practice, how significant are the differences in these methods of apportionment? To answer this question, we present results in Table 3. The data comes from Balinski and Young (2001, p.179). The Table 3 shows how the 15 largest states would have been apportioned seats in the House under the various methods, based on the 2000 census. As we can see from the table, the different methods occasionally produce significantly different apportionments. For example, under the Adams method California would have recieved 50 seats in 2000 but 55 seats under Jefferson's method. As we will demonstrate, these differences can be decisive in close elections.

3 Graphs

We present our findings using a device that we call a "representation graph". This is a simple two-dimensional graph with the number of electors in the EC measured on the horizontal axis, and the proportion of Democratic electors in the EC measured on the vertical axis. The proportion of Republican electors is, obviously, one minus this value.

Given a method of apportionment, a value for k and a value for m (currently, these are 2 and 1 respectively), we use voting data to "graph" how the proportion of Democratic electors in the EC changes as House size increases or decreases. As House size increases (other things equal) then the size of the EC increases and we move rightward along the horizontal axis. Note that whenever the "graph" cuts the horizontal 0.5 line then a different candidate is elected president. Cutting the line from below indicates that a Republican has been replaced by a Democrat, and cutting the line from above means that a Democrat has been replaced by a Republican. The graph itself can shift as we vary k and m (and also the method of apportionment). This allows us to consider a range of potential reforms.

Table 3: Seat allocations in 2000 for a House size of 435 (certain states).

	Population	Quota	Adam	Dean	Hill	Webs	Jeff	Ham
California	33,930,798	52.447	50	52	53	53	55	52
Texas	20,903,994	32.312	31	32	32	32	33	32
New York	19,004,973	29.376	28	29	29	29	30	29
Florida	16,028,890	24.776	24	25	25	25	26	25
Illinois	12,439,042	19.227	19	19	19	19	20	19
Pennsylvania	12,300,670	19.013	19	19	19	19	19	19
Ohio	11,374,540	17.582	17	18	18	18	18	18
Michigan	9,955,829	15.389	15	15	15	15	16	15
New Jersey	8,424,354	13.022	13	13	13	13	13	13
Georgia	8,206,975	12.686	13	13	13	13	13	13
North Carolina	8,067,673	12.470	12	12	13	13	13	13
Virginia	7,100,702	10.976	11	11	11	11	11	11
Massachusetts	6,355,568	9.824	10	10	10	10	10	10
Indiana	6,090,782	9.415	9	9	9	9	9	9
Washington	5,908,684	9.133	9	9	9	9	9	9

In our graphs, we add two additional lines. The first line is a vertical line that indicates the size of the EC in the year that the election took place. The second line is an additional horizontal line, indicated by dots rather than by dashes.[11] This dotted line indicates the Democratic proportion of electors in the EC under the assumption that each state's number of electors in the EC is equal to its population. We can think of this line as a "limit concept"; it represents the Democratic proportion in the EC as the number of seats in the House tends toward infinity. It is important to note that all apportionment methods will coincide at this limit (there is no problem of rounding). The

[11] The dashed line is the 0.5 line.

divisor is 1 for all divisor methods, and each state will be apportioned its population under Hamilton's method. We call this dotted line the S^{∞} line.

Examples

To give a flavour of how our construction works, we present the representation graph for the 2000 presidential election. As mentioned above, Hill's method was used to determine the apportionment in this election.

Figure 1: 2000 election, Hill's method.

Several interesting things emerge from figure 1. First, the referendum paradox of that year can be located in the graph. In the existing case of $k = 2$ and $m = 1$ we can see that the graph crosses the vertical line (which is drawn at the current EC size of 538) beneath the 0.5 line. Bush wins the election. However, we know that Gore obtained more votes (although this is something you cannot directly see from the graph itself). An interesting observation is that if $k = 1$ and $m = 1$ then Gore would have won and the referendum paradox would have been avoided.[12] The graph corresponding to these parameter values (where each state loses 1 from its fixed component) crosses the vertical line above the 0.5 line. We can also see that increasing the size of the house would have produced a win for Gore - the S^{∞} line lies above the 0.5 line. Of course, when k is large (such as $k = 10$) it takes longer for the expanded House size to offset the effect of Bush winning lots of small states. However, this must happen eventually. As we can see, the federal outcome ($k = 1$ and $m = 0$) sees Bush winning comfortably.[13] We should emphasise that nobody's vote is changing throughout this exercise, the EC is changing and a different candidate is elected president.

We know that a referendum paradox occurred under Hill's method in 2000. Would another method have avoided this paradox without changing either k, m or the size of the House? To answer this question, we present figure 2.

[12]This would have been true irrespective of the method of apportionment.

[13]Again, this would have been true irrespective of the method of apportionment.

Figure 2 is similar to figure 1, the exception being that we compare the performance of all apportionment methods in figure 2 under the assumption that $k = 2$ and $m = 1$ (the current situation). The interesting thing to note is how the representation graphs representing the various methods cross the vertical line. All of them, with the exception of the Jefferson method, cross the line beneath the 0.5 line. This means that a referendum paradox would have occured under all apportionment methods with the exception of Jefferson's. This might tempt us into thinking that Jefferson's method can avoid the paradox. Unfortunately, this is not the case.

Figure 2: 2000 election, different apportionment methods ($k = 2, m = 1$).

As we can see from the following figure (figure 3), the use of Jefferson's method in the 1888 election as opposed to Hamilton's would have produced a referendum paradox. This demonstrates that all methods of apportionment can generate a referendum paradox.[14]

Figure 3: 1888 election, referendum paradox under Jefferson's method.

It is worth concluding this section by emphasing the following point. In the 2000 election, the methods of apportionment always appear to be "close" to one another in terms of the outcome of the presidential election. However, this is an artifact of the requirement that $k = 2$ and $m = 1$. If we consider the "unconstrained" case where $k = 0$ and $m = 0$ then the methods of apportionment diverge substantially in terms of the outcome when the size of the House is small. We demonstrate this in figure 4.

Figure 4: 2000 election, different apportionment methods ($k = 0, m = 0$).

[14] In fact, as we can see from figure 2, even Jefferson's method would have produced a paradox had the size of the House been smaller.

4 No robust solution

Is there a reform that will always ensure that a referendum paradox can be avoided? We have seen in the previous section that changing the method of apportionment does not help. All apportionment methods are susceptible to the paradox. What about changing the size of the House? We have already seen that increasing the size of the House sufficiently would have prevented the paradox in the 2000 election. Is this all we need to do? It turns out that the answer to this is no. We can generate a paradox under perfect proportionality (where $k = 0$ and a state's proportional component equals its population).

Before we demonstrate this, we identify some of the strange behaviour that can arise when the size of the House increases. Figure 5 shows the representation graph for the presidential election of 1996. Figure 6 shows the representation graph for the presidential election of 1992.

Of course, both of these elections were won by the Democratic candidate, Bill Clinton. In 1996, Clinton obtained 49.2% of the popular vote as opposed to Republican Bob Dole's 40.1%. This gave him 379 votes in the EC compared to 159 for Dole. This represents 70.4% of the EC vote (0.704 on the vertical axis of our graph). Notice, however, that as the size of the House increases and we approach the S^{∞} line, then Clinton's share of the EC vote increases *even further*. It moves away from his share of the popular vote.

The same is true of the 1992 election. Clinton won 43% of the popular vote as opposed to Republican George H.W. Bush's 37.4%.[15] This gave Clinton 370 votes in the EC compared to 168 for Bush. This represents 68.77% of the EC vote (0.688 on the vertical axis of our graph). Again, as the size of the House increases and we approach the S^{∞} line, then Clinton's share of the EC vote increases further. It moves away from his share of the popular vote.[16]

Figure 5: 1996 election, Hill's method.

Figure 6: 1992 election, Hill's method.

[15] Ross Perot obtained a creditable 18.9% share of the popular vote in this election.

[16] As can be seen from inspecting figures 5 and 6, in both of these elections the "federalism" scenario would have brought the EC outcome closer to the true vote share.

An example of this phenomenon from a Republican perspective comes from the 1896 election, which the Republicans won. They received 51.1% of the popular vote, compared with 45.8% for the Democrats. The EC margin of victory was large, 271 votes to 176. However, as we increase the size of the House the Republican share of the EC vote increases, moving further away for their share of the popular vote (just like in 1992 and 1996). We illustrate this in figure 7.

We can see from these examples that increasing the size of the House does not always produce the outcome that you expect.

Figure 7: 1896 election, Hamilton's method.

Before we present our key example, we first discuss an election in which reducing the size of the House produces a referendum paradox even though one did not exist originally. The election of interest is the 1976 presidential election.

In the 1976 election, Democrat Jimmy Carter defeated Republican Gerald Ford. Carter received 50.08% of the popular vote, whereas Ford received 48.02%. The election was, therefore, very close. Carter received 297 EC votes compared to Ford's 240.[17] This corresponds to 55.2% of the EC vote. Clearly, there is no referendum paradox in this election.

However, as we can see in figure 8, a referendum paradox *would have occured that year* had the House size been smaller. This is true under all methods of apportionment and not just Hill's.

We have seen that a referendum paradox can arise when the House size is low, and also that strange behaviour can arise when the House size increases. Our most surprising finding, however, comes from the 1916 election. In this election Woodrow Wilson for the Democrats defeated Charles Evan Hughes for the Republicans. Wilson obtained 49.2% of the popular vote compared with 46.1% for Hughes. Wilson also obtained 277 votes in the EC compared with 254 for Hughes. This represents 52.1% of the total (0.521 as measured

[17] One faithless elector from Washington, Mike Padden, gave Ronald Reagan his electoral vote instead of Ford. Apparently Padden did this to indicate his support for Reagan's pro-life position, not as a protest against Ford.

on the vertical axis of our graph). Wilson won 30 states, compared to 18 for Hughes. Clearly, there is no referendum paradox here. What is surprising is that when we increase the size of the House then *Hughes wins the election.* We can see in figure 9 that the S^{∞} line lies beneath the 0.5 line.[18] What this means is that increasing the size of the House is not, in general, a way to avoid a referendum paradox. A larger house would have prevented the paradox in 2000 *but would have caused one in 1916.*

Figure 8: 1976 election, Hill's method.

Figure 9: 1916 election, Webster's method.

Why does this happen? The answer lies with the "winner-takes-all" rule. To give some intuition, consider the following example.

Imagine that there is a situation of perfect proportionality where each state's EC vote is equal to the size of its population and $k = 0$. Assume that there are 3 states with 10 voters in each state. The following table indicates how the voters vote in the states.

Table 5: A hypothetical example.

States	**Republican votes**	**Democratic votes**
A.	6	4
B.	8	2
C.	0	10
Total	**14**	**16**

As we can see, under the "winner-takes-all" rule the Republicans receive 10 EC votes each from state A and state B whereas the Democrats receive 10 EC votes from state C. The Republicans win the election by 20 EC votes to 10. However, there is a referendum paradox. 16 people vote Democrat compared to 14 for the Republicans. This simple observation accounts for the strange behaviour we observed in 1996 and 1992 (the Democratic share

[18]Webster's method was used to determine the apportionment in 1916.

of the EC vote moves away from their share of the popular vote as House size increases). It also accounts for the potential referendum paradox in 1916 that arises when we increase the size of the House.

We conclude our analysis by considering the other two elections in which a referendum paradox occured, the elections of 1888 and 1876.

1888 and 1876

The 1888 election is a textbook case. Cleveland, the Democratic candidate, obtained more votes than Harrison, the Republican candidate. However, Cleveland obtained only 168 EC votes compared with 233 for Harrison. There was a referendum paradox, like in 2000. Hamilton's method was used to determine the apportionment and the representation graph for this election is given in figure 10.

Figure 10: 1888 election, Hamilton's method.

An important difference between the 2000 election and 1888 election is that in the latter if we change k, m or the method of apportionment, then there *always* exists a paradox. This is unlike the 2000 election. The paradox exists under perfect proportionality and also under federalism. This is a striking feature of this election.

Our final election is the one in 1876. Rutherford Hayes, a Republican, became president with only one EC vote more than Samuel Tilden, a Democrat. Hayes won by 185 EC votes compared with 184 votes for Tilden. However, Tilden obtained 51.5% of the popular vote compared with 48.4% for Hayes.

With this election, one can understand the importance of the choice of the method of apportionment. The method used was Hamilton's method, which, in this particular case, produced the same apportionment as Webster's method. 283 seats were to be apportioned given the 1870 census, but 9 more were added for the 1876 election in addition to an extra 3 seats for the new state of Colorado. Since there were 37 states, there were 74 senators and so the total number of electors was 369. Surprisingly, in these new conditions, the apportionment used does not correspond to the Hamilton or Webster methods but corresponds to Dean's method. Unfortunately for Tilden, he would have won had either the Hamilton or Webster method been used. Note that when we change the number of seats in our representation graph, we assume that the method used is Hamilton's method, which is not the case for 369 seats.

As we can see, Tilden was really unlucky since he would have won with slightly fewer seats and also with slightly more seats.

Figure 11: 1876 election, Hamilton's method.

5 Conclusion

If presidents were elected by tossing a coin, then that would be regarded as undemocratic and unacceptable. Fortunately, this is not the case in the United States. The outcome of the election does depend on how individuals vote. It is not determined randomly. Nor is it arbitrary in the sense that a different candidate can *always* be elected for some possible EC architecture. However, one conclusion of this paper is that presidential elections lack robustness in the sense that parameters that should not matter (and intuitively feel "irrelevant") can change the outcome. This is surely troubling. Moreover, sometimes a "small" change is all that is needed to alter the election outcome.

Criticism of the EC is most acute when a referendum paradox occurs. As we have seen, such events are rare in U.S. history but when they occur the democratic legitimacy of the elected president is inevitably undermined. If this is considered to be undesirable then the question might be asked: is it possible to reform the EC so that a referendum paradox can never arise? We have shown in this paper that the answer is no. Although it is possible to demonstrate this point mathematically using hypothetical votes, we have been able to demonstrate it using actual voting data from past elections. In other words, we have given an empirical proof of a social choice impossibility theorem.

Our most surprising finding is that increasing the size of the House can trigger a paradox when none existed originally. Although it is easy to understand how this can happen with a simple example, the fact that an actual election had this property is rather remarkable. What seems like a simple solution is, in fact, no solution at all.

Of course, one effective policy would be to abolish the EC and elect the president directly on the basis of the popular vote. It is unlikely, however, that such a reform would ever be enacted. It is probably more politically feasible to pass a reform that reduces the *likelihood* of a referendum paradox, while retaining the institution of the EC itself. One way of achieving this has already been hinted at in this paper - adopt nationwide the Maine/Nebraska mechanism as a way of incorporating an element of proportionality into the allocation of a state's EC votes. Although a referendum paradox can still occur under this system, Merlin and Senne (2008) demonstrate that the probability of a paradox arising is strictly lower than under the current "winner-takes-all" rule. Naturally, there are many possible alternatives to the current system for electing presidents but discussing them all goes beyond the scope of this paper.[19] We will say, however, that probability calculations in the spirit of Merlin and Senne (2008) are possible with the parameter variations that we consider in this paper. Therefore, although none of the reforms we have considered can eliminate the possibility of a referendum paradox, we can calculate the likelihood of a paradox for various values of k, m and House size. In doing so, we would not directly be comparing methods as in Merlin and Senne, but we would be able to say which values minimise the probability of a paradox given certain assumptions about voter behaviour.[20] We leave this question for future research.

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[19] An interesting debate with a variety of perspectives can be found at www.signonsandiego.com/electoral-college

[20] A standard "impartial culture" assumption could be used, for example.

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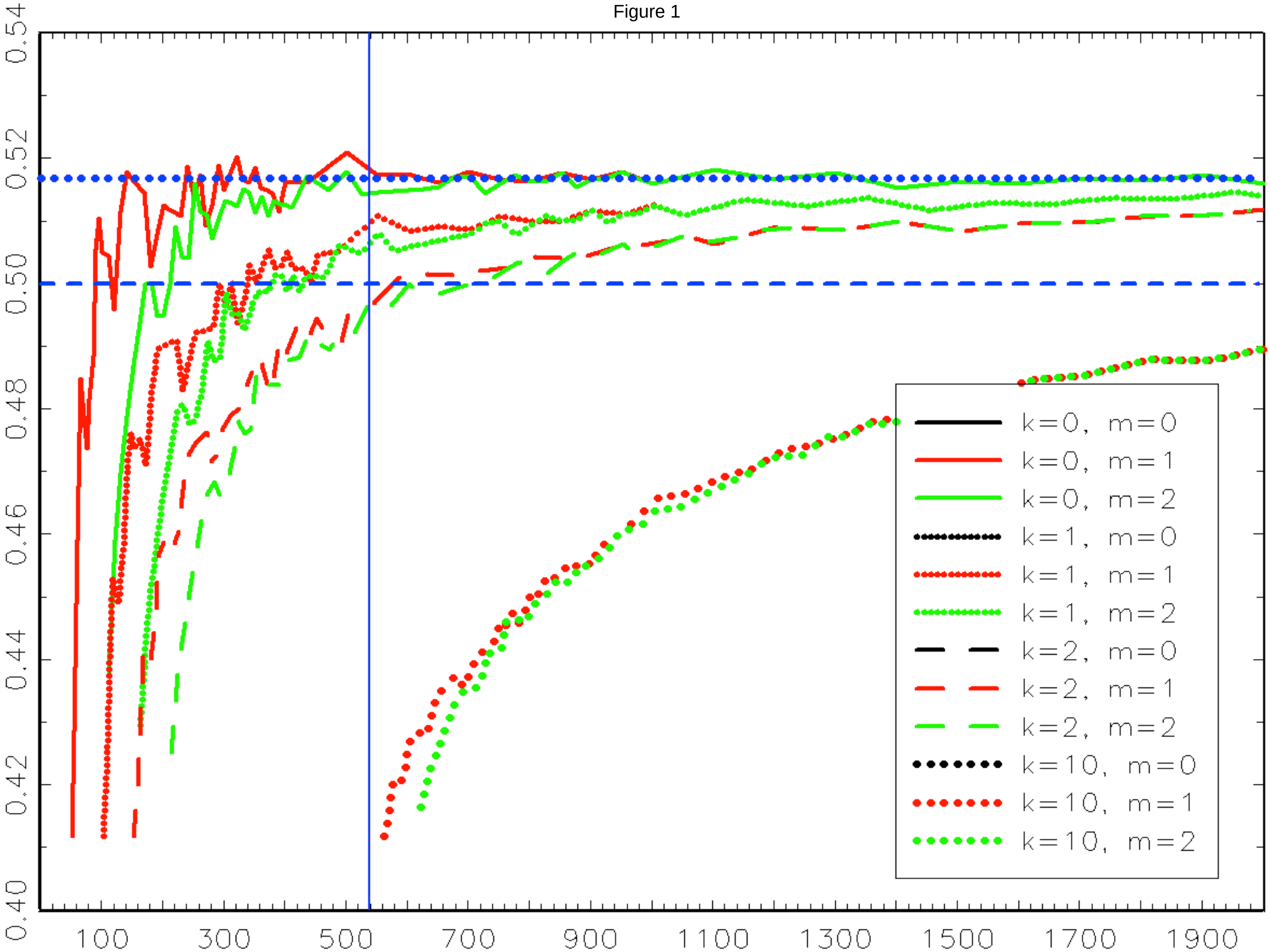
Figure 1



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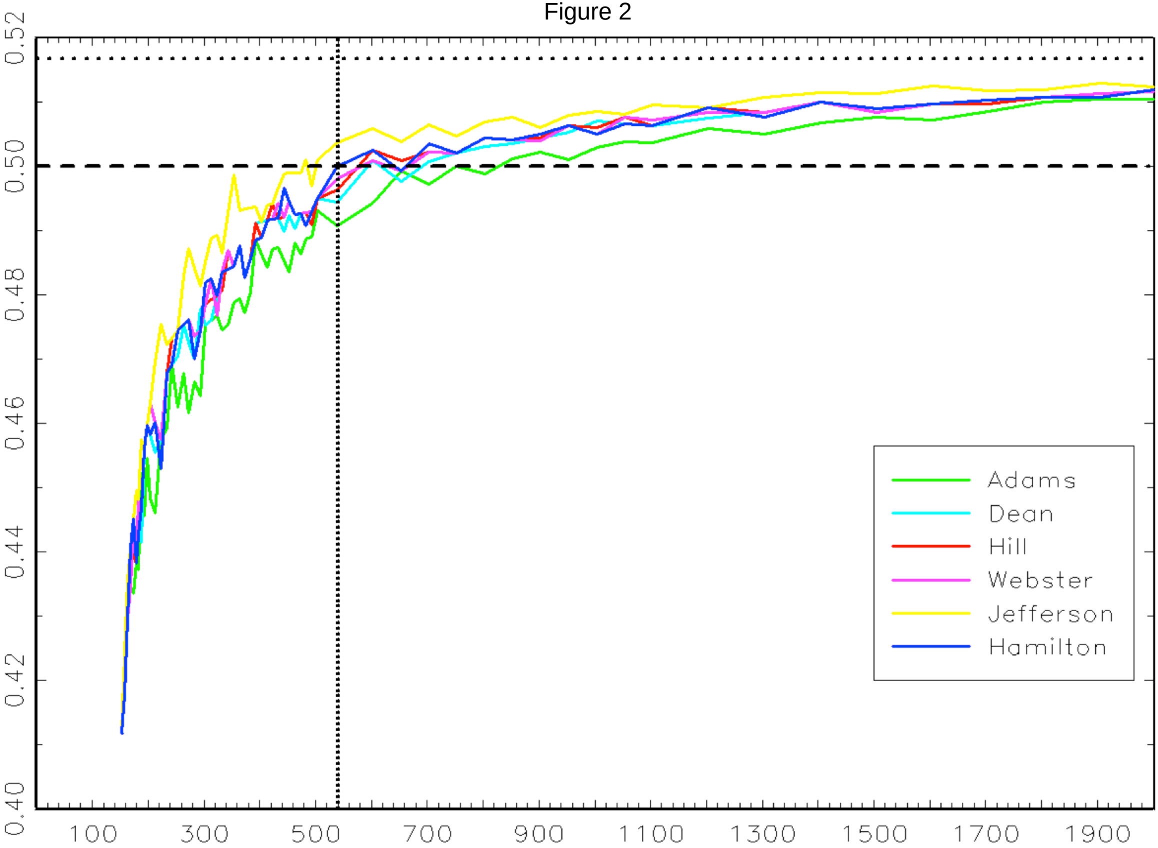
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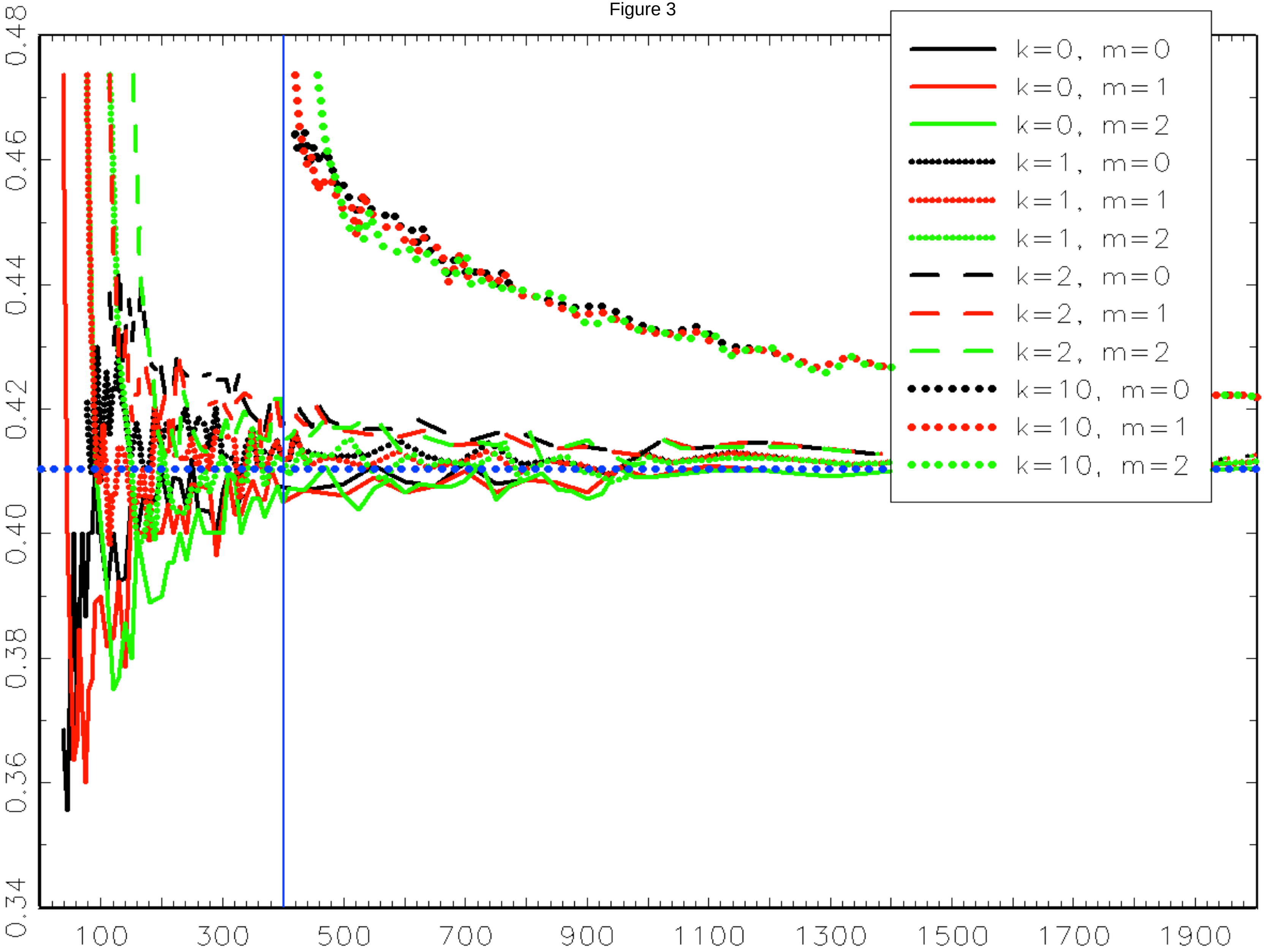

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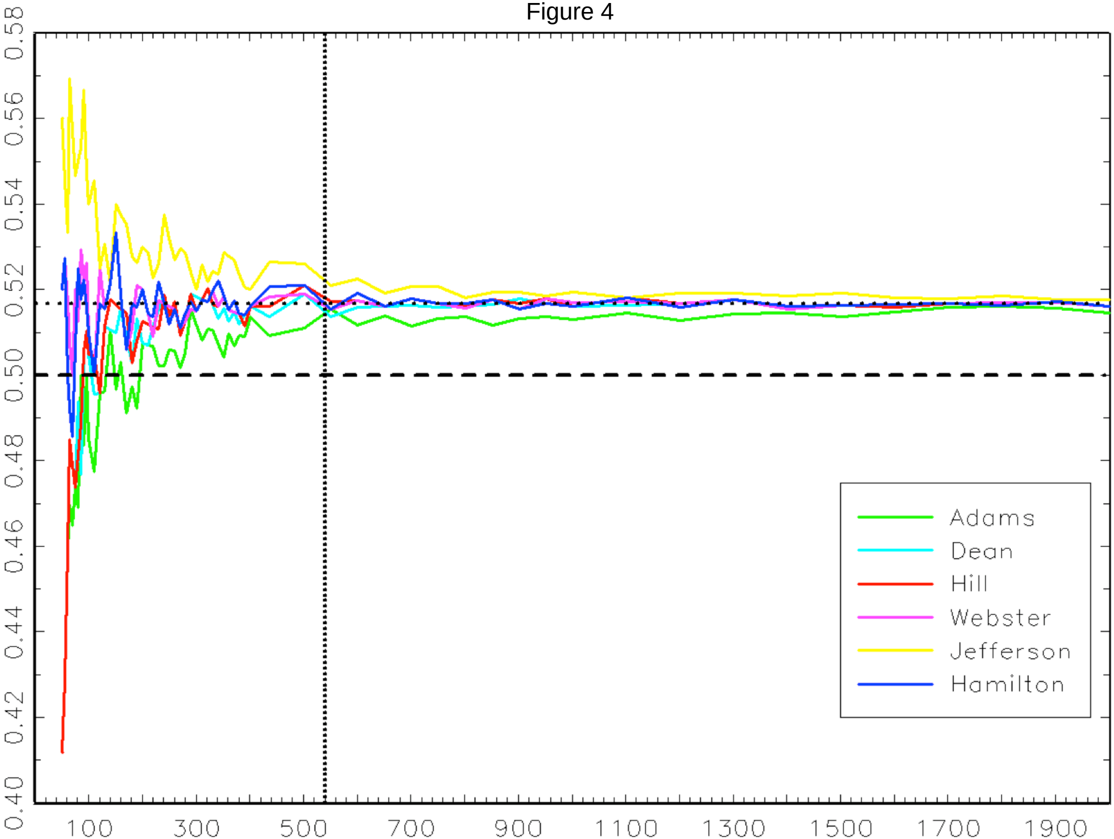
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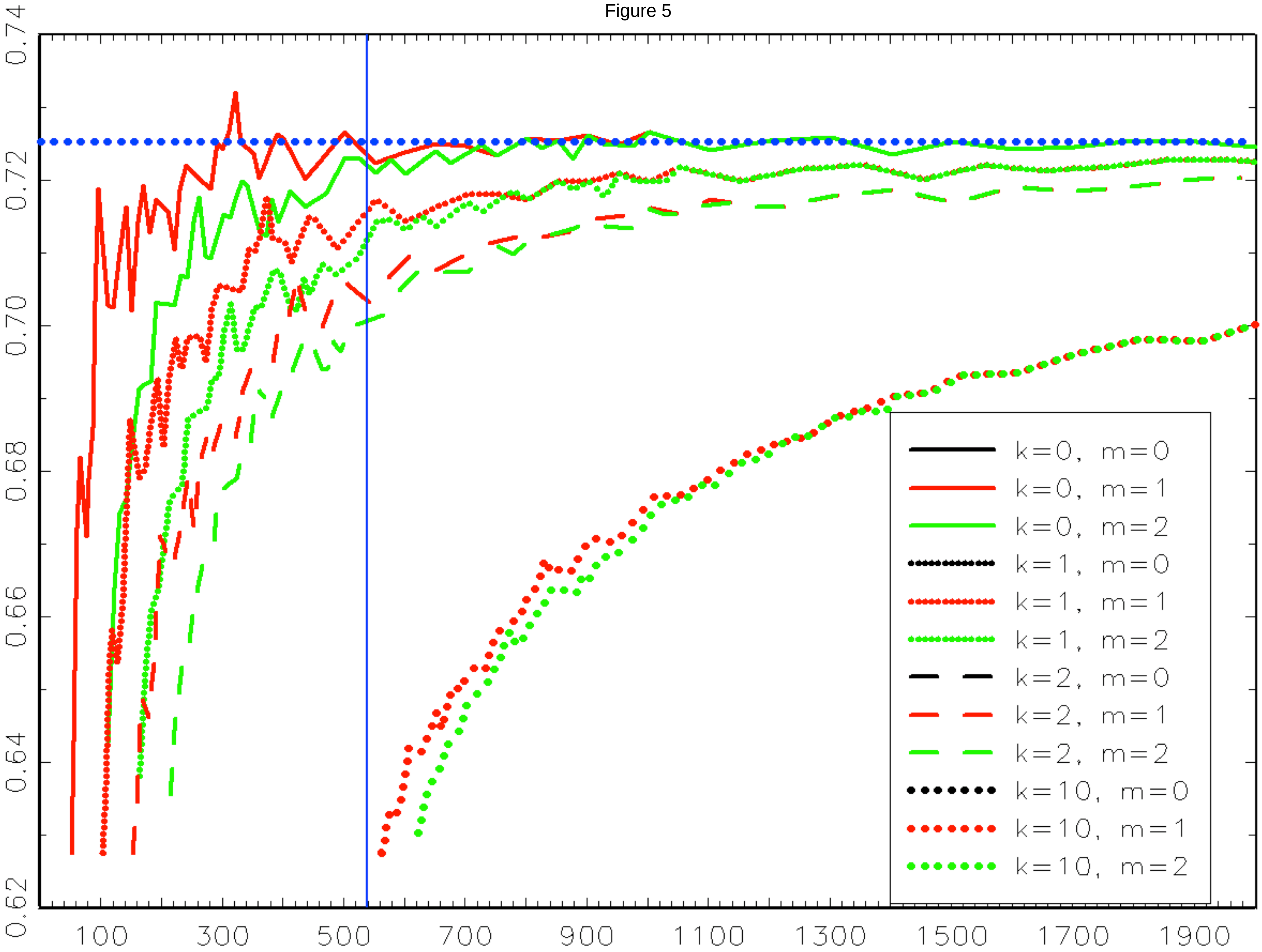

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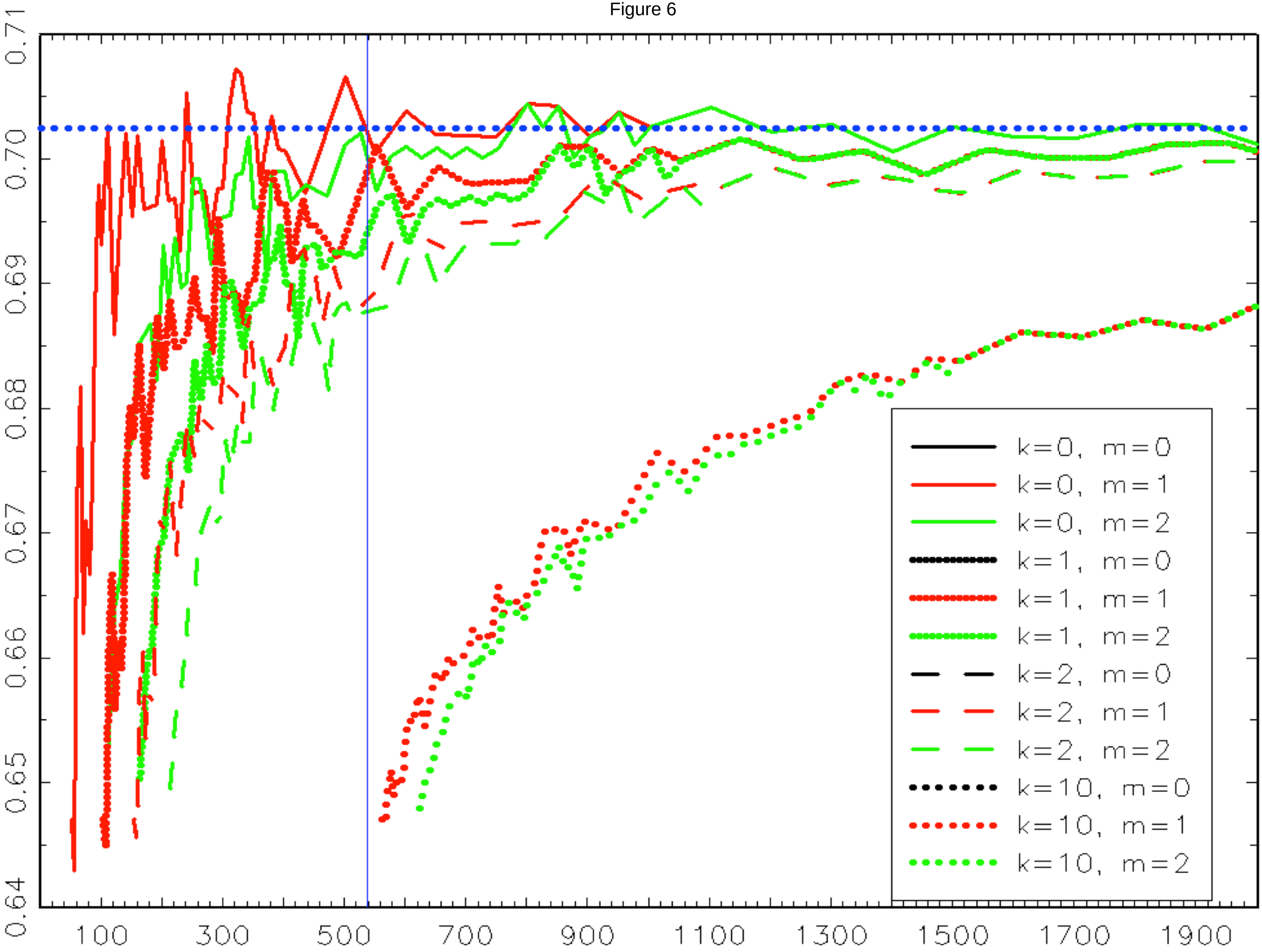
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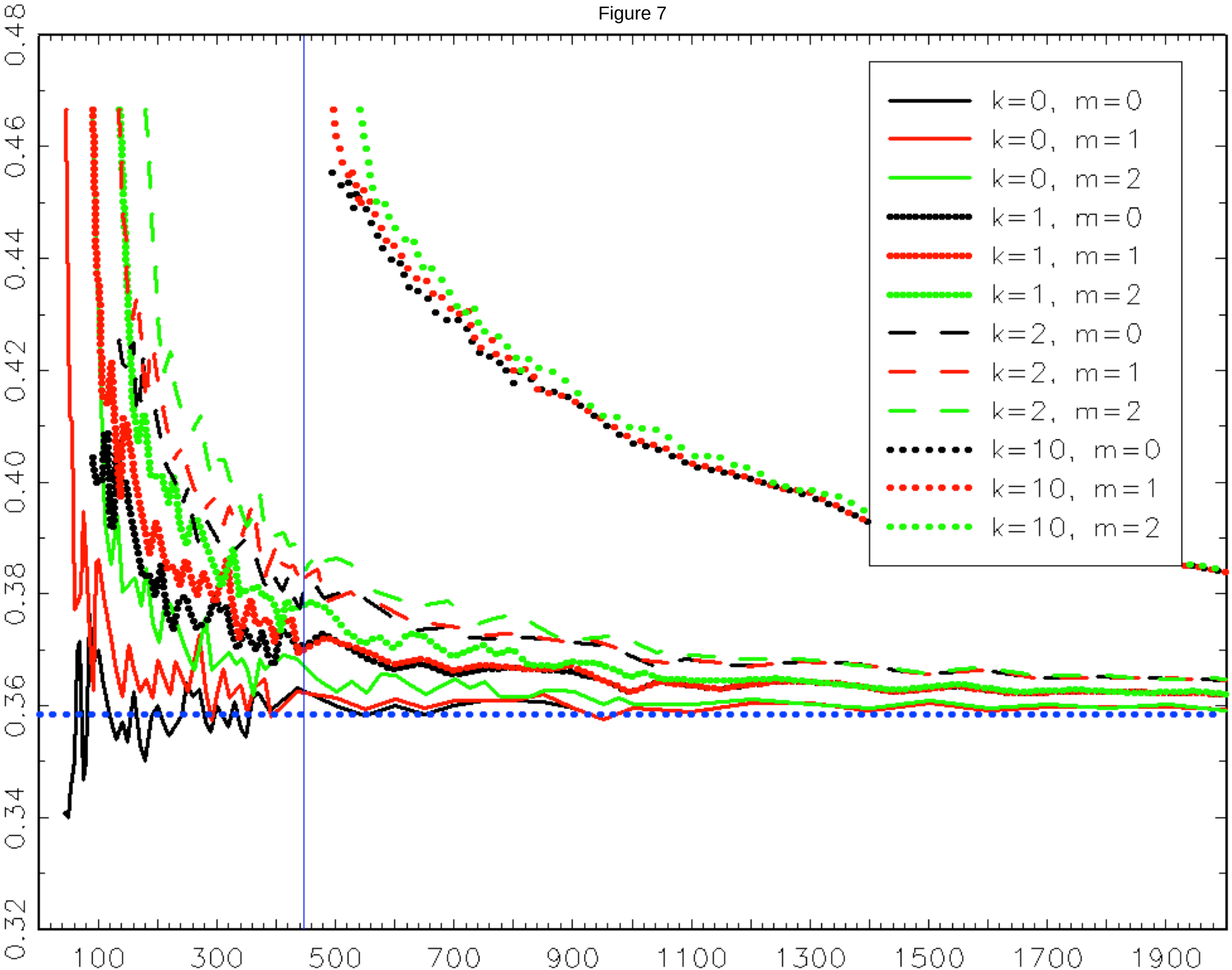

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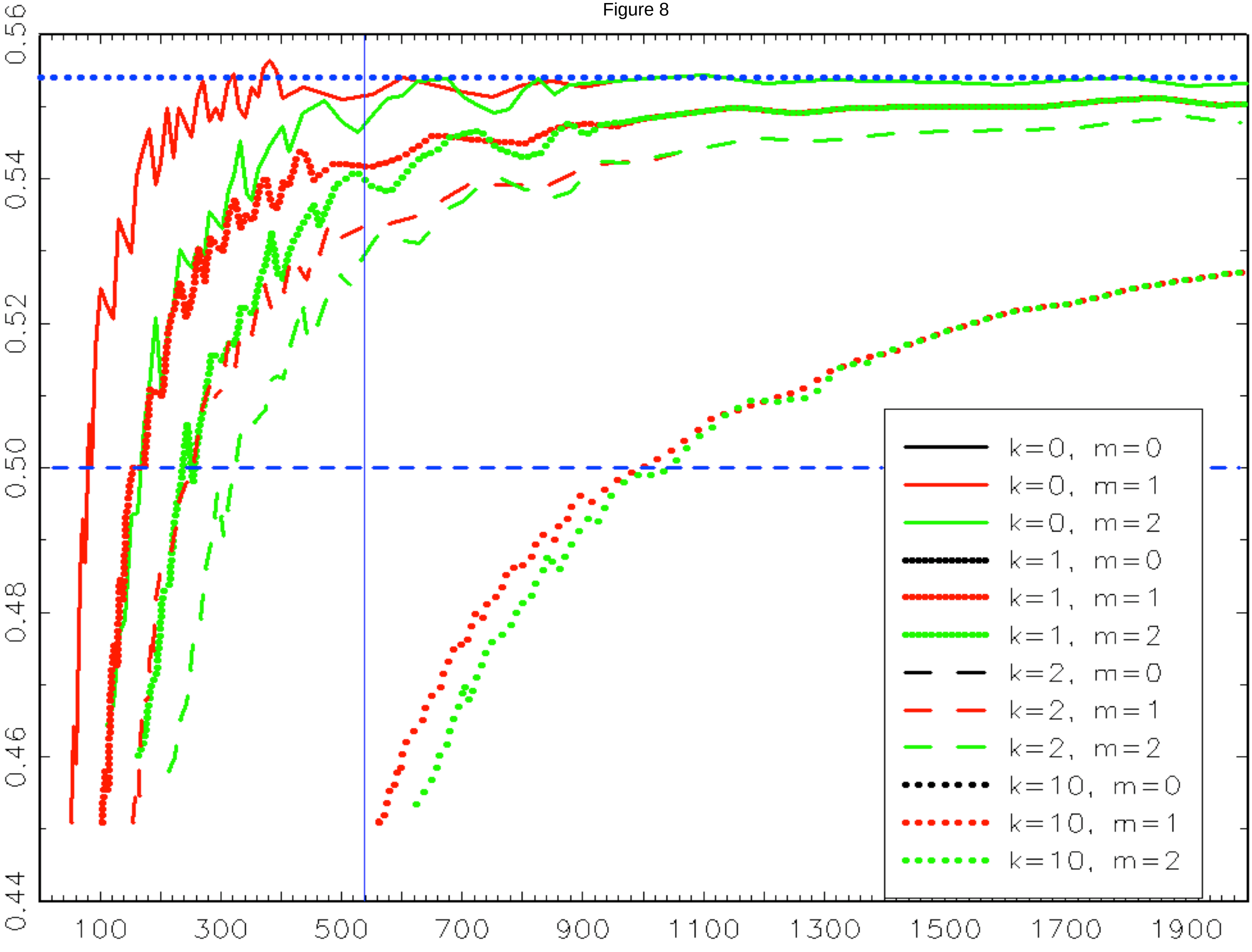

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Figure 9

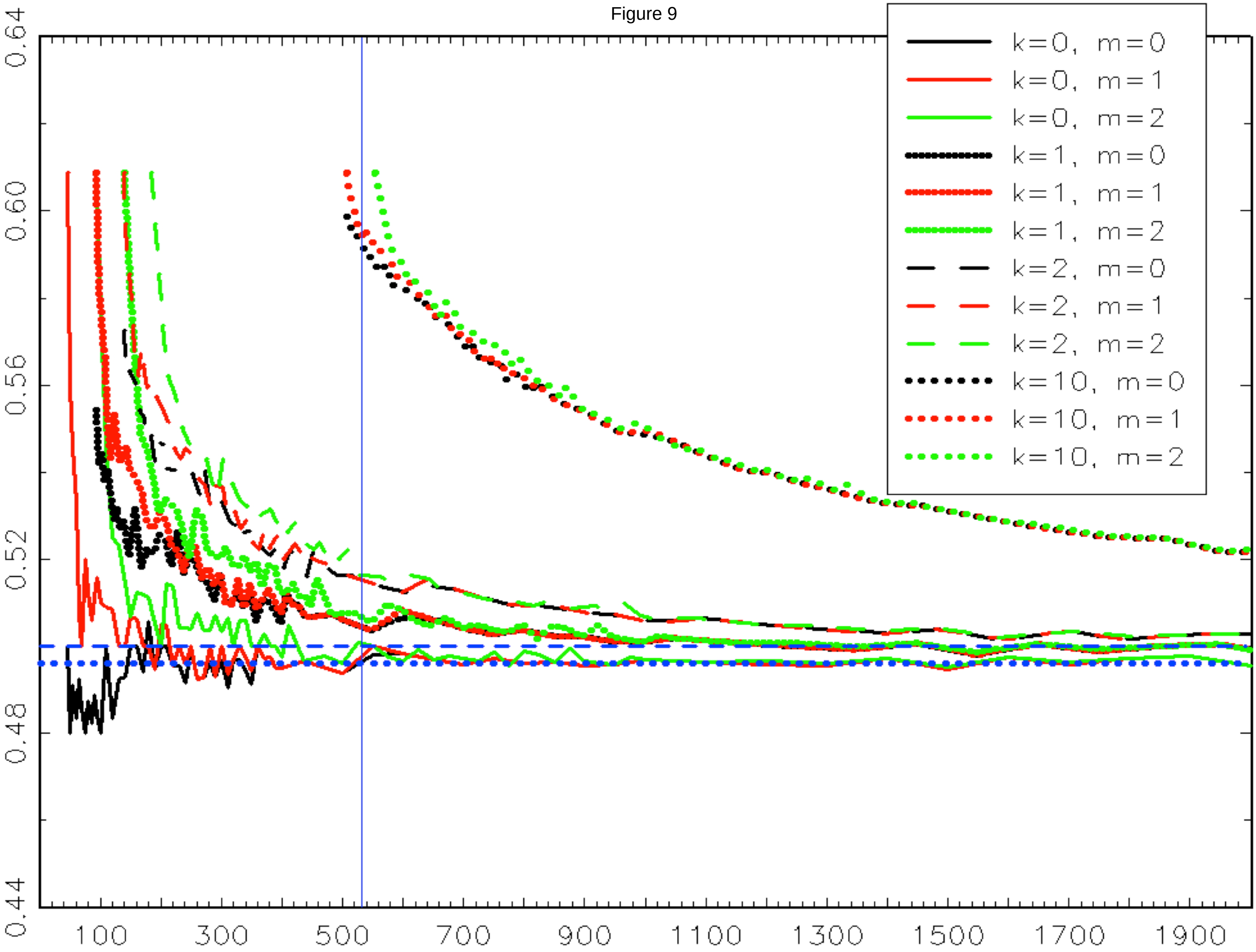
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Figure 10

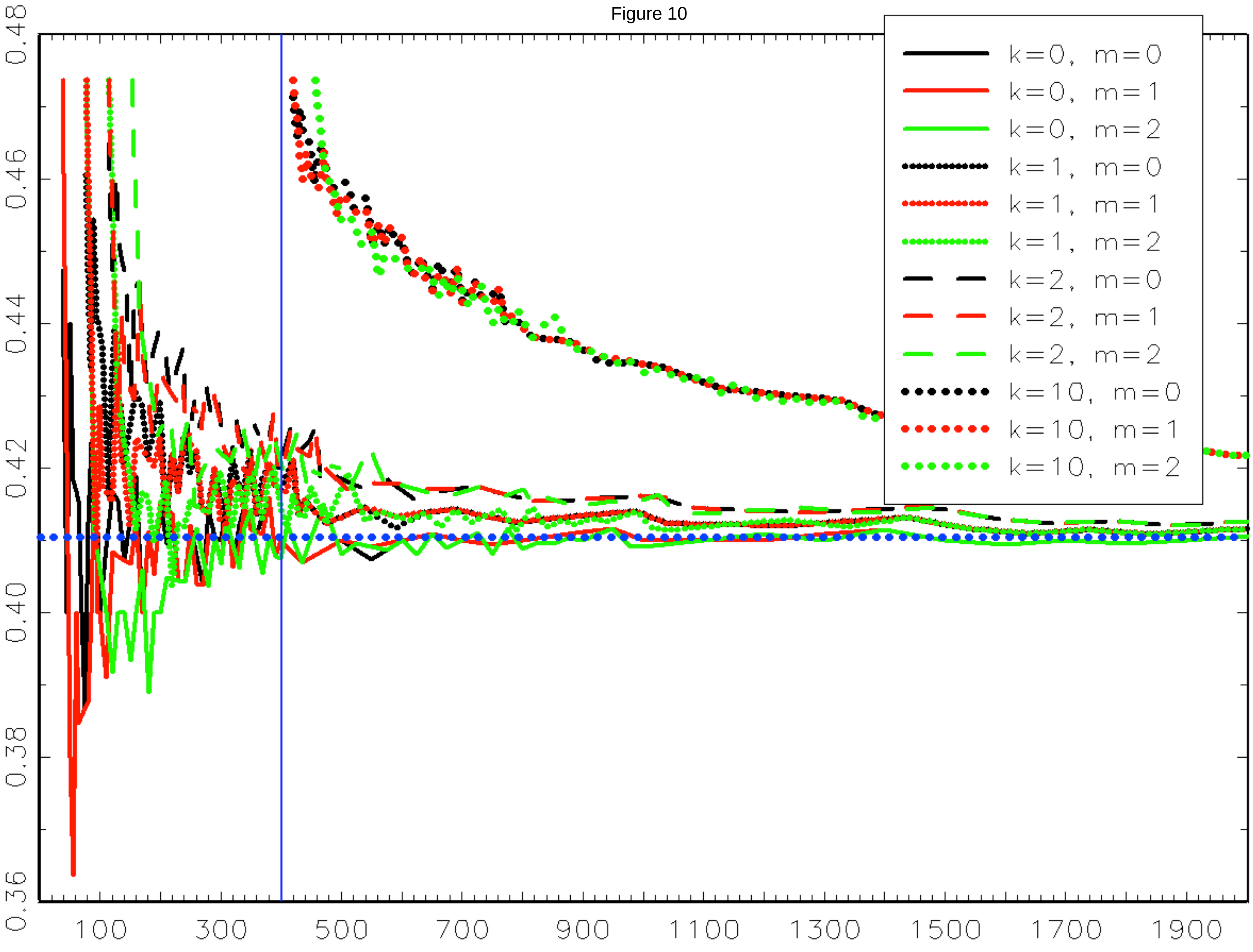

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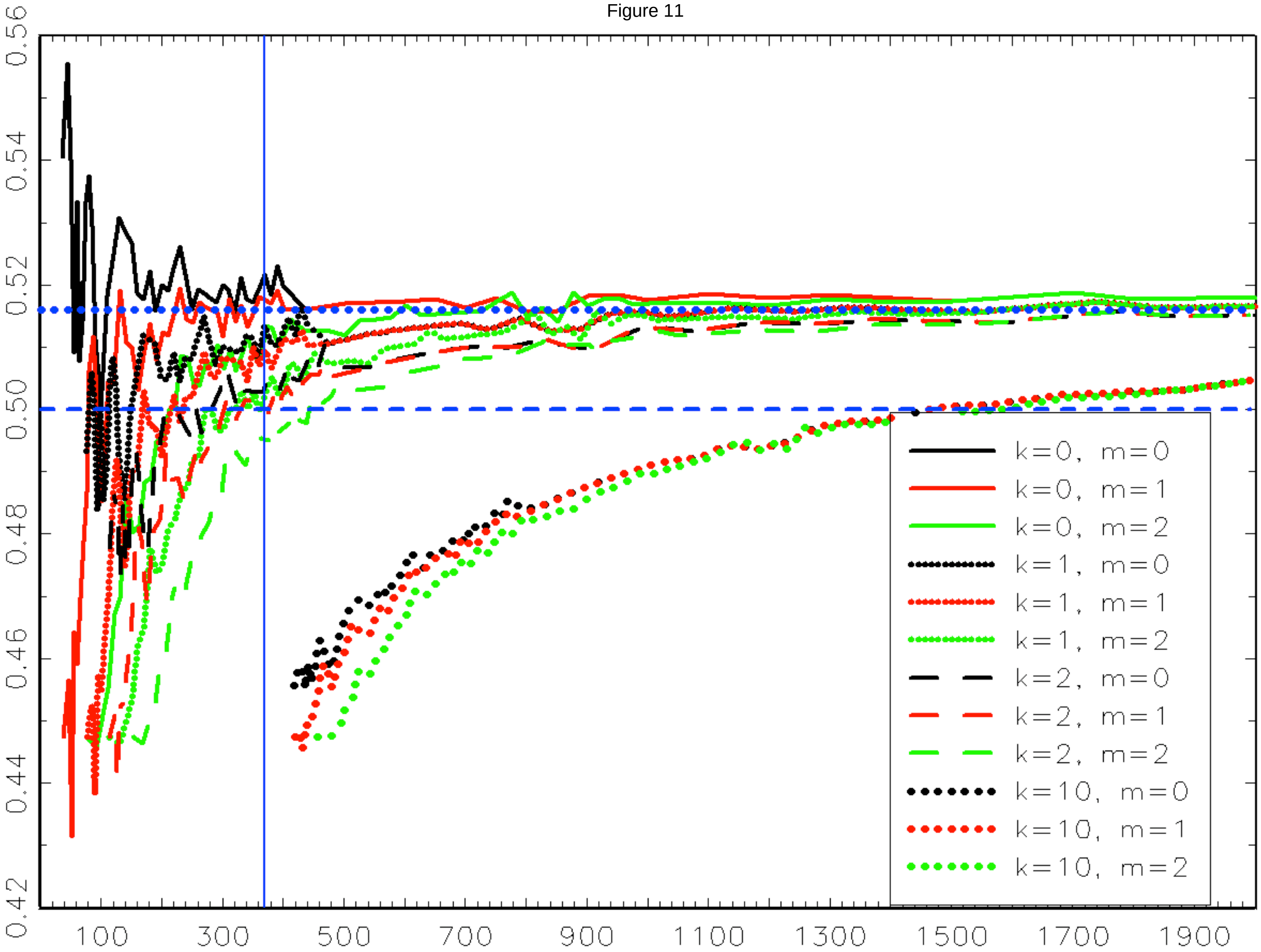
Figure 11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