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Fabrice Barthelemy
Jean-Luc Prigent

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Real Estate Portfolio Management: Optimization under Risk Aversion

Fabrice Barthélémy* and Jean-Luc Prigent*

**ThEMA, University of Cergy-Pontoise, Cergy-Pontoise, France
E-mails: fabrice.barthelemy@u-cergy.fr; jean-luc.prigent@u-cergy.fr

Abstract

This paper deals with real estate portfolio optimization when investors are risk averse. In this framework, we determine several types of optimal times to sell a diversified real estate and analyze their properties. The optimization problem corresponds to the maximization of a concave utility function defined on the terminal value of the portfolio. We extend previous results (Baroni *et al.*, 2007, and Barthélémy and Prigent, 2009), established for the quasi linear utility case, where investors are risk neutral. We consider four cases. In the first one, the investor knows the probability distribution of the real estate index. In the second one, the investor is perfectly informed about the real estate market dynamics. In the third case, the investor uses an intertemporal optimization approach which looks like an American option problem. Finally, the buy-and-hold strategy is considered. For these four cases we analyze numerically the solutions that we compare with those of the quasi linear case. We show that the introduction of risk aversion allows to better take account of the real estate market volatility. We also introduce the notion of compensating variation to better compare all these solutions.

Key Words Real estate portfolio, Optimal holding period, Risk aversion, Real estate market volatility.

JEL Classification C61, G11, R21.

1. Introduction

Knowing the real estate holding period is important for investment in commercial real estate portfolios. However, results about optimality of holding period are almost exclusively empirical. Hendershott and Ling (1984), Gau and Wang (1994) or Fisher and Young (2000) show that, for the US, the holding durations depend mainly on tax laws. Brown and Geurts (2005) deal with small residential investment. Through a sample of apartment buildings of between 5 and 20 units over the period 1970-1990 in the city of San Diego, they find empirically how long an investor must own an apartment building. The average holding period is around five years. For the UK market, Rowley, Gibson and Ward (1996) prove the existence of ex ante expectations about holding periods, related to depreciation or obsolescence factors. As illustrated by Brown (2004), the risk peculiar to real estate investments can justify the behaviour of real estate investors. However, Geltner and Miller (2001) show that CAPM is not a convenient tool to understand portfolio management. Note also that values of ex post holding periods are higher than those usually claimed by investors, as illustrated by Collett, Lizieri and Ward (2003)[1].

However, this kind of empirical study does not allow to precise conclusions about the relation between asset volatility and optimal holding period. Baroni *et al.* (2007a, b)[2] determine the optimal holding period when it has to be chosen at initial time. They suppose that the marketing period risk corresponds to time-on-the-market that determines how long it takes to sell an asset once it is put on the market to be sold. It is an exogenous random variable. To better model, the times to sell are endogenously determined from the optimization problems. Barthélémy and Prigent (2009) examine also the determination of the optimal holding period (optimal time to sell in a real estate portfolio) by introducing additional criteria such the American option approach. Nevertheless, they assume that the investor is risk-neutral. Due to this latter hypothesis, the volatility of the real estate asset either has no impact on the optimal holding period or its role is only implicit.

In this paper, our aim is to better emphasize the impact of the volatility. For this purpose, we consider that the investor maximizes his expected utility at maturity over a given time period and is risk-averse. In this standard approach, time horizon and risk aversion are the key parameters. It is typically assumed that the individual preferences are described by utility functions with constant absolute risk aversion (CARA) and hyperbolic absolute risk aversion (HARA), which will be examined in this paper[3]. Our results show in particular that the relative risk aversion plays a key role to evaluate the monetary loss from not having access to the "best" horizon. This feature has to be related to previous works about the influence of risk aversion as shown by Kallberg and Ziemba (1983). We also examine the robustness of our results with respect to the utility specification.

Our model provides solutions whose properties can potentially explain most of the previous empirical results. First, we determine the optimal holding period when it has to be chosen at initial date, extending previous results of Baroni *et al.* (2007a, b). The investor is assumed to know probability distribution of real estate asset. We illustrate what are the impacts of the risk aversion, the real asset value and the volatility on the selling strategies. First we determine this latter one when the investor is perfectly informed about the growth rate dynamics but must choose his strategy only at initial time. However, usually such a solution is not time consistent since the same determination of optimal time to sell at a future date leads to a different solution. Second, we study the best ideal case where the investor knows exactly the price dynamics, as soon as a new period starts. In that case, he can immediately choose the best time to sell the asset. This approach provides the upper bound of the present value of the portfolio as a function of holding period policy. Indeed, the present value is maximized using perfect foresight. We use this special framework as a benchmark. Finally, we determine the optimal holding period according to the American option approach. In this context, at each time during a given management period, the investor compares the present expected utility of portfolio value with the maximal expected utility he could have if he would keep the asset. We show that the investor must sell as soon as the present utility is higher than its expectation.

Additionally, we introduce the notion of compensating variation to evaluate the monetary loss of not having the “best” portfolio. As shown in de Palma and Prigent (2008, 2009), the compensating variation allows to measure the adequacy of a given portfolio to investor’s utility.

The structure of the paper is laid out as follows: Section 2 provides a survey about expected utility theory, risk aversion and compensating variation. Section 3 presents the continuous-time framework and the optimal time to sell we get in the neutral risk investor case. Results for the optimal holding period when the date must be chosen at initial time is developed in section 4 for three different utility functions. Section 5 gives a theoretical framework for other portfolio strategies, as the perfectly informed investor, the American option solution and the buy-and-hold strategy. These approaches are compared in Section 6 by using compensating variations.

2. Utility functions, risk aversion and compensating variation

In this section, first we recall basic notions about utility functions and risk aversion. Then, we detail the concept of compensating variation.

2.1. Expected utility theory and measures of risk aversion

The aim of expected utility (EU) theory is to model problems of decision under uncertainty by means of a functional representation of preferences over lotteries. These latter ones are composed of all possible events and their corresponding probabilities. Preferences reflect the “degree of satisfaction” which results from the individual's choice. The objective of rational individuals with respect to this criterion is thus to maximize this function. Expected utility theory assumes that preference relation is a linear relation of the probabilities[4]. In other words, there exists a utility function u(.) defined (up to a positive monotonic transformation) over the outcome space Ω with values in R such that, for any lotteries $L^a = \{(\omega_1, p_1^a),...,(\omega_m, p_m^a)\}$ and $L^b = \{(\omega_1, p_1^b),...,(\omega_m, p_m^b)\}$, the following equivalence holds:

$$L^a \geq L^b \Leftrightarrow \sum_{i=1}^{m} u(\omega_i) p_i^a \geq \sum_{i=1}^{m} u(\omega_i) p_i^b.$$

Risk aversion. The empirical observation of individuals suggests that usually investors are risk-averse. Therefore, investment in a risky asset implies that the latter must provide returns that are significantly higher than those corresponding to risk-free investment. To illustrate the notion of risk-aversion, Friedman and Savage (1948) introduce the following model: let X be a random variable with only two possible values x_1 and x_2. Let p be the probability that the value be x_1 and $(1-p)$ the probability that the value be x_2. Let $u(.)$ be a utility function defined over the possible outcomes. Then consider the two following lotteries, L^a and L^b: the lottery L^a yields $E[X]$ with a probability of 1; the lottery L^b pays x_1 with probability p and x_2 with probability $(1-p)$. These two lotteries have the same expected payoff. However, a risk-averse investor prefers L^a to L^b. It means that we have:

$$U[L^a] = u(E[X]) \ \geq \ U[L^b] = E[u(X)].$$

Previous property is satisfied for all lotteries if and only if the utility $u(.)$ is concave.[5] Risk aversion is also linked to the concept of "certainty equivalent". Consider the "safe" lottery, denoted by $C[X]$. It yields the same utility level as the lottery L^b. The investor is indifferent between receiving the certain amount $C[X]$ and playing the lottery L^b, with expected payoff $C[X]$. The inequality $C[X] < E[X]$ corresponds to risk aversion. The gap $\pi(X) = E[X] - C[X]$ has been introduced by Pratt (1964) as the risk premium. It is equal to the amount that the investor would be willing to pay in order to benefit from a risk-free payoff. In the expected utility framework, risk-aversion is entirely characterized by assumptions about the utility function u(.):

1) Investors are risk-averse if $C[X] < E[X]$, or equivalently if $\pi(X) \geq 0$, for any random variable *X*. It is equivalent to concavity of function $u(.)$.
2) Investors are risk-neutral if $C[X] = E[X]$, or equivalently if $\pi(X) = 0$, for any random variable *X*. It is equivalent to linearity of function $u(.)$.
3) Investors are risk-loving if $C[X] \geq E[X]$, or equivalently if $\pi(X) \leq 0$, for any random variable *X*. It is equivalent to convexity of function $u(.)$.

The Arrow-Pratt Measures of Risk-Aversion. The measure of individual's degree of risk-aversion is a thorny problem. To solve it, one way is to analyze the risk premium

π and to examine its relationship with the concavity of the utility function (see Pratt, 1964; Arrow, 1965). The definition of the notion of "more risk aversion" can be made as follows: Let $u(.)$ and $v(.)$ be two utility functions. The preference associated with $u(.)$ is said to exhibit "more risk-aversion" than that associated with $v(.)$ if the risk premia satisfy the following relationship $\pi_u(X) \geq \pi_v(X)$ for any random variable *X*. Then, if $u(.)$ and $v(.)$ are continuous, positive, and twice-differentiable, the following properties allow us to define the notion of exhibiting "more risk-aversion":

1) The derivatives of the two utility functions are such that:

$$-\left(\frac{u''}{u'}\right)(x) > -\left(\frac{v''}{v'}\right)(x), \text{for all } x \text{ in } \mathbb{R}.$$

2) There exists a concave function, Φ, such that: $u(x) = \Phi(v(x))$, for all x in $\mathbb{R}$.

3) The risk premia satisfy: $\pi_u(X) \geq \pi_v(X)$, for any random variable *X*.

The ratio $-u''(E[X])/u'(E[X])$ can be considered as a measure of risk-aversion[6]. It is positive as the utility function $u(.)$ is increasing and concave. The term $A(x) = -u''(x)/u'(x)$ is called the Arrow-Pratt Absolute Risk Aversion (ARA). The Arrow-Pratt Relative Risk Aversion (RRA) is defined as $R(x) = -x\,u''(x)/u'(x)$. It allows to take the level of individual wealth into account.

2.2. Standard utility functions

The specification of utility functions remains a tough problem. Positing a utility function is restrictive, even if we estimate its associated parameters, given that different utility functions have different behavioral implications (regarding the latter with respect to investment and saving strategies, see Gollier, 2001, Gollier, Eeckhoudt and Schlessinger, 2005 and de Palma and Prigent, 2009). The preceding definitions of risk-aversion allow us to characterize standard utility functions, and in particular the HARA class of utility functions.

Hyperbolic Risk Aversion. A utility function $u(.)$ is of type HARA ("hyperbolic absolute risk aversion") if the inverse of absolute risk-aversion is a linear function of wealth. HARA utility functions, $u(.)$, are written as follows: $u(x) = a(b + (x/c))^{1-c}$, where $u(.)$ is defined over the domain $b + (x/c) > 0$. The parameters *a*, *b* and *c* are constants such that: $a(1-c)/c > 0$. The associated ARA $A(x)$ is given by: $A(x) = (b + (x/c))^{-1}$, the inverse of which is indeed a linear function of wealth, *x*. Note that the condition $a(1-c)/c > 0$ allows us to conclude that $u' > 0$ and $u'' < 0$.

Three sub-classes of functions are typically distinguished:[7]

1. The Quadratic Utility Function. If $c=-1$, the utility function $u(.)$ is quadratic. For $u(.)$ to be positive, the domain here is restricted to the interval $]-\infty, b[$. The ARA of a quadratic utility function is increasing in wealth ("increasing absolute risk aversion", IARA). This implies that the risk premium $\pi(.)$ is increasing, which is a fairly counter-intuitive property, and which indicates the limits of the application of this function (despite the simplicity of its use in the determination of optimal portfolios, for example).

2. "Constant absolute risk aversion" (CARA). As the parameter c tends to infinity, we obtain $A(x)=A$, where A is a constant. In this case, the utility function $u(.)$ is of the form: $u(x)=-(\exp[-Ax]/A)$. Note that the RRA here increases with wealth.

3. "Constant relative risk aversion" (CRRA). When $b=0$, then $R(x)=c$ is constant and $u(.)$ is of the type: $u(x)=x^{1-c}/(1-c)$ if $c>1$, $\ln[x]$ if $c=1$. This type of function exhibits decreasing absolute risk-aversion (DARA).

2.3. Compensating variation

The ratio of expected utilities characterizes the investor's choice behaviour but it is only a qualitative criterion since utilities are defined up to affine transformations. In what follows, we use instead a quantitative index of investor's satisfaction based on the standard economic concept of compensating variation. As illustrated in de Palma and Prigent (2008, 2009), the notion of "compensating variation" is very useful to evaluate the monetary loss of not having the "best" portfolio. The utility loss from not having access to a "better" portfolio is provided by the compensating variation measure. If an investor with risk aversion γ and initial investment V_0 faces a choice between two (random) horizons $T^{(1)}$ and $T^{(2)}$, he has to compare the two expected utilities $E[U_\gamma(V_{T^{(i)}});V_0]$. Assume that horizon $T^{(2)}$ provides higher utility than maturity $T^{(1)}$. If the investor selects maturity $T^{(1)}$ instead of $T^{(2)}$, he will get the same expected utility provided that he invests an initial amount $\widehat{V}_0 \geq V_0$ such that:

$$E\left[U_\gamma(V_{T^{(1)}});\widehat{V}_0\right]=E\left[U_\gamma(V_{T^{(2)}});V_0\right] \tag{1}$$

Therefore, this investor requires (theoretically) a monetary compensation that can be evaluated by means of the ratio $\widehat{V}_0/V_0$. This amount is in line with the certainty equivalent concept in expected utility analysis. It can be viewed as an implicit initial investment necessary to keep the same level of expected utility.

3. Continuous-time model and risk neutral investor

In this section, the time of sale is pre-set, committed irrevocably at time 0, based on the expected dynamics of the portfolio value and its cash flow. The real estate portfolio value is defined as the sum of the discounted free cash flows (FCF) and the discounted terminal value (the selling price). Denote *k* as the weighted average cost of capital (WACC), which is used to discount the different free cash flows, and the terminal value. We assume that the free cash flow grows at a constant rate g [8].

3.1. Continuous-time model

As Baroni *et al.* (2007a), we suppose that the price dynamics, which corresponds to the terminal value of a diversified portfolio (for instance a real estate index), follows a geometric Brownian motion:

$$\frac{d\widetilde{P}_t}{\widetilde{P}_t} = \mu dt + \sigma dW_t,$$

where W_t is a standard Brownian motion. We have:

$$\tilde{P}_t = P_0 \exp\left[\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma W_t\right]. \qquad (2)$$

This equation assumes that the real estate return can be modelled as a simple diffusion process where parameters μ and σ are respectively equal to the trend and to the volatility. The expected return of the asset at time t is given by:

$$E\left[\frac{\widetilde{P}_t}{P_0}\right] = \exp(\mu t). \qquad (3)$$

Then the future real estate index value at time *t*, discounted at time 0, can be expressed as:

$$P_t = P_0 \exp\left[\left(\mu - k - \tfrac{1}{2}\sigma^2\right)t + \sigma W_t\right] \text{ with } E\left[\frac{P_t}{P_0}\right] = \exp\left(\left[\mu - k\right]t\right). \qquad (4)$$

Denote by FCF_0 the initial value of the free cash flow. The continuous-time version of the sum of the discounted free cash flows FCF_s is equal to:

$$C_t = \int_0^t FCF_s e^{-ks} ds = \int_0^t FCF_0 e^{-[k-g]s} ds, \qquad (5)$$

which leads to

$$C_t = \frac{FCF_0}{k-g}\left(1 - e^{-[k-g]t}\right). \qquad (6)$$

Introduce the real estate portfolio value process V, which is the sum of the discounted free cash flows and the future real estate index value at time *t*, discounted at time 0:

$$V_t = C_t + P_t = c\left(1 - e^{-at}\right) + P_0\, \mathrm{e}^{\left[\left(\mu - k - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right]}. \tag{7}$$

We determine the portfolio value $V_{\overline{T}}$ for a given maturity $\overline{T}$. This assumption on the time horizon allows to take account of selling constraints before a limit date. The higher $\overline{T}$, the less stringent this limit. Additionally, this hypothesis allows the study of buy-and-hold strategies (see section 6). The future portfolio value at maturity, discounted at time 0, is given by:

$$V_{\overline{T}} = \frac{FCF_0}{k - g}\left(1 - e^{-[k-g]\overline{T}}\right) + P_0\, \mathrm{e}^{\left[\left(\mu - k - \frac{1}{2}\sigma^2\right)\overline{T} + \sigma W_{\overline{T}}\right]}. \tag{8}$$

The portfolio value $V_{\overline{T}}$ is the sum of a deterministic component and a Lognormal random variable.

In what follows, we determine the optimal solution at time 0, for a given maturity $\overline{T}$ and for an investor maximizing expected utility. First recall that the sum of the discounted free cash flows is always increasing due to the cash accumulation over time. Second, we have to analyze the expected utility of the future real estate index value at time *t*, discounted at time 0: if the price return μ is higher than the WACC *k*, then, the optimal solution for the linear utility case is simply equal to the maturity $\overline{T}$. Thus, in what follows, we consider the case $\mu < k$. Consequently, not selling the asset implies a higher cumulated cash but a smaller discounted expected terminal value $P_0\, \mathrm{e}^{(\mu - k)t}$. Hence, the investor has to choose between more (discounted) flows and less expected discounted index value. We also focus on the sub case $g < \mu$, which corresponds to empirical data. We investigate two main numerical cases:

- Case 1: $\mu = 4.4\%, \sigma = 5\%, g = 3\%, k = 8.4\%, P_0 = 100, FCF_0 = 100/22$. It corresponds to an early selling, due in particular to weak expected return of the real estate asset.

- Case 2: $\mu = 6\%, \sigma = 5\%, g = 2\%, k = 9.5\%, P_0 = 100, FCF_0 = 100/15$. It corresponds to a late selling, due in particular to higher expected return of the real estate asset.

3.2 Computation with the linear utility function: (see Barthélémy and Prigent, 2009)

The optimization problem is:

$$Max_{t \in [0,\overline{T}]} E\left[V_t\right]. \tag{9}$$

Since the expectation of V_t is equal to:

$$E\left[V_t\right] = \frac{FCF_0}{k - g}\left(1 - e^{-[k-g]t}\right) + P_0 e^{[\mu - k]t}, \tag{10}$$

we deduce:

$$\frac{\partial E\left[V_t\right]}{\partial t} = FCF_0 e^{-[k-g]t} + P_0\left(\mu - k\right)e^{[\mu-k]t}. \tag{11}$$

Then, the optimal holding period is determined as follows.

Case 1: The initial price P_0 is smaller than $\frac{FCF_0}{k-\mu}e^{-(k-\mu)\overline{T}}$.

Then, the optimal time to sell T^* corresponds to the maturity $\overline{T}$. Since *the Price Earning Ratio* (PER) $\frac{P_0}{FCF_0}$ is too small (< $\frac{e^{-(k-\mu)\overline{T}}}{k-\mu}$), the sell is not relevant before maturity.

Case 2: The initial price P_0 lies between the two values $\frac{FCF_0}{k-\mu}e^{-(k-\mu)\overline{T}}$ and $\frac{FCF_0}{k-\mu}$.

Then, the optimal time to sell T^* is solution of the following equation:

$$\frac{\partial E\left[V_t\right]}{\partial t} = 0. \tag{12}$$

From Equation (11), we deduce[9]:

$$T^* = \frac{1}{\mu - g}\ln\left(\frac{FCF_0}{P_0} \times \frac{1}{k-\mu}\right). \tag{13}$$

In particular, note that T^* is a decreasing function of the initial price P_0 and of the difference between the index return μ and the growth rate g of the free cash flows. This latter property was empirically observed by Brown and Geurts (2005). It means that investors sell property sooner when values rise faster than rent.

Case 3: The initial price P_0 is higher than $\frac{FCF_0}{k-\mu}$.

Then, the optimal time to sell T^* corresponds to the initial time 0. Since the PER $\frac{P_0}{FCF_0}$ is sufficiently large (> $\frac{1}{k-\mu}$), there is no reason to keep the asset P. As an illustration, the cumulative value C_t of the FCF_t values, of the expectation of the index value $E[P_t]$ and the expectation of the portfolio value $E[V_t]$ are displayed in Figure 1. We consider two sets of parameter values for a 20 year management period ($\overline{T} = 20$).

We note that the discounted expected value V_t of the portfolio is concave. The parameter values imply that the optimal holding period, T^*, is respectively equal to 9.13 years and 16.11 years. For these two examples, the optimal time to sell T^* is smaller than the maturity $\overline{T}$. In the second example, the discounted portfolio value varies up to 20%[10]. Knowing the optimal time to sell T^*, which is deterministic, the probability distribution of the discounted portfolio value V_{T^*} can be determined. The value V_{T^*} is equal to:

$$V_{T^*} = \frac{FCF_0}{(k-g)}\left(1 - e^{-(k-g)T^*}\right) + P_0 \exp\left[\left(\mu - k - 1/2\sigma^2\right)T^* + \sigma W_{T^*}\right].$$

Denote $A = \frac{FCF_0}{(k-g)}\left(1 - e^{-(k-g)T^*}\right)$ the cumulative discounted free cash flow value at T^*. Since, from (13), the optimal time to sell satisfies:

$$T^* = \frac{1}{\mu - g}\ln\left(\frac{FCF_0}{P_0\left(k-\mu\right)}\right),$$

then, we deduce:

$$A = \frac{FCF_0}{(k-g)}\left(1 - \left[\frac{FCF_0}{P_0\left(k-\mu\right)}\right]^{\left(-\frac{k-g}{\mu-g}\right)}\right),$$

and the cdf $F_{V_{T^*}}$ of V_{T^*} is given by:

$$F_{V_{T^*}}(v) = \begin{cases} 0, \text{if } v \le A \\ N\left[\frac{1}{\sigma\sqrt{T^*}}\left[\ln\left(\frac{v-A}{P_0}\right) - \left(\left(\mu - k - 1/2\sigma^2\right)T^*\right)\right]\right], \text{if } v > A \end{cases} \tag{14}$$

where N denotes the cdf of the standard Gaussian distribution.

4. Optimal time to sell *T**, chosen at time 0

4.1 Computation with the quadratic utility function

4.1.1 Model with the quadratic utility function

The expected utility of the portfolio value at time *t* in the case of the quadratic utility function is

$$E\left[U\left(V_t\right)\right] = E\left(V_t\right) - \frac{\lambda}{2}E\left(V_t^2\right) \tag{15}$$

$$E\left[U\left(V_t\right)\right] = C_t + E[P_t] - \frac{\lambda}{2}E\left(C_t^2 + 2C_t\, E[P_t] + E[P_t]^2\right)$$

Using (8) we get

$$V_t^2 = c^2\left(1 - e^{-at}\right)^2 + 2c\left(1 - e^{-at}\right)P_0\, \mathrm{e}^{\left[\left(\mu - k - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right]} + P_0^2\, \mathrm{e}^{\left[2\left(\mu - k - \frac{1}{2}\sigma^2\right)t + 2\sigma W_t\right]} \tag{16}$$

and knowing that

$$E\left[e^{\left(A - \frac{1}{2}B^2\right)t + BW_t}\right] = e^{At}, \text{ where } W_t \sim N\left(0, \sqrt{t}\right),$$

we deduce that the expectation of (16) is equal to:

$$E\left[V_t^2\right] = c^2\left(1-e^{-at}\right)^2 + P_0^2\,\mathrm{e}^{2\left(\mu-k-\frac{1}{2}\sigma^2\right)t} + 2c\left(1-e^{-at}\right)P_0\,\mathrm{e}^{(\mu-k)t}$$

which leads to the following expected value of the portfolio at time t described in (15):

$$E\left[U\left(V_t\right)\right] = c\left(1-e^{-at}\right) + P_0\,\mathrm{e}^{(\mu-k)t} - \frac{\lambda}{2}\left[c^2\left(1-e^{-at}\right)^2 + 2c\left(1-e^{-at}\right)P_0\,\mathrm{e}^{(\mu-k)t} + P_0^2\,\mathrm{e}^{2\left(\mu-k-\frac{1}{2}\sigma^2\right)t}\right]$$

For instance, Figure 1 and 2 illustrate the impact of λ on the portfolio utility function according to the selling time. Notice that $\lambda = 0$ corresponds to the risk neutral case studied in Barthélémy and Prigent (2009) and presented in section 3.2. In case 1, the optimal time to sell is decreasing with the level of the risk aversion λ (see Figure 1). In case 2, the effect of λ on the optimal time to sell is opposite, even if the portfolio utilities decrease (see Figure 1).
In case 1, when λ is too high (around 0.0082), $T^* = 0$, while in case 2, $T^* = 20$ (see Figure 2). Case 1 represents a portfolio with a high *Price Earning Ratio* (PER). The importance in the portfolio is given to the selling price whose value is sensitive to the risk aversion. At the contrary, case 2 represents a situation with a low PER. The importance is then attributed to the rents.

4.1.2 Limit of the quadratic utility function

If risk aversion raises, the expected utility can be no longer monotone with respect to the selling time. The quadratic utility function is not clearly defined for all the values of λ. If λ becomes high enough the utility function is then a decreasing function for values higher than $1/\lambda$. Indeed its derivative function is equal to

$$\frac{\partial U\left(x\right)}{\partial x} = \frac{\partial}{\partial x}\left(x - \frac{\lambda}{2}x^2\right) = 1 - \lambda x$$

which is null for $x = 1/\lambda$.
For instance, with $U\left(x\right) = x - 0.0025x^2, (\lambda = 0.005)$, x is constrained to be less than 200, because $U\left(200+a\right) < U\left(200\right), \forall a > 0$ (as illustrated by Figure 3).

In this case $A(x) = -u''(x)/u'(x) = \dfrac{\lambda}{1-\lambda x}$ and $R(x) = -x\,u''(x)/u'(x) = \dfrac{\lambda x}{1-\lambda x}$.
Setting $R(x) = \beta$ leads to $x = \left(\dfrac{\beta}{1+\beta}\right)\dfrac{1}{\lambda}$. Usual values of relative risk aversion lie between 1 and 10. For instance, in case 2, the function becomes convex with $\lambda = 0.015$. This arises even with $\sigma = 0$ as observed on Figure 4. If $\lambda = 0.009$, the maximal value of the expected utility is reached at $x = 1/0.09 = 111.11$ (see Figure 5a). Table 1 shows that for $t \geq 5$ years, the portfolio expected value is greater than 111.11 the limit computed above. Hence, even if $V_6 = 113.27$ is higher than $V_4 = 109.97$, the corresponding expected value is lower, $U(V_6) < U(V_4)$ as illustrated in Figure 5b. There are two extrema, one maximum and one minimum, the latter being induced by the bias in the utility function.

4.1.3 Analysis of T^*

In order to analyze the solution T^*, we have to compute the first derivative of the expected utility:

$$\frac{\partial E\left[U\left(V_t\right)\right]}{\partial t} \quad = \quad ac\,e^{-at} + P_0\left(\mu - k\right)\mathrm{e}^{(\mu-k)t}$$

There is no explicit function for the solution T^* but for the range where the utility function is concave, the derivative function evaluated at T^* is null. Let $d(t) = \frac{\partial E\left[U\left(V_t\right)\right]}{\partial t}$, then $d\left(T^*\right) = 0$. Figure 6 illustrates this feature for two risk aversion levels and for each case (early or late selling).

More generally, the derivative function may be expressed as a function of k, g, μ, FCF_0, m, σ and λ:

$$d(t, k, g, \mu, FCF_0, m, \sigma, \lambda)$$

We may study the implicit functions for the two main parameters of interest being σ and λ:

$$d(t, \lambda) = 0 \Rightarrow t(\lambda)$$

which enables to study $T^*(\lambda)$. Figure 7 illustrates this function $T^*(\lambda)$ for case 1 and Figure 8 for case 2 (the analysis is the one made at the end of section 4.1.1 about the PER).

We have seen that for a given initial endowment V_0, the utility function is no more increasing for high risk aversion. In this case, the study of the implicit function will not be relevant. Hence, we will not study the optimal time to sell according to the different parameters when considering the quadratic utility function. This will be done for the CARA and the CRRA functions used to take account of the risk aversion of the investor.

4.2 Computation with the CARA utility function

4.2.1 CARA utility maximization

We have:

$$E\left[U\left(V_t\right)\right] = -\tfrac{1}{\alpha}E\left[e^{-\alpha C_t}\,e^{-\alpha P_t}\right] = -\tfrac{1}{\alpha}e^{-\alpha c\left(1-e^{-at}\right)}E\left[\,e^{-\alpha P_0 \mathrm{e}^{\left[\left(\mu-k-\frac{1}{2}\sigma^2\right)t+\sigma W_t\right]}}\,\right]$$

At time t, we get:

$$E\left[\,e^{-\alpha P_t}\,\right] = \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\alpha P_0 \mathrm{e}^{\left[\left(\mu-k-\frac{1}{2}\sigma^2\right)t+\sigma\sqrt{t}\,x\right]}}\; e^{-x^2/2}\,\mathrm{d}x.$$

Let us denote $h(t,x) = \left(\mu - k - 1/2\sigma^2\right)t + \sigma\sqrt{t}\,x$, which gives:

$$E\left[\,e^{-\alpha P_t}\,\right] = \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\alpha P_0 \mathrm{e}^{h(t,x)}}\; e^{-x^2/2}\,\mathrm{d}x$$

Its first derivative is equal to:

$$\frac{\partial}{\partial t}E\left[\, e^{-\alpha P_t}\,\right]=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\tfrac{\partial}{\partial t}\left(e^{-\alpha P_0\, \mathrm{e}^{h(t,x)}}\right)\, e^{-x^2/2}\,\mathrm{d}x$$

which corresponds to

$$\frac{\partial}{\partial t}E\left[\, e^{-\alpha P_t}\,\right]=-\tfrac{1}{\sqrt{2\pi}}\alpha P_0\int_{-\infty}^{+\infty} e^{-\alpha P_0\left[\mathrm{e}^{h(t,x)}\right]+h(t,x)}\left(\mu-k-\tfrac{1}{2}\sigma^2+\tfrac{1}{2\sqrt{t}}\sigma\, x\right)\, e^{-x^2/2}\,\mathrm{d}x$$

For the case 1, the utility function for a risk neutral investor leads logically to the same optimal solution as found in section 3: $\alpha=0, T^*=9.131$. This is of course the same result for case 2, where for $\alpha=0, T^*=16.109$.

In the case of early selling (case 1), the optimal time to sell $T^*(\alpha)$ decreases when the risk aversion increases. In this case, the rents are relatively low comparatively to the asset price. Hence, an increase in the risk aversion makes the late selling “more risky” from the point of view of the investor, as the portfolio benefits come more from the terminal value. In order to reduce this risk, the asset has to be sold earlier as illustrated in Figure 9. This is the same analysis as the one made for the quadratic utility function represented on Figure 1.
In the contrary, when considering the case of late selling (case 2), the optimal time to sell $T^*(\alpha)$ increases with the risk aversion. The cash flows induce by the rents are relatively high comparatively to the one of the terminal value. In the arbitrage between rent cash flows and selling price, the cash flows become more and important as the risk aversion increases (remember the rents are deterministic). In order to get a higher portfolio value, the asset has to be sold later as illustrated in Figure 10. We find the same qualitative analysis as the one made for the quadratic utility function illustrated on Figure 2.

4.2.2 Volatility and risk aversion

We analyze the evolution of optimal time to sell $T^*(\alpha,\sigma)$ when the level of risk aversion and the volatility may change. Figure 11 (for case 1) and Figure 12 (for case 2), clearly show how these two parameters modify the optimal time to sell in the same way. In the case of early selling (case 1), the optimal time to sell decreases with the risk aversion and the volatility. The volatility reinforces the risky effect on the selling price (as seen previously on Figure 9). In the case of late selling (case 2), the optimal time to sell increases with these two parameters. As in case 1, more volatility implies a more risky terminal value. But, as the strategy focuses on the rents component of the portfolio, the asset has to be sold later.
Functions $T^*(\alpha,\sigma)$ may be analyzed by considering fixed one of these two parameters. Figure 13 examines the impact of the volatility level on the relation $T^*(\alpha)$. For case 1, the decreasing shape is more and more pronounced as the volatility increases. When the volatility level is quite small ($\sigma=0.01$), the risky effect on the selling price is low and the optimal time to sell is relatively constant whatever the risk aversion level. With higher volatilities, the optimality tends to be a sell a time 0 (in order to avoid a potential loss in the terminal value).
For case 2, the increasing of $T^*(\alpha)$ is more and more important as the volatility increases. As for case 1, a low volatility ($\sigma=0.01$) implies a relatively constant

optimal time to sell (as in the risk neutral case). With higher volatilities, the optimality tends to be a sell a time $\overline{T}$. Figure 14 illustrates the evolution of $T^*(\sigma)$ for a given risk aversion level.

4.2.3 Risk aversion and initial price P_0

Whatever the cases (1 or 2, see Figure 15), $T^*(P_0)$ is a decreasing function of the initial price P_0. Considering a higher initial price P_0 for a given level of rents, implies an increase of *the Price Earning Ratio* (PER) $\frac{P_0}{FCF_0}$. Then, as more importance is given to the terminal value, the optimal time to sell decreases in order to take account of the increase of the risk in the selling price.
At the contrary, when $\frac{P_0}{FCF_0}$ decreases, the rents have a higher weight in the portfolio value, and the optimal time to sell increases. At the limit, if the PER is too small, the sell is not relevant before maturity. This analysis will be extended with the CRRA utility function.

4.3 Computation with the CRRA utility function

4.3.1 CRRA utility maximization

The expected utility is given by:

$$\begin{aligned} E\left[U\left(V_t\right)\right] &= \tfrac{1}{1-\gamma} E\left[\left(C_t + P_t\right)^{1-\gamma}\right] \\ &= \tfrac{1}{1-\gamma} E\left[\left(c\left(1-e^{-at}\right) + P_0\, \mathrm{e}^{\left(\mu-k-\frac{1}{2}\sigma^2\right)t+\sigma W_t}\right)^{1-\gamma}\right] \end{aligned}$$

and the first derivative is equal to:

$$\frac{\partial}{\partial t} E\left[U(V_t)\right] = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(C_t + P_0\, e^{h(t,x)}\right)^{-\gamma} \left[ac\, e^{-at} + P_0\, e^{h(t,x)} \left(\mu - k - \tfrac{1}{2}\sigma^2 + \tfrac{1}{2\sqrt{t}}\sigma\, x\right)\right] e^{-x^2/2}\, \mathrm{d}x$$

Previous formula does not allow to get explicit relations between the optimal selling time T^* and various parameters such as the relative risk aversion γ, the volatility σ and the initial real estate asset value P_0. In what follows, we begin by examining the shape of the utility function according to the relative risk aversion γ. Then, we study the impact of the volatility. Figure 16 for case 1, and Figure 17 for case 2 lead to the same qualitative results as the ones described with the use of the quadratic or the CARA utility functions.

Volatility and risk aversion

Figures 18 to 20 lead to the same qualitative results obtained with the CARA utility function.

4.3.2 Risk aversion (γ) and initial price P_0

With this function, we have the same results as those described in section 4.2.3. We develop these results analyzing the variation in the initial price P_0 differently. According to the initial price (the rents remaining constant), we may change the situation from an early selling to a late selling and vice versa.

For instance, in case 1, with an initial price P_0 equal to 105, the optimal time to sell T^* is a decreasing function of the risk aversion γ (with $\gamma = 0.5, T^* = 9$, with $\gamma = 2, T^* = 8$ and with $\gamma = 10, T^* = 0$). This is true for each value of P_0 equal or greater than 100 (the reference value), as this case illustrates an early selling situation. Let us notice, the higher P_0, the more decreasing $T^*(\gamma)$. There exists a value of P_0 below which the effect of the risk aversion is reversed: we have then a late selling situation. For instance, with an initial price P_0 equal to 95, the optimal time to sell T^* is an increasing function of the risk aversion γ (with $\gamma = 0.5, T^* = 12.9$, with $\gamma = 2, T^* = 13.1$ and with $\gamma = 10, T^* = 16.5$).

The same analysis may be made with case 2. For values of P_0 less than 100, the optimal time to sell T^* is a increasing function of the risk aversion γ. For instance an initial price of 90 leads to with $\gamma = 0.5, T^* = 18.6$, with $\gamma = 2, T^* = 18.8$ and with $\gamma = 10, T^* = 20$).

There exists a pivotal initial price point from which the impact of the risk aversion on the optimal to sell is changing. This is illustrated on Figure 22.

5. Other optimal times to sell for a risk averse investor

*5.1 Perfectly informed investor T***

In this section, the investor is supposed to have a perfect foresight about the entire future price path. Trajectories are random (the investor does not choose the realized path) but, at time 0, the whole path is known. Therefore, the investor can maximize with respect to this trajectory. Thus, the optimal solution is deterministic conditionally to this information. Nevertheless, the path is unknown just before time 0. Consequently, the optimal time to sell is a random variable. This 'ideal' framework is not realistic but provides an upward benchmark. Note that, since the investor is rational, his utility function is increasing. Therefore, since the path is known, the maximization of the utility of his portfolio value is equivalent to the maximization of a linear utility. This means that we recover previous solution provided in Barthélémy and Prigent (2009). In what follows, we recall the distributions of the optimal holding period T^{**} and of the optimal value $V_{T^{**}}$. Barthélémy and Prigent (2009) provide an explicit formula by means of a mild approximation. Introduce the function G defined by:

$$G(m,y,t) = 1 - \frac{1}{2} Erfc\left(\frac{y}{\sqrt{2t}} - m\frac{\sqrt{t}}{\sqrt{2}}\right) - \frac{1}{2} e^{2my} Erfc\left(\frac{y}{\sqrt{2t}} + m\frac{\sqrt{t}}{\sqrt{2}}\right),$$

where the function $Erfc$ is given by:

$$Erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2} du.$$

Denote also

$$A(v) = \frac{FCF_0}{v} + \mu - k - 1/2\sigma^2, \text{and } B(v) = \ln\left(\frac{v}{P_0}\right).$$

Then, the approximated cdf of V_{T**} is given explicitly by:

$$P[V_{T**} \le v] = \begin{cases} 0, \text{ for } v < P_0, \\ G\left(\frac{A(v)}{\sigma}, \frac{B(v)}{\sigma}, \overline{T}\right) \text{for } v > P_0 \end{cases}. \tag{15}$$

The probability that the real estate portfolio value is higher than P_0 is equal to 1. Thus, whatever the path, the investor receives at least P_0. Indeed, if all the future discounted portfolio values are lower than the initial price, he knows he has to sell at time 0 and then receives exactly P_0.

*5.2 American optimal selling time T****

In this third case, we allow that the investor may choose the optimal time to sell, according to market fluctuations and information from past observations. In this case, he faces an "American" option problem. Recall that the investor preferences are modelled by means of utility function. At any time *t* before selling, he compares the utility of the present value P_t with the maximum of the future utility value he expects given the available information at time *t* (mathematically speaking he computes the maximum expected utility of his portfolio on all $\mathcal{J}_{t,\overline{T}}$-measurable stopping times τ). It means that he decides to sell at time *t* only if the utility of his portfolio value at this time is higher than the maximal expected utility that he can expect to reach if he does not sell at this time t. Thus, he has to compare $U_\gamma(C_t + P_t)$ with $\sup_{\tau \in \mathcal{J}_{t,\overline{T}}} E\left[U_\gamma(C_\tau + P_\tau) | \mathcal{J}_t\right]$, where C_s denotes the FCF value at time s.

Intuitively, the optimal time T^{***} must be the first time at which the utility $U_\gamma(C_t + P_t)$ is "sufficiently" high. At this price level, the future free cash flows (received in case of no sell) will not be high enough to balance an index value lower than the price P_t at time t (the expected index value decreases with time as the discounted trend $\mu - k$ is negative). The optimal time T^{***} corresponds exactly to the first time at which the asset price P_t is higher than a deterministic level (see Appendix C). This result generalizes the case considered in Barthélémy and Prigent (2009) where the investor has a linear utility. In that case, he sells directly the asset if the price P_0 is higher than $\frac{FCF_0}{k-\mu}$. Then, since the return of the discounted free cash flows is equal to $e^{-(k-g)t}$, the price P_t has to be compared with the value $\frac{FCF_0}{k-\mu} e^{-(k-g)t}$.

5.2.1. The American option problem

Denote by $\mathrm{V}(x,t)$ the following value function:

$$\mathrm{V}(x,t) = \sup_{\tau \in F_{t,\bar{T}}} E\left[U_\gamma \left(C_\tau + P \right)_\tau \left| P_t = x \right. \right].$$

Note that we always have $\mathrm{V}(x,t) \geq U_\gamma\left(C_t + x\right)$, since $\tau = t \in \mathrm{J}_{t,\bar{T}}$ and, in that case, $\mathrm{V}(x,t) = U_\gamma\left(C_t + x\right)$.

As usual for American options[11], two "regions" have to be considered:

- *The continuity region:*

$$C = \left\{ (x,t) \in R^+ \times [0,\bar{T}] \middle| \mathrm{V}(x,t) > U_\gamma\left(C_t + x\right) \right\}$$

- *The stopping region:*

$$S = \left\{ (x,t) \in R^+ \times [0,\bar{T}] \middle| \mathrm{V}(x,t) = U_\gamma\left(C_t + x\right) \right\}$$

The first optimal stopping time T_t^{***} after time t is given by

$$T_t^{***} = \inf\left\{ u \in [t,\bar{T}] \middle| \mathrm{V}(P_u,u) = U_\gamma\left(C_u + P_u\right) \right\}.$$

Then:

$$T_t^{***} = \inf\left\{ u \in [t,\bar{T}] \middle| u_t \notin C \right\}.$$

5.2.2 Computation of the value function V

To determine T_t^{***}, we have to calculate $\mathrm{V}(x,t)$.

We have to compute:

$$\sup_{\tau \in \mathrm{J}_{t,\overline{T}}} E\left[U_\gamma\left(C_t + c\left(e^{-at} - e^{-a\tau}\right) + P_t \exp\left[\left(\mu - k - 1/2\sigma^2\right)(\tau - t) + \sigma(W_\tau - W_t)\right]\right) \middle| P_t = x\right].$$

In particular, we have to search for the value τ_t^* for which the maximum

$$\sup_{\tau \in \mathrm{J}_{t,\overline{T}}} E\left[U_\gamma\left(C_t + c\left(e^{-at} - e^{-a\tau}\right) + x \exp\left[\left(\mu - k - 1/2\sigma^2\right)(\tau - t) + \sigma(W_\tau - W_t)\right]\right)\right].$$

is achieved.

This problem is the dynamic version of the determination of T^* presented in Section 4.

Introduce the function $f_{t,x}$ defined by:

$$f_{t,x,z,\gamma}(\theta) = U_\gamma\left(C_t + c\left(e^{-at} - e^{-a(t+\theta)}\right) + x \exp\left[\left(\mu - k - 1/2\sigma^2\right)\theta + \sigma\sqrt{\theta} z\right]\right).$$

This function is strictly increasing with respect to x. We have to solve:

$$\sup_{\theta \in \mathrm{J}_{t,\overline{T}}} E\left[f_{t,x,z,\gamma}(\theta)\right].$$

Case 1. The optimal solution is equal to the maturity $\overline{T}$.

Then:

$$\mathrm{V}(x,t) = E\left[U_\gamma\left(C_t + c\left(e^{-at} - e^{-a\overline{T}}\right) + x \exp\left[\left(\mu - k - 1/2\sigma^2\right)\left(\overline{T} - t\right) + \sigma(W_{\overline{T}} - W_t)\right]\right)\right].$$

Using Jensen inequality, we deduce in that case that $\mathrm{V}(x,t) > U_\gamma\left(C_t + x\right)$, for all three standard utility functions (since they are concave and strictly increasing).

Case 2: The optimal solution lies strictly between t and $\overline{T}$.

Then, the optimal time τ_t^* is equal to $(t + \theta^*)$, where θ^* is the solution of the following equation:

$$\frac{\partial E\left[f_{t,x,z,\lambda}\right]}{\partial \theta}(\theta) = 0. \tag{25}$$

$$\mathrm{V}(x,t) = E\left[U_\gamma\left(C_t + c\left(e^{-at} - e^{-a\tau_t^*}\right) + x \exp\left[\left(\mu - k - 1/2\sigma^2\right)\left(\tau_t^* - t\right) + \sigma(W_{\tau_t^*} - W_t)\right]\right)\right].$$

Case 3: The optimal time τ_t^* corresponds to the present time t, and

$$\mathrm{V}(x,t) = U_\gamma\left(C_t + x\right).$$

Consequently, from the three previous cases, we deduce the value of $\mathrm{V}(x,t)$.

Finally, the American optimal time T^{***} is determined by:

$$T^{***} = \inf\left\{ t \in [0,\overline{T}] \middle| \, \mathrm{V}(P_t,t) = U_\gamma\left(C_t + P_t\right) \right\}.$$

Therefore, we can check that $\mathrm{V}(P_t,t) = U_\gamma\left(C_t + P_t\right)$ if and only if:

$$\frac{\partial E\left[f_{t,P_t,z,\lambda} \right]}{\partial \theta}(\theta) < 0, \forall \theta \in \left[0, \overline{T} - t\right].$$

Thus, we have:

$$T^{***} = \inf\left\{ t \in [0,\overline{T}] \middle| \, P_t \geq l\left(t, FCF_0, k, g, \mu\right) \right\},$$

where $l\left(t, FCF_0, k, g, \mu\right)$ is determined from optimality condition of 0 being optimal for the first problem (see section 4).

6. Comparison of the three optimal strategies and the buy-and hold one

We examine both the probability distributions of the optimal times to sell T^*, T^{**} and T^{***} and the corresponding discounted portfolio values V_{T^*}, $V_{T^{**}}$ and $V_{T^{***}}$. We introduce also the comparison with the buy-and hold portfolio $V_{\overline{T}}$. The sensitivity analysis is done with respect to the volatility σ and to the maturity $\overline{T}$. Note that, since the portfolio value $V_{T^{**}}$ dominates the other ones, its cdf is always below the cdf of the others ("first order stochastic dominance"). We also examine compensating variations.

6.1. Cdf of V_T according to T

Figures 23 and 24 provide respectively cdf and pdf of portfolio values according to selling time.

For numerical case 1, we note if the investor sells too quickly (for instance T=3), the probability to get high returns is smaller than for optimal strategy T*. Additionally, the optimal strategy dominates stochastically the buy and hold strategy at the first order.

For numerical case 2, the investor must sell lately. The optimal strategy clearly dominates at the first-order the too early selling strategy (T=3).

6.2. Compensating variations: computations

Recall that, if an investor with risk aversion γ and initial amount V_0 must select one of the two (random) horizons $T^{(1)}$ and $T^{(2)}$, he has to compare the two expected utilities $E[U_\gamma(V_{T^{(i)}});V_0]$. In what follows, we suppose that horizon $T^{(2)}$ provides higher utility than maturity $T^{(1)}$. If the investor selects maturity $T^{(1)}$ instead of $T^{(2)}$, he will get the same expected utility provided that he invests an initial amount $\widehat{V}_0 \geq V_0$ such that:

$$E[U_\gamma(V_{T^{(1)}});\widehat{V}_0] = E[U_\gamma(V_{T^{(2)}});V_0]$$

Recall also that, at any time t of the management period $\left[0,\overline{\mathrm{T}}\right]$, the "portfolio" value is given by:

$$V_t = C_t + P_t,$$

with

$$C_t = \frac{FCF_0}{k-g}\left(1-e^{-[k-g]t}\right) \text{ and } P_t = P_0 \exp\left[\left(\mu - k - 1/2\sigma^2\right)t + \sigma W_t\right]$$

Introduce the returns $R_{T^{(1)}} = V_{T^{(1)}} / \widehat{V}_0$ and $R_{T^{(2)}} = V_{T^{(2)}} / V_0$.

6.2.1 *The compensating variation for the quadratic case*

Suppose that the investor's utility U is of quadratic. Function U is equal to:

$$U(v) = v - \lambda\frac{v^2}{2}, \text{ with } \lambda>0.$$

If we fix the level of risk aversion λ, then Relation $E[U_\lambda(V_{T^{(1)}});\widehat{V}_0] = E[U_\lambda(V_{T^{(2)}});V_0]$ is equivalent to:

$$\widehat{V}_0 E\left[R_{T^{(1)}}\right] - \frac{\lambda}{2}\widehat{V}_0^{\,2} E\left[R^2_{T^{(1)}}\right] = V_0 E\left[R_{T^{(2)}}\right] - \frac{\lambda}{2}V_0^{\,2} E\left[R^2_{T^{(2)}}\right].$$

The previous relation provides the expression of the compensating variation for the quadratic case, through the resolution of the following polynomial equation:

$$\frac{\lambda V_0}{2}x^2 E\left[R^2_{T^{(1)}}\right] - xE\left[R_{T^{(1)}}\right] + E\left[R_{T^{(2)}}\right] - \frac{\lambda V_0}{2}E\left[R^2_{T^{(2)}}\right] = 0,$$

where x denotes the possible values of the compensating variation $\widehat{V}_0 / V_0$. Set:

$$\Delta = \left(E\left[R_{T^{(1)}}\right]\right)^2 - 2\lambda V_0 E\left[R^2_{T^{(1)}}\right]\left(E\left[R_{T^{(2)}}\right] - \frac{\lambda V_0}{2}E\left[R^2_{T^{(2)}}\right]\right).$$

Then, we deduce:

$$\frac{\widehat{V}_0}{V_0} = \frac{E\left[R_{T^{(1)}}\right] + \sqrt{\Delta}}{\lambda V_0 E\left[R^2_{T^{(1)}}\right]}.$$

Since the relative risk aversion is increasing for the quadratic case, it is not surprising that the compensating variation depends on the wealth level V_0.

6.2.2 *The compensating variation for the CARA case*

Suppose that the investor's utility U is of CARA type. Function U is equal to:

$$U(v) = -\frac{e^{-av}}{a}, \text{ with } a>0.$$

Then, if a is fixed, Relation $E[U_a(V_{T^{(1)}});\widehat{V}_0] = E[U_a(V_{T^{(2)}});V_0]$ is equivalent to:

$$E\left[e^{-a\hat{V}_0 R_{T^{(1)}}}\right] = E\left[e^{-aV_0 R_{T^{(2)}}}\right].$$

which yields to an implicit relation between V_0 and $\hat{V}_0$. [12]

6.2.3 The compensating variation for the CRRA case

Suppose that the investor's utility U is of CRRA type. Function U is equal to:

$$U(v) = \frac{v^{1-\gamma}}{1-\gamma}, \text{ with } \gamma>0.$$

Then, Relation $E[U_\gamma(V_{T^{(1)}});\hat{V}_0] = E[U_\gamma(V_{T^{(2)}});V_0]$ is equivalent to:

$$\hat{V}_0^{1-\gamma} E\left[R_{T^{(1)}}^{1-\gamma}\right] = V_0^{1-\gamma} E\left[R_{T^{(2)}}^{1-\gamma}\right],$$

which yields to:

$$\frac{\hat{V}_0}{V_0} = \left(\frac{E\left[R_{T^{(2)}}^{1-\gamma}\right]}{E\left[R_{T^{(1)}}^{1-\gamma}\right]}\right)^{\left(\frac{1}{1-\gamma}\right)}.$$

The previous relation provides the expression of the compensating variation for the CRRA case.

6.2.4 The three compensating variations between T and* $\overline{\mathrm{T}}$.

Recall that T^* is necessarily better than $\overline{\mathrm{T}}$ since it is optimal among all deterministic dates. Therefore, we have to search $\hat{V}_0$ such that the following equality holds:

$$E[U_\gamma(V_{\overline{T}});\hat{V}_0] = E[U_\gamma(V_{T^*});V_0]$$

Recall that $h(t,x) = \left(\mu - k - 1/2\sigma^2\right)t + \sigma\sqrt{t}\,x$. Thus, we get:

$$E[U_\gamma(V_{\overline{T}});\hat{V}_0] = E[U_\gamma(V_{T^*});V_0]$$

$$\Leftrightarrow$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} U_\gamma(c(1-e^{-at}) + P_0\,\mathrm{e}^{h(\overline{T},x)};\hat{V}_0)\,e^{-x^2/2}\,\mathrm{d}x$$

$$=$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} U_\gamma(c(1-e^{-at}) + P_0\,\mathrm{e}^{h(T^*,x)};V_0)\,e^{-x^2/2}\,\mathrm{d}x$$

This latter equation can be numerically solved for the three types of utility functions.

6.2.5 The three compensating variations between respectively T and T** and* $\overline{\mathrm{T}}$ *and T**.*

Recall that *T*** is necessarily better than both *T** and $\overline{\mathrm{T}}$ since it is optimal among all possible dates. Therefore, we have to search $\widehat{V}_0$ such that the following equalities hold:

i) Compensation between T* and T**:

$$E[U_\gamma(V_{T^*});\widehat{V}_0] = E[U_\gamma(V_{T^{**}});V_0]$$

This is equivalent to:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} U_\gamma(c(1-e^{-at}) + P_0\,\mathrm{e}^{h(T^*,x)};\widehat{V}_0)\,e^{-x^2/2}\,\mathrm{d}x = \int_{P_0}^{+\infty} U_\gamma(v;V_0)\,f_{V_{T^{**}}}dv$$

ii) Compensation between $\overline{\mathrm{T}}$ and T**:

$$E[U_\gamma(V_{\overline{T}});\widehat{V}_0] = E[U_\gamma(V_{T^{**}});V_0]$$

This is equivalent to:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} U_\gamma(c(1-e^{-at}) + P_0\,\mathrm{e}^{h(\overline{T},x)};\widehat{V}_0)\,e^{-x^2/2}\,\mathrm{d}x = \int_{-\infty}^{+\infty} U_\gamma(v;V_0)\,f_{V_{T^{**}}}dv$$

6.3 Compensating variations: comparisons for the CARA case

In what follows, we illustrate the compensating variations in the CRRA case (the most usual case). We compare the utility of the portfolio optimal selling time corresponding to time T^* with all those corresponding to the other possible selling times, including of course the maturity $\overline{T}$ itself. We investigate the two numerical base cases as previously (see Figures 25 and 26, respectively for case 1 and case 2).

We analyze on the same graphic four compensating variation curves, for four different risk aversion levels ($\gamma = 0.5$, $\gamma = 2$, $\gamma = 5$ and $\gamma = 10$). As well, for each case, four volatility values are considered ($\sigma = 0.01$, $\sigma = 0.05$, $\sigma = 0.1$ and $\sigma = 0.2$, named respectively Fig a, Fig b, Fig c and Fig d).

For numerical case 1, by considering a low volatility level (Figure 25a), the impact of a wrong selling date is underlined. The optimal time to sell is around 9.16, whatever the risk aversion level (because of the low volatility value). The four curves (according to the risk aversion level) are nearly the same. Selling at time 0 could be seen as a cost of 2% (management cost for instance). Additionally, the cost increases with the gap between the selling date and the optimal time to sell. With a volatility of 5% (Figure 25b), the optimal times to sell are significantly different according to the risk aversion level (*T**=9.2 with $\gamma = 0.5$ while *T**=0 with $\gamma = 5$ or $\gamma = 10$). The "theoretical" management cost could reach 15% with a volatility of 10% (Figure 25c) and even 60% (Figure 25d) with a volatility of 20%. The latter can be seen as 3% per year “theoretical” management cost during 20 years.

For numerical case 2, “theoretical” management costs are higher for the small volatility levels (Figure 26).

7. Conclusion

This paper emphasizes the impact of the real estate market volatility on optimal holding period. For this purpose, the investor is assumed to be risk-averse, which is an usual assumption when dealing with portfolio optimization. Three kinds of optimal times are considered. The first one supposes that the investor can only choose the optimal time to sell at the initial date. The second one assumes that the investor is perfectly informed. This is not too realistic but provides an upward benchmark. Finally, we consider the American option framework, which is more “rational” since the investor is allowed to take account of intertemporal management and cumulative information.

For each of these models, the optimal times to sell and portfolio values are analyzed and compared, using various parameter values of the real estate markets, in particular the volatility index and the portfolio maturity. We show that risk aversion allows actually to better take account of the volatility, according to the ratio of cash flows upon real estate price. The higher the cash flows, the longer the optimal times.

However, the result with respect to the volatility is more mixed. According to specific assumptions on other parameters such as the risk aversion and the real asset value, the optimal time to sell can be increasing or decreasing w.r.t. the volatility. We illustrate also the comparison by means of (monetary) compensating variation.

Further research would introduce other functional representation of utility function such as an additive representation w.r.t. current time. This would allow separation between utility and free cash flows (for example, if they are used to consume) and utility on real asset value at the selling time. Other models such as recursive utility or utility with loss aversion and probability deformation may be also examined.

References

[1] Anderson, S., de Palma, A., Thisse, J.-F. (1992): *Discrete Choice Theory of Product Differentiation*, MIT Press, Cambridge, MA.

[2] Arrow, K.J., (1965): *Aspects of the Theory of Risk-Bearing*, Helsinki: Yrj Hahnsson Foundation.

[3] Barthélémy, F., and J.-L. Prigent, (2009): "Optimal Time to Sell in Real Estate Portfolio Management", *Journal of Real Estate Finance and Economics*, 38, 59-87.

[4] Baroni, M., Barthélémy, F., and M. Mokrane, (2007a): "Monte Carlo Simulations versus DCF in Real Estate Portfolio Valuation", *Property Management*, 25(5), 462-486.

[5] Baroni, M., Barthélémy, F., and M. Mokrane, (2007b): "Optimal Holding Period for a Real Estate Portfolio", *Journal of Property Investment and Finance*, 25(6), 603-625.

[6] Bond, S.A., Hwang, S., Lin, Z. and K.D. Vandell, (2007): Marketing Period Risk in a Portfolio Context: Theory and Empirical Estimates from the UK Commercial Real Estate Market, *Journal of Real Estate Finance and Economics*, Vol. 34, 447-461.

[7] Borodin, A. N., and P. Salminen, (2002): *Handbook of Brownian Motion (Facts and Formulae)*, Birkhäuser.

[8] Brown, R.J., (2004): "Risk and private Real Estate Investments", *Journal of Real Estate Portfolio Management*, Vol. 10, 113.

[9] Brown, R.J., and T. Geurts, (2005): "Private Investor Holding Period", *Journal of Real Estate Portfolio Management*, Vol. 11, 93-104.

[10] Collett, D, Lizieri, C., and C.W.R. Ward, (2003): "Timing and the Holding Periods of Institutional Real Estate", *Real Estate Economics*, Vol. 31, 205-222.

[11] de Palma, A. and J.-L. Prigent, (2008): "Utilitarianism and fairness in portfolio positioning", *Journal of Banking and Finance*, Vol. 32, 1648-1660.

[12] de Palma, A. J.-L. Prigent, (2009): "Standardized versus customized portfolio: a compensating variation approach", *Annals of Operations Research*, 165, 161-185.

[13] Dexter, A.S., Yu, J.N.W. and Ziemba, W.T., (1980): "Portfolio selection in a lognormal market when the investor has a power utility: computational results, in Stochastic Programming", M.A.H. Dempster (Ed.), Academic Press, New-York, 507-523.

[14] Elliott, R.J., and Kopp, P.E., (1999): *Mathematics of Financial Markets*, Springer Finance series, Springer.

[15] Elton, E., Gruber, M. (1995): *Modern Portfolio Theory and Investment Analysis*, J. Wiley, New York.

[16] Fisher, J., and M. Young, (2000): "Institutional Property Tenure: evidence from the NCREIF database", *Journal of Real Estate Portfolio Management*, Vol. 6, 327-338.

[17] Friedman, M., Savage, L. (1948): "Utility analysis of choices involving risk", *Journal of Political Economics*, 56, 279-304.

[18] Gau, G., and K. Wang, (1994): "The Tax-induced Holding Periods and Real Estate Investors: Theory and Empirical Evidence", *Journal of Real Estate Finance and Economics*, Vol. 8, 71-86.

[19] Gollier : Gollier, C, (2001): The *Economics of Risk and Time*. The MIT Press, Cambridge, MA.

[20] Geltner, D. M., and N.G. Miller, (2007): *Commercial Real Estate Analysis and Investments*, Cengage/South-Western, Cincinnati.

[210] Hendershott, P., and D. Ling, (1984): "Prospective Changes in Tax Law and the Value of Depreciable Real Estate", *AREUEA Journal*, Vol. 12, 297-317.

[22] Kallberg, J.G. and W.T. Ziemba, (1983): "Comparison of alternative utility functions in portfolio selection problems", *Management Science*, 29 (11), 1257-1276.

[23] Karatzas, I., and S. Shreve, (2001): *Methods of Mathematical Finance*, Springer.

[24] McFadden, D. (2001): "Economic choices", *American Economic Review*, 91(3), 351-378.

[25] Oksendal, B., (2007): *Stochastic Differential Equations: An introduction with applications*, 6th ed. 2003. (Corr. 4th) , Springer-Verlag.

[26] Pratt, J.W., (1964): "Risk aversion in the small and in the large", *Econometrica*, 32, 122-136.

[27] Prigent, J-L., (2007): *Portfolio Optimization and Performance Analysis*, Chapman & Hall, Florida.

[28] Rothschild, M., Stiglitz, J. E. (1970): "Increasing risk: I. A definition", *Journal of Economic Theory*, 2, 225-243.

[29] Rothschild, M., Stiglitz, J. E. (1971): "Increasing risk: II. Its economic consequences", *Journal of Economic Theory,* 3, 66-84.

[30] Rowley, A., V. Gibson, and C. Ward, (1996): "Quality of Urban Design: a Study of the Involvement of Private Property Decision-Makers in Urban Design", Royal Institution of Chartered Institution of Chartered Surveyors, London.

[31] von Neumann, J., Morgenstern, O. (1944): *Theory of Games and Economic Behavior*, Princeton University Press.

[1] Using the database of properties provided by IPD in the UK over an 18-year period, their empirical analysis shows that the median holding period is about seven years.

[2] These optimization problems are specific to real estate investments and differ from standard financial portfolio management problems (see Karatzas and Shreve, 2001, or Prigent, 2007). First, the asset is not liquid (not divisible). Second, the control variable is the time to sell and not the usual financial portfolio weights (see Oksendal 2007, for a related problem about optimal time to invest in a project with an infinite horizon).

[3] See Elton and Gruber (1995) for a discussion about the estimation of risk aversion parameters.

[4] The independence axiom has been implicitly introduced by von Neumann and Morgenstern (1944). This criterion allows the characterization of the expected utility. This axiom implies the utility function U defined over the lotteries is linear with respect to the probabilities that the different events occur.

[5] The concavity of the utility function yields the following inequality, known as Jensen's inequality:

$$\forall \lambda \in [0,1], \forall a \in \mathbb{R}, \forall b \in \mathbb{R}, \quad \lambda u(a) + (1-\lambda) u(b) \leq u(\lambda a + (1-\lambda) b).$$

[6] Consider the variance $\sigma_X^2 = E[(X - E[X])^2]$. By using a Taylor's expansion, we deduce: $E[u(X)] \simeq u(E[X]) + (1/2) u''(E[(X)]) E[(X - E[X])^2)$

We also have: $u(E[X] - \pi[X]) \simeq u(E[X]) - u'(E[(X)]) \pi[(X)]$,

By definition, $u(E[X] - \pi[X]) \simeq E[u(X)])$

Finally, we have:

$\pi[X] \simeq -(u''(E[X]) / u'(E[X])) \sigma^2$ and $\pi[X] \simeq -((u''(E[X]))/(u'(E[X])))\sigma_X^2$.

[7] See Gollier (2001) for main definitions and properties of utility functions.

[8] This assumption allows explicit solutions for the probability distributions of the optimal times to sell and of the optimal portfolio values. The introduction of stochastic rates would lead to only simulated solutions.

[9] This is the continuous-time version of the solution of Baroni *et al.* (2007b).

[10] We can also examine how the solution depends on the index value P_0. For example, proportional transaction costs imply a reduction of P_0. For instance, for the case 2, a tax of 5% leads to an optimal time to sell T^* equal to 17.39 years, instead of 16.11 years when there is no transaction cost. With a 10% tax, the solution becomes 18.74 years. This is in line with the empirical results showing that high transaction costs imply longer holding periods (see for example Collet *et al.*, 2003).

[11] See Elliott and Kopp (1999, p. 193).

[12] This equation can be analyzed through Laplace transforms of Lognormal distributions.