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Overstating: A tale of two cities*

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Abstract

This work presents a rationale for the prevalent limits to voters' information disclosure in electoral settings. When allowed to express an intensity of preferences, strategic voters overstate in equilibrium of large multicandidate elections. Due to these overstatements, the set of voting equilibria of elections held under different voting rules coincide: the voting rules are strategically equivalent. Voters need not anymore overstate in electorates with few voters. However, enlarging the set of available grades does not significantly alter the set of possible winners in such elections.

KEYWORDS: Strategic voting, voting equilibria, intensity of preferences.

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Shouting and voting are believed to be activities of a different sort. Whereas the latter often goes hand in hand with the idea of finding an agreement (democracy and peace), the former is recurrently associated with the lack of agreement (conflict and wars). Indeed, to put an end to a dispute, one might suggest to switch from shouting to voting. However, even if such a distinction seems to be common sense, both activities have been remarkably close to each other in the past. For instance, Spartans shouted to elect senators to the Gerousia, Sparta's Council of Elders. Each "voter" was allowed to shout as much as he wanted for each of the different candidates. The candidate who had "the most and loudest" acclamations was elected senator[1]. Several centuries later, Venetian oligarchs used a related procedure to elect their Dogi. Instituted in 1268 and used until 1789, the Venetian system allowed voters to express their opinion about each candidate. A voter was given three balls to indicate approval, disapproval or *dubbio*[2]. Another variation of the Spartan method is simply the modern voting rule known as Approval Voting (AV), often advocated for its flexibility since it emerged in the literature in the mid 70s. As it is well known, a voter can either approve or not each of the candidates in an election held under AV.

Is shouting so different than voting? The previous question can be rephrased in formal voting theory by asking: should a voting rule allow the voters to express their intensity of preferences? The most used voting rule, Plurality Voting (henceforth PV[3]), only allows voters to disclose some limited information concerning one of the candidates running for the election (that is to approve her). Hence, a voter cannot express his degree of support for the candidate to whom he gives his votes. This property of Plurality Voting may lead to inefficiencies. To see this, consider a divided society in which the majority prefers some candidate: she will be elected independently of the minority intensity of preferences. Our work explores one of the solutions proposed to solve such inefficiencies: incorporating intensity of preferences at the level of the ballots and the voting rule. To do so, we consider a class of voting rules which incorporates Plurality Voting, Approval Voting and various scoring methods, and which allows more or less the expression of preference intensity by the voter.

For a voter to be able to express preference intensity ("to shout"), it must be the case that the set of ballots is rich enough, and that ballots are counted in a way that makes transparent to the voter how expressing a more intense preference for a candidate impacts this candidate's aggregate score. Therefore *additive* voting rules, in which a ballot is a list of points that the voter is affording to the candidates, and where points for each candidate are simply added, form a nice and natural family of voting rules to raise the question of expressivity. (A formal definition of this family of voting rules is provided in the next section.) With a rich set of ballots, a voter can suggest to the social planner through his vote *how much* he supports a candidate with respect to the others. Interestingly enough, the most studied voting methods such as PV, AV or the Borda rule are not that rich.

We analyze such an issue in the context of strategic voting, that is assuming that voters strategically cast their votes in order to maximize their expected utility (abstention is allowed). This approach raises a well-known methodological problem due to the poor predictive power of the Nash equilibrium concept in voting games without uncertainty.[4] To tackle this problem on large elections, one has to focus on settings in which voters face some uncertainty with respect to the consequences of their own vote. The first and simplest model in this direction has been proposed by Myerson and Weber (1993) [19]: for any pair of candidates, the voter considers that there is a positive probability that his vote is pivotal on this pair, but some of these probabilities are vanishingly small compared to others. Our work is built on this model. We prove that when one extends each of the previously mentioned rules by allowing intensities, the new, more flexible rule leads to the *same voting equilibria* and hence to the same set of possible winners. We denote this equivalence of voting equilibria by saying that both rules are *strategically equivalent*. This is due to the simple structure of the expected utility gains of voters: there is a linear relation between the points given to a candidate and the probability of changing the outcome of the election in favor of this candidate against any other candidate. The uncertainty faced by voters in the model plays the role of the shouts Spartans heard in the senatorial elections: we can imagine that the surrounding noise forced Spartans to either shout as much as they could or to remain silent[5].

The first consequence of strategic equivalence is that, in our terms, shouting is voting. The rules used in two cities lead to the same set of voting equilibria: in other words, Approval Voting and Evaluative Voting (EV)[6] are strategically equivalent. EV is the extension of AV in which the voter evaluates each candidate independently on the same numerical scale. This rule is also called Range Voting, and Approval Voting is Evaluative Voting with the scale $\{0, 1\}$.

The second consequence concerns a different family of voting rules in which no restriction is given over how the grades should be allocated between the different candidates. In an election held under Cumulative Voting (CV), a natural extension of Plurality Voting, a voter is endowed with a finite number of points, and he is allowed to distribute them freely between the different candidates. Different authors[7] have discussed such a method as it gives a high degree of flexibility to the voter. With such a voting rule, voters have the possibility of overstating their vote: that is to give the highest possible amount of points to only one of the candidates. We prove that this is indeed the case in equilibrium, implying that both PV and CV are strategically equivalent.

The third, more general, consequence applies to any voting rule. The voting rule $ext(V)$ is said to be an extension of the voting rule V if any ballot under $ext(V)$ is a linear transformation of a ballot under V. Note that both a voting rule and its extension are ordinally equivalent as a voter can convey the same ordinal information to the social planner under both of them, even though, more detailed (cardinal) information is allowed under the extension. Following similar reasonings as the ones previously explained, it can be shown that a voting rule and its extension are strategically equivalent.

The described equivalence between voting equilibria described is valid along the lines of the theory of large elections proposed by Myerson and Weber (1993) [19]. Nevertheless, small elections (that is elections with few voters) raise new questions. For instance, the information available to voters might be much more detailed in a small election than in a mass election, implying that such theory is of scant interest in the former case. In order to investigate whether the previous claims are robust in environments with few voters, we use trembling-hand perfection à la Selten (1975) [25] (due to the previously mentioned lack of predictive power of Nash equilibrium in voting environments). In a perfect equilibrium, the information voters have can be intricate as voters make uncorrelated mistakes when casting their ballots. Due to this intricate (and detailed, when compared to a large election) information, voters need not have an interest to systematically overstate. Indeed, we provide an example of an election held under EV with a reduced number of voters in which *the unique best response for a voter is not overstating.* The other interesting feature we analyze in environments with few voters is the set of possible winners in equilibrium. Even though we cannot fully characterize the set of perfect equilibria (due to the lack of structure of voters' mistakes) we are able to show that the equilibria under which "unappealing" candidates win the election both exist under a voting rule and its extension. In other words, the extension of the voting rule does not seem to refine the set of possible winners of the election.

This paper is organized as follows. Section 1 presents the basics of the model. Section 2 describes the concept of voting equilibrium in a large election; and Section 3 presents the strategic equivalence between the above-mentioned voting rules. Section 4 presents the results concerning the environments with few voters, and Section 5 gives concluding comments.

1 The setting

There are $\mathcal{N}$ voters in the election. Each voter has a type t that determines his strict preferences over the set of candidates $\mathcal{K} = \{1, 2, \ldots, \mathcal{K}\}$. The preferences of a voter with type t (a t-voter) is denoted by $u_t = (u_t(k))_{k\in\mathcal{K}}$. Thus, for a given t, $u_t(j) > u_t(k)$ implies that t-voters strictly prefer candidate j to candidate k. All types t belong to a finite set of types $\mathcal{T}$. The distribution of types is denoted by $r = (r(t))_{t\in\mathcal{T}}$ with $\sum_t r(t) = 1$: in other words, $r(t)$ represents the share of t-voters.

Within this work, we stick to the comparison of additive rules: a ballot is a vector

$b = (b_1, b_2, \ldots, b_{\mathcal{K}})$ where b_k is the number of points given to candidate k, to be added to elect the candidate with the largest score. We denote by e_j the ballot that assigns one point to candidate j and zero points to the rest of the candidates. Each voter must choose a ballot b from a finite set of possible ballots denoted by B.

For instance, in an election held under PV, voters can give one point to at most one candidate. Formally, we say that

$$b \text{ is a } PV \text{ ballot if } \forall\, j \in \mathcal{K},\ b_j \in \{0,1\} \textit{ and } \text{there is at most one } b_j \neq 0.$$

One can write:

$$B_{PV} = \{b \in \{0,1\}^{\mathcal{K}}, b_j \neq 0 \ \text{ for at most one } j\}.$$

Similarly, an AV ballot consists of a vector that lists whether each candidate has been approved or not. Hence, we say that

$$b \text{ is an } AV \text{ ballot if } \forall\, j \in \mathcal{K},\ b_j \in \{0,1\}.$$

One can write:

$$B_{AV} = \{0,1\}^{\mathcal{K}}.$$

Under Borda rule, a voter must assign $\mathcal{K}-1$ points to a candidate, $\mathcal{K}-2$ points to another candidate and so on. Then,

$$b \text{ is a Borda ballot if for each } l \in \{0,1\},\ \ b_j \in \{0,1,...,l(\mathcal{K}-1)\}$$
$$\textit{and } b_{(s)} - b_{(s+1)} = l \text{ for each } m\ ,$$

in which $b_{(s)}$ stands for the s^{th} highest valuation. One can write:

$$B_{BV} = \{b \in \{0,1,...,\mathcal{K}-1\}^{\mathcal{K}} : \sum_j b_j = \mathcal{K}(\mathcal{K}-1)/2\} \cup \{0\}.$$

The focus of this work is mainly on the previously described voting rules which represent the three main studied additive rules. For simplicity, we will refer to the voting rules presented within this work by:

1. the set of available grades I a voter can assign to each candidate.

2. the restrictions over how the grades must be placed (if any).

The aim of this work is to understand whether extending the sets I would modify the set of voting equilibria. We say that a voting rule with the set of available grades $\{0, \ldots, i\}$ allows better representation of the intensity of preferences than the same voting rule in which the set of available grades equals $\{0, \ldots, j\}$ with $i > j$.

2 Large Elections

We assume that each voter maximizes his expected utility to determine which ballot in the set B he will cast. In this model, his vote has an impact in his payoff if it changes the winner of the election. Therefore, a voter needs to estimate the probability of these situations: the pivot outcomes. We say that two candidates are tied if their vote totals are equal. Furthermore, let H denote the set of all unordered pairs of candidates. We denote a pair $\{i, j\}$ in H as ij with $ij = ji$.

For each pair of candidates i and j, the ij-pivot probability p_{ij} is the probability of the outcome perceived by the voters that candidates i and j will be tied for first place in the election. A voter perceives that the probability that he will change the winner of the election from candidate i to candidate j by casting ballot b with $b_i \geq b_j$ to be linearly proportional to $b_i - b_j$, and that the constant of proportionality (the ij-pivot probability) is the same for the perceived chance of changing the winner from j to i if $b_j \geq b_i$ [8].

A vector listing the pivot probabilities for all pairs of candidates is denoted by $p = (p_{ij})_{ij \in H}$. This vector p is assumed to be identical and common knowledge for all voters in the election. A voter with ij-pivot probability p_{ij} anticipates that submitting the ballot b can change the winner of the election from candidate j to candidate i to be $p_{ij} \max\{b_i - b_j, 0\}$.

Let $E_t[b]$ denote the expected utility gain of a t-voter from casting ballot b when p is the common vector of pivot probabilities:

$$E_t[b] = \sum_{ij \in H} (b_i - b_j) \cdot p_{ij} \cdot [u_t(i) - u_t(j)]. \qquad (1)$$

The expected utility gain from casting ballot b equals the expected utility of casting ballot b minus the expected utility of abstaining. Focusing on utility gains simplifies notation.

A (voting) strategy is a probability distribution σ over the set B that summarizes the voting behavior of voters of each type. For any ballot b and any type t, $\sigma(b \mid t)$ is the probability that a t-voter casts ballot b. Therefore,

$$\tau(b) = \sum_{t \in \mathcal{T}} r(t)\sigma(b \mid t)$$

is the share of the electorate who cast ballot b. Hence, the (expected) score of candidate k is

$$S(k) = \sum_{b \in B} b_k \tau(b).$$

The set of likely winners of the election contains the candidates whose expected score $S(k)$ is maximal given the strategy σ.

Myerson and Weber (1993) [19] assume that voters expect candidates with lower expected scores to be less likely serious contenders for first place than candidates with higher expected scores. In other words, if the expected score for some candidate l is strictly higher than the expected score for some candidate k, then the voters would perceive that candidate l's being tied with any third candidate m is much more likely than candidate k's being tied for first place with candidate m[9].

Definition 1. *Given a (voting) strategy σ and any $0 < \varepsilon < 1$, a pivot probability vector p satisfies the* ordering condition *for ε (with respect to σ) if, for every three distinct candidates i, j and k:*

$$S(i) > S(j) \Longrightarrow p_{jk} \leq \varepsilon p_{ik}.$$

Besides, Myerson and Weber (1993) [19] assume that the probability of three (or more) candidates being tied for first place is infinitesimal in comparison to the probability of a two-candidate tie.

Definition 2. *The strategy σ is a (voting)* equilibrium *of the game if and only if, for every positive number ε, there exists some vector p of positive pivot probabilities that satisfies the ordering condition and such that, for each ballot b and for each type t,*

$$\sigma(b \mid t) > 0 \implies b \in \arg\max_{d \in B} E_t[d].$$

It should be stressed that, in this definition, the pivot probabilities p_{ij} are supposed to be constant when the voter contemplates casting one ballot or the other. This point will play an important role in the next section. It is justified when the number of voters is large for, in that case, the voter cannot change with his single vote the order of magnitude of these probabilities. It can be shown that the set of equilibria is non-empty[10].

Finally, an important concept in our model should be defined: the equivalence between equilibria under different voting rules.

Definition 3. *An equilibrium σ_U of an election held under a voting rule U is equivalent to an equilibrium σ_V of the same election held under V if and only if*

1. *the pivot probabilities satisfy the same ordering and*

2. *the relative scores of the candidates coincide.*

The sets of voting equilibria of an election held under two voting rules U and V are equivalent if for any voting equilibrium of the election held under U (resp. V), there exists an equivalent voting equilibrium of the election held under V (resp. U)

Definition 4. *Two voting rules are strategically equivalent if and only if their set of voting equilibria are equivalent.*

We will pay special attention to the set of possible winners W_V that arise under a voting rule V. A possible winner is a candidate who wins the election in equilibrium with positive probability. We say that the set of possible winners of an election held under the voting rule V is such that

$$W_V = \{k \in \mathcal{K} \mid \text{There exists an equilibrium } \sigma \text{ in which } S(k) \text{ is maximal}\}.$$

It is simple to see that if the equilibria of an election held under two different voting rules are equivalent, the set of possible winners under both rules is equal. However, the converse need not be true.

It is noteworthy that the definition of strategic equivalence used is rather demanding. It requires more than the set of possible winners being the same under two voting rules. This demanding definition reinforces our results as we show that this strong version of equivalence holds in the Myerson-Weber setting.

3 Applications

3.1 Overstating

Let b and b' denote two ballots in an election held some voting rule. We let v stand for the vector of length k such that $v = b - b'$.

Lemma 1 (Overstating)**.** *For $v = b - b'$, $E_t[b] - E_t[b'] = E_t[v]$, which implies that:*

1. $E_t[b] > E_t[b'] \iff E_t[b+v] > E_t[b'+v]$.
2. $E_t[b] = E_t[b'] \iff E_t[b+v] = E_t[b'+v]$.

Proof. By assumption, the coefficients p_{ij} and the utilities $u_t(i)$ are independent of the voter's choice when casting his ballot. Thus, the lemma is an immediate consequence of the fact that formula (1) is linear with respect to b. □

3.2 One man, Many extended votes.

We first study the impact of allowing intensity of preferences under AV with which a voter can give at most one point to each of the candidates. This section proves that giving the possibility to give more points to each of the candidates (EV) does not change the set of equilibria and hence the possible winners of the elections.

Under Evaluative Voting, a voter can assign up to m points to each candidate for some positive m. Hence,

$$b \text{ is an } EV \text{ ballot if} \; \forall\, j \in \mathcal{K},\; b_j \in \{0, 1, \ldots, m\}.$$

One can write that

$$B_{EV} = \{0, \ldots, m\}^{\mathcal{K}}.$$

Lemma 2. *In an election held under EV, a voter's strict best response is to assign to any given candidate* $\begin{cases} \textit{either the highest score } m \\ \textit{or the lowest score 0.} \end{cases}$

If a voter assigns any other score to a candidate, he is indifferent about giving her any possible score.

Proof. Let b and b^* be two ballots in the set B_{EV} such that

$$b^* - b = e_j.$$

The expression of the expected utility gain (formula (1)) implies that:

$$E_t[b^*] - E_t[b] = E[e_j].$$

Hence, whenever the expected utility gain of voting e_j is strictly positive (strictly negative), the t-voter assigns the highest possible score m (lowest possible score 0) to candidate j. If the t-voter gets a nil expected utility gain of voting e_j, he is indifferent between assigning any number of points to candidate j. □

Prior to stating the strategic equivalence result, we need to establish next proposition: the set of equilibria in which some voters do not overstate is a subset of the equilibria in which every voter overstates. This result is important to state the strategic equivalence result between *AV* and *EV* as it allows us to establish a one-to-one correspondence between overstating equilibria under *EV* and equilibria under *AV*. Roughly speaking, non-overstating equilibria under *EV* "do not matter".

Proposition 1. *In an election held under EV, for any non-overstating equilibrium, there exists an equivalent overstating equilibrium.*

Proof. Let σ be a non-overstating equilibrium such that some t-voter's best response satisfies $\sigma(b \mid t) > 0$ with $b_1 = s \neq 0, m$. By Lemma 2, the t-voter is indifferent between assigning any number of points to candidate 1. The strategy σ^* depicted as follows assigns the same expected number of points to candidate 1 as the strategy σ[11]:

$$\sigma^*(m, b_2, \ldots, b_k \mid t) = \frac{s}{m} \text{ and } \sigma^*(0, b_2, \ldots, b_k \mid t) = 1 - \frac{s}{m},$$

and $\sigma^*(\cdot \mid t') = \sigma(\cdot \mid t') \; \forall t' \neq t,\, t' \in \mathcal{T}$. Besides, for every $\varepsilon > 0$, the pivot probability vector p that justifies the strategy σ also justifies the strategy σ^* as the scores of candidates coincide under both strategies, implying that σ^* is an overstating equilibrium.

All in all, both σ and σ^* are justified by the same pivot probability vector and under both of them, the expected scores of the candidates coincide. There, for any non-overstating equilibrium σ, there exists an equivalent overstating equilibrium σ^*. □

The previous results imply one of the main results of this work: namely, extending or reducing the set of available grades for each candidate does not modify the set of equilibria under Evaluative Voting.

Theorem 1. *EV and AV are strategically equivalent.*

Proof. Let σ be a strategy in an election held under EV in which every voter overstates (in other words, he only casts ballots that assign either 0 or m points to each of the candidates, the set of which is denoted by $\overline{B_{EV}}$). Let us now denote by σ^* a strategy in the same election that satisfies

$$\sigma^*(b^* \mid t) = \sigma(b \mid t), \quad (2)$$

in which each ballot b^* satisfies $b^* = \frac{1}{m}b$, for each of the ballots b cast with positive probability in the strategy σ. Note that the strategy σ^* uniquely assigns positive weight to ballots which are admissible under AV.

Let us now prove that the strategy σ is an equilibrium of the election held under EV if and only if the strategy σ^* is an equilibrium of the election held under AV with both σ and σ^* being equivalent. Let us first prove that casting ballot b is a best response given σ if and only if casting ballot b^* is a best response given σ^*:

$$\begin{aligned} b \in \arg\max_{d \in \overline{B_{EV}}} E_t[d] \iff E_t[b] \geq E_t[d] \ \forall\, d \in \overline{B_{EV}} &\iff mE_t[\frac{1}{m}b] \geq mE_t[\frac{1}{m}d] \ \forall\, d \in \overline{B_{EV}} \\ &\iff E_t[b^*] \geq E_t[d^*] \ \forall\, d^* \in B_{AV} \\ &\iff b^* \in \arg\max_{d^* \in B_{AV}} E_t[d^*]. \end{aligned}$$

Besides, given that the strategy σ^* satisfies (2), the scores of the candidates $S(\cdot)$ given σ and $S^*(\cdot)$ given σ^* satisfy

$$S^*(k) = \frac{1}{m}S(k) \ \forall\, k \in \mathcal{K},$$

and whence the relative scores of candidates coincide under both strategies.

In order to finish the proof, it remains to be proved that pivot probabilities satisfy the same ordering in both equilibria σ and σ^*. However, as the relative scores of candidates coincide in both equilibria, the same pivot probabilities vector p satisfies the same ordering condition under both σ and σ^*. We have proved so far that σ is an overstating equilibrium with EV if and only if there exists an equivalent equilibrium σ^* under AV. But the previous equivalence finishes the proof as by Lemma 1, any non-overstating equilibrium is equivalent to an overstating equilibrium in an election held under EV. □

3.3 One man, One extended vote.

We now move into the study the impact of allowing intensity of preferences under PV in which a voter can only vote for one candidate. We show that the set of equilibria under PV is equivalent to the set of equilibria under two families of voting rules: extended PV and Cumulative Voting. Under extended PV a voter can give at most m points to a unique candidate. Under Cumulative Voting, a voter is endowed m points and is allowed to distribute them freely among the different candidates.

In an election held under extended PV, voters can give up to m points to at most one candidate. Formally, we say that

$$b \text{ is an extended } PV \text{ ballot if } b_j \in \{0, \ldots, m\} \textit{ and } \text{there is at most one } b_j \neq 0,$$

which implies that the set of available ballots under extended PV satisfies

$$B_{EPV} = \{b \in \{0, \ldots, m\}^{\mathcal{K}},\ b_j \neq 0 \text{ for at most one } j\}.$$

Lemma 3. *In an election held under extended PV, a voter's strict best response only includes ballots in which one candidate gets m points while the rest of them get zero points.*

Proof. Let b and b^* denote two ballots in the set B_{EPV} such that

$$b^* - b = e_j.$$

As $E_t[b^*] - E_t[b] = E_t[e_j]$, voters either assign the minimum or the maximum number of points to candidate j if $E_t[e_j] \neq 0$. To see why no indifferences are allowed (in contrast with the case of EV), let us assume that $E_t[e_j] = 0$. By formula (1), we know that, for some positive integer s,

$$E_t[b^*] = E_t[se_j] = sE_t[e_j] = 0 = E_t[b].$$

The previous equality implies that both b and b^* are strictly dominated as for instance giving one point to his preferred candidate h, gives a strictly positive expected utility gain to a t-voter. Formally, denoting by b' such ballot, we can write

$$E_t[b'] = \sum_{j \neq h} (1 - 0) \cdot p_{jh} \cdot (u_t(h) - u_t(j)) > 0 = E_t[b^*] = E_t[b].$$

We can therefore infer that casting ballots b and b^* is not a best response: under extended PV, a voter's best response only include overstating ballots. □

Theorem 2. *PV and extended PV are strategically equivalent.*

Proof. The proof is analogous to the one of Theorem 1. As in an election held under extended PV there only exist overstating equilibria, defining strategies σ and σ^* in a similar manner, it can be shown that both PV and extended PV are strategically equivalent. □

In an election held under Cumulative Voting, a voter can assign up to m points to each candidate for some positive m with the restriction that the sum of the points he can assign to each of the candidates is at most m. Hence,

$$b \text{ is a } CV \text{ ballot if } b_j \in \{0, 1, \ldots, m\} \forall\, j \in \mathcal{K}$$
$$\textit{and} \sum_{j \in \mathcal{K}} b_j \leq m \text{ for some positive integer } m.$$

One can define the set of available ballots under Cumulative Voting as follows

$$B_{CV} = \{b \in \{0, \ldots, m\}^{\mathcal{K}}, \sum_{j \in \mathcal{K}} b_j \leq m \text{ for some positive integer } m.\}.$$

Definition 5. *Candidate j is a top candidate for a t-voter if and only if $E_t[e_j] \geq \max_{k \in \mathcal{K}} E_t[e_k]$.*

Lemma 4. *In an election held under CV, a voter's best response uniquely includes ballots in which only top candidates get positive scores.*

Proof. Let σ be a strategy in an election held under CV in which the t-voter's best response is such that $\sigma(b \mid t) > 0$ with $b_1 = s \neq 0, m$. Let us suppose that candidate 1 is the unique top candidate for t-voters. Hence, we can write that $E_t[e_1] - E_t[e_j] > 0$ for each $j \neq 1$, $j \in \mathcal{K}$. Let b^* and b be two CV ballots such that

$$b^* - b = e_1 - e_i.$$

We can write that

$$E_t[b^*] - E_t[b] = E_t[e_1] - E_t[e_i] > 0, \quad (3)$$

as a consequence of formula (1). Hence, assigning the total number of points to candidate 1 is the unique best response for t-voters as inequality (3) holds independently of the value of b_1. Hence, a voter's best response under Cumulative Voting coincides with a voter's best response under extended Plurality voting ballots whenever there is a unique top candidate for every t-voter.

Let us consider the case in which some t-voter has two top candidates, i and j. The formula (1) implies that only candidate i and j can get positive scores. Indeed, any ballot b with

$$b_i + b_j = m \text{ and } b_k = 0 \ \forall\, k \neq i, j,$$

strictly dominates any other ballot under CV for the t-voter, proving the claim. $\square$

Theorem 3. *CV and PV are strategically equivalent.*

Proof. The proof is analogous to the one of Proposition 1. It suffices to show that for every non-overstating equilibrium, there exists an equivalent overstating equilibrium in an election held under CV. Indeed, it is sufficient as the set of overstating ballots under CV coincides with the set of overstating ballots under extended PV. Therefore, CV and extended PV are strategically equivalent as voters' best responses only include overstating ballots in elections held under extended PV (Lemma 3). This in turn implies that CV and PV are strategically equivalent as both PV and extended PV are strategically equivalent (Theorem 2) and hence finishes the proof. □

3.4 A strategically equivalent extension

The tendency of voters to overstate their vote when allowed to do so is more general than the one depicted in the cases of AV and PV. Indeed, this phenomenon applies to any voting rule in the following sense.

For any voting rule V, we denote the finite set of possible ballots by B in which ballots are vectors such that $b = (b_1, b_2, \ldots, b_{\mathcal{K}})$.

Definition 6. *The extension of the voting rule V is denoted by $ext(V)$. A ballot b is in the set $B_{ext(V)}$ if and only if there exists a ballot $b^* \in B$ such that $b = l \cdot b^*$ for some positive integer $l = 0, 1, \ldots, m$.*

Formally,

$$B_{ext(V)} = \bigcup_{l=0}^{m} l \cdot B.$$

The set of voting equilibria is not modified by this extension of the set of possible ballots as next result shows (no proof is provided as it is analogous to the one of Theorem 1).

Theorem 4. *V and $ext(V)$ are strategically equivalent.*

A simple application of Theorem 4 applies to the well-known case of Borda rule. We propose an extended version for the Borda rule in which a voter can assign $l(\mathcal{K}-1)$ points to a candidate, $l(\mathcal{K}-2)$ points to another candidate and so on for any integer $l \in \{0, \ldots, m\}$ (rather than $\mathcal{K}-1$ to a candidate, $\mathcal{K}-2$ to another one and so on with Borda rule). This extension preserves the spirit of Borda rule in the sense that there is always the same difference between two consecutive scores assigned to candidates.

Formally, for some positive integer m,

$$b \text{ is an extended Borda ballot if for each } l \in \{0, \ldots, m\},\ \ b_j \in \{0, 1, ..., l(\mathcal{K}-1)\}$$
$$\textit{and } b_{(s)} - b_{(s+1)} = l \text{ for each } s,$$

in which $b_{(s)}$ stands for the s^{th} highest valuation. Note that an extended Borda ballot is a Borda ballot whenever $l \in \{0,1\}$. One can write:

$$B_{EBV} = \{b \in \{0,1,...,l(\mathcal{K}-1)\}^{\mathcal{K}} : \text{for each } l \in \{0,\ldots,m\},\ \sum_j b_j = lK(\mathcal{K}-1)/2\} \cup \{0\}.$$

It is simple to see that Borda and extended Borda rule are strategically equivalent since extended Borda satisfies Definition 6 and hence the logic of Theorem 4 applies.

4 Small Elections

The previous results can then be summarized as follows:

- the set of equilibria under a voting rule V and its extension $ext(V)$ are equivalent and
- therefore the set of possible winners W_V and $W_{ext(V)}$ are identical under both rules.

These results are a consequence of the model used in which voters' perceptions over the impact of their ballots in switching the winner of the election have a very specific shape. Even if such a theory is far from perfect, it fits particularly well the study of mass elections. As shown by further developments of the theory[12] , more formal models give, roughly speaking, similar predictions depending on whether the ordering condition is satisfied. However, it seems that neither the specific shape of expected utility nor the ordering condition are particularly relevant for studying voting in committees (that is voting with few voters). Indeed, in a committee, the information a voter knows can be much more detailed than in a large election. Hence in order to address the previous issues in environments with few voters, we focus on trembling-hand perfection.

The formal definition of trembling-hand perfection is as follows:

Definition 7. *A completely mixed strategy $\sigma^{\varepsilon}_{\mathcal{N}}$ is an ε-perfect equilibrium in an $\mathcal{N}$-voters game if*

$$\forall\, i \in \mathcal{N},\ \forall\, b^i, \bar{b}^i \in B,\ \textit{if } U_i(b^i, \sigma^{\varepsilon}_{\mathcal{N}}) > U_i(\bar{b}^i, \sigma^{\varepsilon}_{\mathcal{N}})\ ,\ \textit{with } \sigma^{\varepsilon}(\bar{b}^i) \leq \varepsilon,$$

in which $U_i(b)$ denotes the payoff of voter i given the strategy combination b. We refer to the strategy combination $\sigma_{\mathcal{N}}$ as a perfect equilibrium if there exists a sequence $\{\sigma^{\varepsilon}_{\mathcal{N}}\}$ of ε-perfect equilibria converging (for $\varepsilon \to 0$) to σ.

4.1 Overstating need not be optimal

Let us consider a voting game in which there are three candidates $\mathcal{K} = \{1,2,3\}$ and three different types $\mathcal{T} = \{\mathtt{a},\mathtt{b},\mathtt{c}\}$, with cardinal utilities given by:

$$u_{\mathtt{a}} = (3,1,0),\quad u_{\mathtt{b}} = (0,3,1) \text{ and } u_{\mathtt{c}} = (0,1,3).$$

There are seven voters in the electorate. Voters 1 and 2 have type `a`, voters 3 and 4 have type `b` and voters 5, 6 and 7 have type `c`.

We consider Evaluative Voting in which voters can give up to two points to each of the candidates.

Proposition 2. *In an election held under EV, voters' unique best responses need not be overstating in a perfect equilibrium.*

Proof. See the appendix. □

This proposition shows that Lemma 1 does not hold in a perfect equilibrium. Indeed, a perfect equilibrium is the limit of completely mixed strategies of the voters that arise as a consequence of uncorrelated mistakes of the voters. Hence, voters' expected utility is not anymore "smooth" as it is by assumption in the large elections model. Even though the definition of strategic equivalence used within this work does not directly apply to the perfect equilibrium context, it seems intuitive that the equivalence between *EV* and *AV* does not anymore hold.

4.2 Possible Winners remain unchanged

We now address the issue of the set of possible winners in an election. To do so, we give a proposition which extends a previous result of De Sinopoli (2000) [7] (which focused in Plurality Voting). We show that any candidate who is not a Condorcet loser can win the election under Plurality Voting, extended Plurality Voting and Cumulative Voting.

Prior to stating it, we need the definition of Condorcet loser.

Definition 8. *Candidate k' is a Condorcet loser if*

$$\#\{i \in \mathcal{N} \mid u_i(k) > u_i(k')\} > \#\{i \in \mathcal{N} \mid u_i(k') > u_i(k)\} \;\; \forall\, k \,\in\, \mathcal{K} \setminus k'.$$

Proposition 3. *In an election held under either PV, extended PV or CV with at least 4 voters, for every candidate k who is not a Condorcet loser there exists a perfect equilibrium in which k wins the election.*

Proof. Let 1 and 2 be two candidates who are not Condorcet losers. Let us divide the voters in two groups: the voters who prefer candidate 1 to candidate 2, $V(1,2) = \{i \in \mathcal{N} \mid u_i(1) > u_i(2)\}$, and the remaining ones $V(2,1) = \{i \in \mathcal{N} \mid u_i(2) > u_i(1)\}$. Under both extended PV and CV, a voter can assign up to m points to a single candidate. Under PV, the proof remains unchanged with the constraint that $m = 1$. Consider the mixed strategy d^ε such that for every voter $i \in V(1,2)$, where η_i denotes the mixed strategy of voter that assigns equal probability to all his pure strategies with obvious notations,

$$d_i^\varepsilon = (1 - \varepsilon - \varepsilon^2)(m, 0, \ldots, 0) + \varepsilon(0, m, 0 \ldots, 0) + \varepsilon^2 \eta_i,$$

and such that for every voter $i \in V(2,1)$,

$$d_i^\varepsilon = (1-\varepsilon-\varepsilon^2)(0,m,\ldots,0) + \varepsilon(m,0,0\ldots,0) + \varepsilon^2\eta_i.$$

For each voter, the pivot event which becomes infinitely more likely as ε tends towards zero is one in which candidates 1 and 2 are involved.[13] Hence, each voter plays his best response with probability higher than ε in the sequence of mixed strategies d^ε. Besides, as ε approaches zero, every voter in the set $V(1,2)$ votes for candidate 1, and every other voter votes for candidate 2, which implies that either candidate 1 or candidate 2 wins the election, proving the claim. □

The previous result implies that extending the set of available grades in the case of PV does not refine in a relevant way the set of possible winners of elections with few voters.

The reason why the equilibrium depicted by the Proposition 3 can be constructed is simple. For any pair of candidates 1 and 2 (who are not Condorcet losers), we split the electorate in two blocs: the ones who prefer candidate 1 to candidate 2 (the partisans of candidate 1) and the ones who prefer candidate 2 to candidate 1 (the partisans of candidate 2). Let us assume that partisans of candidate 1 assign her the maximum number of points whereas partisans of candidate 2 behave in the same manner with respect to candidate 2. Each of the two blocs is homogenous in the sense that each voter makes the same mistakes. Hence, when casting his ballot, a voter knows almost surely that, provided being pivotal, his vote will break the close race between candidates 1 and 2. Therefore, it is a best response for the partisans of a candidate to assign her the maximum number of points, proving that this is an equilibrium. The three voting rules analyzed in the Proposition 3 share the feature that a voter can assign the total number of points to a single candidate, leading to the construction of this "almost-everything-can-happen" type of result.

The Majority Preferred Candidate In order to conclude our investigation in the case of a reduced number of voters, we focus on the majority preferred candidate situation, in a similar spirit to the one depicted by Nuñez (2010) [21]. Let us consider a voting game held under Evaluative Voting. There are three types of voters in the electorate:

$$u_{\mathtt{a}} = (3,0,1), u_{\mathtt{b}} = (1,3,0) \text{ and } u_{\mathtt{c}} = (1,0,3),$$

with voters 1,2 being of type a, voters 3 to 7 being of type b and voters 8 to 10 of the third type. We will refer to candidate 2 as the *majority preferred candidate* as 5 voters over 10 rank him first. Candidate 1 is only ranked as a first option by two over ten voters in the election but it can nevertheless be elected at equilibrium in elections held under EV and AV as shown by next result.

Proposition 4. *There exists a perfect equilibrium in which candidate 1 is the unique winner of the election held under both EV and AV.*

Proof. Under EV, a voter can assign up to m points to a single candidate. Consider the mixed strategy e^{ε} where η_i denotes the mixed strategy of a voter that assigns equal probability to all his pure strategies,

$$\begin{aligned}
e_i^{\varepsilon} &= (1-\varepsilon-\varepsilon^2)(m,0,0)+\varepsilon^2\eta_i \text{ with } i=1,2\\
e_i^{\varepsilon} &= (1-\varepsilon-\varepsilon^2)(m,m,0)+\varepsilon(0,0,m)+\varepsilon^2\eta_i \text{ with } i=3,\dots,7,\\
e_i^{\varepsilon} &= (1-\varepsilon-\varepsilon^2)(0,0,m)+\varepsilon^2\eta_i \text{ with } i=8,\dots,10.
\end{aligned}$$

For each voter $i=1,2$, the pivot event which becomes infinitely more likely as ε tends towards zero is $(4m,3m,5m)$ so that it is a strict best response to vote only for his first-ranked candidate. Similarly, for each voter $i=3,\dots,7$, the pivot event which becomes infinitely more likely as ε tends towards zero is $(4m,2m,5m)$ so that it is a strict best response to vote for his first-ranked and his second-ranked candidate. Finally, the event that determines voters $i=8,9,10$'s best responses is $(5m,3m,4m)$ and hence their unique best response is to cast ballot $(0,0,m)$. Besides, as ε approaches zero, candidate 1 wins the election as every voter who votes for candidate 2 also votes for candidate 1, proving the claim. $\square$

The bottom-line of this example is that even if we do not provide a characterization of possible winners under Evaluative Voting, enlarging the set of possible grades does not remove the coordination problems already present under Approval Voting. Hence, one can intuitively think that the set of possible winners should not be too refined by EV (when compared to AV), if at all. Similar coordination problems as the ones illustrated by Proposition 4 have been already identified by Nuñez (2010) [21] in the case of AV. The logic of this unattractive equilibrium boils down to voters' anticipations. In a certain manner, AV performs better than PV in preference aggregation as, with the former voting rule the voter does not face the classical trade-off between voting for his preferred candidate and voting for his preferred likely winner (the wasted-vote effect). However, this property of AV (and of EV) may not be enough to ensure a correct preference aggregation in every election. If the majority of voters anticipate that their preferred candidate is not included in the most probable pivot outcome, this may lead to the election of an unappealing candidate. Indeed, due to their anticipations, the majority of voters favors their preferred likely winner by assigning her the maximum number of points and at the same time vote for their preferred candidate, leading to the election of the former candidate.

5 Conclusion

We have pointed at an unnoticed and important consequence of the theory of strategic voting, which explains the prevalent limits to voters' information disclosure in electoral settings. When strategic voters are allowed to overstate, they do so in a large election.

These overstatements imply that neither the set of voting equilibria nor the set of possible winners is affected when giving the possibility of expressing intensity of preferences to voters. In other words, there is no difference between shouting and voting when voters act strategically.

This equivalence between the sets of equilibria does not hold anymore in an context with a reduced number of voters, using trembling-hand perfection as equilibrium concept. However, the fact that voting equilibria are not equivalent does not imply that the set of winners of elections held under a voting rule and its respective extension do not coincide. Indeed, we provide some (partial) results that suggest that extending the set of available grades to voters does not refine the set of possible winners of the election.[14]

Our results imply that the research agenda on the strategic analysis of voting rules should focus more on the restrictions over how the grades must be placed than on the number of available grades. An interesting extension of the present work would be to understand whether similar results apply in multi-seat elections in which voters have to distribute their votes.

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Notes

[1] See Girard (2010) [10] for a detailed account of the Spartan voting method.

[2] The italian *dubbio* corresponds to the English doubt. According to Lines (1986) [15], this doubt is roughly equivalent to an abstention as "a doubt vote, if it ever did exist in doge elections, would essentially be a no vote".

[3] In an election held under PV, a voter is allowed to give at most one point to at most one candidate. The candidate with the most votes wins the election. The most used rule for presidential elections is Plurality with a Runoff (Blais (1997) [2]), but we here restrict attention to one-round voting systems.

[4] For instance, take an election held under Plurality Voting with more than three voters. Every voter voting for the same candidate is an equilibrium, as one vote cannot change the outcome of the election.

[5] Actually, we do not know this. Notice that the "Spartan shout" was not anonymous. This lack of anonymity might have triggered social stigmas associated with excessive shouting, unpopular candidates and related issues. In the present paper, votes are anonymous and voters derive utility only from the outcome of the election.

[6] Axiomatic work has advocated the use of Evaluative Voting and has referred to it mainly as Relative Utilitarianism. See Karni (1998) [12], Dhillon and Mertens (1999) [8], Segal (2000) [24] and d'Aspremont and Gevers (2002) [6].

[7] See Sawyer and McRae (1962) [23], Brams (1975) [4], Nitzan (1985) [20], Cox (1990) [5] and Gerber et. al (1998) [9].

[8] This is roughly equivalent to assume that the probability of candidates i and j being tied for first place is the same as the probability of candidate i being in first place one vote ahead candidate j (and both candidates above the rest of the candidates), which is in turn the same one as the probability of candidate j being in first place one vote ahead candidate i. Myerson and Weber (1993) [19] justify this assumption by arguing that it seems reasonable when the electorate is large enough. This is not verified in Poisson games, a formal model of Large elections in which the pivot probabilities are derived endogenously from the structure of the game.

[9] This assumption is needed in order to ensure the existence of equilibrium. The results presented within this work do not lie on the ordering of the pivot probabilities.

[10] See Theorem 1, page 105 in Myerson and Weber (1993) [19].

[11] There is a slight abuse of notation within the proof. We implicitly assume that the equilibrium σ is in pure strategies. Similar arguments can be used to extend the proof whenever σ involves that some t-voters play in mixed strategies.

[12] See Myerson (1998) [16], Myerson (2000) [17], Myerson (2002) [18], Laslier (2009) [14], Núñez (2009) [22], Bouton and Castanheira (2010) [3], Goertz and Maniquet (2010) [11], and Nuñez (2010) [21].

[13] If there were less than four voters in the election, a ballot need not be only pivotal between candidates 1 and 2. The restriction concerning the minimal number of voters cannot be dropped.

[14] We have very few observations to back up, or to invalidate, these theoretical results. Laslier and Van der Straeten (2004) [13] report on an experiment comparing EV with the 0 to 10 scale and AV, and Baujard and Igersheim (2010) [1] report on an experiment comparing EV with the 0-1-2 scale, and AV. In both cases it is observed that the outcome of the election (the elected candidate) is the same under the two systems, even if it is not observed that voters concentrate on extreme grades.

6 Appendix

Proof of Proposition 2.

Proof. In an election held under Evaluative Voting in which voters can give up to two points to each of the three candidates, voters have three undominated strategies: to give two points to their favorite candidate, no points to their least preferred candidate and zero, one or two points to their middle ranked candidate. The previous observation is important, as in a perfect equilibrium voters only choose undominated strategies[15]. It is easy to see that the strategy combination

$$f = ((2,0,0),(2,0,0),(0,2,1),(0,2,1),(0,0,2),(0,0,2),(0,0,2))$$

is an undominated equilibrium in which $\mathtt{b}$-voters do not overstate and in which candidate 3 wins the election. Consider the following completely mixed strategy combination f^{ε}, where η_i denotes the mixed strategy of player i which assigns equal probability to all his pure strategies.

$i = 1,2 \;\; f_i^{\varepsilon} = (1 - 27\varepsilon^2)(2,0,0) + 27\varepsilon^2\eta_i$
$i = 3,4 \;\; f_i^{\varepsilon} = (1 - 27\varepsilon^2)(0,2,1) + 27\varepsilon^2\eta_i$
$i = 5,6,7 \;\; f_i^{\varepsilon} = (1 - \varepsilon_1 - \varepsilon_2 - 25\varepsilon^2)(0,0,2) + (\varepsilon_1 - \varepsilon^2)(2,0,0) + (\varepsilon_2 - \varepsilon^2)(2,2,0) + 25\varepsilon^2\eta_i,$

in which $\varepsilon_1 = 1/3(\varepsilon + \varepsilon^2)$ and $\varepsilon_2 = 1/3(2\varepsilon - \varepsilon^2)$.

It is easy to see that, for ε sufficiently close to zero, this is an ε-perfect equilibrium. Suppose all voters other than i choose the strategies prescribed by f. Then, the three undominated strategies of voter i are equivalent. Since for ε going to zero, the probability of voter 5 (the same statement is valid for voters 6 or 7) to tremble towards $(2,0,2)$ or $(2,2,0)$ is infinitely greater than the probability of any other mistake, due to the trembling of one or several players, it is enough to check that in both of these events the limiting strategy is preferred to the other undominated strategy.

For voters 1 and 2, the relevant contingencies which allow them to discriminate between their three undominated strategies is when the behavior of the others is summarized by the vectors $(4,4,6)$ and $(4,6,6)$. Let us denote their probabilities given voter's best responses by $p((4,4,6) \mid f^{\varepsilon}_{-i})$ and $p((4,6,6) \mid f^{\varepsilon}_{-i})$. Furthermore, given voter's best responses, we can write that $2p((4,4,6) \mid f^{\varepsilon}_{-i}) = p((4,6,6) \mid f^{\varepsilon}_{-i})$. Since

$$\begin{aligned} U_1(2,0,0) &= 3/2\, p((4,4,6) \mid f^{\varepsilon}_{-i}) + 4/3\, p((4,6,6) \mid f^{\varepsilon}_{-i}) \\ &= 25/12\, p((4,6,6) \mid f^{\varepsilon}_{-i}) \\ &> U_1(2,1,0), U_1(2,2,0). \end{aligned}$$

Hence, (2,0,0) is the best reply to f^{ε}_{-i}. The same statement is true for voter 2.

For voters 3 and 4, the relevant contingencies can be summarized by the vectors $(6,2,5)$ and $(6,4,5)$. Let us denote their probabilities by $p((6,2,5) \mid f^{\varepsilon}_{-i})$ and $p((6,4,5) \mid f^{\varepsilon}_{-i})$. Furthermore, given voter's best responses, we can write that $2p((6,2,5)|f^{\varepsilon}_{-i}) = p((6,4,5)|f^{\varepsilon}_{-i})$. Since

$$\begin{aligned} U_3(0,2,1) &= 1/2\, p((6,2,5) \mid f^{\varepsilon}_{-i}) + 4/3\, p((6,4,5) \mid f^{\varepsilon}_{-i}) \\ &= 19/12\, p((6,4,5) \mid f^{\varepsilon}_{-i}) \\ &> U_3(0,2,0) = U_3(0,2,2). \end{aligned}$$

the non-overstating strategy is the best reply to f^{ε}_{-i}. The same statement applies for voter 4.

Similarly, one can deduce that for voters $i = 5,6,7$ casting ballot (0,0,2) is a best response against f^{ε}. Indeed, for voters 5, 6 and 7, the relevant contingencies are summarized by the vectors $(6,4,4)$ and $(6,6,4)$. Let us denote their probabilities by $p((6,4,4)|f^{\varepsilon}_{-i})$ and $p((6,6,4)|f^{\varepsilon}_{-i})$. Furthermore, given voter's best responses, we can write that $2p((6,4,4)|f^{\varepsilon}_{-i}] = p((6,6,4) \mid f^{\varepsilon}_{-i})$. Since

$$\begin{aligned} U_5(0,0,2) &= 3/2\, p((6,4,4) \mid f^{\varepsilon}_{-i}) + 4/3\, p((6,6,4) \mid f^{\varepsilon}_{-i}) \\ &= 25/12\, p((6,6,4) \mid f^{\varepsilon}_{-i}) \\ &> U_5(0,1,2), U_5(0,2,2). \end{aligned}$$

the non-overstating strategy is the best reply to f^{ε}_{-i} and similarly for voters 6 and 7.

Hence, $\{f^{\varepsilon}\}$ is a sequence of ε-perfect equilibria. Since f is the limit of f^{ε}, it is a perfect equilibrium in which voters' best responses are not overstating. $\square$