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Fabrice Barthelemy
Mathieu Martin
Bertrand Tchantcho

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Some conjectures on the two main power indices

Fabrice BARTHÉLÉMY*, Mathieu MARTIN † and Bertrand TCHANTCHO‡

Summary The purpose of this paper is to present a structural specification of the Shapley-Shubik and Banzhaf power indices in a weighted voting rule. We compare them in term of the cardinality of the sets of power vectors (PV). This is done in different situations where the quota or the number of seats are fixed or not.

JEL classification: C7, D7.

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1 Introduction

A weighted voting rule is a (social, economical or political) situation where each member of a body (such as a group of shareholders, a council, a committee or a parliament) controls a fixed number of votes called his weight, and a certain number of votes (called the quota) is required to pass a proposal.

Such a rule can be represented as a sequence $[q; w_1, ..., w_n]$ where n is the number of agents or (more generally) voters, w_i is the weight (number of seats) of voter i, and q the relative quota with $q \leq 1$. The total number of weights is denoted by $\bar{w}$, hence $\sum_{i=1}^{n} w_i = \bar{w}$ and we assume that $w_1 \geq w_2 \geq ... \geq w_n$. A set of voters S is said to be winning if $\sum_{i\in S} w_i \geq q\bar{w}$. Furthermore, it is assumed that the complement of a winning set of voters is a losing set, meaning that the relative quota is greater than $\frac{1}{2}$. Particular attention is given to the well-known majority rules for which $q = \frac{1}{2}$. In this case, if the structure of weights is $(w_1, ..., w_n)$, then a set of voters is winning if and only if $\sum_{i\in S} w_i \geq \frac{\bar{w}}{2} + 1$ if $\bar{w}$ is even and $\sum_{i\in S} w_i \geq \frac{\bar{w}+1}{2}$ if $\bar{w}$ is odd[1]. The extent of control that a voter possesses over the decision-making process due to the decision rule alone is referred to as his voting power. In other words, it is his constitutional power (see Felsenthal and Machover, 1998).

*University of Cergy Pontoise, THEMA, F-95000 Cergy-Pontoise, FRANCE. E-mail: fabrice.barthelemy@u-cergy.fr

†University of Cergy Pontoise, THEMA, F-95000 Cergy-Pontoise, FRANCE. E-mail: mathieu.martin@u-cergy.fr

‡University of Yaounde I, Ecole Normale Superieure, Cameroon, PO Box 47 Yaounde, btchantcho@yahoo.fr

[1]Note that the real values of the quota are $q = \frac{1}{2} + \frac{1}{\bar{w}}$ if $\bar{w}$ is even and $q = \frac{1}{2} + \frac{1}{2\bar{w}}$ if $\bar{w}$ is odd. The notation $q = \frac{1}{2}$ is clearer and shorter.

There is an abundant literature on the a priori measure of power of each agent in such a collective decision-making procedure. A complete description of power indices can be found in Felsenthal and Machover (1998), Leech (2002) or in Laruelle and Valenciano[2] (2008). Famous indices include the Shapley-Shubik (1954) index and the (normalized and non normalized) Banzhaf index, both of which are considered by scholars to be pre-eminent by virtue of their properties and various axiomatizations[3]. The Shapley-Shubik index is based on the concept of the pivotal voter while the Banzhaf index relies on the notion of the decisive voter. A voter i is said to be pivotal with respect to a ranking of voters if the set of voters obtained by considering all the voters ranked before i is losing while adding him in that set of voters yields a winning set. On the other hand, a voter i is said to be decisive in a set of voters S if either $i \in S$, S is winning and $S \backslash \{i\}$ is not winning or $i \notin S$, S is not winning and $S \cup \{i\}$ is winning.

These indices do however yield different power vectors even though the relative rankings of voters according to these indices coincide. Indeed, it is well known from Tomiyama (1987) (see Diffo and Moulen, 2002 for a generalization) that in a weighted voting rule, given two voters i and j, i has at least as much power as j with respect to the Shapley-Shubik index if and only if this is the case with respect to the non normalized Banzhaf index. But this induced ranking between voters could be quite different from the one observed regarding the structure of weights. For example, having a positive weight does not ensure having a positive power, different weighting structures may lead to the same voting power and so forth.

While attention has been given to the rankings of voters, nothing so far has been said neither on different power vectors achieved by these indices nor on the total number of vectors achievable. We shall illustrate this in a moment but we can note that it could be interesting to know all possible distributions of power. This can be of use in seeking the most adequate voting rule for a committee of representatives such as the European Council of Ministers, given the number of voters and a structure of weights (Laruelle and Valenciano, 2008). This could also be interesting, in respect of a comparison of both indices, to determine all achievable power vectors and so assess the probability that both indices give the same power structure.

Various methods are available to compute the Banzhaf and the Shapley-Shubik indices. See for example Leech (2002) for a description of each method and their respective interest. Direct enumeration consists of directly applying the definition of the indices. A shortcoming of this approach is the number of voters which should be less than 31. Generating functions, as suggested by Mann and Shapley (1962), make it possible to deal with higher numbers of voters (up to 200) and give an exact result. The Monte Carlo simulations presented by Mann and Shapley (1960) are an approximation

[2]In particular, the authors present the Shapley-Shubik power index in the classical cooperative game theory framework and they show the difficulties of its interpretation.

[3]The normalized Banzhaf is referred to as the Banzhaf-Coleman index while the non normalized is referred to as the Banzhaf-Penrose index.

as are multilinear extensions approximation methods developed by Owen (1972, 1975) and modified by Leech (2003).

But, as far as we know, there is no formula which provides either the list of all achievable power vectors according to Shapley-Shubik and (normalized and non normalized) Banzhaf, or its cardinality. We present in this paper some tables with the number of achievable power vectors for a given number of voters. These numbers are computed from an enumeration we made to get all the different power vectors for the indices mentioned above.

To explain the purpose of this paper, consider the simple 2-voter case. A weighted rule can be written as a sequence $[q; w_1, w_2]$, with (without loss of generality) $w_1 \geq w_2$. We construct a partition of the weighted rules set such that the decisive (or pivotal) voters structure is constant within a given class. From this construction, all the weighted rules belonging to the same class, lead to a unique power vector (PV)[4]. Let us notice that, by construction, the classes are non empty sets, as each weighted rule belongs to one and only one class.

In the 2-voter case, there exist only two classes. The first class is the set of all the weighted rules such that the first voter decides alone, $w_1 \geq q\bar{w}$ (the only winning set of voters is $\{1\}$). The second class consists of all the weighted rules where the first voter may not decide alone, $w_1 < q\bar{w}$ (the only winning set of voters is $\{1, 2\}$, as $w_1 \geq w_2$). Whatever the weighted rule, it belongs to one of these two classes, which implies that there are, at most, two different power vectors. In fact, the cardinality of the power vectors set is equal to 2, for the 2-player case[5].

What happen to the cardinality of the power vectors set when there are constraints on the total number of seats $\bar{w}$ and/or on the relative quota q? This is the central question that we answer in this paper.

Let us continue with the 2-voter case. If the total number of seats $\bar{w}$ and the relative quota q are fixed, there exists a finite number of weighted rules, which corresponds to the number of vectors (w_1, w_2) such that $w_1 + w_2 = \bar{w}$ and $w_1 \geq w_2$. By contrast, the number of possible weighted rules becomes infinite when at most one of these two values is fixed. This arises because there is an infinity of vectors (w_1, w_2) and/or an infinity of quotas $(1/2 < q \leq 1)$. Hence, the number of possible weighted rules is greater than two (as soon as $\bar{w} \geq 3$) when both $\bar{w}$ and q are fixed and infinite otherwise, while, as mentioned previously, there exists only 2 classes.

We illustrate the potential impact of constraints on the cardinality of the power vectors set, considering three situations.

[4]Which may be different for different indices, but for a given index, the PV remains the same.

[5]In the first class, the first voter has all the power, and in the second class the powers of the two voters are equivalent.

First, to illustrate the case where $\bar{w}$ and q are fixed, consider $q = 1/2$ and $\bar{w} = 3$. It is worth noting for a fixed $\bar{w}$, several distributions of w_i may exist. For instance, the vectors $(3, 0)$ and $(2, 1)$ lead to $\bar{w} = 3$. Hence, there are two possible weighted rules, $[\frac{1}{2}; 3, 0]$ and $[\frac{1}{2}; 2, 1]$, which belong to the first class. Hence, only one PV is available. Let us remark that fixing $\bar{w}$ and q does not imply necessarily a smaller number of PVs. Considering $q = 1/2$ and $\bar{w} = 4$, three weighted rules are available. Both $[\frac{1}{2}; 4, 0]$ and $[\frac{1}{2}; 3, 1]$ belong to the first class while $[\frac{1}{2}; 2, 2]$ belong to the second class: the two PVs are available.

Second, to illustrate the case where $\bar{w}$ is not fixed, consider $q = \frac{1}{2}$. For instance, $[\frac{1}{2}; 3, 0]$ belongs to the first class (where $\bar{w} = 3$) and $[\frac{1}{2}; 2, 2]$ belongs to the second class (where $\bar{w} = 4$). Hence, the two classes are non empty sets. In fact, the number of non empty classes is always equal to two[6] when $\bar{w} \geq 2$.

Third, to illustrate the case where q is not fixed, consider $\bar{w} = 4$. The two classes are non empty sets since for instance $[\frac{2}{3}; 3, 1]$ belongs to the first class, while $[\frac{1}{2}; 2, 2]$ belongs to the second class. In fact, the number of non empty classes is always equal[7] to two when $\bar{w} \geq 2$.

In this paper, we study the four different situations, described in Table 1, obtained by different conditions on q and $\bar{w}$, previously illustrated with the 2-voter case. Complete answers to the main questions are given for the 2, 3 and 4 voter case. In particular, whenever the quota is fixed, the number of achieved power vectors for the Shapley-Shubik and both the non normalized and normalized Banzhaf indices coincide. Meanwhile when the quota is not fixed, in general the number of power vectors achieved by the non normalized Banzhaf index is greater than that achieved by the normalized Banzhaf index, this later being at least as large as the number of achieved power vectors via Shapley-Shubik. These quite surprising results are confirmed using a computer program for more voters.

Table 1: The four situations

Situation 1: q and $\bar{w}$ are not fixed	*Situation 3*: q is not fixed while is $\bar{w}$ fixed.
Situation 2: q is fixed and is $\bar{w}$ is not	*Situation 4*: q and $\bar{w}$ are fixed

The paper is organized as follows: section 2 presents some analytical results for 2, 3 and 4 voters, section 3 presents tables involving more players obtained thanks to the use of a computer.

[6] When $\bar{w} = 1$, there exists only one weighted rule, $[\frac{1}{2}; 1, 0]$, and one of the two classes is then empty.

[7] When $\bar{w} = 1$, all the weighted rules, $[q; 1, 0]$, whatever the quota, belong to the same class.

2 The analytical case: 2, 3 and 4 voters

Throughout, n is the number of voters. Let $[q; w_1, ..., w_n]$ be a weighted rule: the set of winning set of voters S such that $\sum_{i \in S} w_i \geq q\bar{w}$ will be referred to as W. The characteristic function of the rule denoted by v is defined by:

$$v(S) = \begin{cases} 1 \text{ if } S \in W \\ 0 \text{ if } S \notin W \end{cases}$$

The Shapley-Shubik index (SSI) is given by the following formula[8]

$$\phi_i = \sum_{S \subseteq N} \frac{(|S|-1)!(n-|S|)!}{n!}[v(S) - v(S \backslash \{i\})]$$

The vector $(\phi_1, \phi_2, ..., \phi_n)$ is hereafter called the SSI power vector (PV).

The non normalized Banzhaf index (BI') of voter i is

$$\beta'_i = \frac{\sum_{S \subseteq N}[v(S) - v(S \backslash \{i\})]}{2^{n-1}}$$

The vector $(\beta'_1, \beta'_2, ..., \beta'_n)$ is called the power vector PV for BI'.

The normalized Banzhaf index (BI) is

$$\beta_i = \frac{\sum_{S \subseteq N}[v(S) - v(S \backslash \{i\})]}{\sum_{j \in N} \sum_{S \subseteq N}[v(S) - v(S \backslash \{j\})]}$$

The vector $(\beta_1, \beta_2, ..., \beta_n)$ is the PV for BI.

We shall denote by $SSI(n, q, \bar{w})$ (respectively $BI(n, q, \bar{w})$ and $BI'(n, q, \bar{w})$) the set of possible Shapley-Shubik (respectively normalized and non normalized Banzhaf) power vectors when the number of voters is n, the (relative) quota is q and the total number of seats is $\bar{w}$.

If either of the parameters q and $\bar{w}$ is not fixed, it will be replaced in the notation above with a point. For example, the set of Shapley-Shubik power vectors when the relative quota is not fixed is $SSI(n, ., \bar{w})$ while the set of non normalized Banzhaf power vector when $\bar{w}$ is not fixed is $BI'(n, q, .)$. It is worth noting that

$$SSI(n, ., \bar{w}) = \bigcup_{q \geq \frac{1}{2}} SSI(n, q, \bar{w})$$

$$BI'(n, q, .) = \bigcup_{\bar{w} \geq 1} BI'(n, q, \bar{w}).$$

$SSI(n, ., .)$ is simply denoted $SSI(n)$, and similar notations for $BI(n)$ and $BI'(n)$.

We assume in this section that n is equal to 2, 3 or 4.

[8]The notation $|A|$ represents the cardinal of the set A.

In each case we show that for a fixed quota, all the three numbers coincide, that is,

$$|SSI(n,q,.)| = |BI'(n,q,.)| = |BI(n,q,.)|,\ n = 2,3,4$$

and

$$|SSI(n,q,\bar{w})| = |BI'(n,q,\bar{w})| = |BI(n,q,\bar{w})|,\ n = 2,3,4$$

But when the quota is not fixed, the three numbers differ as follows

$$|SSI(3)| = |BI(3)| < |BI'(3)|$$

and

$$|SSI(4)| < |BI(4)| < |BI'(4)|$$

The values of $|SSI(n,.,\bar{w})|$, $|BI(n,.,\bar{w})|$ and $|BI'(n,.,\bar{w})|$ are given as a function of $\bar{w}$ which lead to:

$$|SSI(3,.,\bar{w}| = |BI'(3,.,\bar{w})| < |BI(3,.,\bar{w})|,\ \text{for } \bar{w} \geq 3$$

and

$$|SSI(4,.,\bar{w}| < |BI'(4,.,\bar{w})| < |BI(4,.,\bar{w})|,\ \text{for } \bar{w} \geq 10$$

2.1 The 2-voter case

Let us start with the obvious case $n = 2$ and denote $\mathcal{R}_2$ the set of all voting rules. As seen in the introduction, even if its cardinality is infinite, the relevant partition of $\mathcal{R}_2$ contains only two classes of weighted rules, denoted, $\mathcal{C}_i(2)$, for $i = 1,2$. For q and $\bar{w}$ given, let us define the classes as follows:

$\mathcal{C}_1(2,q,\bar{w}) = \{[q,w_1,w_2] : w_1 \geq q\bar{w}\}$

$\mathcal{C}_2(n,q,\bar{w}) = \{[q,w_1,w_2] : w_1 < q\bar{w}\}$

Then define, for $i = 1,2$, $C_i(2)$ the set of all the weighted rules belonging to class i:

$$\mathcal{C}_i(2) = \bigcup_{q \geq \frac{1}{2}} \bigcup_{\bar{w} \geq 1} \mathcal{C}_i(2,q,\bar{w})$$

By the definition of a partition, $\mathcal{R}_2 = \mathcal{C}_1(2) \cup \mathcal{C}_2(2)$ and $\mathcal{C}_1(2) \cap \mathcal{C}_2(2) = \emptyset$. The constraints on w_1, w_2 and q, the corresponding PV for the three power indices of interest and an example of weighted rule, are reported in Table 2.

Table 2: The two different weighted rules for $n = 2$, with examples.

Classes of weighted rules	SSI	BI'	BI	Examples
$w_1 \geq q\bar{w}$	$\phi_1 = (1,0)$	$\beta_1' = (1,0)$	$\beta_1 = (1,0)$	$[\frac{1}{2};2,1]$
$w_1 < q\bar{w}$	$\phi_2 = (\frac{1}{2},\frac{1}{2})$	$\beta_2' = (\frac{1}{2},\frac{1}{2})$	$\beta_2 = (\frac{1}{2},\frac{1}{2})$	$[1;2,1]$

It is easy to check that if q is fixed, when $\bar{w}$ is fixed or not, then[9]

$$|SSI(n,q,\bar{w})| = |BI(n,q,\bar{w}))| = |BI'(n,q,\bar{w})| \quad \textit{(Situation 4)}$$
$$|SSI(n,q,.)| = |BI(n,q,.)| = |BI'(n,q,.)| \quad \textit{(Situation 2)}$$

Depending on q, we may have for example $|SSI(n,q,.)| = 1$ or $|SSI(n,q,.)| = 2$. If q is not fixed, then[9]

$$|SSI(2)| = |BI(2)| = |BI'(2)| \quad \textit{(Situation 1)}$$
$$|SSI(n,.,\bar{w})| = |BI(n,.,\bar{w})| = |BI'(n,.,\bar{w})| \quad \textit{(Situation 3)}$$

2.2 The 3-voter case

In this case a weighted rule can be written as $[q, w_1, w_2, w_3]$, with $q \geq \frac{1}{2}$, $w_1 \geq w_2 \geq w_3$ and $w_1 + w_2 + w_3 = \bar{w}$. Let $\mathcal{R}_3$ denotes the set of all weighted rules. As in the 2-voter case, its cardinality is infinite. In order to differentiate constant structures of decisive (pivotal) voters, 5 different classes of weighted rules arise. We obtain a partition of $\mathcal{R}_3$ in 5 classes denoted, $\mathcal{C}_i(3)$, for $i = 1, \ldots, 5$. Their notation will depend on whether q and $\bar{w}$ are fixed or not. For a given q and $\bar{w}$, we denote (*Situation 4*):

$\mathcal{C}_1(3,q,\bar{w}) = \{[q,w_1,w_2,w_3] : w_1 \geq q\bar{w}\}$
$\mathcal{C}_2(3,q,\bar{w}) = \{[q,w_1,w_2,w_3] : w_1 + w_3 < q\bar{w} \text{ and } w_1 + w_2 \geq q\bar{w}\}$
$\mathcal{C}_3(3,q,\bar{w}) = \{[q,w_1,w_2,w_3] : w_2 + w_3 \geq q\bar{w}\}$
$\mathcal{C}_4(3,q,\bar{w}) = \{[q,w_1,w_2,w_3] : w_1 < q\bar{w},\ w_2 + w_3 < q\bar{w} \text{ and } w_1 + w_3 \geq q\bar{w}\}$
$\mathcal{C}_5(3,q,\bar{w}) = \{[q,w_1,w_2,w_3] : w_1 + w_2 < q\bar{w}\}$

It is quite obvious that if a given weighted rule $[q, w_1, w_2, w_3]$ does not belong, for instance to $\bigcup_{i=1}^{4} \mathcal{C}_i(3,q,\bar{w})$, then it belongs to $\mathcal{C}_5(3,q,\bar{w})$. We define different sets according to the fact that q and $\bar{w}$ are fixed or not fixed.

For $i = 1, \ldots, 5$, let

$$\mathcal{C}_i(3,q,.) = \bigcup_{\bar{w}\geq 1} \mathcal{C}_i(3,q,\bar{w}) \quad (\textit{Situation 2})$$

$$\mathcal{C}_i(3,.,\bar{w}) = \bigcup_{q\geq \frac{1}{2}} \mathcal{C}_i(3,q,\bar{w}) \quad (\textit{Situation 3})$$

$$\mathcal{C}_i(3) = \bigcup_{q\geq \frac{1}{2}} \bigcup_{\bar{w}\geq 1} \mathcal{C}_i(3,q,\bar{w}) = \bigcup_{q\geq \frac{1}{2}} \mathcal{C}_i(3,q,.) = \bigcup_{\bar{w}\geq 1} \mathcal{C}_i(3,.,\bar{w}) \quad (\textit{Situation 1})$$

Table 3 summarizes the results for the five classes $\mathcal{C}_i(3)$.

Proposition 1 *Assume that $n = 3$. If q and $\bar{w}$ are not fixed, then $|SSI(3)| = |BI(3)| = 4$ and $|BI'(3)| = 5$ (Situation 1).*

[9] This is illustrated in the introduction using different weighted rules.

Table 3: The five different weighted rules for $n = 3$, with examples.

	Classes of weighted rules	SSI	BI'	BI	Examples
1	$w_1 \geq q\bar{w}$	$\phi_1 = (1, 0, 0)$	$\beta'_1 = (1, 0, 0)$	$\beta_1 = (1, 0, 0)$	$[\frac{1}{2}; 2, 0, 0]$
2	$w_1 + w_3 < q\bar{w}$ and $w_1 + w_2 \geq q\bar{w}$	$\phi_2 = (\frac{1}{2}, \frac{1}{2}, 0)$	$\beta'_2 = (\frac{1}{2}, \frac{1}{2}, 0)$	$\beta_2 = (\frac{1}{2}, \frac{1}{2}, 0)$	$[\frac{1}{2}; 1, 1, 0]$
3	$w_2 + w_3 \geq q\bar{w}$	$\phi_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$\beta'_3 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$\beta_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$[\frac{1}{2}; 2, 2, 2]$
4	$w_1 < q\bar{w}$ and $w_2 + w_3 < q\bar{w}$ and $w_1 + w_3 \geq q\bar{w}$	$\phi_4 = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$	$\beta'_4 = (\frac{3}{4}, \frac{1}{4}, \frac{1}{4})$	$\beta_4 = (\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$	$[\frac{1}{2}; 4, 2, 2]$
5	$w_1 + w_2 < q\bar{w}$	$\phi_5 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$\beta'_5 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$\beta_5 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$[1; 1, 1, 1]$

Proof: Since q is not fixed, we have by construction of a partition:

$$\begin{cases} \forall i = 1, ..., 5,\ \mathcal{C}_i(3) \neq \emptyset \\ \forall i \neq j,\ \mathcal{C}_i(3) \cap \mathcal{C}_j(3) = \emptyset \\ \bigcup\limits_{i=1}^{5} \mathcal{C}_i(3) = \mathcal{R}_3 \end{cases}$$

All weighted rules in the same class have the same power vector with respect to any of the power indices studied herein. To show that $|SSI(3)| = |BI(3)| = 4$ we can remark that any rule belonging to $\mathcal{C}_3(3)$ or $\mathcal{C}_5(3)$ yields the power vector $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ with respect to Shapley-Shubik and normalized Banzhaf power indices. On the other hand, it is easy to show that for each i, there exists $\bar{w}$ such that $\mathcal{C}_i(3) \neq \emptyset$. By taking $\bar{w} \geq 5$, one can prove that the following weighted rule Y_i, for $i = 1, \ldots, 5$, are such that $Y_i \in \mathcal{C}_i(n, q, \bar{w})$ (and thus $\mathcal{C}_i(3) \neq \emptyset$):

$Y_1 = [q; \lceil q\bar{w} \rceil, \lceil \frac{\bar{w} - \lceil q\bar{w} \rceil}{2} \rceil, \bar{w} - (\lceil q\bar{w} \rceil + \lceil \frac{\bar{w} - \lceil q\bar{w} \rceil}{2} \rceil)]$[10]

$Y_2 = [q; \lfloor \frac{\bar{w}}{2} \rfloor, \lfloor \frac{\bar{w}}{2} \rfloor, \bar{w} - 2\lfloor \frac{\bar{w}}{2} \rfloor]$

$Y_3 = [q; \bar{w} - 2\lfloor \frac{\bar{w}}{3} \rfloor, \lfloor \frac{\bar{w}}{3} \rfloor, \lfloor \frac{\bar{w}}{3} \rfloor]$

$Y_4 = [q; \lceil q\bar{w} \rceil - 1, \bar{w} - \lceil q\bar{w} \rceil, 1]$

$Y_5 = [q; x_1, x_2, \bar{w} - \lceil q\bar{w} \rceil + 1]$ with $x_1 = \frac{\lceil q\bar{w} \rceil - 1}{2}$ and $x_2 = x_1$ if $\lceil q\bar{w} \rceil$ is odd,
$x_1 = \frac{\lceil q\bar{w} \rceil}{2}$, $x_2 = x_1 - 1$ if $\lceil q\bar{w} \rceil$ is even.

Since β_i are pairwise distinct for $i = 1, 2, 3, 4$, $|BI(3)| = 4$ and likewise, ϕ_i are pairwise distinct for $i = 1, 2, 3, 4$, $|SSI(3)| = 4$. On the other hand, β'_i are pairwise distinct and since $\mathcal{C}_i(3) \neq \emptyset$ for each i, $|BI'(3)| = 5$. ■

Proposition 2 *Assume that $n = 3$. If q is not fixed and $\bar{w}$ is fixed with $\bar{w} \geq 5$, then $|SSI(3, ., \bar{w})| = |BI(3, ., \bar{w})| = 4$ and $|BI'(3, ., \bar{w})| = 5$ (Situation 3).*

Proof: Similar to the above, see also Table 3.

[10] For all x, $\lceil x \rceil$ is the smallest integer greater than or equal to x and $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Note however that $\bar{w} \in \{1, 2, 3, 4\}$ are marginalized cases. It is easy to check that
$|SSI(3, ., 4)| = |BI(3, ., 4)| = |BI'(3, ., 4)| = 4$,
$|SSI(3, ., 3)| = |BI(3, ., 3)| = 3$, and $|BI'(3, ., 3)| = 4$,
$|SSI(3, ., 2)| = |BI(3, ., 2)| = |BI'(3, ., 2)| = 2$. ■

The following proposition deals with the subcase where q is fixed.

Proposition 3 *Assume that $n = 3$ and that q is fixed while $\bar{w}$ is not. Then $|SSI(3, q, .)| = |BI(3, q, .)| = |BI'(3, q, .)|$ (Situation 2).*

Proof: Assume that q is fixed.

1. First we show that $\mathcal{C}_3(3, q, .) \neq \emptyset$ and $\mathcal{C}_5(3, q, .) \neq \emptyset$ is impossible.

   Indeed, assume that q is fixed and that $\mathcal{C}_3(3, q, .) \neq \emptyset$ and $\mathcal{C}_5(3, q, .) \neq \emptyset$.

   Recall that $\mathcal{C}_3(3, q, .) = \bigcup_{\bar{w} \geq 1} \mathcal{C}_3(3, q, \bar{w})$ where $\mathcal{C}_3(3, q, \bar{w}) = \{[q, w_1, w_2, w_3] : w_2 + w_3 \geq q\bar{w}\}$ and $\mathcal{C}_5(3, q, .) = \bigcup_{\bar{w} \geq 1} \mathcal{C}_5(3, q, \bar{w})$ with $\mathcal{C}_5(3, q, \bar{w}) = \{[q, x_1, x_2, x_3] : x_1 + x_2 < q\bar{x}\}$.

   Let $[q, w_1, w_2, w_3] \in \mathcal{C}_3(3, q, .)$ and $[q, x_1, x_2, x_3] \in \mathcal{C}_5(3, q, .)$ (with $\bar{x} = x_1 + x_2 + x_3$). Then $x_1 + x_2 < q\bar{x}$ and $w_2 + w_3 \geq q\bar{w}$. But, $x_1 + x_2 < q\bar{x} \Rightarrow qx_3 > (1 - q)(x_1 + x_2)$, thus $x_3 > \frac{1-q}{q}(x_1 + x_2)$. Since $x_1 \geq x_2 \geq x_3$, it follows that $x_3 \leq \frac{x_1+x_2}{2}$; and therefore $\frac{1-q}{q} < \frac{1}{2}$, that is $q > \frac{2}{3}$.

   On the other hand, $[q, w_1, w_2, w_3] \in \mathcal{C}_3(3, q, .)$ meaning that $w_2 + w_3 \geq q\bar{w}$. This implies that $w_1 \leq \frac{1-q}{q}(w_2 + w_3)$. Thanks to $w_1 \geq w_2 \geq w_3$, we obtain $w_1 \geq \frac{w_2+w_3}{2}$, thus $q \leq \frac{2}{3}$; a contradiction. Finally, $\mathcal{C}_3(3, q, .) \neq \emptyset$ and $\mathcal{C}_5(3, q, .) \neq \emptyset$ is impossible

2. Second, we see from Table 3 that for all $i \in \{1, 2, 3, 4\}, \phi_i$, β_i and β'_i are all pairwise distinct thus, that $|SSI(3, q, .)| = |BI'(3, q, .)| = |BI(3, q, .)|$. ■

Considering the particular case of the majority rule, the number of vectors achieved by these power indices is determined as follows.

Proposition 4 *Assume that $n = 3$. If q is the majority rule, then*

$$|SSI(3, \tfrac{1}{2}, \bar{w})| = |BI'(3, \tfrac{1}{2}, \bar{w})| = |BI(3, \tfrac{1}{2}, \bar{w})| = \begin{cases} 2 \text{ if } \bar{w} = 2 \text{ or if there exists } t \geq 1 : \bar{w} = 2t + 1 \\ 3 \text{ if } \bar{w} = 4 \\ 4 \text{ if there exists } t \geq 3 : \bar{w} = 2t \end{cases}$$

The above result feeds into situations 2 and 4, where the fixed quota is $\frac{1}{2}$. This result which does not present any particular difficulty can be clearly seen in Table 3.

2.3 The 4-voter case

A weighted rule is a sequence $[q, w_1, w_2, w_3, w_4]$, with $q \geq \frac{1}{2}$, $w_1 \geq w_2 \geq w_3 \geq_4$ and $w_1 + w_2 + w_3 + w_4 = \bar{w}$. Denote by $\mathcal{R}_4$ the set of all weighted rules. The partition of this set contains 14 different classes of weighted rules, $\mathcal{C}_i(4)$, for $i = 1, \ldots, 14$. For any q and $\bar{w}$, let:

$$
\begin{aligned}
&\mathcal{C}_1(4,q,\bar{w}) = \{[q,w_1,w_2,w_3,w_4] : w_1 \geq q\bar{w}\}\\
&\mathcal{C}_2(4,q,\bar{w}) = \{[q,w_1,w_2,w_3,w_4] : w_2 + w_3 \geq q\bar{w}\}\\
&\mathcal{C}_3(4,q,\bar{w}) = \{[q,w_1,w_2,w_3,w_4] : w_1 + w_4 \geq q\bar{w},\ w_2 + w_3 + w_4 \geq q\bar{w}\}\\
&\mathcal{C}_4(4,q,\bar{w}) = \{[q,w_1,w_2,w_3,w_4] : w_1 < q\bar{w},\ w_1 + w_4 \geq q\bar{w},\ w_2 + w_3 + w_4 < q\bar{w}\}\\
&\mathcal{C}_5(4,q,\bar{w}) = \{[q,w_1,w_2,w_3,w_4] : w_2 + w_3 \geq q\bar{w},\ w_1 + w_4 < q\bar{w},\ w_2 + w_3 < q\bar{w},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad w_2 + w_3 + w_4 \geq q\bar{w}\}\\
&\mathcal{C}_6(4,q,\bar{w}) = \{[q,w_1,w_2,w_3,w_4] : w_2 + w_3 \geq q\bar{w}, w_1 + w_4 < q\bar{w}, w_2 + w_3 < q\bar{w},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad w_2 + w_3 + w_4 < q\bar{w}\}\\
&\mathcal{C}_7(4,q,\bar{w}) = \{[q,w_1,w_2,w_3,w_4] : w_1 + w_2 \geq q\bar{w},\ w_1 + w_3 < q\bar{w},\ w_2 + w_3 + w_4 \geq q\bar{w}\}\\
&\mathcal{C}_8(4,q,\bar{w}) = \{[q,w_1,w_2,w_3,w_4] : w_1 + w_2 \geq q\bar{w},\ w_1 + w_3 < q\bar{w},\ w_2 + w_3 + w_4 < q\bar{w},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad w_1 + w_3 + w_4 \geq q\bar{w}\}\\
&\mathcal{C}_9(4,q,\bar{w}) = \{[q,w_1,w_2,w_3,w_4] : w_1 + w_2 \geq q\bar{w}, w_1 + w_3 < q\bar{w},\ w_1 + w_3 + w_4 < q\bar{w}\}\\
&\mathcal{C}_{10}(4,q,\bar{w}) = \{[q,w_1,w_2,w_3,w_4] : w_1 + w_2 < q\bar{w},\ w_2 + w_3 + w_4 \geq q\bar{w}\}\\
&\mathcal{C}_{11}(4,q,\bar{w}) = \{[q,w_1,w_2,w_3,w_4] : w_1 + w_2 < q\bar{w},\ w_2 + w_3 + w_4 < q\bar{w},\ w_1 + w_3 + w_4 \geq q\bar{w}\}\\
&\mathcal{C}_{12}(4,q,\bar{w}) = \{[q,w_1,w_2,w_3,w_4] : w_1 + w_2 < q\bar{w},\ w_1 + w_3 + w_4 < q\bar{w},\ w_1 + w_2 + w_4 \geq q\bar{w}\}\\
&\mathcal{C}_{13}(4,q,\bar{w}) = \{[q,w_1,w_2,w_3,w_4] : w_1 + w_2 + w_4 < q\bar{w},\ w_1 + w_2 + w_3 \geq q\bar{w}\}\\
&\mathcal{C}_{14}(4,q,\bar{w}) = \{[q,w_1,w_2,w_3,w_4] : w_1 + w_2 + w_3 < q\bar{w}\}
\end{aligned}
$$

For all i,

$$
\begin{aligned}
&\mathcal{C}_i(4,.,\bar{w}) = \bigcup_{q \geq \frac{1}{2}} \mathcal{C}_i(4,q,\bar{w})\\
&\mathcal{C}_i(4,q,.) = \bigcup_{\bar{w} \geq 1} \mathcal{C}_i(4,q,\bar{w})\\
&\mathcal{C}_i(4) = \bigcup_{q \geq \frac{1}{2}} \bigcup_{\bar{w} \geq 1} \mathcal{C}_i(4,q,\bar{w}) = \bigcup_{q \geq \frac{1}{2}} \mathcal{C}_i(4,q,.) = \bigcup_{\bar{w} \geq 1} \mathcal{C}_i(4,.,\bar{w})
\end{aligned}
$$

Table 4 summarizes the results for the fourteen classes $\mathcal{C}_i(4)$.

Proposition 5 *Assume that $n = 4$. If q and $\bar{w}$ are not fixed then $|SSI(4)| = 11$, $|BI(4)| = 12$ and $|BI'(4)| = 14$ (Situation 1).*

Proof: Since q is not fixed, we have by construction of a partition, as we had for the 3-voter case:

$$
\begin{cases}
\forall i = 1, ..., 14, \mathcal{C}_i(4) \neq \emptyset\\
\forall i \neq j,\ \mathcal{C}_i(4) \cap \mathcal{C}_j(4) = \emptyset\\
\bigcup_{i=1}^{14} \mathcal{C}_i(4) = \mathcal{R}_4
\end{cases}
$$

The different power vectors achieved by the power indices involved are given in Table 4. Recall that if $[q, w_1, w_2, w_3, w_4]$ and $[q', x_1, x_2, x_3, x_4]$ belong to the same class then both rules have the same set of winning voters and thus have the same power vector with respect to any given power index. For instance, from Table 4 we can induce that if a weighted rule belongs to the class $\mathcal{C}_6(4)$ then $\phi_6 = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6}, 0)$, $\beta_6 = (\frac{3}{5}, \frac{1}{5}, \frac{1}{5}, 0)$ and $\beta'_6 = (\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, 0)$. Since $\phi_{11} = \phi_3$, $\phi_{13} = \phi_2$ and $\phi_{14} = \phi_{10}$, $|SSI(4)| = 14 - 3 = 11$. In the same way, as $\beta_{14} = \beta_{10}$ and $\beta_{13} = \beta_2$ we deduce that $|BI(4)| = 12$ and $|BI'(4)| = 14$ (β' are pairwise distinct). ■

We likewise show the following.

Proposition 6 *Assume that $n = 4$. If q is not fixed and $\bar{w}$ is fixed with $\bar{w} \geq 10$ then $|SSI(4, ., \bar{w})| = 11$, $|BI(4, ., \bar{w})| = 12$ and $|BI'(4, ., \bar{w})| = 14$ (Situation 3).*

Proof: As we proved for $n = 3$, it is easy to prove that for $\bar{w} \geq 8$, each class $\mathcal{C}_i(4, ., \bar{w})$ is non empty and we can proceed as in the case where q is not fixed (with a non fixed $\bar{w}$) to get the results. This is summarized and can be seen in Table 5. ■

The proposition above implies that we have $|SSI(4)| < |BI(4)| < |BI'(4)|$ and $|SSI(4, ., \bar{w})| < |BI(4, ., \bar{w})| < |BI'(4, ., \bar{w})|$ for $\bar{w} \geq 10$. The next results deals with the case where q is fixed.

Proposition 7 *Assume that $n = 4$. If q is fixed then $|BI(4, q, .)| = |BI'(4, q, .)|$ and $|BI(4, q, \bar{w})| = |BI'(4, q, \bar{w})|$ (Situations 2 and 4).*

Proof: Assume that $n = 4$ and q is fixed.

Let us begin with $|BI(4, q, .)| = |BI'(4, q, .)|$:

The difference between the cardinality of $BI(4)$ and $BI'(4)$ when q is not fixed arises from the fact that $\beta_{14} = \beta_{10}$ and $\beta_{13} = \beta_2$. Hence, it is sufficient to show that when q is fixed, the two following results:

- $\mathcal{C}_2(4, q, .) \neq \emptyset$ and $\mathcal{C}_{13}(4, q, .) \neq \emptyset$, are not possible simultaneously.
- $\mathcal{C}_{10}(4, q, .) \neq \emptyset$ and $\mathcal{C}_{14}(4, q, .) \neq \emptyset$ is also impossible.

• First, assume that $\mathcal{C}_2(4, q, .) \neq \emptyset$ and $\mathcal{C}_{13}(4, q, .) \neq \emptyset$.

$$\mathcal{C}_2(4, q, .) = \bigcup_{\bar{w} \geq 1} \mathcal{C}_2(4, q, \bar{w}) \text{ with } \mathcal{C}_2(4, q, \bar{w}) = \{[q, w_1, w_2, w_3, w_4] : w_2 + w_3 \geq q\bar{w}\}$$

$$\mathcal{C}_{13}(4, q, .) = \bigcup_{\bar{w} \geq 1} \{[q, w_1, w_2, w_3, w_4] : w_1 + w_2 + w_4 < q\bar{w},\ w_1 + w_2 + w_3 \geq q\bar{w}\}$$

Let $[q, w_1, w_2, w_3, w_4] \in \mathcal{C}_{13}(4, q, .)$ and $[q, x_1, x_2, x_3, x_4] \in \mathcal{C}_2(4, q, .)$ with $\bar{x} = \sum_{i=1}^{4} x_i$. The implications on the w_i's inferred by the fact that $[q, w_1, w_2, w_3, w_4] \in \mathcal{C}_{13}(4, q, .)$ are:

Table 4: The different weighted rules for $n = 4$, with examples.

	Classes of weighted rules	SSI	BI'	BI	Example
1	$w_1 \geq q\bar{w}$	$\phi_1 = (1,0,0,0)$	$\beta'_1 = (1,0,0,0)$	$\beta_1 = (1,0,0,0)$	$[\frac{1}{2}; 4,1,1,1]$
2	$w_2 + w_3 \geq q\bar{w}$	$\phi_2 = (\frac{1}{3},\frac{1}{3},\frac{1}{3},0)$	$\beta'_2 = (\frac{1}{2},\frac{1}{2},\frac{1}{2},0)$	$\beta_2 = (\frac{1}{3},\frac{1}{3},\frac{1}{3},0)$	$[\frac{1}{2}; 2,2,2,1]$
3	$w_1 + w_4 \geq q\bar{w}$, $w_2 + w_3 + w_4 \geq q\bar{w}$	$\phi_3 = (\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6})$	$\beta'_3 = (\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$	$\beta_3 = (\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6})$	$[\frac{1}{2}; 2,1,1,1]$
4	$w_1 < q\bar{w}$,$w_1 + w_4 \geq q\bar{w}$, $w_2 + w_3 + w_4 < q\bar{w}$	$\phi_4 = (\frac{3}{4},\frac{1}{12},\frac{1}{12},\frac{1}{12})$	$\beta'_4 = (\frac{7}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8})$	$\beta_4 = (\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10})$	$[\frac{1}{2}; 4,2,1,1]$
5	$w_1 + w_3 \geq q\bar{w}$, $w_1 + w_4 < q\bar{w}$, $w_2 + w_3 < q\bar{w}$,$w_2 + w_3 + w_4 \geq q\bar{w}$	$\phi_5 = (\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{12})$	$\beta'_5 = (\frac{5}{8},\frac{3}{8},\frac{3}{8},\frac{1}{8})$	$\beta_5 = (\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{12})$	$[\frac{1}{2}; 4,3,2,1]$
6	$w_1 + w_3 \geq q\bar{w}$, $w_1 + w_4 < q\bar{w}$, $w_2 + w_3 < q\bar{w}$,$w_2 + w_3 + w_4 < q\bar{w}$	$\phi_6 = (\frac{2}{3},\frac{1}{6},\frac{1}{6},0)$	$\beta'_6 = (\frac{3}{4},\frac{1}{4},\frac{1}{4},0)$	$\beta_6 = (\frac{3}{5},\frac{1}{5},\frac{1}{5},0)$	$[\frac{1}{2}; 2,1,1,0]$
7	$w_1 + w_2 \geq q\bar{w}$, $w_1 + w_3 < q\bar{w}$,$w_2 + w_3 + w_4 \geq q\bar{w}$	$\phi_7 = (\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6})$	$\beta'_7 = (\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4})$	$\beta_7 = (\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6})$	$[\frac{1}{2}; 2,2,1,1]$
8	$w_1 + w_2 \geq q\bar{w}$, $w_1 + w_3 < q\bar{w}$,$w_2 + w_3 + w_4 < q\bar{w}$,$w_1 + w_3 + w_4 \geq q\bar{w}$	$\phi_8 = (\frac{7}{12},\frac{1}{4},\frac{1}{12},\frac{1}{12})$	$\beta'_8 = (\frac{5}{8},\frac{3}{8},\frac{1}{8},\frac{1}{8})$	$\beta_8 = (\frac{1}{2},\frac{3}{10},\frac{1}{10},\frac{1}{10})$	$[\frac{7}{10}; 5,3,1,1]$
9	$w_1 + w_2 \geq q\bar{w}$, $w_1 + w_3 < q\bar{w}$,$w_1 + w_3 + w_4 < q\bar{w}$	$\phi_9 = (\frac{1}{2},\frac{1}{2},0,0)$	$\beta'_9 = (\frac{1}{2},\frac{1}{2},0,0)$	$\beta_9 = (\frac{1}{2},\frac{1}{2},0,0)$	$[\frac{1}{2}; 2,2,0,0]$
10	$w_1 + w_2 < q\bar{w}$, $w_2 + w_3 + w_4 \geq q\bar{w}$	$\phi_{10} = (\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$	$\beta'_{10} = (\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8})$	$\beta_{10} = (\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$	$[\frac{1}{2}; 1,1,1,1]$
11	$w_1 + w_2 < q\bar{w}$, $w_2 + w_3 + w_4 < q\bar{w}$,$w_1 + w_3 + w_4 \geq q\bar{w}$	$\phi_{11} = (\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6})$	$\beta'_{11} = (\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{4})$	$\beta_{11} = (\frac{2}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5})$	$[\frac{4}{5}; 2,1,1,1]$
12	$w_1 + w_2 < q\bar{w}$, $w_1 + w_3 + w_4 < q\bar{w}$,$w_1 + w_2 + w_4 \geq q\bar{w}$	$\phi_{12} = (\frac{5}{12},\frac{5}{12},\frac{1}{12},\frac{1}{12})$	$\beta'_{12} = (\frac{3}{8},\frac{3}{8},\frac{1}{8},\frac{1}{8})$	$\beta_{12} = (\frac{3}{8},\frac{3}{8},\frac{1}{8},\frac{1}{8})$	$[\frac{4}{5}; 3,3,2,2]$
13	$w_1 + w_2 + w_4 < q\bar{w}$, $w_1 + w_2 + w_3 \geq q\bar{w}$	$\phi_{13} = (\frac{1}{3},\frac{1}{3},\frac{1}{3},0)$	$\beta'_{13} = (\frac{1}{4},\frac{1}{4},\frac{1}{4},0)$	$\beta_{13} = (\frac{1}{3},\frac{1}{3},\frac{1}{3},0)$	$[\frac{6}{7}; 2,2,2,1]$
14	$w_1 + w_2 + w_3 < q\bar{w}$	$\phi_{14} = (\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$	$\beta'_{14} = (\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8})$	$\beta_{14} = (\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$	$[1; 1,1,1,1]$

$$
\begin{aligned}
[q,w_1,w_2,w_3,w_4]\in\mathcal{C}_{13}(4,q,.) \quad &\Rightarrow \quad w_1+w_2+w_4<q\bar{w} \text{ and } w_1\geq w_2\geq w_3\geq w_4\\
&\Rightarrow \quad q>\tfrac{w_1+w_2+w_4}{w_1+w_2+w_3+w_4}\\
&\Rightarrow \quad q>\tfrac{2}{3} \text{ since } 3(w_1+w_2+w_4)>2(w_1+w_2+w_3+w_4)
\end{aligned}
$$

On the other hand, the implications for the x_i's are:

$$
\begin{aligned}
[q,x_1,x_2,x_3,x_4]\in\mathcal{C}_{2}(4,q,.) \quad &\Rightarrow \quad x_2+x_3\geq q\bar{x} \text{ and } x_1\geq x_2\geq x_3\\
&\Rightarrow \quad q\leq\tfrac{x_2+x_3}{x_1+x_2+x_3+x_4},\ x_2+x_3\leq 2x_1\leq 2(x_1+x_4)\\
&\Rightarrow \quad q\leq\tfrac{x_2+x_3}{x_1+x_2+x_3+x_4} \text{ and } 3(x_2+x_3)\leq 2(x_1+x_2+x_3+x_4)\\
&\Rightarrow \quad q\leq\tfrac{x_2+x_3}{x_1+x_2+x_3+x_4}\leq\tfrac{2}{3}
\end{aligned}
$$

which is in contradiction with the previous result. Therefore, if $\mathcal{C}_2(4,q,.)\neq\emptyset$ then $\mathcal{C}_{13}(4,q,.)=\emptyset$.

• Second, assume that $\mathcal{C}_{10}(4,q,.)\neq\emptyset$ and $\mathcal{C}_{14}(4,q,.)\neq\emptyset$. Let $[q,w_1,w_2,w_3,w_4]\in\mathcal{C}_{14}(4,q,.)$ and $[q,x_1,x_2,x_3,x_4]\in\mathcal{C}_{10}(4,q,.)$.

Since $w_1+w_2+w_3<q\bar{w}$, then $w_4>\frac{1-q}{q}(w_1+w_2+w_3)$. Furthermore, $w_1\geq w_2\geq w_3\geq w_4$, thus $w_4\leq\frac{1}{3}(w_1+w_2+w_3)$. This implies that $\frac{1}{3}>\frac{1-q}{q}$ and $q>\frac{3}{4}$.

Furthermore, $x_2+x_3+x_4\geq q\bar{x}$ and $q>\frac{3}{4}$, then $x_1<\frac{1}{4}\bar{x}$, which contradicts $w_1\geq w_2\geq w_3\geq w_4$. From Table 4, we can see that β'_i are pairwise distinct, for $i\in\{1,2,...,14\}\setminus\{13,14\}$ as well as β_i; thus $|BI(4,q,.)|=|BI'(4,q,.)|$.

When $\bar{w}$ is fixed, the result arises from noting that again β'_i are pairwise distinct as well as β'_i. $|BI(4,q,\bar{w})|=|BI'(4,q,\bar{w})|$. ■

Proposition 8 *If $n=4$ and q is fixed then $|SSI(4,q,.)|=|BI(4,q,.)|$ and $|SSI(4,q,\bar{w})|=|BI(4,q,\bar{w})|$ (Situations 2 and 4).*

Proof: Assume that $n=4$ and q is fixed.

Let us begin with $|SSI(4,q,.)|=|BI(4,q,.)|$

Note that $\phi_2=\phi_{13}$ and $\beta_2=\beta_{13}$, $\phi_{10}=\phi_{14}$ and $\beta_{10}=\beta_{14}$. The difference between the number of power vectors achievable by SSI and BI when q is not fixed arises from the fact that $\phi_3=\phi_{11}$ while $\beta_3\neq\beta_{11}$. It is then sufficient to show that when q is fixed, $\mathcal{C}_3(4,q,.)\neq\emptyset$ and $\mathcal{C}_{11}(4,q,.)\neq\emptyset$ is not possible.

$$\mathcal{C}_3(4,q,.)=\bigcup_{\bar{w}\geq 1}\{[q,w_1,w_2,w_3,w_4]: w_1+w_4\geq q\bar{w},\ w_2+w_3+w_4\geq q\bar{w}\}$$

$$
\begin{aligned}
\mathcal{C}_{11}(4,q,.)=\bigcup_{\bar{w}\geq 1}\{[q,w_1,w_2,w_3,w_4]: &\ w_1+w_2<q\bar{w},\ w_2+w_3+w_4<q\bar{w},\\
&\ w_1+w_3+w_4\geq q\bar{w}\}
\end{aligned}
$$

Let $[q,w_1,w_2,w_3,w_4]\in\mathcal{C}_3(4,q,.)$ and $[q,x_1,x_2,x_3,x_4]\in\mathcal{C}_{11}(4,q,.)$, with $\bar{x}=x_1+x_2+x_3+x_4$.

Thanks to $[q,w_1,w_2,w_3,w_4]\in\mathcal{C}_3(4,q,.)$, we have $w_1+w_4\geq q\bar{w}$ (1) and $w_2+w_3+w_4\geq q\bar{w}$ (2). By (1), we get $w_2+w_3\leq(1-q)\bar{w}$. Since $w_2\geq w_3\geq w_4$, $w_4\leq\frac{1}{2}(w_2+w_3)$ and we obtain

$w_4 \leq \frac{1}{2}(1-q)\bar{w}$. By (2), $w_2 + w_3 \geq q\bar{w} - w_4$. Adding (1), it follows that $(1-q)\bar{w} \geq q\bar{w} - w_4$ and $w_4 \geq (2q-1)\bar{w}$. Thus, $\frac{1}{2}(1-q)\bar{w} \geq (2q-1)\bar{w}$ and $q \leq \frac{3}{5}$.

On the other hand, $[q, x_1, x_2, x_3, x_4] \in \mathcal{C}_{11}(4, q, .)$ implies that $x_1 + x_2 < q\bar{x}$, thus $x_1 + x_2 < \frac{3}{5}\bar{x}$ and $x_3 + x_4 > \frac{2}{5}\bar{x}$. By $x_2 + x_3 + x_4 < q\bar{x}$, we have $\bar{x} - x_1 < q\bar{x}$, and $x_1 > (1-q)\bar{x}$. Thus $x_1 > \frac{2}{5}\bar{x}$. Since $x_1 + x_2 < \frac{3}{5}\bar{x}$, then $x_2 < \frac{1}{5}\bar{x}$. But $x_3 + x_4 > \frac{2}{5}\bar{x}$ yields a contradiction with $x_2 \geq x_3 \geq x_4$.

When $\bar{w}$ is fixed, the proof is similar. ■

The two propositions above imply the following obvious corollary.

Corollary 1 *Assume that $n = 4$ and q is fixed. Then $|SSI(4, q, \bar{w})| = |BI'(4, q, \bar{w})| = |BI(4, q, \bar{w})|$ and $|SSI(4, q, .)| = |BI'(4, q, .)| = |BI(4, q, .)|$ (Situations 2 and 4).*

Now, we consider the particular case of the majority rule and we show below that the number of vectors achieved by these power indices is 9 if the number of seats $\bar{w}$ is not fixed.

Proposition 9 *Assume that $n = 4$. If q is the majority rule, then*
$|SSI(4, \frac{1}{2}, .)| = |BI'(4, \frac{1}{2}, .)| = |BI(4, \frac{1}{2}, .))| = 9$ (Situation 2 for $q = \frac{1}{2}$).

Proof: It has already been proved that $|SSI(4, \frac{1}{2}, .)| = |BI'(4, \frac{1}{2}, .)| = |BI(4, \frac{1}{2}, .)|$. We will obtain (for example) $|SSI(4, \frac{1}{2}, .)|$ by determining the cardinality of the set $\{\phi_i, i \in \{1, 2, ..., 14\}\}$ where ϕ_i is the Shapley-Shubik vector of any weighted rule in class i.

(a) First, we will prove that $\mathcal{C}_8(4, \frac{1}{2}, .) = \emptyset$, $\mathcal{C}_{11}(4, \frac{1}{2}, .) = \emptyset$, $\mathcal{C}_{12}(4, \frac{1}{2}, .) = \emptyset$.

- ***case 1:*** Let us show that $\mathcal{C}_8(4, \frac{1}{2}, .) = \emptyset$. Assume that there exists $\bar{w}$ with a structure of weights (w_1, w_2, w_3, w_4) such that $[\frac{1}{2}, w_1, w_2, w_3, w_4] \in \mathcal{C}_8(4, \frac{1}{2}, \bar{w})$: then $w_1 + w_2 \geq \frac{1}{2}\bar{w}$, $w_1 + w_3 < \theta$, $w_1 + w_3 + w_4 \geq \theta$ and $w_2 + w_3 + w_4 < \theta$ with $\theta = \begin{cases} \frac{\bar{w}}{2} + 1 \text{ if } \bar{w} \text{ is even} \\ \frac{\bar{w}+1}{2} \text{ if } \bar{w} \text{ is odd.} \end{cases}$

Since $w_1 + w_3 + w_4 \geq \theta$, then $w_1 + w_3 + w_4 \geq w_2 + a$ (1) with $a = 1$ if $\bar{w}$ is odd and $a = 2$ if $\bar{w}$ is even. Since $w_1 + w_3 < \theta$, then $w_1 + w_3 < w_2 + w_4 + a$ (2). Since $w_2 + w_3 + w_4 < \theta$, then $w_2 + w_3 + w_4 < w_1 + a$ (3). By (2) and (3), $w_2 + w_3 + w_4 - a < w_1 < w_2 - w_3 + w_4 + a$ and $w_3 < a$. Thus $w_3 = 1$ or $w_3 = 0$. If $w_3 = 0$, then $w_4 = 0$ and $w_1 + w_3 < \theta$ and $w_1 + w_3 + w_4 \geq \theta$ are not compatible. Therefore $w_3 = 1$ and $\bar{w}$ is even ($a = 2$). Two structures of weights are possible $(w_1, w_2, 1, 1)$ and $(w_1, w_2, 1, 0)$. If $w_4 = 0$, by (1), $w_1 \geq w_2 + 1$ and by (2) $w_1 < w_2 + 1$, a contradiction. Thus, $w_4 = 1$. By (3), $w_1 > w_2$ and by (2) $w_1 < w_2 + 2$. Therefore, $w_1 = w_2 + 1$ and $\bar{w} = 2w_2 + 3$, a contradiction of $\bar{w}$ is even.

- ***case 2:*** Lets show that $\mathcal{C}_{11}(4, \frac{1}{2}, .) = \emptyset$. Assume on the contrary that $[q, w_1, w_2, w_3, w_4] \in \mathcal{C}_{11}(4, \frac{1}{2}, \bar{w})$: then $w_1 + w_2 < \theta$, $w_2 + w_3 + w_4 < \theta$, $w_1 + w_3 + w_4 \geq \theta$ with $\theta = \frac{\bar{w}}{2} + 1$ if $\bar{w}$ is even and $\theta = \frac{\bar{w}+1}{2}$ if $\bar{w}$ is odd.

Since $w_1 + w_2 < \theta$, then $w_1 + w_2 < w_3 + w_4 + a$ (1) with $a = 1$ if $\bar{w}$ is odd and $a = 2$ if $\bar{w}$ is even. Since $w_2 + w_3 + w_4 < \theta$, then $w_2 + w_3 + w_4 < w_1 + a$ (2). Since $w_1 + w_3 + w_4 \geq \theta$, then $w_1 + w_3 + w_4 \geq w_2 + a$ (3). By (1) and (2), $w_2 < a$ and $a = 2$, thus $w_2 = 1$. Indeed, $w_1 + w_2 < \theta$ and $w_1 + w_3 + w_4 \geq \theta$ are not compatible if $w_2 = 0$. Thus $\bar{w}$ is even. By (1) and (3), $w_2 - w_3 - w_4 + a \leq w_1 < -w_2 + w_3 + w_4 + a$ and $w_2 < w_3 + w_4$. Therefore $w_3 = w_4 = 1$ and w_1 is odd. By (1), we obtain $w_1 < 3$ and $w_1 = 1$. It is not compatible with (2).

***-case** 3:* Let us show that $\mathcal{C}_{13}(4, \frac{1}{2}, .) = \emptyset$. Assume on the contrary that $[q, w_1, w_2, w_3, w_4] \in \mathcal{C}_{13}(4, \frac{1}{2}, \bar{w})$: then $w_1 + w_2 < \theta$, $w_1 + w_3 + w_4 < \theta$, $w_1 + w_2 + w_4 \geq \theta$ with $\theta = \frac{\bar{w}}{2} + 1$ if $\bar{w}$ is even or $\theta = \frac{\bar{w}+1}{2}$ if $\bar{w}$ is odd.

Since $w_1 + w_2 < \theta$, then $w_1 + w_2 < w_3 + w_4 + a$ (1) with $a = 1$ if $\bar{w}$ is odd and $a = 2$ if $\bar{w}$ is even. Since $w_1 + w_3 + w_4 < \theta$, then $w_1 + w_3 + w_4 < w_2 + a$ (2). By (1) and (2), $w_1 + w_3 + w_4 - a < w_2 < -w_1 + w_3 + w_4 + a$ and $w_1 < a$. Since $w_1 \neq 0$, $w_1 = 1$ and $\bar{w}$ is even. Two structures of weights are then possible: $(1, 1, 0, 0)$ which is not compatible with (1) and $(1, 1, 1, 1)$ which is not compatible with (2).

Hence, $\mathcal{C}_8(4, \frac{1}{2}, .) = \emptyset$, $\mathcal{C}_{11}(4, \frac{1}{2}, .) = \emptyset$, $\mathcal{C}_{12}(4, \frac{1}{2}, .) = \emptyset$.
In Table 4 majority rules are proposed belonging to the sets $\mathcal{C}_i(4, \frac{1}{2}, .) \neq \emptyset$ for $i \neq 8, 11, 12$. This shows that $\forall i \neq 8, 11, 12$, $\mathcal{C}_i(4, \frac{1}{2}, .) \neq \emptyset$.

(b) Second, we have $\phi_{13} = \phi_2$ and $\phi_{14} = \phi_{10}$ implying that the cardinality of the set $\{\phi_i, i \in \{1, 2, ..., 14\}\}$ is at most 9. But table 8 shows that $\phi_i \neq \phi_j$ $\forall i, j \in \{1, 2, 3, 4, 5, 6, 7, 9, 10\}$ thus $|SSI(4, \frac{1}{2}, .)| = 9$. More explicitly, we give below the value of $|SSI(4, \frac{1}{2}, \bar{w})|$ when the quota which is fixed is the majority rule and the number of seats is fixed (see table 7).

Proposition 10 *Assume that $n = 4$. If $q = \frac{1}{2}$ (the majority rule), then*

$$|SSI(4, \tfrac{1}{2}, \bar{w})| = |BI(4, \tfrac{1}{2}, \bar{w})| = |BI'(4, \tfrac{1}{2}, \bar{w})| = \begin{cases} 2 \text{ if } \bar{w} \in \{2, 3\} \\ 3 \text{ if there exists } t \geq 2 : \bar{w} = 2t + 1 \\ 4 \text{ if } \bar{w} = 4 \\ 6 \text{ if } \bar{w} = 6 \\ 8 \text{ if } \bar{w} = 8 \text{ or there exists } t \geq 1 : \bar{w} = 4t + 6 \\ 9 \text{ if there exists } t \geq 1 : \bar{w} = 4t + 8 \end{cases}$$

3 More voters

The purpose of this section is to confirm the 2, 3 and 4-voter cases: when the quota is not fixed, the number of PV is different with SSI, BI' and BI, always in the same order

$|SSI(n)| < |BI(n)| < |BI'(n)|$ (*Situation 1*) and

$|SSI(n,.,\bar{w})| < |BI(n,.,\bar{w})| < |BI'(n,.,\bar{w})|$ (*Situation 3*).

Furthermore, when the quota is fixed, we have

$|SSI(n,q,.)| = |BI(n,q,.| = |BI'(n,q,.)|$ (*Situation 2*) and

$|SSI(n,q,\bar{w})| = |BI(n,q,\bar{w})| = |BI'(n,q,\bar{w})|$ (*Situation 4*).

These results are obtained through systematic enumeration on a computer[11]. Tables 5 and 6 correspond to *Situation 3*: all quotas are permitted that is the quota is not fixed while the number of seats $\bar{w}$ is fixed. The number of PV is given for $\bar{w} \leq 45$. Let us note that the number of PV is not monotonic with $\bar{w}$. For instance, there are 57 PV for SSI when $\bar{w} = 20$ and $n = 5$ while there are only 56 PV when $\bar{w} = 21$. Remark also that the number of PV increases quickly, which explains why the analytical approach is only used for 2, 3 and 4 voters.

Thanks to Tables 7 and 8 we tend to *Situation 1* since these tables are cumulative with respect to Tables 5 and 6[12]. However we obtain only a trend since it is not possible to obtain the sets $SSI(n,.,\bar{w})$, $BI(n,.,\bar{w})$, and $BI'(n,.,\bar{w})$ when $\bar{w}$ becomes too high.

For majority rules[13], Tables 9 and 10 present some results concerning *Situations 2* and *Situations 4*. Table 10 is the cumulative[14] approach of Table 9. The distinction between the different power indices is not necessary since the cardinality of the sets $SSI(n,\frac{1}{2},\bar{w})$, $BI'(n,\frac{1}{2},\bar{w})$ and $BI(n,\frac{1}{2},\bar{w})$ is always the same. Thus, only one column is given in our tables.

All these tables confirm our analytical results developed in section 2 and enables us to present the four following conjectures:

Conjecture 1 *$|SSI(n)| < |BI(n)| < |BI'(n)|$ for $n \geq 4$.*

Conjecture 2 *$|SSI(n,.,\bar{w})| < |BI(n,.,\bar{w})| < |BI'(n,.,\bar{w})|$ for $n \geq 4$ and $\bar{w} > x$, with $x = 10$ for $n = 4$, $x = 9$ for $n = 5$ and $x = 5$ for $n \geq 6$.*

Conjecture 3 *$|SSI(n,q,.)| = |BI(n,q,.)| = |BI'(n,q,.)|$.*

Conjecture 4 *$|SSI(n,q,\bar{w})| = |BI(n,q,\bar{w})| = |BI'(n,q,\bar{w})|$.*

[11]For a description of the computational method, see Barthélémy and Martin (2008).

[12]We compute $|\bigcup_{x\leq\bar{w}} SSI(n,.,x)|$, $|\bigcup_{x\leq\bar{w}} BI(n,.,x)|$ and $|\bigcup_{x\leq\bar{w}} BI'(n,.,x)|$.

[13]Equivalent results with different quotas were obtained but are omitted here.

[14]We compute $|\bigcup_{x\leq\bar{w}} SSI(n,\frac{1}{2},x)|$, $|\bigcup_{x\leq\bar{w}} BI(n,\frac{1}{2},x)|$ and $|\bigcup_{x\leq\bar{w}} BI'(n,\frac{1}{2},x)|$.

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Table 5: *PVs when q is not constrained*

	$n=3$			$n=4$			$n=5$			$n=6$		
$\bar{w}$	SSI	BI	BI'	SSI	BI	BI'	SSI	BI	BI'	SSI	BI	BI'
1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	4	3	3	4	3	3	4	3	3	4
4	4	4	4	5	5	6	5	5	6	5	5	6
5	4	4	5	6	7	8	7	8	11	7	8	11
6	4	4	5	9	9	10	11	12	13	12	13	16
7	4	4	5	9	10	12	14	16	19	16	20	23
8	4	4	5	11	11	13	21	21	23	26	28	30
9	4	4	5	10	11	13	23	25	30	33	36	43
10	4	4	5	11	12	14	30	31	35	49	51	55
11	4	4	5	11	12	14	32	35	40	58	62	70
12	4	4	5	11	12	14	39	42	45	78	82	88
13	4	4	5	11	12	14	38	42	47	92	98	107
14	4	4	5	11	12	14	46	50	55	118	124	131
15	4	4	5	11	12	14	45	49	54	130	139	149
16	4	4	5	11	12	14	49	53	58	163	170	178
17	4	4	5	11	12	14	50	54	59	177	186	196
18	4	4	5	11	12	14	53	57	62	220	230	239
19	4	4	5	11	12	14	52	56	61	232	242	253
20	4	4	5	11	12	14	53	57	62	273	286	295
21	4	4	5	11	12	14	52	56	61	283	294	305
22	4	4	5	11	12	14	53	57	62	330	342	352
23	4	4	5	11	12	14	53	57	62	341	353	364
24	4	4	5	11	12	14	53	57	62	383	398	408
25	4	4	5	11	12	14	53	57	62	384	397	408
26	4	4	5	11	12	14	53	57	62	435	451	461
27	4	4	5	11	12	14	53	57	62	425	440	451
28	4	4	5	11	12	14	53	57	62	464	479	489
29	4	4	5	11	12	14	53	57	62	466	480	491
30	4	4	5	11	12	14	53	57	62	490	508	519
31	4	4	5	11	12	14	53	57	62	490	506	517
32	4	4	5	11	12	14	53	57	62	510	530	540
33	4	4	5	11	12	14	53	57	62	503	521	532
34	4	4	5	11	12	14	53	57	62	521	539	550
35	4	4	5	11	12	14	53	57	62	516	534	545
36	4	4	5	11	12	14	53	57	62	531	550	561
37	4	4	5	11	12	14	53	57	62	527	546	557
38	4	4	5	11	12	14	53	57	62	533	552	563
39	4	4	5	11	12	14	53	57	62	529	548	559
40	4	4	5	11	12	14	53	57	62	534	553	564
41	4	4	5	11	12	14	53	57	62	534	553	564
42	4	4	5	11	12	14	53	57	62	535	554	565
43	4	4	5	11	12	14	53	57	62	535	554	565
44	4	4	5	11	12	14	53	57	62	536	555	566
45	4	4	5	11	12	14	53	57	62	535	554	565

Table 6: *PVs when q is not constrained*

	$n=7$			$n=8$			$n=9$			$n=10$		
$\bar{w}$	SSI	BI	BI'	SSI	BI	BI'	SSI	BI	BI'	SSI	BI	BI'
1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	4	3	3	4	3	3	4	3	3	4
4	5	5	6	5	5	6	5	5	6	5	5	6
5	7	8	11	7	8	11	7	8	11	7	8	11
6	12	13	16	12	13	16	12	13	16	12	13	16
7	17	21	27	17	21	27	17	21	27	17	21	27
8	28	32	34	29	33	38	29	33	38	29	33	38
9	38	45	52	40	50	57	41	51	62	41	51	62
10	59	63	68	64	72	77	66	77	82	67	78	87
11	78	84	95	88	100	111	93	111	122	95	117	128
12	111	116	122	131	141	147	141	157	163	146	168	174
13	143	151	164	177	190	205	197	220	235	207	239	254
14	188	194	205	248	254	266	282	295	307	302	325	337
15	234	243	261	319	330	352	378	398	424	412	446	472
16	298	314	323	420	438	450	515	534	546	574	605	618
17	365	379	396	539	555	580	680	700	732	775	808	843
18	462	480	490	700	724	737	894	921	938	1049	1078	1095
19	541	554	579	872	888	921	1157	1178	1219	1380	1405	1452
20	666	689	703	1100	1131	1149	1478	1513	1534	1786	1823	1850
21	768	792	814	1350	1380	1411	1886	1923	1966	2326	2369	2417
22	947	967	985	1685	1718	1741	2381	2423	2450	2972	3022	3052
23	1072	1094	1120	2028	2051	2096	2984	3008	3069	3802	3831	3902
24	1299	1328	1345	2509	2549	2574	3721	3775	3805	4794	4855	4891
25	1418	1453	1478	2943	2989	3032	4560	4615	4676	6020	6088	6158
26	1716	1753	1773	3621	3675	3702	5639	5713	5744	7510	7597	7635
27	1854	1901	1930	4218	4265	4317	6853	6901	6987	9344	9395	9502
28	2190	2244	2262	5084	5158	5185	8344	8442	8475	11489	11601	11641
29	2366	2403	2432	5861	5901	5963	10020	10062	10162	14126	14170	14301
30	2779	2846	2868	7079	7166	7200	12191	12304	12352	17302	17445	17502
31	2937	2985	3017	7997	8050	8112	14418	14468	14567	20995	21047	21184
32	3419	3485	3508	9573	9661	9698	17368	17483	17535	25463	25600	25667
33	3582	3656	3686	10759	10842	10908	20419	20517	20632	30687	30803	30954
34	4129	4205	4229	12717	12821	12865	24352	24480	24548	36879	37030	37125
35	4286	4369	4402	14137	14232	14310	28308	28385	28535	43981	44049	44269
36	4924	5026	5048	16720	16862	16898	33723	33904	33956	52721	52929	52992
37	5037	5128	5161	18382	18482	18562	38820	38913	39067	62326	62407	62631
38	5722	5836	5861	21609	21762	21806	45959	46157	46224	74194	74423	74505
39	5838	5958	5992	23652	23768	23870	52590	52692	52903	87190	87230	87569
40	6505	6650	6675	27407	27593	27641	61627	61875	61950	102846	103142	103241
41	6647	6784	6817	29787	29929	30023	70001	70131	70334	119988	120058	120386
42	7430	7578	7605	34749	34945	34999	82254	82531	82620	141645	141958	142084
43	7466	7618	7652	37200	37347	37458	92344	92469	92723	163602	163651	164061
44	8244	8412	8438	43038	43252	43310	107713	107996	108102	191696	192006	192163
45	8282	8467	8501	46172	46354	46474	120961	121140	121404	221155	221310	221754

Table 7: *Cumulative number of PVs when q is not constrained*

	$n=3$			$n=4$			$n=5$			$n=6$			$n=7$		
$\bar{w}$	SSI	BI	BI'	SSI	BI	BI'	SSI	BI	BI'	SSI	BI	BI'	SSI	BI	BI'
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	4	3	3	4	3	3	4	3	3	4	3	3	4
4	4	4	5	5	5	7	5	5	7	5	5	7	5	5	7
5	4	4	5	6	7	9	7	8	12	7	8	12	7	8	12
6	4	4	5	9	10	12	11	13	17	12	14	20	12	14	20
7	4	4	5	10	11	13	15	18	23	17	22	29	18	23	33
8	4	4	5	11	12	14	22	25	30	27	34	41	29	38	48
9	4	4	5	11	12	14	27	31	36	38	47	56	43	58	70
10	4	4	5	11	12	14	34	38	43	57	67	76	68	86	99
11	4	4	5	11	12	14	38	42	47	74	84	94	98	118	135
12	4	4	5	11	12	14	45	49	54	101	112	122	146	168	185
13	4	4	5	11	12	14	48	52	57	127	138	148	201	226	243
14	4	4	5	11	12	14	51	55	60	159	170	180	273	297	316
15	4	4	5	11	12	14	52	56	61	187	198	209	358	381	403
16	4	4	5	11	12	14	53	57	62	226	237	248	466	494	516
17	4	4	5	11	12	14	53	57	62	256	267	278	582	612	634
18	4	4	5	11	12	14	53	57	62	299	310	321	739	772	794
19	4	4	5	11	12	14	53	57	62	331	342	353	898	931	956
20	4	4	5	11	12	14	53	57	62	367	379	390	1101	1139	1164
21	4	4	5	11	12	14	53	57	62	394	406	417	1312	1355	1380
22	4	4	5	11	12	14	53	57	62	427	441	452	1583	1627	1654
23	4	4	5	11	12	14	53	57	62	449	463	474	1833	1881	1910
24	4	4	5	11	12	14	53	57	62	475	491	502	2167	2221	2250
25	4	4	5	11	12	14	53	57	62	488	505	516	2477	2542	2571
26	4	4	5	11	12	14	53	57	62	501	519	530	2860	2928	2958
27	4	4	5	11	12	14	53	57	62	511	530	541	3219	3305	3336
28	4	4	5	11	12	14	53	57	62	520	539	550	3669	3770	3801
29	4	4	5	11	12	14	53	57	62	526	545	556	4065	4171	4202
30	4	4	5	11	12	14	53	57	62	530	549	560	4578	4692	4724
31	4	4	5	11	12	14	53	57	62	533	552	563	5040	5158	5191
32	4	4	5	11	12	14	53	57	62	535	554	565	5568	5696	5729
33	4	4	5	11	12	14	53	57	62	536	555	566	6043	6186	6220
34	4	4	5	11	12	14	53	57	62	536	555	566	6608	6759	6793
35	4	4	5	11	12	14	53	57	62	536	555	566	7090	7246	7280
36	4	4	5	11	12	14	53	57	62	536	555	566	7671	7843	7877
37	4	4	5	11	12	14	53	57	62	536	555	566	8145	8330	8364
38	4	4	5	11	12	14	53	57	62	536	555	566	8664	8866	8900
39	4	4	5	11	12	14	53	57	62	536	555	566	9122	9341	9375
40	4	4	5	11	12	14	53	57	62	536	555	566	9614	9862	9896
41	4	4	5	11	12	14	53	57	62	536	555	566	10016	10283	10317
42	4	4	5	11	12	14	53	57	62	536	555	566	10478	10761	10795
43	4	4	5	11	12	14	53	57	62	536	555	566	10879	11175	11209
44	4	4	5	11	12	14	53	57	62	536	555	566	11276	11589	11623
45	4	4	5	11	12	14	53	57	62	536	555	566	11615	11937	11971

Table 8: *Cumulative number of PVs when q is not constrained*

	$n=8$			$n=9$			$n=10$		
$\bar{w}$	SSI	BI	BI'	SSI	BI	BI'	SSI	BI	BI'
1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2
3	3	3	4	3	3	4	3	3	4
4	5	5	7	5	5	7	5	5	7
5	7	8	12	7	8	12	7	8	12
6	12	14	20	12	14	20	12	14	20
7	18	23	33	18	23	33	18	23	33
8	30	39	52	30	39	52	30	39	52
9	45	63	78	46	64	83	46	64	83
10	73	98	114	75	103	123	76	104	128
11	109	142	162	114	156	180	116	162	190
12	170	210	230	181	236	260	186	251	279
13	247	295	317	271	344	370	282	374	404
14	359	407	432	405	485	514	429	538	571
15	496	545	574	582	672	709	628	761	802
16	679	736	765	829	929	966	915	1070	1112
17	901	959	991	1142	1249	1291	1293	1468	1518
18	1201	1264	1297	1567	1683	1726	1826	2014	2065
19	1548	1606	1648	2101	2212	2267	2511	2699	2764
20	1992	2058	2100	2788	2906	2963	3405	3604	3673
21	2509	2580	2624	3639	3765	3828	4555	4762	4837
22	3176	3251	3298	4736	4867	4935	6046	6265	6346
23	3903	3980	4034	6036	6163	6247	7889	8099	8204
24	4850	4938	4994	7702	7845	7936	10258	10487	10602
25	5889	5991	6047	9678	9837	9931	13178	13426	13545
26	7192	7298	7356	12156	12325	12422	16856	17118	17242
27	8620	8741	8804	15067	15252	15364	21339	21611	21761
28	10402	10541	10605	18708	18921	19034	26965	27266	27419
29	12279	12416	12488	22872	23078	23211	33681	33968	34154
30	14664	14805	14883	28098	28320	28461	42088	42405	42599
31	17161	17303	17387	34049	34265	34422	52059	52366	52589
32	20194	20348	20433	41313	41551	41713	64289	64626	64857
33	23397	23566	23658	49572	49832	50011	78761	79124	79375
34	27290	27467	27563	59635	59908	60097	96452	96828	97100
35	31259	31439	31539	70803	71072	71287	116978	117339	117663
36	36178	36379	36481	84509	84811	85032	142092	142494	142831
37	41168	41377	41486	99676	99987	100223	171134	171559	171915
38	47059	47279	47396	117785	118111	118365	206006	206452	206828
39	53122	53352	53482	137861	138191	138486	246259	246689	247154
40	60377	60637	60770	161972	162334	162642	294585	295069	295554
41	67496	67773	67915	188040	188413	188745	349429	349915	350444
42	76185	76476	76623	219495	219899	220240	415336	415874	416417
43	84824	85130	85279	253545	253951	254326	490004	490504	491136
44	94780	95102	95255	293595	294014	294416	578185	578690	579372
45	104891	105223	105383	337305	337736	338168	678326	678880	679612

Table 9: *Number of PVs according to $\bar{w}$ and n with $q = 1/2$*

$\bar{w}$ \ n	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	2	2	2	2	2	2	2	2
4	3	4	4	4	4	4	4	4
5	2	3	4	4	4	4	4	4
6	4	6	7	8	8	8	8	8
7	2	3	5	6	7	7	7	7
8	4	8	11	13	14	15	15	15
9	2	3	7	10	12	13	14	14
10	4	8	14	19	22	24	25	26
11	2	3	7	12	17	20	22	23
12	4	9	19	29	36	41	44	46
13	2	3	7	17	27	34	39	42
14	4	8	21	38	52	63	70	75
15	2	3	7	19	36	49	60	67
16	4	9	24	51	76	97	112	123
17	2	3	7	20	48	73	94	109
18	4	8	25	63	105	142	171	193
19	2	3	7	21	60	102	139	167
20	4	9	26	77	145	208	259	300
21	2	3	7	21	76	146	210	261
22	4	8	24	85	183	284	371	443
23	2	3	7	21	85	186	289	376
24	4	9	27	102	243	402	545	666
25	2	3	7	21	100	251	417	563
26	4	8	24	109	304	539	765	963
27	2	3	7	21	112	324	573	804
28	4	9	26	119	374	715	1062	1375
29	2	3	7	21	119	400	767	1129
30	4	8	25	122	445	924	1437	1921
31	2	3	7	21	125	486	1010	1551
32	4	9	26	129	536	1208	1958	2689
33	2	3	7	21	132	604	1361	2169
34	4	8	24	125	625	1525	2593	3665
35	2	3	7	21	132	713	1732	2891
36	4	9	27	134	732	1934	3434	4987
37	2	3	7	21	133	846	2242	3903
38	4	8	24	126	814	2367	4432	6642
39	2	3	7	21	135	979	2812	5136
40	4	9	26	133	916	2896	5687	8788
41	2	3	7	21	135	1105	3489	6679
42	4	8	25	130	1008	3522	7257	11539
43	2	3	7	21	135	1249	4343	8684
44	4	9	26	131	1120	4306	9279	15152
45	2	3	7	21	135	1419	5424	11323

Table 10: *Cumulative number of PVs according to $\bar{w}$ and n with $q = 1/2$*

$\bar{w}$ \ n	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3
4	4	5	5	5	5	5	5	5
5	4	6	7	7	7	7	7	7
6	4	8	10	11	11	11	11	11
7	4	8	12	14	15	15	15	15
8	4	9	16	20	22	23	23	23
9	4	9	17	24	28	30	31	31
10	4	9	20	32	39	43	45	46
11	4	9	20	35	47	54	58	60
12	4	9	24	45	64	76	83	87
13	4	9	24	49	74	93	105	112
14	4	9	26	61	96	126	145	157
15	4	9	26	63	108	148	178	197
16	4	9	27	76	139	195	240	270
17	4	9	27	77	153	227	288	333
18	4	9	27	90	193	296	381	448
19	4	9	27	90	207	338	452	543
20	4	9	27	105	260	436	592	718
21	4	9	27	105	277	493	695	863
22	4	9	27	115	336	624	896	1126
23	4	9	27	115	347	688	1035	1336
24	4	9	27	126	422	865	1323	1725
25	4	9	27	126	436	951	1518	2034
26	4	9	27	132	521	1180	1915	2594
27	4	9	27	132	530	1279	2169	3023
28	4	9	27	136	623	1571	2713	3818
29	4	9	27	136	629	1684	3048	4421
30	4	9	27	137	727	2052	3776	5535
31	4	9	27	137	729	2173	4203	6350
32	4	9	27	138	840	2634	5175	7883
33	4	9	27	138	843	2782	5734	9003
34	4	9	27	138	949	3332	6997	11086
35	4	9	27	138	950	3476	7668	12553
36	4	9	27	138	1067	4156	9316	15363
37	4	9	27	138	1067	4326	10182	17337
38	4	9	27	138	1169	5106	12262	21060
39	4	9	27	138	1169	5264	13271	23577
40	4	9	27	138	1270	6189	15899	28465
41	4	9	27	138	1270	6349	17138	31730
42	4	9	27	138	1350	7404	20427	38105
43	4	9	27	138	1350	7544	21873	42245
44	4	9	27	138	1433	8790	25997	50515
45	4	9	27	138	1433	8951	27784	55872