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**Abstract**

Building upon the classical concept of Hölder continuity and the notion of "continuous implementation" introduced in Oury and Tercieux (2009), we define Hölder continuous implementation. We show that, under a richness assumption on the payoff profiles (associated with outcomes), the following full characterization result holds for finite mechanisms: a social choice function is Hölder continuously implementable if and only if it is fully implementable in rationalizable messages.

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# 1 Introduction

There are many ways of implementing a social choice function. Partial Nash implementation, which is widely used both in theoretical and applied works, is often seen as a quite weak requirement[1]. At the other extreme stands full implementation in rationalizable messages which is known to have a lot of bite. To put it informally, we assume, in the present paper, that the social planner has some “small doubts”about his model and we show that full implementation in rationalizable messages is necessary (and sufficient) for partial Nash implementation in this context.

More precisely, we follow the approach introduced in Oury and Tercieux (2009) (hereafter OT) and require that in any model that embeds the initial model, there exists an equilibrium that yields “the desired outcome”, not only at all types of the initial model but also at all types “close”to initial types. To formalize closeness, we use the method introduced by Harsanyi (1967) and developed in Mertens and Zamir (1985). Each type in the initial model is mapped into a hierarchy of beliefs. Then, following the interim approach due to Weinstein and Yildiz (2007)(hereafter WY), we define a notion of “nearby”type. This notion, formally described by the product topology in the universal type space, captures the restrictions on the modeler's ability to observe the players' (high order) beliefs.

If our approach is similar to that of OT, two important differences between their setup and ours must be pointed out. First, as in WY, we assume that for each state of nature, the payoff profiles (associated with each outcome) may slightly differ from those corresponding to the benchmark model the social planner has in mind. (This richness assumption will be explained in more details in the next paragraph.) Second, for a fixed model (that embeds the initial one), our definition of a “satisfactory”equilibrium is different from that defined in OT. On the one hand, we do not require partial implementation in strict (or pure) Nash equilibrium on the initial model. On the other hand, our continuity condition is (slightly) stronger than that of OT since, in the present paper, “Hölder continuity”is required. We do believe that this latter restriction is very weak: this technical point is more precisely presented and discussed in Subsection 2.3.

This paper establishes the following full characterization result for finite mechanisms: a social choice function $f$ is (partially) Hölder continuously implementable

[1] For instance, in complete information settings and with more than three players, any social choice function can be partially implemented.

if and only if it is fully implementable in rationalizable messages. If this result is reminiscent of that of WY, our technical contribution is very different from the contagion argument which is used in their proof. This is due to the specificity of mechanism design. Let us be more precise. In the model of WY, a set of actions is fixed (and common knowledge). Hence, a parameter (that is, a "state of nature") may be identified with a function that maps *action profiles* to payoff profiles. By contrast, in the present implementation setting, only the set of *outcomes* is initially fixed and a parameter may be identified with a function that maps *outcomes* to payoff profiles[2]. The richness assumption of WY, which specifies that each action of each player can be strictly dominant for some parameter value, makes it immediate to break the ties for best reply in favor of a desired action. Indeed, it suffices to allow the type to put slightly higher probability on the payoff function at which this action is strictly dominant. To put it another way, to "obtain strictness"-which is necessary for contagion-, one can perturb the first-order belief of a type slightly. Such a possibility does not exist in our context.

# 2 Setup

We consider a finite set $\mathcal{I} = \{1, ..., I\}$ of players. Each agent $i$ has a utility function $u_i : A \times \Theta^{\star\star} \to \mathbb{R}$ where $\Theta^{\star\star}$ is the set of states of nature and $A$ is the finite set of outcomes. A model $\mathcal{T}$ is a pair $(T, \kappa)$ where $T = T_1 \times ... \times T_I$ is a type space and $\kappa(t_i) \in \Delta(\Theta^{\star\star} \times T_{-i})$ denotes the associated beliefs for each $t_i \in T_i$. The social planner has an initial finite model in mind which we denote by $\bar{\mathcal{T}}$ and desires to implement the social choice function $f : \bar{T} \to A$.

## 2.1 Richness assumption

We assume that for each $\bar{t}_i \in \bar{T}_i$, the support of the distribution $\mathrm{marg}_{\Theta^{\star\star}}\kappa(\bar{t}_i)$ is finite and we set:

$$\Theta := \bigcup_{i=1}^{I} \bigcup_{\bar{t}_i \in \bar{T}_i} \mathrm{supp}(\mathrm{marg}_{\Theta^{\star\star}}\kappa(\bar{t}_i)).$$

Let us also assume that for each $\theta \in \Theta$, it is possible to slightly "perturb" the payoffs associated with $\theta$. More precisely:

[2]In this paper, we maintain the assumption that sending a message is costless. See OT for implications of continuity when this assumption is relaxed.

**Assumption 1** *For each outcome $a \in A$, each player $i \in \mathcal{I}$ and each state of nature $\theta \in \Theta$, there exists $\theta^{\star\star}(\theta, a, i) \in \Theta^{\star\star}$ such that:*

$$u_i(a, \theta^{\star\star}(\theta, a, i)) > u_i(a, \theta),$$

*and,*

$$u_i(a', \theta^{\star\star}(\theta, a, i)) = u_i(a', \theta),$$

*for all $a' \neq a$.*

To illustrate this assumption, consider the case where $\Theta^{\star\star}$ is simply an index for the profile of payoff functions. For instance, take $\Theta^{\star\star} = \Theta_1^{\star\star} \times \ldots \times \Theta_I^{\star\star}$ with $\Theta_i^{\star\star} \subseteq [0,1]^{|A|}$, for each $i$ and $u_i(\theta^{\star\star}, a) = \theta_i^{\star\star}(a)$, for each $(i, a, \theta^{\star\star})$. Assumption 1 may then be restated as follows: for each $\theta \in \Theta$, there exists an open set $V(\theta)$ in $[0,1]^{|A|I}$ such that $V(\theta) \subseteq \Theta^{\star\star}$ and $\theta \in V(\theta)$.

In the rest of the paper, we will only consider states of natures belonging to the finite set $\Theta^\star$ defined by:

$$\Theta^\star := \Theta \cup \{\bigcup_{\theta \in \Theta} \bigcup_{a \in A} \bigcup_{i \in \mathcal{I}} \theta^{\star\star}(\theta, a, i)\},$$

that is, we restrict ourselves to $\Theta$ and all the relevant perturbations around it. In addition, we write for each $\theta \in \Theta$, $\theta = (\theta, \tilde{\theta}^0)$ and $\theta^{\star\star}(\theta, a, i) = (\theta, \tilde{\theta}_i^a)$, for each $a \in A$ and $i \in \mathcal{I}$. Finally, we denote $A^\star = \{0\} \cup A$ and set: $\Theta^\star = \Theta \times \tilde{\Theta}$ where $\tilde{\Theta} = \bigcup_{i \in \mathcal{I}} \bigcup_{a \in A^\star} \tilde{\theta}_i^a$.

## 2.2 Partial implementation

A mechanism specifies a message set for each agent and a mapping from message profiles to outcomes. More precisely, we write $M$ as an abbreviation for $\prod_{i \in \mathcal{I}} M_i$ (where $M_i$ is the message set of player $i$) and for each $i$, $M_{-i}$ for $\prod_{j \neq i} M_j$. (Similar abbreviations will be used throughout the paper for analogous objects.) A mechanism $\mathcal{M}$ is a pair $(M, g)$ where $M$ is finite and the outcome function $g : M \to A$ assigns to each message profile $m$ an alternative $g(m) \in A$. By a slight abuse of notations, given a space $X$, we will sometimes note $x$ for the degenerate distribution in $\Delta(X)$ assigning probability 1 to $\{x\}$; $g$ will also be extended to lotteries, i.e., given $\alpha \in \Pi_{i \in \mathcal{I}} \Delta(M_i)$, $g(\alpha)$ denotes the lottery $\sum_{m \in M} \alpha(m) g(m)$.

For each mechanism $\mathcal{M}$ and model $\mathcal{T}$, we write $U(\mathcal{M}, \mathcal{T})$ for the induced incomplete information game. In this game, a (behavioral) strategy of a player $i$ is any measurable function $\sigma_i : T_i \to \Delta(M_i)$. Given any type $t_i$ and any strategy profile $\sigma$, we write $\pi_i(\cdot \mid t_i, \sigma) \in \Delta(\Theta^\star \times M_{-i})$ for the joint distribution on the underlying uncertainty and the other players' messages induced by $t_i$ and $\sigma$. We define the best response correspondence by:

$$BR(\sigma|t_i) = \text{argmax}_{m_i \in M_i} \sum_{\theta^\star,\, m_{-i}} \pi_i(\theta^\star, m_{-i}|t_i, \sigma) u_i(g(m_i, m_{-i}), \theta^\star),$$

for each player $i$, type $t_i$ and strategy profile $\sigma$.

**Definition 1** *A profile of strategies $\sigma = (\sigma_1, ..., \sigma_I)$ is a Bayes Nash equilibrium in $U(\mathcal{M}, \mathcal{T})$ iff for each $i \in \mathcal{I}$ and $t_i \in T_i$, the support of $\sigma_i(t_i)$ is included in $BR(\sigma|t_i)$.*

**Definition 2** *The social choice function f is partially implementable iff there exists a finite mechanism $\mathcal{M} = (M, g)$ and an equilibrium $\sigma$ in $U(\mathcal{M}, \bar{T})$ such that for each $\bar{t} \in \bar{T}$: $g(\sigma(\bar{t})) = f(\bar{t})$.*

## 2.3 Hölder continuous implementation

For each metric space $X$, $x \in X$ and $\delta > 0$, we write respectively $B_\delta(x)$ and $\bar{B}_\delta(x)$ for the open and the closed balls of radius $\delta$ about $x$. For each integer $n$, $\mathbb{R}^n$ will always be endowed with the max norm, i.e.:

$$||x|| = \max\,(x^1, \ldots, x^n),$$

for each $x \in \mathbb{R}^n$. In addition, we define for each $n$: $\Delta^{n-1} := \{x \in \mathbb{R}^n_+ \,\big|\, \sum_{\ell=1}^n x^\ell = 1\}$. A (mixed) outcome $a \in \Delta(A)$ is viewed as a point in $\Delta^{|A|-1}$. (Hence, in our setting, the maximal distance between two outcomes $a, a' \in \Delta(A)$ is equal to one.)

We also define a topology on types. We first recall the notion of hierarchy of beliefs. Given a model $(T, \kappa)$ and a type $t_i$ in type space $T_i$, we can compute the belief of $t_i$ on $\Theta^\star$ by setting:

$$h_i^1(t_i) = \text{marg}_{\Theta^\star} \kappa(t_i),$$

which is called the "first-order belief" of $t_i$. We can compute the second-order belief of type $t_i$, i.e. his belief about $(\theta^\star, h_1^1(t_1), ..., h_I^1(t_I))$, by setting

$$h_i^2(t_i)(F) = \kappa(t_i)(\big\{(\theta^\star, t_{-i}) \mid (\theta, h_1^1(t_1), ..., h_I^1(t_I)) \in F\big\}),$$

for each measurable set $F \subseteq \Theta^\star \times \Delta(\Theta^\star)^I$. Proceeding iteratively in this way, we can compute an entire hierarchy of beliefs $h_i(t_i) = (h_i^1(t_i), h_i^2(t_i), ..., h_i^k(t_i), ...)$. The set of all belief hierarchies for which it is common knowledge that the beliefs are coherent (i.e., each player knows his beliefs and his beliefs at different orders are consistent with each other) is the universal type space (see Mertens and Zamir (1985) and Brandenburger and Dekel (1993)). We denote by $\mathcal{T}_i^\star$ the set of player $i'$s hierarchies of belief in this space and write $\mathcal{T}^\star = \prod\limits_{i \in \mathcal{I}} \mathcal{T}_i^\star$. Each $\mathcal{T}_i^\star$ is endowed with the product topology: a sequence of types $\{t_i^n\}_{n=0}^\infty$ converges to a type $t_i$, if, for each $k$, $h_i^k(t_i^n)$ converges toward $h_i^k(t_i)$ in the topology of weak convergence of measures. The product topology is metrized by the following distance $d(h_i(t_i[n]), h_i(t_i)) := \sum\limits_{k=1}^{\infty} \frac{1}{2^k} d^k(h_i^k(t_i[n]), h^k(t_i))$ where the metric $d^k(.,.)$ on the $k$th level beliefs (i.e. on $\Delta(X_{k-1})$) is one that metrizes the topology of weak convergence of measures[3]. For any type profile $t \in T$ and $\delta \in (0, 1]$, we write: $\bar{B}_\delta(t) = \prod\limits_{i \in I} \bar{B}_\delta(t_i)$.

We are now in position to state a formal definition of Hölder continuous implementation.

**Definition 3** *Fix a countable model $\mathcal{T}$ and a mechanism $\mathcal{M}$. We say that an equilibrium $\sigma$ in $U(\mathcal{M}, \mathcal{T})$ is Hölder continuous if there exists $\alpha > 0$ such that for each $\delta \in (0, 1)$, $\bar{t} \in \bar{T}$ and $t \in \bar{B}_\delta(\bar{t})$,*

$$||g(\sigma(t)) - f(\bar{t})|| \leq \frac{\delta^\alpha}{\alpha}. \tag{1}$$

**Definition 4** *The social choice function $f$ is Hölder continuously implementable if there exists a finite mechanism $\mathcal{M}$ such that for each countable model $\mathcal{T}$, there exists an Hölder continuous equilibrium $\sigma$ in $U(\mathcal{M}, \mathcal{T})$.*

Notice that for each model $\mathcal{T} \supseteq \bar{\mathcal{T}}$, if an equilibrium $\sigma$ in $U(\mathcal{M}, \mathcal{T})$ is Hölder continuous, then we must have:

$$g(\sigma(\bar{t})) = f(\bar{t}),$$

for each $\bar{t} \in \bar{T}$. In other words, Hölder continuous implementation implies partial implementation.

Of course, assuming Hölder continuous implementation is stronger than merely assuming that for each model $\mathcal{T}$, there exists an equilibrium $\sigma$ in $U(\mathcal{M}, \mathcal{T})$ such that

[3] For example, the Prokhorov metric.

for each type profile $\bar{t} \in \bar{T}$ and each sequence of type profiles $\{t^n\}_{n\in\mathbb{N}}$ with $t^n \to \bar{t}$, we have: $\lim_{n\to\infty} g(\sigma(t^n)) \to f(\bar{t})$. However, since the parameter $\alpha$ can be chosen arbitrarily small, we believe that the technical restriction induced by our definition is quite weak. In addition, if for some specific reason (due to the context), Hölder continuity seems too strong, this condition can be replaced by any other (ordered) family of moduli of continuity. For instance, our characterization results holds (and the proof is exactly the same) if we replace Equation (1) by:

$$||g(\sigma(t)) - f(\bar{t})|| \leq \frac{1}{\alpha}(\frac{-1}{\ln(\delta)})^{\alpha},$$

which corresponds to a weaker requirement than Hölder continuity.

# 3 Characterization Result

Let us first recall the definition of interim correlated rationalizability given in Dekel, Fudenberg and Morris (2006, 2007). Pick any profile of types $t$ drawn from some arbitrary model $\mathcal{T} = (T, \kappa)$. For each $i$ and $t_i$, set $R_i^0(t_i \mid \mathcal{M}, \mathcal{T}) = M_i$, and define the sequence $\{R_i^k(t_i \mid \mathcal{M}, \mathcal{T})\}_{k\in\mathbb{N}}$ iteratively as follows. For each integer $k$, message $m_i \in M_i$ belongs to $R_i^k(t_i \mid \mathcal{M}, \mathcal{T})$ if and only if there exists some belief $\pi_i \in \Delta(\Theta^\star \times T_{-i} \times M_{-i})$ such that $m_i$ is a best response to $\pi_i$ and:

1. $\text{marg}_{\Theta^\star \times T_{-i}} \pi_i = \kappa(t_i)$
2. $\pi_i(\theta^\star, t_{-i}, m_{-i}) > 0 \Longrightarrow m_{-i} \in R_{-i}^{k-1}(t_{-i} \mid \mathcal{M}, \mathcal{T})$,

where $R_{-i}^{k-1}(t_{-i} \mid \mathcal{M}, \mathcal{T})$ stands for $\prod_{j\neq i} R_j^{k-1}(t_j \mid \mathcal{M}, \mathcal{T})$. The set of all rationalizable messages for player $i$ (of type $t_i$) is

$$R_i^\infty(t_i \mid \mathcal{M}, \mathcal{T}) = \bigcap_{k=0}^{\infty} R_i^k(t_i \mid \mathcal{M}, \mathcal{T}).$$

Finally, for each type profile $t \in T$, we set:

$$R^\infty(t \mid \mathcal{M}, \mathcal{T}) = \prod_{i=1}^{I} R_i^\infty(t_i \mid \mathcal{M}, \mathcal{T}).$$

**Definition 5** *A social choice function is fully implementable in rationalizable messages if there is a finite mechanism $\mathcal{M}$ such that for each $\bar{t} \in \bar{T}$ and $m \in R^\infty(\bar{t} \mid \mathcal{M}, \bar{\mathcal{T}})$, we have: $g(m) = f(\bar{t})$.*

We now give our characterization result.

**Theorem 1** *The social choice function $f$ is Hölder continuously implementable if and only if it is fully implementable in rationalizable messages by a finite mechanism.*

**Proof.** The proof of the "if part" of Theorem 1 is quite short and as follows. Assume that $f : \bar{T} \to A$ is fully implementable in rationalizable messages by a finite mechanism $\mathcal{M} = (M, g)$. Lemma 1 below is a direct consequence of Dekel, Fudenberg, Morris (2006).

**Lemma 1** *For each integer $n$, there exists some $\bar{\delta}(n) > 0$ satisfying the following property for each model $\mathcal{T}$. For all $\bar{t} \in \bar{T}$ and $t \in T$ with $t \in \bar{B}_{\bar{\delta}(n)}(\bar{t})$:*

$$R^n(t \mid \mathcal{M}, \mathcal{T}) \subseteq R^n(\bar{t} \mid \mathcal{M}, \bar{\mathcal{T}}).$$

On the other hand, since $M$ and $\bar{T}$ are finite, there exists an integer $\bar{n}$ such that for each $\bar{t} \in \bar{T}$:

$$R^\infty(\bar{t} \mid \mathcal{M}, \bar{\mathcal{T}}) = R^{\bar{n}}(\bar{t} \mid \mathcal{M}, \bar{\mathcal{T}}).$$

Notice that there exists $\alpha^\star > 0$ such that: $\frac{1}{\alpha^\star}(\bar{\delta}(\bar{n}))^{\alpha^\star} = 1$. Now, pick some model $\mathcal{T} = (T, \kappa)$ and some Bayes Nash equilibrium[4] $\sigma$ in $U(\mathcal{M}, \mathcal{T})$. We show that $\sigma$ is $\alpha^\star$-continuous, that is, for each $\bar{t} \in \bar{T}$, $\delta \in (0, 1)$ and $t \in \bar{B}_\delta(\bar{t})$:

$$||g(\sigma(t)) - f(\bar{t})|| \leq \frac{\delta^{\alpha^\star}}{\alpha^\star}. \tag{2}$$

By construction of $\alpha^\star$, for each $\delta \geq \bar{\delta}(\bar{n})$, Equation 2 is trivially satisfied. In addition, for each $\delta < \bar{\delta}(\bar{n})$, $t \in \bar{B}_\delta(\bar{t})$ and $m(t) \in \text{supp}(\sigma(t))$, we have: $m(t) \in R^\infty(t \mid \mathcal{M}, \mathcal{T})$. Consequently, $m(t) \in R^{\bar{n}}(t \mid \mathcal{M}, \mathcal{T})$. Since $t \in \bar{B}_{\bar{\delta}(\bar{n})}(\bar{t})$, we obtain applying Lemma 1 above: $m(t) \in R^{\bar{n}}(\bar{t} \mid \mathcal{M}, \bar{\mathcal{T}}) = R^\infty(\bar{t} \mid \mathcal{M}, \bar{\mathcal{T}})$. Hence, the fact that $\mathcal{M}$ fully implements $f$ in rationalizable messages implies that: $g(m(t)) = f(\bar{t})$, which concludes the proof.

We now move to the "only if" part of Theorem 1. We need Theorem 2 below whose proof is presented in Section 3 and which concerns continuous implementa-

[4]The existence of a Bayes Nash equilibrium can be proved using Kakutani-Fan-Glicksberg's fixed point theorem. The space of strategy profiles is compact in the product topology. Using the fact that $u_i : A \times \Theta \to \mathbb{R}$ is bounded (since $A$ and $\Theta$ are finite), all the desired properties of the best response correspondence (in particular upper hemicontinuity) can be proved.

tion for finite type spaces. (Indeed, observe that when type spaces are finite, assuming Hölder continuous implementation is equivalent to merely requiring partial implementation[5].) We first introduce two additional definitions.

**Definition 6** *Fix a decreasing function $e : \mathbb{N} \to \mathbb{R}_+$ with $\lim_{N\to\infty} e(N) = 0$. For each finite mechanism $\mathcal{M}$ and finite type space $\mathcal{T}$, we say that an equilibrium $\sigma$ in $U(\mathcal{M}, \mathcal{T})$ is e-continuous if for each $\bar{t} \in \bar{T}$ and each $t \in \bar{B}_{\frac{1}{N}}(\bar{t})$, we have: $g(\sigma(t)) \in \bar{B}_{e(N)}(f(\bar{t}))$.*

**Definition 7** *Fix a decreasing function $e : \mathbb{N} \to \mathbb{R}_+$ with $\lim_{N\to\infty} e(N) = 0$. The s.c.f $f$ is e-continuously implementable iff there exists a finite mechanism $\mathcal{M}$ such that for all finite type spaces $\mathcal{T}$, there is an e-continuous equilibrium $\sigma$ in $U(\mathcal{M}, \mathcal{T})$.*

**Theorem 2** *There exists a function $e : \mathbb{N} \to \mathbb{R}^+$ with $\lim_{N\to\infty} e(N) = 0$ such that social choice function $f$ is e-continuously implementable only if it is fully implementable in rationalizable messages.*

**Proof.** See Section 3. ∎

Now, assume that $f$ is Hölder continuously implementable on countable type spaces by some mechanism $\mathcal{M}$. Then, there must exist an integer $\alpha^\star$ such that for each finite model $\mathcal{T}$, there exists an $\alpha^\star$-Hölder continuous equilibrium $\sigma$ in $U(\mathcal{M}, \mathcal{T})$.(Indeed, assume that it is not the case. This means that for each integer $q > 0$ there exists a finite model $\mathcal{T}(q)$ such that there is no $\frac{1}{q}$- Hölder continuous equilibrium $\sigma$ in $U(\mathcal{M}, \mathcal{T}(q))$. Now set: $T = \bigcup_{q\in\mathbb{N}} T(q)$. Notice that for each equilibrium $\sigma$ in $U(\mathcal{M}, \mathcal{T})$ and each $q$, the restriction $\sigma_{|T(q)}$ of the equilibrium $\sigma$ to the type space $T(q)$ is also an equilibrium. Hence, there cannot exist an Hölder-continuous equilibrium $\sigma$ in $U(\mathcal{M}, \mathcal{T})$, a contradiction.) Consequently, by Theorem 2 (setting $e(N) = \frac{1}{\alpha^\star}(\frac{1}{N})^{\alpha^\star}$), we know that $f$ is fully implementable in rationalizable messages. ∎

[5] More precisely, for each finite model $\mathcal{T}$, there exists $\delta^\star \in (0, 1)$ such that for each $\bar{t} \in \bar{T}$:

$$B_\delta(\bar{t}) \cap T = \emptyset.$$

It suffices then to choose $\alpha^\star$ such that:

$$\frac{(\delta^\star)^{\alpha^\star}}{\alpha^\star} \geq 1,$$

and the condition of $\alpha^\star$-Hölder continuity is trivially satisfied for the finite model $\mathcal{T}$.

# 4 Proof of Theorem 2

Assume that there exists a finite mechanism $\bar{\mathcal{M}} = (\bar{M}, g)$ and a decreasing function $e : \mathbb{N} \to [0,1]$ with $\lim_{N\to\infty} e(N) = 0$ such that for each finite type space $\mathcal{T}$, there exists an $e$-continuous equilibrium $\sigma$ in $U(\bar{\mathcal{M}}, \mathcal{T})$.

We restrict our attention to mechanisms $\mathcal{M} = (M, g)$ where $g$ is the outcome function of mechanism $\bar{\mathcal{M}}$ and $M = \Pi_{i\in\mathcal{I}} M_i$ with $M_i \subseteq \Delta(\bar{M}_i)$, for each $i$. The core of the proof uses a geometric argument. For each $i$, each message $m_i \in \Delta(\bar{M}_i)$ is identified with a point in the coordinate system of $\mathbb{R}^{|\bar{M}_i|}$ associated with $\bar{M}_i = \{\bar{m}_i^1, \ldots, \bar{m}_i^{|\bar{M}_i|}\}$. That is, for each $k = 1, \ldots, |\bar{M}_i|$, the message $\bar{m}_i^k$ is identified with the point of $\mathbb{R}^{|\bar{M}_i|}$ whose all components are equal to 0 except for the $k$-th one, which is equal to 1. Similarly, we put an arbitrary order on $\bar{M}_{-i}$ and write $\bar{M}_{-i} = \{\bar{m}_{-i}^1, \ldots, \bar{m}_{-i}^{|\bar{M}_{-i}|}\}$. Each $m_{-i} \in \Delta(\bar{M}_{-i})$ is identified with a point in the coordinate system of $\mathbb{R}^{|\bar{M}_{-i}|}$ associated with $\bar{M}_{-i}$.

For any subset $S$, we write $|S|$ for its cardinal. We also write Aff$(S)$ for the affine hull of $S = \{s_1, ..., s_m\} \subseteq \mathbb{R}^n$, i.e:

$$\mathrm{Aff}(S) := \left\{ x \in \mathbb{R}^n \,\middle|\, \exists\, \alpha \in \mathbb{R}^m \text{ s.t. } x = \sum_{i=1}^{m} \alpha_i s_i \text{ and } \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

We denote the convex hull of $S$ by Co$(S)$, i.e:

$$\mathrm{Co}(S) := \left\{ x \in \mathbb{R}^n \,\middle|\, \exists\, \alpha \in \mathbb{R}_+^m \text{ s.t. } x = \sum_{i=1}^{m} \alpha_i s_i \text{ and } \sum_{i=1}^{m} \alpha_i = 1 \right\}.$$

In a standard way, we write dim$(S)$ for the dimension of Aff$(S)$. Finally, we introduce the following definition.

**Definition 8** *The message space $M = \Pi_{i=1}^{I} M_i \subset \Pi_{i\in\mathcal{I}} \Delta(\bar{M}_i)$ has no redundant messages if for each player $i$ and each $m_i$, $m_i' \in M_i$, there exists $m_{-i}(m_i, m_i') \in M_{-i}$ such that:*

$$g(m_i, m_{-i}(m_i, m_i')) \neq g(m_i', m_{-i}(m_i, m_i')).$$

## 4.1 Intuition of the proof

Let us give a brief sketch of the proof. For simplicity, we assume here that there are only two players and that the initial model $\bar{\mathcal{T}}$ is a complete information one. The arguments are quite similar in the general setting.

Intuitively, the starting point of the proof is as follows. Fix a message space, $M = \Pi_{i\in\mathcal{I}} M_i \subset \Pi_{i\in\mathcal{I}}\Delta(\bar{M}_i)$, with no redundant message, a player $i$, a state of nature $\theta \in \Theta$ and two messages $m_i, m_i' \in M_i$. Write[6] $S(m_i, m_i', \theta)$ for the set of messages $m_j \in \Delta(M_j)$ such that, at state $\theta$, the (expected) payoff of player $i$ associated with lottery $g(m_i, m_j)$ is equal to the one associated with lottery $g(m_i', m_j)$. By the multilinearity of the (expected) payoff function, $S(m_i, m_i', \theta)$ is the intersection of $\Delta(M_j)$ with an affine subspace. Of course, if $m_i$ and $m_i'$ are not payoff equivalent, the dimension of $S(m_i, m_i', \theta)$ is *strictly* smaller than that of $M_j$. Now, assume that $m_i$ and $m_i'$ are payoff equivalent. Since $M$ has no redundant message, there must exist $m_j(m_i, m_i') \in M_j$ such that $g(m_i, m_j(m_i, m_i')) \neq g(m_i', m_j(m_i, m_i'))$. Therefore, using our local richness assumption, it is also possible to slightly perturb the payoffs of the outcomes at state $\theta$ in such a way that, when $m_j(m_i, m_i')$ is played, the (expected) payoffs of player $i$ associated with $m_i$ and $m_i'$ are no longer equal. Since the message spaces are finite, it is possible to perturb the payoffs in such a way that there is no pair of payoff equivalent messages. Hence, with these perturbed payoffs, for each $i$ and $m_i, m_i' \in M_i$, the dimension of $S(m_i, m_i', \theta)$ is *strictly* smaller than that of $M_j$.

This means that, if for some type space $\mathcal{T}$ and some equilibrium $\sigma$, the dimension of $\sigma_j(T_j)$ is equal to that of $M_j$, then, for each $\theta \in \Theta$, one can build a type $t_i(\theta)$ and an associated belief $\kappa(t_i(\theta)) \in \Delta(\Theta^\star \times T_j)$ such that :

- (i) $\text{marg}_{\Theta^\star}\kappa(t_i(\theta))[\theta, \tilde{\theta}^0]$ is close to one;
- (ii) The best-response of type $t_i(\theta)$ against $\sigma_j$ is a singleton.

Similarly, we show that if, in addition, there exist $t_j \in T_j$ and $m_j \in M_j$ with $\sigma_j(t_j) = m_j$, then, for each $\theta \in \Theta$, one can build a type $t_i(\theta, m_j)$ and a belief $\kappa(t_i(\theta, m_j)) \in \Delta(\Theta^\star \times T_j)$ such that (i) and (ii) above are satisfied and:

- (iii) Type $t_i(\theta, m_j)$ puts a probability close to one on message $m_j$ when $\sigma_j$ is played;

With this intuition in mind, we proceed as follows. The proof has four steps. In the first step (Proposition 1), we build a "sufficiently small"message space, $M^\star = \Pi_{i\in\mathcal{I}} M_i^\star \subseteq \Pi_{i\in\mathcal{I}}\Delta(\bar{M}_i)$, and a "sufficiently large"model $\mathcal{T}^0$ such that for each $e$-continuous equilibrium $\sigma$ in $U(\mathcal{M}^\star, \mathcal{T}^0)$ and each player $i$: $\dim(\sigma_i(T_i^0)) = \dim(M_i^\star)$.

[6] Since our formal setting is more general, the notations used in the core of the proof are different.

Proposition 2 then establishes in a formal way the intuition we explained above to generate "strictness".

In the third step of the proof (Proposition 3), we build a model $\tilde{\mathcal{T}} \supset \mathcal{T}^0$ which will serve as a starting point for the contagion argument of Step 4. More precisely, fix an equilibrium[7] $\sigma$ in $U(\mathcal{M}^\star, \mathcal{T}^0)$. Applying our argument on strictness (Propositions 1 and 2), we build for each player $i$, a type $t_i^1$ such that the best response of $t_i^1$ when $\sigma$ is played in $U(\mathcal{M}^\star, \mathcal{T}^0)$ is a singleton, denoted $m_i^1$. Notice that, for each $\theta \in \Theta$, the best reply of player $j$ to $m_i^1$ at state $\theta$ is not necessarily unique. However, applying Propositions 1 and 2 again and the upper semi-continuity of the best-response correspondence, we show that it is possible to build a type $t_j^2(\theta)$ which puts a very high probability on type $t_i^1$ and on state $\theta$ and whose best response against $\sigma$ is a singleton, denoted $m_j^2(\theta)$, which is also a best-reply to $m_i^1$ at $\theta$. Set $M_j^2(\sigma) := m_j^1 \cup \{\bigcup_{\theta \in \Theta} m_j^2(\theta)\}$. By the same argument as above, for each $\mu_i \in \Delta(M_j^2(\sigma))$, we can build a type $t_j^3(\theta, \mu_i)$ whose unique best response is the message $m_j^3(\theta, \mu_i)$ which is also a best-response to the belief $\mu_i$ at $\theta$. Hence, proceeding iteratively until a fixed point is reached (recall that $M^\star$ is finite), we build a type space $\tilde{T}$ and a message space $\tilde{M}(\sigma)$ such that, for each $i$, the following two properties are satisfied:

1. *Closedness:* For each belief $\mu_i \in \Delta(\tilde{M}_j(\sigma))$, there is a message $m_i \in \tilde{M}_i(\sigma)$ such that $m_i$ is a best reply to $\mu_i$ at $\theta$.

2. *Full range:* For each $m_i \in \tilde{M}_i(\sigma)$, there is a type $t_i \in \tilde{T}_i$ whose unique best response (when $\sigma$ is played in $U(\mathcal{M}^\star, \mathcal{T}^0)$) is $m_i$.

Now, fix a rationalizable message $m_i$ in the complete information game associated with $\theta$ and $\tilde{M}(\sigma)$. We show in the last step of the proof (Proposition 4) that it is possible to build a type $t_i$ which plays $m_i$ as unique best-reply (when $\sigma$ is played in $U(\mathcal{M}^\star, \mathcal{T}^0)$) and is arbitrarily close to the complete information type associated with $\theta$. To see why, first notice that by Point 1 above (*Closedness*), $m_i$ is also rationalizable at $\theta$ when the message space is $M^\star$. The sequel of the proof is then similar to the contagion argument used in Weinstein and Yildiz (2007). (To change a best response into a strict best response, we use Point 2 above (*Full range*).)

[7] Of course, there may exist infinitely many $e$-continuous equilibria in $U(\mathcal{M}^\star, \mathcal{T}^0)$. To avoid technicalities, we neglect the issue of cardinality of the type spaces. Details are provided in the proof.

## 4.2 Step 1

In this first step of the proof, topological arguments are used to establish Proposition 1 below.

**Proposition 1** *There exist a finite message space $M^\star = \Pi_{i=1}^I M_i^\star \subset \Pi_{i=1}^I \Delta(\bar{M}_i)$ and a finite model $\mathcal{T}^0$ such that the following three conditions are satisfied:*

1. *The mechanism $\mathcal{M}^\star = (M^\star, g)$ allows for e-continuous implementation,*

2. *The message space $M^\star$ has no redundant message,*

3. *For each e-continuous equilibrium $\sigma$ in $U(\mathcal{M}^\star, \mathcal{T}^0)$ and each player $i$:*

$$\text{Aff}(\sigma_{-i}(T_{-i}^0)) = \text{Aff}(M_{-i}^\star).$$

**Proof.** For each player $i$, we use the Hausdorff metric on the set of finite message spaces $M_i \subseteq \Delta(\bar{M}_i)$. More precisely, for each $M_i, M_i' \subseteq \Delta(\bar{M}_i)$, the distance between $M_i$ and $M_i'$ is defined by:

$$d(M_i, M_i') := \max(\max_{m_i \in M_i} \min_{m_i' \in M_i'} ||m_i - m_i'||, \max_{m_i' \in M_i'} \min_{m_i \in M_i} ||m_i - m_i'||),$$

where we recall that $||m_i - m_i'||$ is the distance associated with the max norm on $\Delta^{|\bar{M}_i|-1}$.

We define for each $i$ and $p_i = 0, \dots, |\bar{M}_i| - 1$, the subsets $L_i(p_i)$ and $\bar{L}_i(p_i)$ as follows. We set: $L_i(|\bar{M}_i| - 1) = \bar{M}_i$ and for each $p_i = 0, \dots, |\bar{M}_i| - 2$, a message set $M_i$ belongs to $L_i(p_i)$ if and only if there exists $M_i' \in \bar{L}_i(p_i + 1)$ (where $\bar{L}_i(p_i + 1)$ is the closure of $L_i(p_i + 1)$ using the topology induced by the Hausdorff distance) and an affine subspace $E$ of $\text{Aff}(M_i')$ such that: $\text{Co}(M_i) = E \cap \text{Co}(M_i')$. In addition, for each vector $\vec{p} = (p_1, \dots, p_I)$, we let $\bar{L}(\vec{p}) = \Pi_{i=1}^I \bar{L}_i(p_i)$ and for each integer $p$, we let $\bar{L}(p)$ be the union of the sets $\bar{L}(\vec{p})$ with $\vec{p}$ satisfying $\sum_{i=1}^I p_i = p$. Claim 1 provides basic properties of the family of subspaces $\{\bar{L}_i(p_i)\}_{p_i=0}^{p_i=|\bar{M}_i|-1}$.

**Claim 1** *For each $i$ and $p_i = 0, \dots, |\bar{M}_i| - 1$:*

*(i) There exists $K(p_i)$ such that each message set $M_i \in \bar{L}_i(p_i)$ satisfies: $|M_i| \leq K(p_i)$ (and may thus be identified with an element of $(\Delta^{|\bar{M}_i|-1})^{K(p_i)}$);*

*(ii) The set $\bar{L}_i(p_i)$ is compact in $(\Delta^{|\bar{M}_i|-1})^{K(p_i)}$;*

*(iii) For each $M_i \in \bar{L}_i(p_i)$, we have:* $\dim(M_i) \leq p_i$.

**Proof.** See Appendix. ∎

We set $\bar{K} := \max_{i\in\mathcal{I}} \max_{p_i=0,\ldots,|\bar{M}_i|-1} K(p_i)$ and in the sequel of this subsection, we restrict our attention to messages spaces $M$ where for each player $i$, the cardinality of $M_i$ is bounded by $\bar{K}$. In addition, each message space $M$ is identified with an element of $\Pi_{i=1}^{I}(\Delta^{|\bar{M}_i|-1})^{\bar{K}}$. Moreover, for each finite model $\mathcal{T}$, each message space $M$ and each player $i$ a strategy of $i$ is identified with an element of $(\Delta^{\bar{K}-1})^{|T_{-i}|}$. The proof of Claim 2 below is standard.

**Claim 2** *Fix a finite model $\mathcal{T}$. The following two properties are satisfied:*

1. *The set of message spaces allowing e-continuous implementation in $\mathcal{T}$ is compact in $\Pi_{i=1}^{I}(\Delta^{|\bar{M}_i|-1})^{\bar{K}}$.*

2. *For each message space $M$ allowing e-continuous implementation, the set of e-continuous equilibria in $U(\mathcal{M}, \mathcal{T})$ is compact.*

**Proof.** See Appendix. ∎

Using compactness of $\bar{L}(p)$ and Point 1 of Claim 2 above (Point 2 will be used in the proof of Proposition 3), we establish Lemma 2 below.

**Lemma 2** *Fix an integer $p$. If, for each $p' \leq p$, there is no message space in $\bar{L}(p')$ allowing for e-continuous implementation, then there exists a finite model $\mathcal{T}(p)$ for which, for each $p' \leq p$ and each message space $M \in \bar{L}(p')$, there is no e-continuous equilibrium in $U((M, g), \mathcal{T}(p))$.*

**Proof.** See Appendix. ∎

Now, let $p^\star$ be the smallest integer $p$ such that there exists a message space belonging to $\bar{L}(p)$ and allowing for $e$-continuous implementation. (Since $\bar{M}$ allows for $e$-continuous implementation, such a $p^\star$ is well-defined.) Let $M(p^\star) \in \bar{L}(p^\star)$ be a message space allowing for $e$-continuous implementation. We establish that the mechanism $\mathcal{M}(p^\star) = (M(p^\star), g)$ and the model $\mathcal{T}(p^\star - 1)$ (with the notation of Lemma 2 above) satisfy Point 3 of Proposition 1.

**Lemma 3** *For each e-continuous equilibrium $\sigma$ in $U(\mathcal{M}(p^\star), \mathcal{T}(p^\star - 1))$:*

$$\mathrm{Aff}(\sigma_{-i}(T_{-i}(p^\star - 1))) = \mathrm{Aff}(M_{-i}(p^\star)), \tag{3}$$

*for each player $i$.*

**Proof.** See Appendix. The intuition is as follows. We show that if there exists an $e$-continuous equilibrium $\sigma$ which does not satisfy Equation (3), then there must exist a message space $M \in \bar{L}(p^\star - 1)$ such that $\sigma$ is also an $e$-continuous equilibrium in the game $U(\mathcal{M}, \mathcal{T}(p^\star - 1))$, a contradiction with the definition of $\mathcal{T}(p^\star - 1)$. ■

We now conclude the proof of Proposition 1. For each player $i$, if there are some redundant messages in $M_i(p^\star)$, then we eliminate all of them but one. This procedure yields $M^\star$.

We show that the mechanism $M^\star$ and the model $\mathcal{T}(p^\star - 1)$ satisfy the three conditions of Proposition 1. Regarding Point 1, notice that if for some model $\mathcal{T}$, $\sigma$ is an equilibrium in $U(\mathcal{M}(p^\star), \mathcal{T})$, then we can find an equilibrium $\sigma'$ in $U(\mathcal{M}^\star, \mathcal{T})$ such that for each $i$ and $t_i \in T_i$, $g(\sigma(t_i), \cdot) = g(\sigma'(t_i), \cdot)$. Since $\mathcal{M}(p^\star)$ allows for $e$-continuous implementation, this means that the mechanism $(M^\star, g)$ also allows for $e$-continuous implementation. Claim 3 below (whose proof is standard and relegated to the Appendix) establishes that Point 2 is satisfied.

**Claim 3** *The message space $M^\star$ has no redundant message.*

**Proof.** See Appendix. ■

Finally, regarding Point 3, fix an $e$-continuous equilibrium $\sigma \in U(\mathcal{M}^\star, \mathcal{T}(p^\star - 1))$. The strategy profile $\sigma$ must also be an $e$-continuous equilibrium in $U(\mathcal{M}(p^\star), \mathcal{T}(p^\star - 1))$. (Indeed, if $m_i' \in M_i(p^\star) \setminus M_i^\star$, then there exists $m_i(m_i') \in M_i^\star$ such that: $g(m_i(m_i'), m_{-i}) = g(m_i', m_{-i})$ for each $m_{-i} \in M_{-i}^\star$.) Consequently, by Lemma 3 above, for each player $i$ :

$$\text{Aff}(\sigma_{-i}(T_{-i}(p^\star - 1)) = \text{Aff}(M_{-i}(p^\star)) \supseteq \text{Aff}(M_{-i}^\star). \tag{4}$$

Since $\sigma$ is a strategy profile in $U(\mathcal{M}^\star, \mathcal{T}(p^\star - 1))$, we also have:

$$\text{Aff}(\sigma_{-i}(T_{-i}(p^\star - 1))) \subseteq \text{Aff}(M_{-i}^\star). \tag{5}$$

Equations (4) and (5) together establish Point 3, which concludes the proof of Proposition 1[8]. ■

[8] In the sequel of the proof, we follow the notations of Proposition 1 and write $\mathcal{T}^0$ for the model $\mathcal{T}(p^\star - 1)$.

### 4.3 Step 2

We now move to the second step of the proof. For each player $i$, let us define the mapping $f^1_{-i} : \Theta \to \bar{M}_{-i}$ by: $f^1_{-i}(\theta) = \bar{m}^1_{-i}$, for each $\theta \in \Theta$. In addition, for each $k = 2, \ldots, |\bar{M}_{-i}|$, and $\theta \in \Theta$, we define the mapping $f^{k,\theta}_{-i} : \Theta \to \bar{M}_{-i}$ by:

$$f^{k,\theta}_{-i}(\theta) = \bar{m}^k_{-i},$$

and,

$$f^{k,\theta}_{-i}(\theta') = \bar{m}^1_{-i},$$

for each $\theta' \neq \theta$. We write $\bar{\Phi}_{-i}$ for the set of mappings from $\Theta$ to $\Delta(\bar{M}_{-i})$ and identify each $\phi_{-i} \in \bar{\Phi}_{-i}$ with a point in the coordinate system of $\mathbb{R}^{|\Theta|(|\bar{M}_{-i}|-1)+1}$ associated with the family of mappings $\mathcal{F}_{-i} = f^1_{-i} \cup \{\bigcup_{\theta\in\Theta} \bigcup_{k=2}^{|\bar{M}_{-i}|-1} f^{k,\theta}_{-i}\}$. Finally, we write $\Phi^\star_{-i}$ for the set of mappings from $\Theta$ to $[\mathrm{Aff}(M^\star_{-i}) \cap \Delta(\bar{M}_{-i})]$.

For each $\vec{\varepsilon} = (\varepsilon^a)_{a\in A} \in \mathbb{R}^{|A|}$, we define $u^{\vec{\varepsilon}}_i : A \times \Theta \to \mathbb{R}$ by $u^{\vec{\varepsilon}}_i(a, \theta) = u_i(a, \theta) + \varepsilon^a$, for each $a \in A$ and $\theta \in \Theta$. For each $\bar{t}_i \in \bar{T}_i$ and $\vec{\varepsilon} \in \mathbb{R}^{|A|}$, we define the expected utility function $Eu^{\vec{\varepsilon}}_{\bar{t}_i}$ by:

$$Eu^{\vec{\varepsilon}}_{\bar{t}_i}(m_i, \phi_{-i}) = \sum_{\theta\in\Theta} \kappa(\bar{t}_i)[\theta] u^{\vec{\varepsilon}}_i(g(m_i, \phi_{-i}(\theta)), \theta),$$

for all $m_i \in M^\star_i$ and $\phi_{-i} \in \Phi^\star_{-i}$. Similarly, we define the best-response correspondence $BR^{\vec{\varepsilon}}_{\bar{t}_i}$ by:

$$BR^{\vec{\varepsilon}}_{\bar{t}_i}(\phi_{-i}) = \arg\max_{m_i \in M^\star_i} Eu^{\vec{\varepsilon}}_{\bar{t}_i}(m_i, \phi_{-i}),$$

for all $\phi_{-i} \in \Phi^\star_{-i}$. We write $Eu_{\bar{t}_i}$ and $BR_{\bar{t}_i}$ for the expected utility function and the best response correspondence associated with the vector $\vec{\varepsilon} \in \mathbb{R}^{|A|}$ where $\varepsilon^a = 0$ for each $a \in A$. (We will follow similar notations for similar notions in the sequel of the proof.)

In the third step of the proof (Proposition 3), we will associate a mapping $\phi_{-i}$ and a vector $\vec{\varepsilon}$ with a type. Since type spaces are discrete in our setting, we introduce the following notion of discretization. For each player $i$ and integer $N$, we write $\Upsilon_{-i}(N)$ for the set of elements $x \in \Delta^{\dim(\Phi^\star_{-i})}$ such that, for each $\ell = 0, \ldots, \dim(\Phi^\star_{-i})$, $x^\ell$ is a multiple of $1/N$. For each $S_{-i} = \{s^0_{-i}, ..., s^{\dim(\Phi^\star_{-i})}_{-i}\} \subset \Phi^\star_{-i}$, we also note $\Upsilon(N, S_{-i})$ for the $N$-discretization of $\mathrm{Co}(S_{-i})$, i.e.:

$$\Upsilon(N, S_{-i}) := \left\{ y \in \mathrm{Co}(S_{-i}) \,\middle|\, \exists\, x \in \Upsilon_{-i}(N) \text{ s.t. } y = \sum_{\ell=0}^{\dim(\Phi^\star_{-i})} x^\ell s^\ell_{-i} \right\}.$$

We write $\mu_{-i}$ for the operator that maps each subset $Y \subseteq \Phi^\star_{-i}$ to its Lebesgue measure[9] in $\Phi^\star_{-i}$. Finally, for each $\bar{r} > 0$, we write $\mathcal{S}_{-i}(\bar{r})$ for the set of subsets $S_{-i} \subset \Phi^\star_{-i}$ such that:

1. $|S_{-i}| = \dim(\Phi^\star_{-i}) + 1$, and

2. $\mu_{-i}(\mathrm{Co}(S_{-i})) \geq \bar{r}$.

Proposition 2 states in a formal way the intuition we presented in subsection 3.1 as the starting point of the proof.

**Proposition 2** *For each $\bar{r} > 0$, there exist $\vec{\varepsilon}(\bar{r}) \in \mathbb{R}^{|A|}_+$ with $\sum_{a\in A} \varepsilon^a \leq 1$ and $N^\star$ such that for each $\bar{t}_i \in \bar{T}_i$ and each $S_{-i} \in \mathcal{S}_{-i}(\bar{r})$, there is $\phi_{-i} \in \Upsilon(N^\star, S_{-i})$ satisfying:*

1. *$BR^{\vec{\varepsilon}(\bar{r})}_{\bar{t}_i}(\phi_{-i})$ is a singleton, and*

2. *$BR^{\vec{\varepsilon}(\bar{r})}_{\bar{t}_i}(\phi_{-i}) \subseteq BR_{\bar{t}_i}(\phi_{-i})$.*

**Proof.** Fix $\bar{t}_i \in \bar{T}_i$. For any $\vec{\varepsilon} \in \mathbb{R}^{|A|}$, we define the equivalence relation $\sim^{\vec{\varepsilon}}_{\bar{t}_i}$ on $M^\star_i$ by $m_i \sim^{\vec{\varepsilon}}_{\bar{t}_i} m'_i$ if and only if

$$Eu^{\vec{\varepsilon}}_{\bar{t}_i}(m_i, \phi_{-i}) = Eu^{\vec{\varepsilon}}_{\bar{t}_i}(m'_i, \phi_{-i}),$$

for all $\phi_{-i} \in \Phi^\star_{-i}$. We write $[m_i]^{\vec{\varepsilon}}_{\bar{t}_i}$ for the equivalent class to which $m_i$ belongs. For each $m_i$ , $m'_i \in M^\star_i$ with $m'_i \notin [m_i]^{\vec{\varepsilon}}_{\bar{t}_i}$, define the subset $\Xi^{\vec{\varepsilon}}_{\bar{t}_i}(m_i, m'_i) \subset \Phi^\star_{-i}$ by:

$$\Xi^{\vec{\varepsilon}}_{\bar{t}_i}(m_i, m'_i) := \{\phi_{-i} \in \Phi^\star_{-i} | Eu^{\vec{\varepsilon}}_{\bar{t}_i}(m_i, \phi_{-i}) = Eu^{\vec{\varepsilon}}_{\bar{t}_i}(m'_i, \phi_{-i})\}.$$

Finally, we write $\Xi^{\vec{\varepsilon}}_{\bar{t}_i}$ for the union of the subsets $\Xi^{\vec{\varepsilon}}_{\bar{t}_i}(m_i, m'_i)$ over $m_i, m'_i \in M^\star_i$ with $m'_i \notin [m_i]^{\vec{\varepsilon}}_{\bar{t}_i}$. Claim 4 will be used in the proofs of Lemmas 4 and 5. In the sequel, for simplicity, we write "affine hyperplane of $\Phi^\star_{-i}$" for the intersection of an affine hyperplane of $\mathrm{Aff}(\Phi^\star_{-i})$ with $\Phi^\star_{-i}$.

**Claim 4** *For each $i \in \mathcal{I}$, $\bar{t}_i \in \bar{T}_i$ and $\vec{\varepsilon} \in \mathbb{R}^{|A|}$, the set $\Xi^{\vec{\varepsilon}}_{\bar{t}_i}$ is included in the union of $\frac{|M^\star_i|(|M^\star_i|-1)}{2}$ affine hyperplanes of $\Phi^\star_{-i}$.*

[9] More precisely, define an affine basis $\mathcal{B}_{-i}$ in $\Phi^\star_{-i}$ and an affine application $\mathcal{L}$ associating to each $\phi_{-i} \in \Phi^\star_{-i}$ its coordinates in the coordinate system associated with $\mathcal{B}_{-i}$. Write $\mu^L_{-i}$ for the operator associated with the Lebesgue measure in $\mathbb{R}^{\dim(\Phi^\star_{-i})}$ normalized by $\mu^L_{-i}(\mathcal{L}(\Phi^\star_{-i})) = 1$. Finally, define $\mu_{-i}$ by $\mu_{-i} = \mu^L_{-i} \circ \mathcal{L}$.

**Proof.** We establish that for each $m_i, m_i'$ with $m_i' \notin [m_i]^{\vec{\varepsilon}}_{\bar{t}_i}$:

$$\dim(\Xi^{\vec{\varepsilon}}_{\bar{t}_i}(m_i, m_i')) < \dim(\Phi^\star_{-i}).$$

By the multilinearity of the expected utility function $Eu^{\vec{\varepsilon}}_{\bar{t}_i}$, we have:

$$\Xi^{\vec{\varepsilon}}_{\bar{t}_i}(m_i, m_i') = \text{Aff}(\Xi^{\vec{\varepsilon}}_{\bar{t}_i}(m_i, m_i')) \cap \Phi^\star_{-i}.$$

Now notice that if $\dim(\Xi^{\vec{\varepsilon}}_{\bar{t}_i}(m_i, m_i')) = \dim(\Phi^\star_{-i})$, then:

$$\text{Aff}(\Xi^{\vec{\varepsilon}}_{\bar{t}_i}(m_i, m_i')) = \text{Aff}(\Phi^\star_{-i}),$$

and hence: $\Xi^{\vec{\varepsilon}}_{\bar{t}_i}(m_i, m_i') = \text{Aff}(\Phi^\star_{-i}) \cap \Phi^\star_{-i} = \Phi^\star_{-i}$, a contradiction with the assumption that $m_i' \notin [m_i]^{\vec{\varepsilon}}_{\bar{t}_i}$. ■

Now, for each $i$, $\bar{t}_i \in \bar{T}_i$ and $\vec{\varepsilon} \in \mathbb{R}^{|A|}$, set:

$$\Gamma^{\vec{\varepsilon}}_1(\bar{t}_i) := \{\phi_{-i} \in \Phi^\star_{-i} \mid BR^{\vec{\varepsilon}}_{\bar{t}_i}(\phi_{-i}) \text{ is not a singleton}\},$$

and,

$$\Gamma^{\vec{\varepsilon}}_2(\bar{t}_i) := \{\phi_{-i} \in \Phi^\star_{-i} | BR^{\vec{\varepsilon}}_{\bar{t}_i}(\phi_{-i}) \nsubseteq BR_{\bar{t}_i}(\phi_{-i})\}.$$

Since $M^\star$ has no redundant message, one can show that for each $\bar{\varepsilon} > 0$, there exists $\vec{\varepsilon}(\bar{\varepsilon}) \in \mathbb{R}^{|A|}$ with $\|\vec{\varepsilon}(\bar{\varepsilon})\| < \bar{\varepsilon}$ such that for each $\bar{t}_i \in \bar{T}_i$ and $m_i, m_i' \in M^\star_i$: $[m_i]^{\vec{\varepsilon}(\bar{\varepsilon})}_{\bar{t}_i} \neq [m_i']^{\vec{\varepsilon}(\bar{\varepsilon})}_{\bar{t}_i}$. Hence, Claim 4 enables us to establish Lemma 4 below.

**Lemma 4** *Fix $\bar{\varepsilon} > 0$. There exists $\vec{\varepsilon}(\bar{\varepsilon}) \in \mathbb{R}^{|A|}_+$ with $\|\vec{\varepsilon}(\bar{\varepsilon})\| < \bar{\varepsilon}$ such that for all players $i$ and all types $\bar{t}_i$, the set $\Gamma^{\vec{\varepsilon}}_1(\bar{t}_i)$ is included in the union of at most $\frac{|M^\star_i|(|M^\star_i|-1)}{2}$ affine hyperplanes of $\Phi^\star_{-i}$.*

**Proof.** See Appendix. ■

We use the fact that, by Claim 4, the set $\Xi_{\bar{t}_i}$ is included in $\frac{|M^\star_i|(|M^\star_i|-1)}{2}$ affine hyperplanes of $\Phi^\star_{-i}$ to establish Lemma 5 below.

**Lemma 5** *For each $\delta > 0$, there exists $\bar{\varepsilon}(\delta) > 0$ such that for each $\vec{\varepsilon} \in \mathbb{R}^{|A|}$ with $\|\vec{\varepsilon}\| \leq \bar{\varepsilon}(\delta)$ and $\bar{t}_i \in \bar{T}_i$, the set $\Gamma^{\vec{\varepsilon}}_2(\bar{t}_i)$ is included in the $\delta$-neighborhood of the union of at most $\frac{|M^\star_i|(|M^\star_i|-1)}{2}$ affine hyperplanes of $\Phi^\star_{-i}$.*

**Proof.** See Appendix. ■

For each $\bar{r} > 0$, when $\delta$ is very small, the measure of the $\delta$-neighborhood of any affine hyperplane of $\Phi^\star_{-i}$ is very small when compared to that of any set $S_{-i} \in \mathcal{S}_{-i}(\bar{r})$. This is the intuition of Lemma 6 below.

**Lemma 6** *For each $\bar{r} > 0$, there exist $N^\star$ and $\delta(\bar{r}) > 0$ such that for each $S_{-i} \in \mathcal{S}_{-i}(\bar{r})$ and affine hyperplane $H$ of $\Phi^\star_{-i}$:*

$$\frac{\mid \Upsilon(N^\star, S_{-i}) \cap B_{\delta(\bar{r})}(H) \mid}{\mid \Upsilon(N^\star, S_{-i}) \mid} \leq \frac{1}{|M_i^\star|(|M_i^\star - 1|) + 1}.$$

**Proof.** See Appendix. ■

We now conclude the proof of Proposition 2. Fix $\bar{r} > 0$. We choose $N = N^\star$ and $\delta = \delta(\bar{r})$ (as defined in Lemma 6), $\bar{\varepsilon} = \min(\frac{1}{|A|}, \bar{\varepsilon}(\delta))$ (where $\bar{\varepsilon}(\delta)$ is as defined in Lemma 4) and $\vec{\varepsilon}(\bar{r}) = \vec{\varepsilon}(\bar{\varepsilon})$ (as defined in Lemma 5).

Let us check that $\vec{\varepsilon}(\bar{r})$ and $N^\star$ satisfy the conditions of Proposition 2. First notice that since $||\vec{\varepsilon}(\bar{r})|| < \frac{1}{|A|}$, we have: $\sum_{a \in A} \varepsilon^a(\bar{r}) \leq 1$. Now, fix some $\bar{t}_i \in \bar{T}_i$. By Lemmas 4 and 5, the set $\Gamma_1^{\vec{\varepsilon}(\bar{r})}(\bar{t}_i) \cup \Gamma_2^{\vec{\varepsilon}(\bar{r})}(\bar{t}_i)$ is included in the union of the $\delta$-neighborood of at most $|M_i^\star|(|M_i^\star| - 1)$ affine hyperplanes of $\Phi^\star_{-i}$. Consequently, for each $S_{-i} \in \mathcal{S}_{-i}(\bar{r})$, applying Lemma 6, we have:

$$\frac{\mid \Upsilon(N^\star, S_{-i}) \cap \{\Gamma_1^{\vec{\varepsilon}(\bar{r})}(\bar{t}_i) \cup \Gamma_2^{\vec{\varepsilon}(\bar{r})}(\bar{t}_i)\} \mid}{\mid \Upsilon(N^\star, S_{-i}) \mid} \leq \frac{|M_i^\star|(|M_i^\star| - 1)}{|M_i^\star|(|M_i^\star| - 1) + 1} < 1.$$

This means that there exists $\phi_{-i} \in \Upsilon(N^\star, S_{-i})$ which does not belong to $\Gamma_1^{\vec{\varepsilon}(\bar{r})}(\bar{t}_i) \cup \Gamma_2^{\vec{\varepsilon}(\bar{r})}(\bar{t}_i)$, i.e., which is such that the two conditions of Proposition 2 are satisfied. ■

## 4.4 Step 3

We now use Propositions 1 and 2 to build a model $\tilde{\mathcal{T}}$ which contains the model $\mathcal{T}^0$ (as defined in Proposition 1) and which will be the starting point of the contagion argument used in Proposition 4.

**Proposition 3** *There exists a finite type space $\tilde{\mathcal{T}}$ such that for each e-continuous equilibrium $\sigma$ in $U(\mathcal{M}^\star, \tilde{\mathcal{T}})$, there is a message space $\tilde{M}(\sigma) = \Pi_{i=1}^I \tilde{M}_i(\sigma) \subseteq M^\star$ satisfying the following two properties:*

1. (Closedness)*: If $\phi_{-i} \in \Phi^\star_{-i}$ is such that $\phi_{-i}(\theta) \in \Delta(\tilde{M}_{-i}(\sigma))$, for each $\theta \in \Theta$, then we have:*

$$\tilde{M}_i(\sigma) \cap BR_{\bar{t}_i}(\phi_{-i}) \neq \emptyset,$$

*for each $\bar{t}_i \in \bar{T}_i$.*

2. (Full Range)*: For each $m_i \in \tilde{M}_i(\sigma)$, there exists $\tilde{t}_i(m_i, \sigma) \in \tilde{T}_i$ such that*

$$BR_i(\sigma_{-i} \mid \tilde{t}_i(m_i, \sigma)) = \{m_i\}.$$

**Proof.** We need to introduce some additional notations. For each $i$, we set:

$$\Omega_{-i} = \{\omega_{-i} \subseteq (T^0_{-i})^{\Theta} \text{ s.t. } |\omega_{-i}| = \dim(\Phi^{\star}_{-i}) + 1\}.$$

We write $\Sigma^0$ for the set of $e$-continuous equilibria in $U(\mathcal{M}^{\star}, \mathcal{T}^0)$. With a slight abuse of notations, for each $\sigma_{-i} \in \Sigma^0_{-i}$ and $\omega_{-i} \in \Omega_{-i}$, we note $\sigma_{-i} \circ \omega_{-i} = \{\sigma_{-i} \circ \omega^0_{-i}, \ldots, \sigma_{-i} \circ \omega^{\dim(\Phi^{\star}_{-i})}_{-i}\}$ for the subset of $\Phi^{\star}_{-i}$ associated with $\sigma_{-i}$ and $\omega_{-i}$. Notice that $\sigma_{-i} \circ \omega_{-i}$ contains (at most) $\dim(\Phi^{\star}_{-i}) + 1$ elements. Using Proposition 1 and the fact that $\Sigma^0$ is compact (Claim 2, Point 2), we establish Lemma 7 below.

**Lemma 7** *There exists $\bar{r} > 0$ satisfying the following property. For each e-continuous equilibrium $\sigma$ in $U(\mathcal{M}^{\star}, \mathcal{T}^0)$ and each player $i$, there exists $\omega_{-i} \in \Omega_{-i}$ such that $\sigma_{-i} \circ \omega_{-i} \in \mathcal{S}_{-i}(\bar{r})$.*

**Proof.** See Appendix. ■

Recall that $\vec{\varepsilon}(\bar{r})$ (as defined in Proposition 2) belongs to $\mathbb{R}^{|A|}_{+}$ and is such that $\sum_{a \in A} \varepsilon^a(\bar{r}) \leq 1$. For each $\theta \in \Theta$, we write $z(\theta, a, i) = u_i(a, \theta^{\star\star}(\theta, a, i)) - u_i(a, \theta)$ and set:

$$\varepsilon^a_i(\bar{r}, \theta) = \frac{\varepsilon^a(\bar{r})}{z(\theta, a, i)},$$

and,

$$\varepsilon^0_i(\bar{r}, \theta) = 1 - \sum_{a \in A} \varepsilon^a_i(\bar{r}, \theta).$$

**Lemma 8** *Fix $\bar{t}_i \in \bar{T}_i$ and $\omega_{-i} \in \Omega_{-i}$. There exists a type space $T_i(\bar{t}_i, \omega_{-i})$ such that for each e-continuous $\sigma$ in $U(\mathcal{M}^{\star}, \mathcal{T}^0)$, $\phi_{-i} \in \Upsilon(N^{\star}, \sigma_{-i}(\omega_{-i}))$ and $\bar{t}_i \in \bar{T}_i$, there is a type $t_i(\phi_{-i}) \in T_i(\bar{t}_i, \omega_{-i})$ such that the belief $\pi_i(\cdot|t_i(\phi_{-i}), \sigma) \in \Delta(\Theta^{\star} \times M^{\star}_{-i})$ satisfies:*

$$\pi_i(\theta, \tilde{\theta}^a_i, m_{-i}|t_i(\phi_{-i}), \sigma) = \kappa(\bar{t}_i)[\theta]\varepsilon^a_i(\bar{r}, \theta)\phi_{-i}(m_{-i}|\theta),$$

*for all $a \in A^{\star}$, $\theta \in \Theta$ and $m_{-i} \in M^{\star}_{-i}$.*

**Proof.** See Appendix. ■

We now fix some $\bar{t}_i \in \bar{T}_i$ for each player $i$ and define:

$$T^1_i = T^0_i \cup \bigcup_{\omega_{-i} \in \Omega_{-i}} T_i(\bar{t}_i, \omega_{-i}).$$

We write $\mathcal{T}^1$ for the belief-closed model associated with the type space $\Pi^I_{i=1} T^1_i$. Using Point 1 of Proposition 2, we establish that this model satisfies the following property.

**Lemma 9** *For each e-continuous equilibrium $\sigma$ in $U(\mathcal{M}^\star, \mathcal{T}^1)$ and each player $i$, there exists a type $t_i(\sigma) \in T_i^1 \setminus T_i^0$ such that $BR(\sigma \mid t_i(\sigma))$ is a singleton.*

**Proof.** See Appendix. ■

Using Points 1 and 2 of Proposition 2 and proceeding in a way similar to that of Lemma 9, we establish Lemma 10 below.

**Lemma 10** *There is a family of models $\{\mathcal{T}^n\}_{n\geq 1}$ with $\mathcal{T}^n \subseteq \mathcal{T}^{n+1}$ satisfying the following two properties for each integer $n \geq 2$ and each e-continuous equilibrium $\sigma$ in $U(\mathcal{M}^\star, \mathcal{T}^n)$ :*

1. *For each $i$, the set $M_i^n(\sigma) := \{m_i \in M_i^\star \mid \exists\, t_i \in T_i^n \setminus T_i^0 \text{ s.t. } BR(\sigma|t_i) = \{m_i\}\}$ is non-empty.*
2. *For all $\bar{t}_i \in \bar{T}_i$ and $\phi_{-i} \in (\Delta(M_{-i}^{n-1}(\sigma_{|T^{n-1}}))^\Theta$ (where $\sigma_{|T^{n-1}}$ is the restriction of $\sigma$ to $T^{n-1}$), there is a type $t_i \in T_i^n$ such that $BR(\sigma|t_i)$ is a singleton included in $BR_{\bar{t}_i}(\phi_{-i})$.*

**Proof.** See Appendix. ■

Finally, we show that the model $\tilde{\mathcal{T}} := \mathcal{T}^{\sum_{i=1}^I |M_i^\star|+1}$ satisfies the two properties stated in Proposition 3. Fix an $e$-continuous equilibrium $\sigma$ in $U(\mathcal{M}^\star, \tilde{\mathcal{T}})$. Notice that for each integer $n \in \{1, ..., \sum_{i=1}^I |M_i^\star|\}$, $M^n(\sigma_{|T^n}) \subseteq M^{n+1}(\sigma_{|T^{n+1}})$. Since the message space $M^\star$ is finite, this means that there must exist $\tilde{n} \in \{2, ..., \sum_{i=1}^I |M_i^\star| + 1\}$ such that $M^{\tilde{n}}(\sigma_{|T^{\tilde{n}}}) = M^{\tilde{n}-1}(\sigma_{|T^{\tilde{n}-1}})$.

Set $\tilde{M}(\sigma) := M^{\tilde{n}}(\sigma_{|T^{\tilde{n}}})$. By construction, the Full range condition of Proposition 3 (Point 2) is satisfied. Let us check that the Closedness condition of Proposition 3 (Point 1) also holds. Fix $i \in \mathcal{I}$, $\bar{t}_i \in \bar{T}_i$ and $\phi_{-i} \in (\Delta(\tilde{M}_{-i}(\sigma)))^\Theta$. By definition of $\tilde{n}$, $\phi_{-i}$ is also an element of $(\Delta(M_{-i}^{\tilde{n}-1}(\sigma_{|T^{\tilde{n}-1}})))^\Theta$. Hence, by Lemma 10, there is a type $t_i \in T_i^{\tilde{n}} \subseteq \tilde{T}_i$ such that $BR(\sigma|t_i)$ is a singleton included in $BR_{\bar{t}_i}(\phi_{-i})$. We must have: $BR(\sigma|t_i) \in M^{\tilde{n}}(\sigma_{|T^{\tilde{n}}}) = \tilde{M}_i(\sigma)$. Consequently, $BR_{\bar{t}_i}(\phi_{-i}) \cap \tilde{M}_i(\sigma) \neq \emptyset$, as claimed. ■

## 4.5 Step 4

The proof of Proposition 4 is notationally involved and is relegated to the Appendix, but the key ideas are simple. Let us sketch them. Write $\tilde{\Sigma}$ for the set of $e$-continuous equilibria in $U(M^\star, \tilde{T})$ (where $\tilde{\mathcal{T}}$ is the model defined in Proposition 3) and define the following equivalence classes on $\tilde{\Sigma}$. For each $\sigma, \sigma' \in \tilde{\Sigma}$, $\sigma' \in [\sigma]$ if and only if:

1. $\tilde{M}(\sigma') = \tilde{M}(\sigma)$, and,

2. For each $m_i \in \tilde{M}_i(\sigma')$: $\tilde{t}_i(m_i, \sigma') = \tilde{t}_i(m_i, \sigma)$ (with the notations of Proposition 3).

Using Point 1 of Proposition 3 (*closedness*), one can show that for each $e$-continuous equilibrium $\sigma$ in $U(\mathcal{M}^\star, \tilde{\mathcal{T}})$, each type profile $\bar{t}$, and each message profile $m \in R^\infty(\bar{t}|\tilde{\mathcal{M}}(\sigma), \bar{\mathcal{T}})$, $m$ also belongs to $R^\infty(\bar{t}|\mathcal{M}^\star, \bar{\mathcal{T}})$. Now, fix an integer $N$. Using a contagion argument similar[10] to that used in Weinstein and Yildiz (2007), one can build a model $\mathcal{T}^N \supset \tilde{T}$ satisfying the following property. For each $[\sigma] \in [\tilde{\Sigma}]$, each $\bar{t} \in \bar{T}$ and each $m \in R^\infty(\bar{t}|\tilde{M}([\sigma]), \bar{T})$, there exists a type[11] $t(m, [\sigma], N)$ such that:

(i) We have: $t(m, [\sigma], N) \in \bar{B}_{\frac{1}{N}}(\bar{t})$.

(ii) At each equilibrium $\sigma^N$ in $U(M^\star, \mathcal{T}^N)$ satisfying $\sigma^N_{|\tilde{T}} \in [\sigma]$:

$$\sigma^N(t(m, [\sigma], N)) = \{m\}.$$

Since $M^\star$ allows for $e$-continuous implementation, there exists an $e$-continuous equilibrium $\sigma^N$ in $U(M^\star, \mathcal{T}^N)$. Setting $M^N := \tilde{M}(\sigma^N_{|\tilde{T}})$, we obtain the following proposition.

**Proposition 4** *There exists a family of message spaces $\{M^N\}_{N\in\mathbb{N}}$ (with $M^N = \Pi_{i\in\mathcal{I}} M_i^N \subseteq M^\star$, for each $N$) satisfying the following property: for each $N$, $\bar{t} \in \bar{T}$ and $m \in R^\infty(\bar{t}\,|\mathcal{M}^N, \bar{\mathcal{T}})$, $g(m) \in \bar{B}_{e(N)}(f(\bar{t}))$.*

**Proof.** See Appendix. ■

We now conclude the proof of Theorem 1. Let $M$ be an accumulation point of the sequence $\{M^N\}_{N\in\mathbb{N}}$. (Such an accumulation point exists since $M^\star$ is finite.) We establish that the finite mechanism $\mathcal{M} = (M, g)$ fully implements $f$ in rationalizable messages. Pick some $\bar{t} \in \bar{T}$ and $m \in R^\infty(\bar{t}|\mathcal{M}, \bar{T})$. For each $N$, there exists $N' \geq N$ such that: $M = M^{N'}$. Consequently, by Proposition 4, $g(m) \in \bar{B}_{e(N')}(f(\bar{t})) \subseteq \bar{B}_{e(N)}(f(\bar{t}))$. Since $e(N)$ tends toward zero as $N \to \infty$, we obtain: $g(m) = f(\bar{t})$, which concludes the proof. □

[10]We use Point 2 (*Full range*) of Proposition 3 to breaks ties and change a best response into a strict best response.

[11]Notations are slightly different in the core of the proof.

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# A Proofs of Step 1

**Proof of Claim 1.** Fix a player $i$. Notice that the set $\bar{M}_i$ satisfies points (i), (ii) and (iii) of Claim 1. We proceed by induction and assume that for some $p_i > 0$, these three properties are satisfied. There exists[12] an increasing function $v : \mathbb{N} \to \mathbb{N}$ such that for each finite set $S \subset \mathbb{R}^{|\bar{M}_i|}$ and each affine subspace $E$ of $\text{Aff}(S)$, the number of vertices of the convex polytope $P = \text{Co}(S) \cap E$ is smaller than $v(|S|)$. Let us write $K(p_i - 1) = f(K(p_i))$. By the induction hypothesis, for each $M_i \in L(p_i - 1)$, we must have: $|M_i| \leq K(p_i - 1)$. By continuity, this property is also true for each $M_i' \in \bar{L}_i(p_i - 1)$, which establishes point (i). Consequently, $\bar{L}_i(p_i - 1)$ may be viewed as a subset of $(\Delta^{|\bar{M}_i|-1})^{K(p_i-1)}$. Now, fix a sequence $\{M_i^n\}_{n \in \mathbb{N}} \subseteq L_i(p_i - 1)$ which tends toward $M_i \in (\Delta^{|\bar{M}_i|-1})^{K(p_i-1)}$ in the topology associated with the max norm. The Hausdorff distance between $M_i^n$ and $M_i$ tends toward zero as $n \to \infty$. Thus, $M_i \in \bar{L}_i(p_i - 1)$, which establishes point (ii). Finally, regarding point (iii), since for each $M_i' \in L_i(p_i - 1)$ there must exist $M_i'' \in \bar{L}_i(p_i)$ and an affine subspace $E_i$ of $\text{Aff}(M_i'')$ such that $\text{Co}(M_i') = E_i \cap \text{Co}(M_i'')$, we must have: $\text{Aff}(M_i') = E_i$ which implies: $\dim(M_i') < \dim(M_i'')$. Therefore, by the induction hypothesis: $\dim(M_i') \leq p_i - 1$. By continuity, we also have: $\dim(M_i') \leq p_i - 1$, for each $M_i' \in \bar{L}_i(p_i - 1)$, which concludes the proof. □

**Proof of Claim 2.** Fix a model $\mathcal{T}$ and a sequence of message spaces $\{M^n\}_{n \in \mathbb{N}}$ with $M^n \to M$ (in the topology associated with the max norm on $\Pi_{i=1}^I (\Delta^{|\bar{M}_i|-1})^{\bar{K}}$). In addition, assume that there exists an $e$-continuous equilibrium $\sigma^n$ in $U((M^n, g), \mathcal{T})$ for each $n \in \mathbb{N}$. Recall that for each $n$, we view $\sigma^n$ as an element of $\Pi_{i=1}^I (\Delta^{\bar{K}})^{T_i}$. Since this set is compact, taking a subsequence if necessary, there exists a strategy profile $\sigma^\star$ in $U((M, g), \mathcal{T})$ such that for each $i$, each $t_i$ and each $q = 1, ..., \bar{K} : \sigma_i^n(m_{i,q}^n \mid t_i) \to \sigma_i^\star(m_{i,q} \mid t_i)$. We establish that the strategy profile $\sigma^\star$ is an $e$-continuous equilibrium

[12] To check this, fix a finite set $S \subseteq \mathbb{R}^{\bar{M}_i}$ and a hyperplan $H$ of $\text{Aff}(S)$. Each vertex of the convex polytope $P = \text{Co}(S) \cap H$ must either be a vertex of $\text{Co}(S)$ (i.e., an element of $S$) or the intersection of an edge of $\text{Co}(S)$ and $H$. Hence, the number of vertices of $P$ is bounded by $\frac{|S|(|S|+1)}{2}$. Now notice that for each affine subspace $E$ of $\text{Aff}(S)$, there exists a finite sequence $\{H^d\}_{d=1}^D$ with $D \leq |\bar{M}_i|$ such that $H_D$ is an affine hyperplane of $\text{Aff}(S)$, for each $d = 1, \ldots, D-1$, $H_d$ is an affine hyperplane of $\text{Aff}(H_{d+1}) \cap \ldots \cap H_D \cap \text{Co}(S)$ and:

$$E \cap \text{Co}(S) = H_1 \cap \ldots \cap H_D \cap \text{Co}(S).$$

.

in $U((M,g),\mathcal{T})$.

Notice first that $\sigma^\star$ is an equilibrium in $U((M,g),\mathcal{T})$. Indeed, fix $t_i \in T_i$. Using the fact that $\sigma^n$ is an equilibrium in $U(\mathcal{M}^n,\mathcal{T})$, we have:

$$\sum_{\theta^\star\in\Theta^\star}\sum_{t_{-i}\in T_{-i}}\kappa(t_i)[\theta^\star,t_{-i}]u_i(g(\sigma_i^n(t_i),\sigma_{-i}^n(t_{-i})),\theta^\star) \geq \sum_{\theta^\star\in\Theta^\star}\sum_{t_{-i}\in T_{-i}}\kappa(t_i)[\theta^\star,t_{-i}]u_i(g(m_q^n,\sigma_{-i}^n(t_{-i})),\theta^\star),$$

for all $q \in \{1,\ldots,\bar{K}\}$. Since $M^n \to M$, taking the limit as $n$ goes to infinity, we get

$$\sum_{\theta^\star\in\Theta^\star}\sum_{t_{-i}\in T_{-i}}\kappa(t_i)[\theta^\star,t_{-i}]u_i(g(\sigma_i^\star(t_i),\sigma_{-i}^\star(t_{-i})),\theta^\star) \geq \sum_{\theta^\star\in\Theta^\star}\sum_{t_{-i}\in T_{-i}}\kappa(t_i)[\theta^\star,t_{-i}]u_i(g(m_q,\sigma_{-i}^n(t_{-i})),\theta^\star)$$

for all $q \in \{1,\ldots,\bar{K}\}$.

Now pick $\bar{t} \in \bar{T}$, $N$ and $t \in \bar{B}_{\frac{1}{N}}(\bar{t})$. Since for each $n$, $\sigma^n$ is $e$-continuous in $U((M^n,g),\mathcal{T})$, we have: $g(\sigma^n(t)) \in \bar{B}_{e(N)}(f(\bar{t}))$. Since $M^n \to M$, we have: $g(\sigma^n(t)) \to g(\sigma^\star(t))$, establishing that $\sigma^\star$ is $e$-continuous.

We conclude the proof of Claim 2. Regarding Point 1, the fact that $\sigma^\star$ is an $e$-continuous equilibrium in $U(\mathcal{M},\mathcal{T})$ clearly means that $M$ allows for $e$-continuous implementation in $\mathcal{T}$. Regarding Point 2, let $M$ be a message space allowing $e$-continuous implementation in $\mathcal{T}$ and let $\{\sigma^n\}_{n\in\mathbb{N}}$ be a sequence of $e$-continuous equilibria in $U(\mathcal{M},\mathcal{T})$ converging toward some strategy profile $\sigma^\star$. Applying the argument above with $M^n = M$ for each $n$, we know that $\sigma^\star$ is an $e$-continuous equilibrium in $U(\mathcal{M},\mathcal{T})$. □

**Proof of Lemma 2.** Assume that for each $p' \leq p$, there is no message space in $\bar{L}(p')$ allowing for $e$-continuous implementation. This means that there is a collection $\mathbf{T}(p)$ of finite models such that for each $p' \leq p$ and message space $M \in \bar{L}(p')$, there exists a model $\mathcal{T}(M) \in \mathbf{T}(p)$ such that there is no $e$-continuous equilibrium in $U((M,g),\mathcal{T}(M))$. For each $\mathcal{T} \in \mathbf{T}(p)$, we note $\mathcal{W}(\mathcal{T})$ for the union of all message spaces $M$ in $\bigcup_{p'=0}^{p}\bar{L}(p')$ for which there is no $e$-continuous equilibrium in $U((M,g),\mathcal{T})$. By Claim 2 (Point 1), $\mathcal{W}(\mathcal{T})$ is open in $\prod_{i=1}^{I}(\Delta^{|\bar{M}_i-1|})^{\bar{K}}$. In addition, by definition of $\mathbf{T}(p)$, the collection $\{\mathcal{W}(\mathcal{T})\}_{\mathcal{T}\in\mathbf{T}(p)}$ is a covering of $\bigcup_{p'=0}^{p}\bar{L}(p')$. Since $\bigcup_{p'=0}^{p}\bar{L}(p')$ is compact, we can extract a finite family $\mathbf{T}^\star(p) \subseteq \mathbf{T}(p)$ such that $\{\mathcal{W}(\mathcal{T})\}_{\mathcal{T}\in\mathbf{T}^\star(p)}$ is a covering of $\bigcup_{p'=0}^{p}\bar{L}(p')$. Now set:

$$\mathcal{T}(p) = \bigcup_{\mathcal{T}\in\mathbf{T}^\star(p)}\mathcal{T}.$$

We prove that $\mathcal{T}(p)$ satisfies the property described in Lemma 2. Fix some message space $M \in \bigcup_{p'=0}^{p} \bar{L}(p')$ and some equilibrium $\sigma$ in $U((M,g),\mathcal{T}(p))$. For each $\mathcal{T} \in \mathbf{T}^\star(p)$, the restriction $\sigma_{|\mathcal{T}}$ of $\sigma$ to the type space $T$ is an equilibrium in $U((M,g),\mathcal{T})$. Hence, by definition of $\mathbf{T}^\star(p)$ there must exist some $\mathcal{T} \in \mathbf{T}^\star(p)$ such that $\sigma_{|\mathcal{T}}$ is not $e$-continuous, which establishes that $\sigma$ is not $e$-continuous. □

**Proof of Lemma 3.** We first establish that for each $e$-continuous equilibrium $\sigma$ in $U(\mathcal{M}(p^\star),\mathcal{T}(p^\star-1))$ and each player $i$:

$$\text{Aff}(\sigma_i(T_i(p^\star-1)) = \text{Aff}(M_i(p^\star)). \tag{6}$$

Proceed by contradiction and assume that there exists an $e$-continuous equilibrium $\sigma$ in $U(\mathcal{M}(p^\star),\mathcal{T}(p^\star-1))$ such that for some player $i$, $\text{Aff}(\sigma_i(T_i(p^\star-1))$ is *strictly* included in $\text{Aff}(M_i(p^\star))$. Write $p_i^\star$ for the smallest integer $p_i$ such that $M_i(p^\star) \in \bar{L}_i(p_i)$ and $M_i'$ for the message set such that : $\text{Co}(M_i') = \text{Aff}(\sigma_i(T_i(p^\star)) \cap \text{Co}(M_i(p^\star))$. Since $\dim(M_i(p^\star)) \geq 1$, by Claim 1, $p_i^\star \geq 1$ and thus the set $L_i(p_i^\star - 1)$ is well-defined. By construction, $M_i' \in L_i(p_i^\star-1)$ and $(M_i', M_{-i}(p^\star)) \in \bar{L}(p^\star-1)$.

Notice that since $\sigma_i(T_i(p^\star-1)) \subseteq \text{Co}(M_i(p^\star))$, we must also have: $\sigma_i(T_i(p^\star-1)) \subseteq \text{Co}(M_i')$. Consequently, if we define a new mechanism by replacing $M_i(p^\star)$ by $M_i'$, we can build for each $t_i$, $\tilde{\sigma}_i(t_i) \in \Delta(M_i')$ such that $g(\sigma_i(t_i),\cdot) = g(\tilde{\sigma}_i(t_i),\cdot)$. Since $\sigma_i$ is a best response to $\sigma_{-i}$ in $U((M_i(p^\star), M_{-i}(p^\star), g), \mathcal{T}(p^\star-1))$ and $\text{Co}(M_i') \subseteq \text{Co}(M_i(p^\star))$, $\tilde{\sigma}_i$ must also be a best response to $\sigma_{-i}$ in $U((M_i', M_{-i}(p^\star), g), \mathcal{T}(p^\star-1))$. Hence, the strategy profile $(\tilde{\sigma}_i, \sigma_{-i})$ is an $e$-continuous equilibrium in $U((M_i', M_{-i}(p^\star), g), \mathcal{T}(p^\star - 1))$, a contradiction with the definition of $\mathcal{T}(p^\star-1)$. This establishes Equation (6).

Now, recall that for each strategy profile $\sigma$ in $U(\mathcal{M}(p^\star),\mathcal{T}(p^\star-1))$ and each player $i$, we have: $M_{-i}(p^\star) = \Pi_{j\neq i} M_j(p^\star)$ and $\sigma_{-i}(T_{-i}(p^\star-1)) = \Pi_{j\neq i}\sigma_i(T_i(p^\star-1))$. Thus, Claim 5 below and Equation (6) together imply:

$$\text{Aff}(\sigma_{-i}(T_{-i}(p^\star-1))) = \text{Aff}\left(\prod_{j\neq i}\text{Aff}\left(\sigma_j(T_j(p^\star-1))\right)\right) = \text{Aff}\left(\prod_{j\neq i}\text{Aff}\left(M_j(p^\star)\right)\right) = \text{Aff}\left(M_{-i}(p^\star)\right),$$

which concludes the proof of Lemma 3. □

**Claim 5** *For each player $i$, let $Y_i$ be a finite set $\{y_i^1, ..., y_i^{\ell_i}\} \subseteq \mathbb{R}^{\bar{M}_i}$ and $Y_{-i} = \Pi_{j\neq i} Y_i = \{y_{-i}^1, ..., y_{-i}^{\ell_{-i}}\}$ (with $l_{-i} = \Pi_{j\neq i} l_j$). We have:*

$$\text{Aff}\left(Y_{-i}\right) = \text{Aff}\left(\prod_{j\neq i}\text{Aff}\left(Y_j\right)\right), \tag{7}$$

*where* $\mathrm{Aff}(Y_{-i})$ *and* $\mathrm{Aff}(\prod_{j\neq i}\mathrm{Aff}(Y_j))$ *are respectively the affine hulls of* $Y_{-i}$ *and* $\prod_{j\neq i}\mathrm{Aff}(Y_j)$ *in* $\mathbb{R}^{|\bar{M}_{-i}|}$ .

**Proof.** Since $Y_{-i}\subseteq\prod_{j\neq i}\mathrm{Aff}\,(Y_j)$, we have: $\mathrm{Aff}\,(Y_{-i})\subseteq\mathrm{Aff}(\prod_{j\neq i}\mathrm{Aff}\,(Y_j))$. Hence, it suffices to show that $\prod_{j\neq i}\mathrm{Aff}\,(Y_j)\subseteq\mathrm{Aff}\,(Y_{-i})$. (Indeed, this will give: $\mathrm{Aff}(\prod_{j\neq i}\mathrm{Aff}\,(Y_j))\subseteq\mathrm{Aff}(\mathrm{Aff}\,(Y_{-i}))=\mathrm{Aff}\,(Y_{-i})$.) Recall that by construction of $Y_{-i}$ there exists a bijection $\beta$ associating to each vector of integers $k=\prod_{j\neq i}k_j\in\prod_{j\neq i}[1,\ell_j]$ an integer $\beta(k)\in[1,\ell_{-i}]$ such that: $y_{-i}^{\beta(k)}=\prod_{j\neq i}y_j^{k_j}$. Now, note that if $x_{-i}=\prod_{j\neq i}x_j$ is an element of $\prod_{j\neq i}\mathrm{Aff}\,(Y_j)$, then for each $j\neq i$, there exists $\alpha_j\in\mathbb{R}^{\ell_j}$ (with $\sum_{k=1}^{\ell_j}\alpha_j^k=1$) such that $x_j=\sum_{k=1}^{\ell_j}\alpha_j^k y_j^k$. In other terms, $x_{-i}=\sum_{k=1}^{\ell_{-i}}\alpha_{-i}^k y_{-i}^k$ where the vector $\alpha_{-i}\in\mathbb{R}^{\ell_{-i}}$ is such that for each integer $k\in[1,\dots\ell_{-i}]$, $\alpha_{-i}^k=\prod_{j\neq i}\alpha_j^{\beta^{-1}(k)^j}$ (where $\beta^{-1}(k)^j$ is the $j$th component of the preimage of $k$ by bijection $\beta$). Write $j(1),\dots,j(I-1)$ for the $I-1$ opponents to $i$. We have successively:

$$\begin{aligned}\sum_{k=1}^{\ell_{-i}}\alpha_{-i}^k &=\sum_{k=1}^{\ell_{-i}}\prod_{j\neq i}\alpha_j^{\beta^{-1}(k)^j}\\ &=\sum_{k_{j(1)}=1}^{\ell_{j(1)}}\cdots\sum_{k_{j(I-1)}=1}^{\ell_{j(I-1)}}\prod_{j\neq i}\alpha_j^{k_j}\\ &=\sum_{k_{j(1)}=1}^{\ell_{j(1)}}\cdots\sum_{k_{j(I-2)}=1}^{\ell_{j(I-2)}}\prod_{j=j(1)}^{j(I-2)}\alpha_j^{k_j}\,(\sum_{k_{j(I-1)}=1}^{\ell_{j(I-1)}}\alpha_{j(I-1)}^{k_{j(I-1)}}).\end{aligned}$$

Since for each $j\neq i$, $\sum_{k_j=1}^{\ell_j}\alpha_j^{k_j}=1$, this means that:

$$\begin{aligned}\sum_{k=1}^{\ell_{-i}}\alpha_{-i}^k &=\sum_{k_{j(1)}=1}^{\ell_{j(1)}}\cdots\sum_{k_{j(I-2)}=1}^{\ell_{j(I-2)}}(\prod_{j=j(1)}^{j(I-2)}\alpha_j^{k_j})\\ \sum_{k=1}^{\ell_{-i}}\alpha_{-i}^k &=\sum_{k_{j(1)}=1}^{\ell_{j(1)}}\cdots\sum_{k_{j(I-3)}=1}^{\ell_{j(I-3)}}(\prod_{j=j(1)}^{j(I-3)}\alpha_j^{k_j})(\sum_{k_{j(I-2)}=1}^{\ell_{j(I-2)}}\alpha_{j(I-2)}^{k_{j(I-2)}})\\ &=\dots\\ &=\sum_{k_{j(1)}=1}^{\ell_{j(1)}}\alpha_1^{k_{j(1)}}\\ &=1.\end{aligned}$$

Hence, $x_{-i}$ belongs to the affine hull of $Y_{-i}$ in $\mathbb{R}^{|M_{-i}|}$, which concludes the proof. ■

**Proof of Claim 3**. Fix a player $i$ and two messages $m_i, m'_i \in M^\star_i$. By construction, there must exist $m_{-i}(m_i, m'_i) \in M_{-i}(p^\star)$ such that $g(m_i, m_{-i}(m_i, m'_i)) \neq g(m'_i, m_{-i}(m_i, m'_i))$ (Otherwise, one of these two messages would have been eliminated). Since only redundant messages have been eliminated from $M_{-i}(p^\star)$ to obtain $M^\star_{-i}$, this means that there must exist $m'_{-i}(m_i, m'_i) \in M^\star_{-i}$ such that:

$$g(m_i, m'_{-i}(m_i, m'_i)) = g(m_i, m_{-i}(m_i, m'_i)) \neq g(m'_i, m_{-i}(m_i, m'_i)) = g(m'_i, m'_{-i}(m_i, m'_i)),$$

which concludes the proof. □

# B Proofs of Step 2

**Proof of Lemma 4.** We first establish the following claim.

**Claim 6** *Fix $\bar{\varepsilon} > 0$. There exists $\vec{\varepsilon}(\bar{\varepsilon}) \in \mathbb{R}_+^{|A|}$ with $\|\vec{\varepsilon}(\bar{\varepsilon})\| < \bar{\varepsilon}$ such that for all players $i$, all types $\bar{t}_i$ and all messages $m_i$, $m_i' \in M_i^\star$, we have: $[m_i]_{\bar{t}_i}^{\vec{\varepsilon}(\bar{\varepsilon})} \neq [m_i']_{\bar{t}_i}^{\vec{\varepsilon}(\bar{\varepsilon})}$.*

**Proof.** First notice that since $M^\star$ contains no redundant messages, for each $m_i, m_i' \in M_i^\star$, there exists $m_{-i}(m_i, m_i') \in M_{-i}^\star$ such that $g(m_i, m_{-i}(m_i, m_i')) \neq g(m_i', m_{-i}(m_i, m_i'))$.

Now, for each $\vec{\varepsilon} \in \mathbb{R}^{|A|}$, we have :

$$Eu_{\bar{t}_i}^{\vec{\varepsilon}}(m_i, m_{-i}(m_i, m_i')) = Eu_{\bar{t}_i}^{\vec{\varepsilon}}(m_i', m_{-i}(m_i, m_i')), \tag{8}$$

if and only if :

$$\vec{\varepsilon} \cdot (g(m_i, m_{-i}(m_i, m_i')) - g(m_i', m_{-i}(m_i, m_i'))) \\ = \\ Eu_{\bar{t}_i}(m_i', m_{-i}(m_i, m_i')) - Eu_{\bar{t}_i}(m_i, m_{-i}(m_i, m_i')),$$

where we recall that an outcome is identified with a point in $\mathbb{R}^{|A|}$ and $\cdot$ stands for the dot product. This means that the affine subspace of vectors $\vec{\varepsilon} \in \mathbb{R}^{|A|}$ such that Equation (8) holds is of dimension $|A|-1$. Since $M_i^\star$ and $\bar{T}_i$ are finite, this concludes the proof. ■

We now conclude the proof of Lemma 4. By Claim 6, for each type $\bar{t}_i$, the sets of mappings $\phi_{-i} \in \Phi_{-i}^\star$ such that there exists two messages $m_i$, $m_i' \in M_i^\star$ satisfying: $Eu_{\bar{t}_i}^{\vec{\varepsilon}(\bar{\varepsilon})}(m_i, \phi_{-i}) = Eu_{\bar{t}_i}^{\vec{\varepsilon}(\bar{\varepsilon})}(m_i', \phi_{-i})$ is equal to $\Xi_{\bar{t}_i}^{\vec{\varepsilon}(\bar{\varepsilon})}$. Hence, Claim 4 allows to conclude the proof. □

**Proof of Lemma 5.** Fix $\bar{t}_i \in \bar{T}_i$ and $\delta > 0$. By Claim 4, we know that $\Xi_{\bar{t}_i}$ is included in the union of $\frac{|M_i^\star|(|M_i^\star - 1|)}{2}$ affine hyperplanes of $\Phi_{-i}^\star$. Write $B_\delta(\Xi_{\bar{t}_i})$ for the *open* $\delta$-neighborhood of $\Xi_{\bar{t}_i}$ in $\Phi_{-i}^\star$.

For any $m_i, m_i' \in M_i^\star$, the function $\phi_{-i} \mapsto |Eu_{\bar{t}_i}(m_i, \phi_{-i}) - Eu_{\bar{t}_i}(m_i', \phi_{-i})|$ is continuous. Hence, the function associating to each $\phi_{-i} \in \Phi_{-i}^\star$,

$$\min_{m_i, m_i' \text{ s.t.} m_i' \notin [m_i]_{\bar{t}_i}} |Eu_{\bar{t}_i}(m_i, \phi_{-i}) - Eu_{\bar{t}_i}(m_i', \phi_{-i})|$$

is also continuous. Besides, this function is strictly positive for each $\phi_{-i} \in \Phi^\star_{-i} \setminus B_\delta(\Xi_{\bar{t}_i})$. Since $\Phi^\star_{-i} \setminus B_\delta(\Xi_{\bar{t}_i})$ is closed, this implies that there exists $\bar{\varepsilon}(\delta)$ such that for each $\phi_{-i} \in \Phi^\star_{-i} \setminus B_\delta(\Xi_{\bar{t}_i})$:

$$\min_{m_i, m_i' \text{ s.t.} m_i' \notin [m_i]_{\bar{t}_i}} |Eu_{\bar{t}_i}(m_i, \phi_{-i}) - Eu_{\bar{t}_i}(m_i', \phi_{-i})| \geq 2\bar{\varepsilon}(\delta)$$

We deduce that for each $\phi_{-i} \in \Phi^\star_{-i} \setminus B_\delta(\Xi_{\bar{t}_i})$, $m_i, m_i' \in M_i^\star$ with $m_i \in BR_{\bar{t}_i}(\phi_{-i})$ and $m_i' \notin BR_{\bar{t}_i}(\phi_{-i})$, we have:

$$Eu_{\bar{t}_i}(m_i, \phi_{-i}) - Eu_{\bar{t}_i}(m_i', \phi_{-i}) \geq 2\bar{\varepsilon}(\delta). \tag{9}$$

Recall that, by construction, for each $m_i \in M_i^\star$ and $\phi_{-i} \in \Phi^\star_{-i}$,

$$Eu^{\vec{\varepsilon}}_{\bar{t}_i}(m_i, \phi_{-i}) \geq Eu_{\bar{t}_i}(m_i, \phi_{-i}) \geq Eu^{\vec{\varepsilon}}_{\bar{t}_i}(m_i, \phi_{-i}) - \bar{\varepsilon}(\delta),$$

for all $\vec{\varepsilon} \in \mathbb{R}^{|A|}$ with $\|\vec{\varepsilon}\| \leq \bar{\varepsilon}(\delta)$. Consequently, Equation (9) implies:

$$Eu^{\vec{\varepsilon}}_{\bar{t}_i}(m_i, \phi_{-i}) - Eu^{\vec{\varepsilon}}_{\bar{t}_i}(m_i', \phi_{-i}) \geq \bar{\varepsilon}(\delta),$$

which gives: $BR^{\vec{\varepsilon}}_{\bar{t}_i}(\phi_{-i}) \subseteq BR_{\bar{t}_i}(\phi_{-i})$. □

**Proof of Lemma 6.** For each $N$, each subset $D \subseteq \Upsilon_{-i}(N)$ and $S_{-i} = \{s^0_{-i}, \ldots, s^{\dim(\Phi^\star_{-i})}_{-i}\} \in \mathcal{S}_{-i}(\bar{r})$, we set:

$$D(S_{-i}) := \{y \in \Phi^\star_{-i} \mid \exists\, x \in D \text{ s.t. } y = \sum_{\ell=0}^{\dim(\Phi^\star_{-i})} x^\ell s^\ell_{-i}\}.$$

Moreover, for each $S_{-i} \in \mathcal{S}_{-i}(\bar{r})$, we write $\mathcal{D}_{-i}(N, S_{-i})$ for the set of subsets $D \subseteq \Upsilon_{-i}(N)$ such that: $\dim(D(S_{-i})) < \dim(\Phi^\star_{-i})$.

**Claim 7** *There exists $\mathcal{D}_{-i}(N)$ such that: $\mathcal{D}_{-i}(N) = \mathcal{D}_{-i}(N, S_{-i})$, for all $S_{-i} \in \mathcal{S}_{-i}(\bar{r})$. Besides, for each $D \in \mathcal{D}_{-i}(N)$, $|D| \leq (N+1)^{\dim(\Phi^\star_{-i})-1}$.*

**Proof.** Fix $S_{-i} = \{s^0, \ldots, s^{\dim(\Phi^\star_{-i})}\} \in \mathcal{S}_{-i}(\bar{r})$ and $D \in \mathcal{D}_{-i}(N, S_{-i})$. Recall that since $\bar{r} > 0$, the vectors $s^1 - s^0, \ldots, s^{\dim(\Phi^\star_{-i})} - s^0$ are linearly independent. Now notice that for each $y = \sum_{\ell=0}^{\dim(\Phi^\star_{-i})} x^\ell s^\ell \in D(S_{-i})$, we can write $y = s_0 + \sum_{\ell=1}^{\dim(\Phi^\star_{-i})} x^\ell (s^\ell - s^0)$. Hence, the coordinates of $y$ in the coordinate system associated with the affine basis $(s^0, s^1 - s^0, \ldots, s^{\dim(\Phi^\star_{-i})} - s^0)$ are given by $(x^\ell)_{\ell=1}^{\dim(\Phi^\star_{-i})}$. Consequently, $\dim(D(S_{-i})) < \dim(\Phi^\star_{-i})$ if and only if there exists a vector $a \in \mathbb{R}^{\dim(\Phi^\star_{-i})}$ (which is not the null vector) and $b \in \mathbb{R}$ such that: $\sum_{\ell=1}^{\dim(\Phi^\star_{-i})} a^\ell x^\ell = b$ for each $x \in D$. Defining $\mathcal{D}_{-i}(N)$ to be the union (over $a \in \mathbb{R}^{\dim(\Phi^\star_{-i})}$ and $b \in \mathbb{R}$) of the subsets $D \subseteq \Upsilon_{-i}(N)$ such that $\sum_{\ell=1}^{\dim(\Phi^\star_{-i})} a^\ell x^\ell = b$ for each $x \in D$, we obtain: $\mathcal{D}_{-i}(N) = \mathcal{D}_{-i}(N, S_{-i})$.

Regarding the second point of Claim 7, fix $D \in \mathcal{D}_{-i}(N)$ and a vector $y = (y^\ell)_{\ell=1}^{\dim(\Phi^\star_{-i})-1}$ such that for each $\ell$, $y^\ell$ is an integer of the interval $[0, N]$. By the above argument, there exists *at most* one element $x \in D$ such that $x^\ell = \frac{y^\ell}{N}$ for all $\ell = 1, \ldots, \dim(\Phi^\star_{-i}) - 1$. This establishes that for each $D \in \mathcal{D}_{-i}(N)$, $|D| \leq (N+1)^{\dim(\Phi^\star_{-i})-1}$. ■

**Claim 8** *There exists $N^\star$ such that for each $D \in \mathcal{D}_{-i}(N^\star)$:*

$$\frac{|D|}{|\Upsilon_{-i}(N^\star)|} \leq \frac{1}{|M_i^\star|(|M_i^\star| - 1) + 1}.$$

**Proof.** Fix an integer $N \geq \dim(\Phi^\star_{-i})$ and write $\mathrm{Int}(\frac{N}{\dim(\Phi^\star_{-i})})$ for the integer value of $\frac{N}{\dim(\Phi^\star_{-i})}$. Now, fix a vector $y = (y^\ell)_{\ell=1}^{\dim(\Phi^\star_{-i})}$ such that for each $\ell$, $y^\ell$ is an integer of the interval $[0, \mathrm{Int}(\frac{N}{\dim(\Phi^\star_{-i})})]$. Notice that for each $\ell$, $\frac{y^\ell}{N} \leq \frac{1}{\dim(\Phi^\star_{-i})}$. Consequently, $\sum_{\ell=1}^{\dim(\Phi^\star_{-i})} \frac{y^\ell}{N} \leq 1$ and there exists an element $x \in \Upsilon_{-i}(N)$ such that $x^\ell = \frac{y^\ell}{N}$ for all $\ell = 1, \ldots, \dim(\Phi^\star_{-i})$. This implies that

$$|\Upsilon_{-i}(N)| \geq (\mathrm{Int}(\frac{N}{\dim(\Phi^\star_{-i})}) + 1)^{\dim(\Phi^\star_{-i})}.$$

Using Claim 7, we deduce that for each $D \in \mathcal{D}_{-i}(N)$,

$$\frac{|D|}{|\Upsilon_{-i}(N)|} \leq \frac{(N+1)^{\dim(\Phi^\star_{-i})-1}}{(\mathrm{Int}(\frac{N}{\dim(\Phi^\star_{-i})}) + 1)^{\dim(\Phi^\star_{-i})}}.$$

Since the right-hand side of this inequality tends toward 0 as $N$ tends toward infinity, this concludes the proof. ■

**Claim 9** *There exists $\zeta(\bar{r}) > 0$ such that for each $D \subseteq \Upsilon_{-i}(N^\star)$ with $D \notin \mathcal{D}_{-i}(N^\star)$ and each $S_{-i} \in \mathcal{S}_{-i}(\bar{r})$, we have: $\mu_{-i}(\mathrm{Co}(D(S_{-i}))) > \zeta(\bar{r})$.*

**Proof.** Fix $D \subseteq \Upsilon_{-i}(N^\star)$ with $D \notin \mathcal{D}_{-i}(N^\star)$. For each $S_{-i} \in \mathcal{S}_{-i}(\bar{r})$, $\dim(D(S_{-i})) = \dim(\Phi^\star_{-i})$, implying that $\mu_{-i}(\mathrm{Co}(D(S_{-i}))) > 0$. Since the function $\mu_{-i}(\mathrm{Co}(D(\cdot))$ is continuous and the set $\mathcal{S}_{-i}(\bar{r})$ is compact, there exists $\zeta(\bar{r}, D) > 0$ such that for each $S_{-i} \in \mathcal{S}_{-i}(\bar{r})$, we have: $\mu_{-i}(\mathrm{Co}(D(S_{-i}))) > \zeta(\bar{r}, D)$. Now, it suffices to notice that since the set $\Upsilon_{-i}(N^\star)$ is finite, the set $\mathcal{D}_{-i}(N^\star)$ is also finite. ■

We are now in position to conclude the proof of Lemma 6. Choose $\delta(\bar{r}) > 0$ such that for each affine hyperplane $H$ of $\Phi^\star_{-i}$,

$$\mu_{-i}(B_{\delta(\bar{r})}(H)) \leq \zeta(\bar{r}). \tag{10}$$

Let us check that $N^\star$ (as defined in Claim 7) and $\delta(\bar{r})$ satisfy the condition of Lemma 6. For each affine hyperplane $H$ of $\Phi^\star_{-i}$ and $S_{-i} \in \mathcal{S}_{-i}(\bar{r})$, define $D^{(H,S_{-i})} \subseteq \Upsilon_{-i}(N^\star)$ by:

$$D^{(H,S_{-i})}(S_{-i}) = B_{\delta(\bar{r})}(H) \cap \Upsilon(N^\star, S_{-i}).$$

Notice that:

$$\mathrm{Co}(D^{(H,S_{-i})}(S_{-i})) = \mathrm{Co}(B_{\delta(\bar{r})}(H) \cap \Upsilon(N^\star, S_{-i})) \subseteq \mathrm{Co}(B_{\delta(\bar{r})}(H)) = B_{\delta(\bar{r})}(H).$$

Consequently by Equation (10), $\mu_{-i}(\mathrm{Co}(D^{(H,S_{-i})}(S_{-i}))) \leq \zeta(\bar{r})$ and by Claim 9, $D^{(H,S_{-i})} \in \mathcal{D}_{-i}(N^\star)$. Applying Claim 8, we obtain:

$$\frac{|D^{(H,S_{-i})}|}{|\Upsilon_{-i}(N^\star)|} \leq \frac{1}{|M_i^\star|(|M_i^\star| - 1) + 1}.$$

Since $|D^{(H,S_{-i})}| = |B_{\delta(\bar{r})}(H) \cap \Upsilon(N^\star, S_{-i})|$ and $|\Upsilon_{-i}(N^\star)| = |\Upsilon(N^\star, S_{-i})|$, this concludes the proof. □

## C Proofs of Step 3

**Proof of Lemma 7.** We prove Lemma 11 which implies Lemma 7 (setting $\rho = 1$) and will be used in the sequel of the proof of Proposition 3 (more precisely for establishing Lemma 10.)

**Lemma 11** *For each $\rho > 0$, there exists $\bar{r}(\rho) > 0$ satisfying the following property. For each e-continuous equilibrium $\sigma$ in $U(\mathcal{M}^\star, \mathcal{T}^0)$ and each player $i$, there exists $\omega_{-i} \in \Omega_{-i}$ such that for each $\phi^\star_{-i} \in \Phi^\star_{-i}$, the set $S_{-i}(\omega_{-i}, \sigma, \phi^\star_{-i}, \rho)$ defined by:*

$$S_{-i}(\omega_{-i}, \sigma, \phi^\star_{-i}, \rho) := \{\phi_{-i} \in \Phi^\star_{-i} | \exists \phi'_{-i} \in \sigma_{-i} \circ \omega_{-i} \text{ s.t. } \phi_{-i} = (1-\rho)\phi^\star_{-i} + \rho\phi'_{-i}\}$$

*belongs to $\mathcal{S}_{-i}(\bar{r}(\rho))$.*

**Proof.** Fix $\rho > 0$. We first establish the following claim.

**Claim 10** *For each e-continuous equilibrium $\sigma$ in $U(\mathcal{M}^\star, \mathcal{T}^0)$, there exists $\omega_{-i}(\sigma) \in \Omega_i$ such that:*

$$\dim(S_{-i}(\omega_{-i}(\sigma), \sigma, \phi^\star_{-i}, \rho)) = \dim(\Phi^\star_{-i}),$$

*for all $\phi^\star_{-i} \in \Phi^\star_{-i}$.*

**Proof.** Fix an $e$-continuous equilibrium $\sigma$ in $U(\mathcal{M}^\star, \mathcal{T}^0)$ and $\phi^\star_{-i} \in \Phi^\star_{-i}$. Let us set:

$$M_{-i}(\sigma, \phi^\star_{-i}(\theta), \rho) := \{m_{-i} \in \Delta(M^\star_{-i}) | \exists m'_{-i} \in \sigma_{-i}(T^0_{-i}) \text{s.t.} m_{-i} = (1-\rho)\phi^\star_{-i}(\theta) + \rho m'_{-i}\}.$$

In addition, we note $\Phi_{-i}(\sigma)$ for the set of mappings from $\Theta$ to $\sigma_{-i}(T^0_{-i})$ and we write:

$$\Phi_{-i}(\sigma, \phi^\star_{-i}, \rho) := \left\{\phi_{-i} \in \Phi^\star_{-i} | \exists \phi'_{-i} \in \Phi_{-i}(\sigma) \text{ s.t. } \phi_{-i} = (1-\rho)\phi^\star_{-i} + \rho\phi'_{-i}\right\}.$$

Notice that:

$$\Phi_{-i}(\sigma, \phi^\star_{-i}, \rho) = \prod_{\theta \in \Theta} M_{-i}(\sigma, \phi^\star_{-i}(\theta), \rho).$$

By Proposition 1, $\dim(\sigma_{-i}(T^0_i)) = \dim(M^\star_{-i})$. Since $\rho > 0$, this means that[13], for each $\theta \in \Theta$:

$$\dim(M_{-i}(\sigma, \phi^\star_{-i}(\theta), \rho)) = \dim(M^\star_{-i}).$$

[13]To check this, recall that, by definition of $\dim(\sigma_{-i}(T^0_i))$, there must exist a set $\{m^0_{-i,\sigma}, \ldots, m^{\dim(M^\star_{-i})}_{-i,\sigma}\} \subseteq \sigma_{-i}(T^0_{-i})$ such that the vectors $m^1_{-i,\sigma} - m^0_{-i,\sigma}, \ldots, m^{\dim(M^\star_{-i})}_{-i,\sigma} - m^0_{-i,\sigma}$ are linearly independant. Now, fix $\theta \in \Theta$. By construction, the set $\{(1-\rho)\phi^\star_{-i}(\theta) + \rho m^0_{-i,\sigma}, \ldots, (1-\rho)\phi^\star_{-i}(\theta) + \rho m^{\dim(M^\star_{-i})}_{-i,\sigma}\}$ is included in $M_{-i}(\sigma, \phi^\star_{-i}(\theta), \rho)$.

We deduce that:

$$\dim(\Phi_{-i}(\sigma,\phi^\star_{-i},\rho)) = |\Theta|\dim(M^\star_{-i}) = \dim(\Phi^\star_{-i}).$$

Notice that for each subset of $\Phi_{-i}(\sigma,\phi^\star_{-i},\rho)$ containing $\dim(\Phi^\star_{-i})+1$ elements there exists some $\omega_{-i}$ such that $S_{-i}(\omega_{-i},\sigma,\phi^\star_{-i},\rho)$ is equal to this subset. Consequently, by Caratheodory's theorem[14],

$$\sum_{\omega_{-i}\in\Omega_{-i}} \mu_{-i}(S_{-i}(\omega_{-i},\sigma,\phi^\star_{-i},\rho)) \geq \mu_{-i}(\Phi_{-i}(\sigma,\phi^\star_{-i},\rho)).$$

Hence, there must exist $\omega_{-i}(\sigma) \in \Omega_{-i}$ such that: $\dim(S_{-i}(\omega_{-i}(\sigma),\sigma,\phi^\star_{-i},\rho)) = \dim(\Phi^\star_{-i})$. ■

We now conclude the proof of Lemma 11. Recall that we write $\Sigma^0$ for the set of $e$-continuous equilibria in $U(\mathcal{M}^\star,\mathcal{T}^0)$. Define the function $\eta : \Sigma^0 \times \Phi^\star_{-i} \to \mathbb{R}$ by $\eta(\sigma,\phi^\star_{-i}) = \max_{\omega_{-i}\in\Omega_{-i}} \mu_{-i}(S_{-i}(\omega_{-i},\sigma,\phi^\star_{-i},\rho))$, for each $\sigma\in\Sigma^0$ and $\phi^\star_{-i}\in\Phi^\star_{-i}$. By Claim 10, $\eta(\sigma,\phi^\star_{-i})$ is strictly positive for each $\sigma\in\Sigma^0$ and $\phi^\star_{-i}\in\Phi^\star_{-i}$. Recall that by Claim 2, the set $\Sigma^0$ is compact. Since the set $\Phi^\star_{-i}$ is compact[15] and the function $\eta$ is continuous in its two arguments, we deduce that there exists $\bar{r}(\rho)>0$ such that $\eta(\sigma,\phi^\star_{-i}) > \bar{r}(\rho)$ for each $\sigma\in\Sigma^0$ and $\phi^\star_{-i}\in\Phi^\star_{-i}$, which concludes the proof. ■

**Proof of Lemma 8.** Fix $\bar{t}_i \in \bar{T}_i$ and $\omega_{-i}\in\Omega_{-i}$. For each $x\in\Upsilon_{-i}(N^\star)$, we define $t_i(x)$ by:

$$\kappa(t_i(x))[\theta,\tilde{\theta}^a_i,t^0_{-i}] = \kappa(\bar{t}_i)[\theta]\varepsilon^a_i(\bar{r},\theta)\sum_{\ell=0}^{\dim(\Phi^\star_{-i})} x^\ell \mathbf{1}_{\omega^\ell_{-i}(\theta)}(t^0_{-i}),$$

for all $\theta\in\Theta, a\in A^\star$ and $t^0_{-i}\in T^0_{-i}$, (where $\mathbf{1}_{\omega^\ell_{-i}(\theta)}(t^0_{-i}) = 1$ if $t^0_{-i} = \omega^\ell_{-i}(\theta)$ and $\mathbf{1}_{\omega^\ell_{-i}(\theta)}(t^0_{-i}) = 0$ otherwise.) Let us set:

$$T^1_i(\bar{t}_i,\omega_{-i}) = \bigcup_{x\in\Upsilon_{-i}(N^\star)} t_i(x).$$

We establish that $T^1_i(\bar{t}_i,\omega_{-i})$ satisfies the property defined in Lemma 8. Pick some $\sigma\in\Sigma^0$ and $\phi_{-i}\in\Upsilon(N^\star,\sigma_{-i}\circ\omega_{-i})$. By construction, there exists $x(\phi_{-i})\in\Upsilon_{-i}(N^\star)$ such that:

$$\phi_{-i} = \sum_{\ell=0}^{\dim(\Phi^\star_{-i})} x^\ell(\phi_{-i})\sigma_{-i}\circ\omega^\ell_{-i}.$$

[14]Caratheodory's theorem states that if a point $x\in\mathbb{R}^d$ lies in the convex hull of a set $P$, there is a subset $P'$ of $P$ consisting of $d+1$ points such that $x$ lies in the convex hull of $P'$.

[15]Indeed, $\bar{\Phi}_{-i}$ is compact in $\mathbb{R}^{|\Theta|(|\bar{M}_{-i}|+1)}$ and $\Phi^\star$ is a closed subset of $\bar{\Phi}_{-i}$.

For each $t^0_{-i} \in T^0_{-i}$, $\theta \in \Theta$ and $m_{-i} \in M^\star_{-i}$, we write $\sigma_{-i}(m_{-i}|t^0_{-i})$ (resp. $\phi_{-i}(m_{-i}|\theta)$) for the probability assigned to $m_{-i}$ by $\sigma_{-i}(t^0_{-i})$ (resp. $\phi_{-i}(\theta)$). For all $a \in A^\star$, $\theta \in \Theta$ and $m_{-i} \in M^\star_{-i}$, we have successively:

$$\begin{aligned}
\pi_i(\theta, \tilde{\theta}^a_i, m_{-i}|t_i(x(\phi_{-i})), \sigma) &= \sum_{t^0_{-i} \in T^0_{-i}} \kappa(t_i(x(\phi_{-i})))[\theta, \tilde{\theta}^a_i, t^0_{-i}]\sigma_{-i}(m_{-i}|t^0_{-i}) \\
&= \kappa(\bar{t}_i)[\theta]\varepsilon^a_i(\bar{r}, \theta) \sum_{t^0_{-i} \in T^0_{-i}} \sum_{\ell=0}^{\dim(\Phi^\star_{-i})} x^\ell(\phi_{-i}) \mathbf{1}_{\omega^\ell_{-i}(\theta)}(t^0_{-i})\sigma_{-i}(m_{-i}|t^0_{-i}) \\
&= \kappa(\bar{t}_i)[\theta]\varepsilon^a_i(\bar{r}, \theta) \sum_{\ell=0}^{\dim(\Phi^\star_{-i})} x^\ell(\phi_{-i}) \sum_{t^0_{-i} \in T^0_{-i}} \mathbf{1}_{\omega^\ell_{-i}(\theta)}(t^0_{-i})\sigma_{-i}(m_{-i}|t^0_{-i}) \\
&= \kappa(\bar{t}_i)[\theta]\varepsilon^a_i(\bar{r}, \theta) \sum_{\ell=0}^{\dim(\Phi^\star_{-i})} x^\ell(\phi_{-i})\sigma_{-i}(m_{-i}|\omega^\ell_{-i}(\theta)) \\
&= \kappa(\bar{t}_i)[\theta]\varepsilon^a_i(\bar{r}, \theta)\phi_{-i}(m_{-i}|\theta),
\end{aligned}$$

which concludes the proof. □

**Proof of Lemma 9.** Fix $\omega_{-i} \in \Omega_{-i}$ and $\bar{t}_i \in \bar{T}_i$. Notice that for each $\phi_{-i} \in \Upsilon(N^\star, \sigma_{-i} \circ \omega_{-i})$ and $m_i \in M^\star_i$, the expected utility of type $t_i(x(\phi_{-i}))$ (with the notation of Lemma 8) playing $m_i$ against $\sigma_{-i}$ is equal to:

$$\begin{aligned}
&\sum_\theta \kappa(\bar{t}_i)[\theta]u_i[g(m_i, \phi_{-i}(\theta)), \theta] + \vec{\varepsilon}(\bar{r}) \cdot g(m_i, \phi_{-i}(\theta)) \\
&= \sum_\theta \kappa(\bar{t}_i)[\theta]u^{\vec{\varepsilon}(\bar{r})}_i[g(m_i, \phi_{-i}(\theta)), \theta] = Eu^{\vec{\varepsilon}(\bar{r})}_{\bar{t}_i}(m_i, \phi_{-i}).
\end{aligned}$$

Consequently,

$$BR(\sigma \mid t_i(x(\phi_{-i}))) = BR^{\vec{\varepsilon}(\bar{r})}_{\bar{t}_i}(\phi_{-i}). \tag{11}$$

We now show that $T^1_i$ satisfies the property defined in Lemma 9. Fix some $e$-continuous equilibrium $\sigma$ in $U(\mathcal{M}^\star, \mathcal{T}^1)$. Notice that the strategy profile $\sigma_{|T^0}$ is an $e$-continuous equilibrium in $U(\mathcal{M}^\star, \mathcal{T}^0)$. Consequently, by Lemma 7, there exists $\omega_{-i} \in \Omega_{-i}$ such that the set $(\sigma_{|T^0})_{-i} \circ \omega_{-i}$ belongs to $\mathcal{S}_{-i}(\bar{r})$. Hence, applying Point 1 of Proposition 2, we know that there exists $\phi_{-i} \in \Upsilon(N^\star, (\sigma_{|T^0})_{-i} \circ \omega_{-i})$ such that $BR^{\vec{\varepsilon}(\bar{r})}_{\bar{t}_i}(\phi_{-i})$ is a singleton. Using Equation (11) and Lemma 8, we conclude that there exists $t_i \in T^1_i$ such that $BR(\sigma \mid t_i)$ is a singleton. □

**Proof of Lemma 10.** We first establish the following result.

**Claim 11** *For each $M_{-i} \subseteq M^{\star}_{-i}$ and $\bar{t}_i \in \bar{T}_i$, there exists a finite family $\mathbf{F}(\bar{t}_i, M_{-i}) \subseteq (\Delta(M_{-i}))^{\Theta}$ such that for any $\phi_{-i} \in (\Delta(M_{-i}))^{\Theta}$, there exists $\phi^{\star}_{-i} \in \mathbf{F}(\bar{t}_i, M_{-i})$ satisfying:*

$$BR_{\bar{t}_i}(\phi^{\star}_{-i}) = BR_{\bar{t}_i}(\phi_{-i}).$$

**Proof.** Fix $\bar{t}_i$ and $M_{-i} \subseteq M^{\star}_{-i}$. We define the (finite) set $\Lambda_i(\bar{t}_i, M_{-i})$ by:

$$\Lambda_i(\bar{t}_i, M_{-i}) := \{M'_i \subseteq M^{\star}_i | \exists \phi_{-i} \in (\Delta(M_{-i}))^{\Theta} \text{ s. t. } M'_i = BR_{\bar{t}_i}(\phi_{-i})\}.$$

For each $M'_i \in \Lambda_i(\bar{t}_i, M_{-i})$, define $\phi_{-i}(M'_i)$ by: $M'_i = BR_{\bar{t}_i}(\phi_{-i}(M'_i))$. Finally, set:

$$\mathbf{F}(\bar{t}_i, M_{-i}) := \bigcup_{M'_i \in \Lambda_i(\bar{t}_i, M_{-i})} \phi_{-i}(M'_i).$$

■

By upper hemi-continuity of correspondence $BR_{\bar{t}_i}(\cdot)$, there exists $\rho^{\star} \in (0, 1]$ such that for each $i$, $\bar{t}_i \in \bar{T}_i$, $M_{-i} \subseteq M^{\star}_{-i}$ and $\phi^{\star}_{-i} \in \mathbf{F}(\bar{t}_i, M_{-i})$:

$$BR_{\bar{t}_i}(\phi_{-i}) \subseteq BR_{\bar{t}_i}(\phi^{\star}_{-i}),$$

for each $\phi_{-i} = (1 - \rho^{\star})\phi^{\star}_{-i} + \rho^{\star}\phi'_{-i}$ with $\phi'_{-i} \in \Phi^{\star}_{-i}$. For notational simplicity, we write $\vec{\varepsilon}_{\star} \in \mathbb{R}^{|A|}_{+}$ for the vector $\vec{\varepsilon}(\bar{r}(\rho^{\star}))$ (with the notations of Proposition 2 and Lemma 11). Recall that $\vec{\varepsilon}_{\star}$ is such that $\sum_{a \in A} \varepsilon^a_{\star} \leq 1$. For each $\theta \in \Theta$, we set:

$$\varepsilon^a_i(\star, \theta) = \frac{\varepsilon^a_{\star}}{z(\theta, a, i)},$$

and,

$$\varepsilon^0_i(\star, \theta) = 1 - \sum_{a \in A} \varepsilon^a_i(\star, \theta).$$

We build inductively the family of models $\{\mathcal{T}^n\}_{n \geq 1}$ as follows. The first element of this family is the model $\mathcal{T}^1$ defined in Lemma 9. Now fix some integer $n$. For each $e$-continuous equilibrium $\sigma$ in $U(\mathcal{M}^{\star}, \mathcal{T}^n)$, each player $i \in \mathcal{I}$ and each message $m_i \in M^n_i(\sigma)$, we let $t_i(m_i, \sigma)$ be a player $i$'s type in $T^n_i \setminus T^0_i$ satisfying $BR(\sigma | t^n_i(m_i, \sigma)) = \{m_i\}$ and define the following equivalence classes over the set $\Sigma^n$ of $e$-continuous equilibria in $U(\mathcal{M}^{\star}, \mathcal{T}^n)$. For each $\sigma, \sigma' \in \Sigma^n$, $\sigma' \in [\sigma]$ if and only if for each player $i$:

1. We have: $M^n_i(\sigma) = M^n_i(\sigma')$;

2. For each $m_i \in M_i^n(\sigma)$: $t_i^n(m_i, \sigma) = t_i^n(m_i, \sigma')$.

Notice that since $T^n$ is finite, the set of equivalence classes defined above is finite. Fix $i \in \mathcal{I}$, $\bar{t}_i \in \bar{T}_i$ and $\omega_{-i} \in \Omega_{-i}$. For each $\sigma \in \Sigma^n$, $\phi^\star_{-i} \in \mathbf{F}(\bar{t}_i, M^n(\sigma))$ and $x \in \Upsilon_{-i}(N^\star)$, we define $t_i^{n+1}(x, [\sigma], \phi^\star_{-i})$ by:

$$\kappa(t_i^{n+1}(x, [\sigma], \phi^\star_{-i}))[\theta, \tilde{\theta}_i^a, t_{-i}^n(m_{-i}, \sigma)] = (1 - \rho^\star)\kappa(\bar{t}_i)[\theta]\varepsilon_i^a(\star, \theta)\phi^\star_{-i}(m_{-i}|\theta)$$

and:

$$\kappa(t_i^{n+1}(x, [\sigma], \phi^\star_{-i}))[\theta, \tilde{\theta}_i^a, t_{-i}^0] = \rho^\star\kappa(\bar{t}_i)[\theta]\varepsilon_i^a(\star, \theta) \sum_{\ell=0}^{\dim(\Phi^\star_{-i})} x^\ell \mathbf{1}_{\omega^\ell_{-i}[\theta]}(t_{-i}^0),$$

for all $\theta \in \Theta$, $a \in A^\star$, $m_{-i} \in M^\star_{-i}$ and $t_{-i}^0 \in T_{-i}^0$. Notice that since for each $m_{-i} \in M_{-i}^n(\sigma)$, $t_{-i}^n(m_{-i}, \sigma)$ does not belong to $T_i^0$, the belief $\kappa(t_i^{n+1}(x, [\sigma], \phi^\star_{-i}))$ is well-defined. We set:

$$T_i^{n+1}(\bar{t}_i, \omega_{-i}) = \bigcup_{x \in \Upsilon_{-i}(N^\star)} \bigcup_{[\sigma] \in [\Sigma^n]} \bigcup_{\phi^\star_{-i} \in \mathbf{F}(\bar{t}_i, M^n(\sigma))} t_i^{n+1}(x, [\sigma], \phi^\star_{-i}).$$

And finally, for each player $i$:

$$T_i^{n+1} = T_i^n \cup \{\bigcup_{\bar{t}_i \in \bar{T}_i} \bigcup_{\omega_{-i} \in \Omega_{-i}} T_i^{n+1}(\bar{t}_i, \omega_{-i})\}.$$

We establish the following lemma which is very similar to Lemma 8.

**Lemma 12** *Fix $\bar{t}_i \in \bar{T}_i$ and $\omega_{-i} \in \Omega_{-i}$. The type space $T_i^{n+1}(\bar{t}_i, \omega_{-i})$ satisfies the following property. For each e-continuous equilibrium $\sigma$ in $U(\mathcal{M}^\star, \mathcal{T}^n)$, $\phi^\star_{-i} \in \mathbf{F}(\bar{t}_i, M_{-i}^n(\sigma))$ and $\phi'_{-i} \in \Upsilon(N^\star, S_{-i}(\omega_{-i}, \sigma, \phi^\star_{-i}, \rho^\star))$, there is a type $t_i^{n+1}(\phi'_{-i}) \in T_i^{n+1}(\bar{t}_i, \omega_{-i})$ such that:*

$$\pi_i(\theta, \tilde{\theta}_i^a, m_{-i}|t_i^{n+1}(\phi'_{-i}), \sigma) = \kappa(\bar{t}_i)[\theta]\varepsilon_i^a(\star, \theta)\phi'_{-i}(m_{-i}|\theta),$$

*for all $a \in A^\star$, $\theta \in \Theta$ and $m_{-i} \in M^\star_{-i}$.*

**Proof.** Fix some $\sigma \in \Sigma^n$, $\phi^\star_i \in \mathbf{F}(\bar{t}_i, M_{-i}^n(\sigma))$ and $\phi'_{-i} \in \Upsilon(N^\star, S_{-i}(\omega_{-i}, \sigma, \phi^\star_{-i}, \rho^\star))$. By construction, there exists $x(\phi'_{-i}) \in \Upsilon_{-i}(N^\star)$ such that:

$$\phi'_{-i} = (1 - \rho^\star)\phi^\star_{-i} + \rho^\star \sum_{\ell=0}^{\dim(\Phi^\star_{-i})} x^\ell(\phi'_{-i})\sigma_{-i} \circ \omega^\ell_{-i}.$$

For all $a \in A^\star$, $\theta \in \Theta$ and $m_{-i} \in M^\star_{-i}$, the type $t_i^{n+1}(x(\phi'_{-i}), [\sigma], \phi^\star_{-i})$ is such that we have successively:

$$\pi_i(\theta, \tilde{\theta}_i^a, m_{-i} | t_i^{n+1}(x(\phi'_{-i}), [\sigma], \phi^\star_{-i}), \sigma)$$
$$=$$
$$\kappa(\bar{t}_i)[\theta]\varepsilon_i^a(\star, \theta)((1-\rho^\star)\phi^\star_{-i}(m_{-i}|\theta) + \rho^\star \sum_{t^0_{-i} \in T^0_{-i}} \sum_{\ell=0}^{\dim(\Phi^\star_{-i})} x^\ell(\phi'_{-i}) \mathbf{1}_{\omega^\ell_{-i}(\theta)}(t^0_{-i}) \sigma_{-i}(m_{-i}|t^0_{-i}))$$
$$=$$
$$\kappa(\bar{t}_i)[\theta]\varepsilon_i^a(\star, \theta)((1-\rho^\star)\phi^\star_{-i}(m_{-i}|\theta) + \rho^\star \textstyle\sum_{\ell=0}^{\dim(\Phi^\star_{-i})} x^\ell(\phi'_{-i}) \sigma_{-i}(m_{-i}|\omega^\ell_{-i}(\theta)))$$
$$=$$
$$\kappa(\bar{t}_i)[\theta]\varepsilon_i^a(\star, \theta)\phi'_{-i}(m_{-i}|\theta).$$

Hence, the type space $T_i^{n+1}(\bar{t}_i, \omega_{-i})$ satisfies the property described in Lemma 12. ■

We now conclude the proof of Lemma 10. Fix $\omega_{-i} \in \Omega_{-i}$ and $\bar{t}_i \in \bar{T}_i$. Notice that for each $\sigma \in \Sigma^n$, $\phi_i^\star \in \mathbf{F}(\bar{t}_i, M^n(\sigma))$, $\phi'_{-i} \in \Upsilon(N^\star, S_{-i}(\omega_{-i}, \sigma, \phi^\star_{-i}, \rho^\star))$ and $m_i \in M_i^\star$ the expected utility of type $t_i(x(\phi'_{-i}), [\sigma], \phi^\star_{-i})$ (with the notation used in Lemma 12 above) playing $m_i$ against $\sigma_{-i}$ is equal to:

$$\begin{aligned}&\sum_\theta \kappa(\bar{t}_i)[\theta](u_i[g(m_i, \phi'_{-i}(\theta)), \theta] + \vec{\varepsilon}_\star \cdot g(m_i, \phi'_{-i}(\theta))) \\ &= \sum_\theta \kappa(\bar{t}_i)[\theta] u_i^{\vec{\varepsilon}_\star}(g(m_i, \phi'_{-i}(\theta)), \theta) = E u_{\bar{t}_i}^{\vec{\varepsilon}_\star}(m_i, \phi'_{-i}).\end{aligned}$$

Consequently,

$$BR(\sigma \mid t_i(x(\phi'_{-i}), [\sigma], \phi^\star_{-i})) = BR^{\vec{\varepsilon}_\star}_{\bar{t}_i}(\phi'_{-i}). \tag{12}$$

We are now in imposition to show that the model $\mathcal{T}^{n+1}$ satisfies the properties described in Lemma 10. Since $\mathcal{T}^1 \subseteq \mathcal{T}^{n+1}$, by Lemma 9, Point 1 is satisfied. Regarding Point 2, fix some $e$-continuous equilibrium $\sigma$ in $U(\mathcal{M}^\star, \mathcal{T}^{n+1})$, $\bar{t}_i \in \bar{T}_i$ and $\phi_{-i} \in (\Delta(M^n_{-i}(\sigma_{|T^n})))^\Theta$. By Claim 11, there exists $\phi^\star_{-i} \in \mathbf{F}^n(\bar{t}_i, M^n_{-i}(\sigma_{|T^n}))$ such that $BR_{\bar{t}_i}(\phi^\star_{-i}) = BR_{\bar{t}_i}(\phi_{-i})$. By Lemma 11, there exists $\omega_{-i} \in \Omega_{-i}$ such that the set $S_{-i}(\omega_{-i}, \sigma, \phi^\star_{-i}, \rho^\star)$ belongs to $\mathcal{S}(\bar{r}(\rho^\star))$. Hence, by Proposition 2, there exists some $\phi'_{-i} \in \Upsilon(N^\star, S_{-i}(\omega_{-i}, \sigma, \phi^\star_{-i}, \rho^\star))$ such that $BR^{\vec{\varepsilon}_\star}_{\bar{t}_i}(\phi'_{-i})$ is a singleton included in $BR_{\bar{t}_i}(\phi'_{-i})$. Consequently, using Equation (12) and Lemma 12, we know that there is a type $t_i^{n+1} \in T_i^{n+1}$ such that $BR(\sigma \mid t_i^{n+1})$ is a singleton included in $BR_{\bar{t}_i}(\phi'_{-i})$. By construction of $\rho^\star$, this means that $BR(\sigma \mid t_i^{n+1})$ is a singleton included in $BR_{\bar{t}_i}(\phi^\star_{-i})$. By definition of $\phi^\star_{-i}$, this finally implies that $BR(\sigma \mid t_i^{n+1})$ is a singleton included in $BR_{\bar{t}_i}(\phi_{-i})$, which concludes the proof. □

# D Proof of Proposition 4

By definition of a rationalizable message, we know that for each $e$-continuous equilibrium $\sigma$ in $U(\mathcal{M}^\star, \tilde{\mathcal{T}})$ and each $m_i \in R_i^\infty(\bar{t}_i \mid \tilde{\mathcal{M}}(\sigma), \bar{\mathcal{T}})$ (where $\tilde{M}(\sigma)$ is the message space defined in Proposition 3), there must exist a belief $b(\bar{t}_i, m_i, \sigma) \in \Delta(\Theta \times \bar{T}_{-i} \times \tilde{M}_{-i}(\sigma))$ such that:

1. We have: $\text{marg}_{\Theta \times \bar{T}_{-i}}\, b(\bar{t}_i, m_i, \sigma) = \kappa(\bar{t}_i)$;

2. The message $m_i$ is a best response to $\text{marg}_{\Theta \times \tilde{M}_{-i}(\bar{\sigma})}\, b(\bar{t}_i, m_i, \sigma)$ when the message space of player $i$ is restricted to $\tilde{\mathcal{M}}_i(\sigma)$;

3. For each $\theta \in \Theta, \bar{t}_{-i} \in \bar{T}_{-i}$ and $m_{-i} \in \tilde{M}_{-i}(\sigma)$, the probability assigned to $(\theta, \bar{t}_{-i}, m_{-i})$ by the belief $b(\bar{t}_i, m_i, \sigma)$ is strictly positive only if $m_{-i}$ belongs to the set $R_{-i}^\infty(\bar{t}_{-i} \mid \tilde{\mathcal{M}}(\sigma), \bar{\mathcal{T}})$.

For ease of exposition, we sometimes consider $b(\bar{t}_i, m_i, \sigma)$ as a measure over $\Theta \times \bar{T}_{-i} \times \tilde{M}_{-i}(\sigma)$ and sometimes as a measure over $\Theta^\star \times \bar{T}_{-i} \times \tilde{M}_{-i}(\sigma)$ assigning probability one on $\{\tilde{\theta}^0\}$.

We write $\tilde{\Sigma}$ for the set of $e$-continuous equilibria in $U(M^\star, \tilde{T})$ (where $\tilde{\mathcal{T}}$ is the model defined in Proposition 3) and define the equivalence classes over $\tilde{\Sigma}$ as follows. For each $\sigma, \sigma' \in \tilde{\Sigma}$, we say that $\sigma' \in [\sigma]$ if and only if:

1. $\tilde{M}(\sigma') = \tilde{M}(\sigma)$, and,

2. For each $m_i \in \tilde{M}_i(\sigma')$: $\tilde{t}_i(m_i, \sigma') = \tilde{t}_i(m_i, \sigma)$ (with the notations of Proposition 3).

For each model $\mathcal{T} \supseteq \tilde{\mathcal{T}}$ and each $e$-continuous equilibria $\sigma$ in $U(\mathcal{M}^\star, \mathcal{T})$, we will identify $[\sigma]$ with $[\sigma_{|\tilde{T}}]$.

Now fix some equivalence class $[\sigma] \in [\tilde{\Sigma}]$. For each $i \in \mathcal{I}$, $\bar{t}_i \in \bar{T}_i$, $m_i \in R_i^\infty(\bar{t}_i \mid \tilde{\mathcal{M}}(\sigma), \bar{\mathcal{T}})$ and $\varepsilon > 0$, we define inductively the sequence of types $\{\hat{t}_i[\varepsilon, k, [\sigma], \bar{t}_i, m_i]\}_{k\in\mathbb{N}}$ as follows. Type $\hat{t}_i[\varepsilon, 1, [\sigma], \bar{t}_i, m_i]$ is defined by:

$$\kappa(\hat{t}_i[\varepsilon, 1, [\sigma], \bar{t}_i, m_i]) = \kappa(\tilde{t}_i(m_i, \sigma)).$$

And for each $k \geq 2$, $\hat{t}_i[\varepsilon, k, [\sigma], \bar{t}_i, m_i]$ is defined by

$$\kappa(\hat{t}_i[\varepsilon, k, [\sigma], \bar{t}_i, m_i]) = (1-\varepsilon) b(\bar{t}_i, m_i, \sigma) \circ \left(\tau_{-i}^{\varepsilon,k}\right)^{-1} + \varepsilon \kappa(\tilde{t}_i(m_i, \sigma)),$$

where $\left(\tau_{-i}^{\varepsilon,k}\right)^{-1}$ stands for the preimage of the function $\tau_{-i}^{\varepsilon,k} : (\theta^\star, \bar{t}_{-i}, m_{-i}) \longmapsto \left(\theta^\star, \hat{t}_{-i}[\varepsilon, k-1, \sigma, \bar{t}_{-i}, m_{-i}]\right)$.

Lemma 13 below shows that when $\varepsilon$ is sufficiently small and $k$ is sufficiently large, type $\hat{t}_i[\varepsilon, k, [\sigma], \bar{t}_i, m_i]$ is "very close" to type $\bar{t}_i$.

**Lemma 13** *For each integer $N$, there exist $k(N)$ and $\varepsilon(N) > 0$ such that for each $\sigma \in \tilde{\Sigma}, \bar{t}_i \in \bar{T}_i$ and $m_i \in R_i^\infty(\bar{t}_i \mid \tilde{\mathcal{M}}(\sigma), \bar{\mathcal{T}})$, we have:*

$$\hat{t}_i[\varepsilon(N), k(N), [\sigma], \bar{t}_i, m_i] \in \bar{B}_{\frac{1}{N}}(\bar{t}_i).$$

Recall that fonction $h_i$ (defined in Section 2.3) associates to each type $t_i$ its induced hierarchy of beliefs. We first establish the following claim.

**Claim 12** *For all $k \geq 1$ and $k' \geq k$:*

$$h_i^k\left(\hat{t}_i[0, k', [\sigma], \bar{t}_i, m_i]\right) = h_i^k\left(\bar{t}_i\right), \tag{13}$$

*for all $\sigma \in \tilde{\Sigma}$, $\bar{t}_i \in \bar{T}_i$ and $m_i \in R_i^\infty(\bar{t}_i \mid \tilde{\mathcal{M}}(\sigma), \bar{\mathcal{T}})$.*

**Proof.** First notice that Equation (13) holds true at rank $k = 1$. Indeed, for all $k' \geq 1$, $\bar{t}_i \in \bar{T}_i$ and $m_i \in R_i^\infty(\bar{t}_i \mid \tilde{\mathcal{M}}(\sigma), \bar{\mathcal{T}})$ :

$$\begin{aligned} h_i^1\left(\hat{t}_i[0, k', [\sigma], \bar{t}_i, m_i]\right) &= \operatorname{marg}_{\Theta^\star} \kappa(\hat{t}_i[0, k', [\sigma], \bar{t}_i, m_i]) \\ &= \operatorname{marg}_{\Theta^\star} b(\bar{t}_i, m_i, \sigma) \circ \left(\tau_{-i}^{0,k'}\right)^{-1} \\ &= \operatorname{marg}_{\Theta^\star} b(\bar{t}_i, m_i, \sigma) = \operatorname{marg}_{\Theta^\star} \bar{\kappa}(\bar{t}_i) = h_i^1(\bar{t}_i) \end{aligned}$$

where the third and the fourth equalities are by definition of $\tau_{-i}^{0,k'}$ and $b(\bar{t}_i, m_i, [\sigma])$ respectively. Now fix some $k \geq 2$ and let $L$ be the set of all belief profiles of players other than $i$ at order $k-1$. Toward an induction, assume that for all $k' \geq k-1$:

$$h_j^{k-1}(\hat{t}_j[0, k', [\sigma], \bar{t}_j, m_j]) = h_j^{k-1}(\bar{t}_j),$$

for each $j$, $\bar{t}_j \in \bar{T}_j$ and $m_j \in R_j^\infty(\bar{t}_j \mid \tilde{\mathcal{M}}(\sigma), \bar{\mathcal{T}})$. Then for all $k' \geq k$ :

$$\operatorname{proj}_{\Theta^\star \times L} \circ (id_\Theta \times h_{-i}) \circ \tau_{-i}^{0,k'} = \overline{\operatorname{proj}}_{\Theta^\star \times L} \circ \left(id_\Theta \times h_{-i} \times id_{\tilde{M}_{-i}([\sigma])}\right),$$

where $id_\Theta$ (resp. $id_{\tilde{M}_{-i}(\sigma)}$) is the identity mapping from $\Theta$ to $\Theta$ (resp. from $\tilde{M}_{-i}(\sigma)$ to $\tilde{M}_{-i}(\sigma)$) while $\operatorname{proj}_{\Theta^\star \times L}$ (resp. $\overline{\operatorname{proj}}_{\Theta^\star \times L}$) is the projection mapping from $\Theta^\star \times \mathcal{T}^\star$

to $\Theta^\star \times L$ (resp. from $\Theta^\star \times \mathcal{T}^\star \times \tilde{M}_{-i}(\sigma)$ to $\Theta^\star \times L$); hence for all $k' \geq k$, $\bar{t}_i \in \bar{T}_i$ and $m_i \in R_i^\infty(\bar{t}_i \mid \tilde{\mathcal{M}}(\sigma), \bar{\mathcal{T}})$ :

$$\begin{aligned}
\text{marg}_{\Theta^\star \times L}\ \kappa(\hat{t}_i[0,k',[\sigma],\bar{t}_i,m_i]) \circ (id_\Theta \times h_{-i})^{-1} &= \text{marg}_{\Theta^\star \times L}\ b(\bar{t}_i, m_i, \sigma) \circ \left(\tau_{-i}^{0,k'}\right)^{-1} \circ (id_\Theta \times h_{-i})^{-1} \\
&= b(\bar{t}_i, m_i, \sigma) \circ \left(\tau_{-i}^{0,k'}\right)^{-1} \circ (id_\Theta \times h_{-i})^{-1} \circ \left(\text{proj}_{\Theta^\star \times L}\right)^{-1} \\
&= b(\bar{t}_i, m_i, \sigma) \circ \left(id_\Theta \times h_{-i} \times id_{M_{-i}(\bar{\sigma})}\right)^{-1} \circ \left(\overline{\text{proj}}_{\Theta^\star \times L}\right)^{-1} \\
&= \text{marg}_{\Theta^\star \times L}\ b(\bar{t}_i, m_i, \sigma) \circ \left(id_\Theta \times h_{-i} \times id_{\tilde{M}_{-i}(\sigma)}\right)^{-1} \\
&= \text{marg}_{\Theta^\star \times L}\ \kappa(\bar{t}_i) \circ (id_\Theta \times h_{-i})^{-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
h_i^k(\hat{t}_i[0,k',[\sigma],\bar{t}_i,m_i]) &= \delta_{h_i^{k-1}(\hat{t}_i[0,k',[\sigma],\bar{t}_i,m_i])} \times \text{marg}_{\Theta^\star \times L}\ \kappa(\hat{t}_i[0,k',[\sigma],\bar{t}_i,m_i]) \circ (id_\Theta \times h_{-i})^{-1} \\
&= \delta_{h_i^{k-1}(\bar{t}_i)} \times \text{marg}_{\Theta^\star \times L}\ \kappa(\bar{t}_i) \circ (id_\Theta \times h_{-i})^{-1} = h_i^k(\bar{t}_i),
\end{aligned}$$

showing that $h_i^k(\hat{t}_i[0,k',[\sigma],\bar{t}_i,m_i]) = h_i^k(\bar{t}_i)$ for all $k' \geq k$. ■

Since the sets $[\tilde{\Sigma}]$, $\bar{T}$ and $M^\star$ are finite, the following Claim 12 concludes the proof of Lemma 14.

**Claim 13** *For all $k \geq 1$, $\sigma \in \tilde{\Sigma}, \bar{t}_i \in \bar{T}_i$ and $m_i \in \tilde{M}_i(\sigma)$, $h_i\left(\hat{t}_i[\varepsilon,k,[\sigma],\bar{t}_i,m_i]\right)$ tends toward $h_i\left(\hat{t}_i[0,k,[\sigma],\bar{t}_i,m_i]\right)$ as $\varepsilon$ tends toward zero.*

**Proof.** This is a rather direct consequence of the following fundamental result.

**Lemma 14 (Mertens and Zamir (1985) and Brandenburger and Dekel (1993))** *Let $\mathcal{T} = (T, \kappa)$ be any model such that $\Theta^\star \times T$ is complete and separable and $\kappa$ is a continuous function of $t_i$. Then, the mapping $h : T \to \mathcal{T}^\star$ is continuous.*

To apply Lemma 14, we set: $C := \{\bigcup_{q \in \mathbb{N}^\star} \frac{1}{q}\} \cup \{0\}$ and we define the (infinite countable) type space $\hat{T}_i[k,[\sigma],\bar{t}_i,m_i]$ by:

$$\hat{T}_i[k,[\sigma],\bar{t}_i,m_i] := \{\bigcup_{\varepsilon \in C} \hat{t}_i[\varepsilon,k,[\sigma],\bar{t}_i,m_i]\} \cup \tilde{T}_i,$$

Finally, we write $\mathcal{T}[k,[\sigma],\bar{t}_i,m_i]$ for the model associated with the type space $\hat{T}_i[k,[\sigma],\bar{t}_i,m_i] \times \tilde{T}_{-i}$ and we endow the model $\mathcal{T}[k,[\sigma],\bar{t}_i,m_i]$ with the topology associated with the following distance. For all $\varepsilon, \varepsilon' \in C$, the distance between types $\hat{t}_i[\varepsilon,k,[\sigma],\bar{t}_i,m_i]$ and $\hat{t}_i[\varepsilon',k,[\sigma],\bar{t}_i,m_i]$ is equal to $|\varepsilon - \varepsilon'|$. For any $t_i \in \tilde{T}_i$, the distance between $t_i$

and any other type $t'_i \in \hat{T}_i[k,[\sigma],\bar{t}_i,m_i]$ is equal to one. For any $j \neq i$, the distance between two types $t_j$ and $t'_j$ is also equal to one.

It can easily be checked that the model $\mathcal{T}[k,[\sigma],\bar{t}_i,m_i]$ satisfies the conditions of Lemma 14 above. Consequently, the fact that $\hat{t}_i[\varepsilon,k,[\tilde{\sigma}],\bar{t}_i,m_i] \to \hat{t}_i[0,k,[\tilde{\sigma}],\bar{t}_i,m_i]$ as $\varepsilon \to 0$ implies that $h_i\left(\hat{t}_i[\varepsilon,k,[\sigma],\bar{t}_i,m_i]\right)$ tends toward $h_i\left(\hat{t}_i[0,k,[\sigma],\bar{t}_i,m_i]\right)$ as $\varepsilon$ tends toward zero. ■

Now, we fix some integer $N$ and for each $i$, we set:

$$\hat{T}_i^N = \{\bigcup_{\bar{t}_i\in\bar{T}_i}\ \bigcup_{k=1}^{k(N)}\ \bigcup_{[\sigma]\in[\Sigma]}\ \bigcup_{m_i\in R_i^\infty(\bar{t}_i|\tilde{\mathcal{M}}(\sigma),\bar{\mathcal{T}})} \hat{t}_i[\varepsilon(N),k,[\sigma],\bar{t}_i,m_i]\} \cup \tilde{T}_i,$$

(where $k(N)$ and $\varepsilon(N)$ have been defined in Lemma 13). Since the mechanism $\mathcal{M}^\star$ allows for $e$-continuous implementation for all finite type spaces, we know that there exists an $e$-continuous equilibrium $\hat{\sigma}^N$ in the game $U(\mathcal{M}^\star,\hat{\mathcal{T}}^N)$. We set $M^N := \tilde{M}(\hat{\sigma}^N_{|\tilde{T}})$ (with the notations of Proposition 3).

Now pick some $\bar{t} \in \bar{T}$ and $m \in R^\infty(\bar{t} \mid \mathcal{M}^N,\bar{\mathcal{T}})$. Using the fact that $\hat{\sigma}^N$ is $e$-continuous and Lemma 13 above, Lemma 15 below allows to establish that $m \in \bar{B}_{e(N)}(f(\bar{t}))$, which concludes the proof of Proposition 4.

**Lemma 15** *For each $\bar{t}_i \in \bar{T}_i$ and $m_i \in R_i^\infty(\bar{t}_i \mid \mathcal{M}^N,\bar{\mathcal{T}})$, we have:*

$$\hat{\sigma}_i^N(\hat{t}_i[\varepsilon(N),k(N),[\hat{\sigma}^N],\bar{t}_i,m_i]) = \{m_i\}.$$

**Proof.** We show by induction on $k$ that for all $k \in \{1,...,k(N)\}$, $\bar{t}_i \in \bar{T}_i$, and $m_i \in R_i^\infty(\bar{t}_i \mid \mathcal{M}^N,\bar{\mathcal{T}})$:

$$\hat{\sigma}_i^N(\hat{t}_i[\varepsilon(N),k,[\hat{\sigma}^N],\bar{t}_i,m_i]) = \{m_i\}. \tag{14}$$

For $k=1$, by construction, we know that for any $\bar{t}_i \in \bar{T}_i$, and $m_i \in R_i^\infty(\bar{t}_i \mid \mathcal{M}^N,\bar{\mathcal{T}})$:

$$\kappa(\hat{t}_i[\varepsilon(N),1,[\hat{\sigma}^N],\bar{t}_i,m_i]) = \kappa(\tilde{t}_i(m_i,\hat{\sigma}^N_{|\tilde{T}})).$$

Hence, by definition of $\tilde{t}_i(m_i,\hat{\sigma}_{|\tilde{T}})$, the proof of this step is completed. Now fix some $k < k(N)$ and assume that for each $i$, $\bar{t}_i \in \bar{T}_i$, and $m_i \in R_i^\infty(\bar{t}_i \mid \mathcal{M}^N,\bar{\mathcal{T}})$, Equation (14) holds true at rank $k$. We show that the same property holds for $k+1$.

Fix $\bar{t}_i \in \bar{T}_i$, and $m_i \in R_i^\infty(\bar{t}_i \mid \mathcal{M}^N,\bar{\mathcal{T}})$. Recall that type $\hat{t}_i[\varepsilon(N),k+1,\bar{t}_i,m_{-i}]$ is defined by:

$$\kappa(\hat{t}_i[\varepsilon(N),k+1,\bar{t}_i,m_i]) = (1-\varepsilon(N))b(\bar{t}_i,m_i,\sigma^N_{|\tilde{T}})\circ(\tau_{-i}^{\varepsilon(N),k+1})^{-1} + \varepsilon(N)\kappa(\tilde{t}_i(m_i,\hat{\sigma}^N)).$$

Define the belief $\pi_i \in \Delta(\Theta^\star \times \hat{T}^N_{-i} \times M^\star_{-i})$ by:

$$\pi_i = \kappa(\hat{t}_i[\hat{\varepsilon}, k+1, [\hat{\sigma}^N], \bar{t}_i, m_i]) \circ \gamma^{-1}_{\hat{\sigma}^N},$$

where $\gamma_{\hat{\sigma}^N} : (\theta^\star, t_{-i}) \mapsto (\theta^\star, t_{-i}, \hat{\sigma}^N_{-i}(t_{-i}))$, for each $\theta^\star \in \Theta^\star$ and $t_{-i} \in \hat{T}^N_{-i}$. Notice that $\pi_i$ is the belief of type $\hat{t}_i[\hat{\varepsilon}, k+1, [\hat{\sigma}^N], \bar{t}_i, m_i]$ on $\Theta^\star \times \hat{T}^N_{-i} \times M^\star_{-i}$ when the equilibrium $\hat{\sigma}^N$ is played. On the one hand, we have:

$$\mathrm{marg}_{\Theta^\star \times M^\star_{-i}}\ \kappa(\tilde{t}_i(m_i, \hat{\sigma}^N_{|\tilde{T}})) \circ (\gamma_{\hat{\sigma}^N})^{-1} = \pi_i(\cdot|\tilde{t}_i(m_i, \hat{\sigma}^N_{|\tilde{T}}), \hat{\sigma}^N_{|\tilde{T}}). \tag{15}$$

On the other hand, the induction hypothesis implies:

$$\gamma_{\hat{\sigma}^N}(\theta^\star, \hat{t}_{-i}[\varepsilon(N), k, [\hat{\sigma}^N], \bar{t}_{-i}, m_{-i}]) = (\theta^\star, \hat{t}_{-i}[\varepsilon(N), k, [\hat{\sigma}^N], \bar{t}_{-i}, m_{-i}], m_{-i}),$$

for each $\theta^\star \in \Theta^\star$, $\bar{t}_{-i} \in \bar{T}_{-i}$ and $m_{-i} \in R^\infty_{-i}(\bar{t}_{-i}|\mathcal{M}^N, \bar{\mathcal{T}})$. Consequently,

$$\mathrm{marg}_{\Theta^\star \times M^N_{-i}}\ b(\bar{t}_i, m_i, \hat{\sigma}^N_{|\tilde{T}}) \circ \left(\tau^{\varepsilon(N),k+1}_{-i}\right)^{-1} \circ (\gamma_{\hat{\sigma}^N})^{-1} = \ \mathrm{marg}_{\Theta^\star \times M^N_{-i}}\ b(\bar{t}_i, m_i, \hat{\sigma}^N_{|\tilde{T}}). \tag{16}$$

Putting Equations (15) and (16) together we conclude that the belief $\pi_i(\cdot|\hat{t}_i[\varepsilon(N), k+1, [\hat{\sigma}], \bar{t}_i, m_i], \hat{\sigma}^N)$ of type $\hat{t}_i[\varepsilon(N), k+1, [\hat{\sigma}], \bar{t}_i, m_i]$ over $\Theta^\star \times M^N_{-i}$ when the strategy profile $\hat{\sigma}^N$ is played satisfies:

$$\pi_i(\cdot|\hat{t}_i[\varepsilon(N), k{+}1, [\hat{\sigma}^N], \bar{t}_i, m_i], \hat{\sigma}^N) = (1-\varepsilon(N))\ \mathrm{marg}_{\Theta^\star \times M^N_{-i}}\ b(\bar{t}_i, m_i, \hat{\sigma}^N_{|\tilde{T}}) + \varepsilon(N)\pi_i(\cdot|\tilde{t}_i(m_i, \hat{\sigma}^N_{|\tilde{T}}), \hat{\sigma}^N_{\tilde{T}}).$$

Recall that by definition $m_i$ is a best response to the belief $\mathrm{marg}_{\Theta^\star \times M^N_{-i}} b(\bar{t}_i, m_i, \hat{\sigma}^N_{|\tilde{T}})$ when the message space of player $i$ is restricted to $\tilde{\mathcal{M}}_i(\hat{\sigma}^N_{|\tilde{T}}) = M^N_i$. Consequently, by Point 1 of Proposition 3 (*closedness*) (setting $\phi_{-i} = \mathrm{marg}_{\Theta^\star \times M^\star_{-i}} b(\bar{t}_i, m_i, \hat{\sigma}^N_{|\tilde{T}})$), $m_i$ is also a best response to $\mathrm{marg}_{\Theta^\star \times M^N_{-i}} b(\bar{t}_i, m_i, \hat{\sigma}_{|\tilde{T}})$ when the message space of player $i$ is $M^\star_i$. Since the best response to the belief $\pi_i(\cdot|\tilde{t}_i(m_i, \hat{\sigma}^N_{|\tilde{T}}), \hat{\sigma}^N_{|\tilde{T}})$ (when the message space of player $i$ is $M^\star_i$) is the singleton $\{m_i\}$, this establishes that the best response to the belief $\pi_i(\cdot|\hat{t}_i[\varepsilon(N), k+1, [\hat{\sigma}^N], \bar{t}_i, m_i], \hat{\sigma}^N)$ must also be the singleton $\{m_i\}$. Finally, the fact that $\hat{\sigma}^N$ is an equilibrium allows to conclude that Equation (14) holds true at rank $k+1$. ■