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Sincere Scoring Rules

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Abstract

Approval Voting is shown to be the unique scoring rule that leads strategic voters to sincere behavior of three candidates elections in Poisson Games. However, Approval Voting can lead to insincere behavior in elections with more than three candidates.

KEYWORDS: Sincerity, Approval Voting, Scoring rules, Poisson Games.

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1 Introduction

A standard assumption in the voting literature is that voters vote strategically: they cast the ballot that maximizes their expected utility. Among the different voting rules, the strategic voting literature (which aims at comparing voting rules by the set of equilibria they lead to) has mainly focused on scoring rules due to their simple structure. With a scoring rule, the voter assigns a number of points to each of the candidates. Among these scoring rules, the most studied ones are Plurality voting (PV) in which a voter can cast a single vote for at most one candidate and Approval Voting (AV) in which a voter can give one or zero votes independently to each of the candidates.

The criteria used within this work to differentiate between the different scoring rules is sincerity. We say that a ballot is sincere if, given the score $s$ assigned to given candidate $k$, the scores assigned to candidates preferred to $k$ are greater or equal than $s$. Our main purpose is to identify whether there exists any scoring rule under which strategic voting leads to sincere voting. This indeed the case in elections with three candidates: under AV, voters’ best responses are always sincere in equilibrium. To do so, we focus on Poisson voting games, games with a random number of voters proposed by Myerson\cite{Myerson1, Myerson2}. Yet, the most interesting feature is not that AV is sincere in such a framework (which was already true with other types of equilibria refinements such as trembling-hand perfection or elimination of weakly undominated strategies). Thanks to the use of Poisson Games (which are more trackable than standard refinements), we are able to show that any other scoring rule may lead to insincere behavior at equilibrium. However, this positive result concerning AV is only valid in elections with three candidates. Indeed, an example of an election with four candidates is provided in which sincerity is not anymore ensured.

This paper is structured as follows. Section 2 introduces the basic model. Section 3 discusses in detail the sincerity of AV and Section 4 proves that the other scoring rules may not lead to sincerity at equilibrium. Section 5 concludes.

2 The model

A Poisson random variable $\mathcal{P}(n)$ is a discrete probability distribution that depends on a unique parameter which represents its mean. The probability that a Poisson random variable of parameter $n$ takes the value $l$, being $l$ a nonnegative integer is equal to

$$e^{-n} \frac{n^l}{l!}.$$
A Poisson voting Game of expected size $n$ is a game such that the actual number of voters taking part in the election is a random variable drawn from a Poisson distribution with mean $n$. This assumption represents the uncertainty faced by voters w.r.t. the number of voters that show up the day of the election. The probability distribution and its parameter $n$ are common knowledge.

Each voter has a type $t$ in set of types $T$ that defines his cardinal preferences over the set of candidates $K$. A voter’s payoff only depends on the candidate who is elected. The preferences of a voter with a type $t$ are denoted by $u_t = (u_t(k))_{k \in K}$. Thus, for a given $t$, $u_t(j) > u_t(k)$ implies that $t$-voters strictly prefer candidate $j$ to candidate $k$. Each voter’s type is independently drawn from $T$ according to the distribution of types denoted by $r = (r(t))_{t \in T}$ \(^2\). In other words, $r(t)$ represents the probability that a voter randomly drawn from the population has type $t$.

A finite Poisson game of expected size $n$ is then represented by $(K, T, n, r, u)$. The expression “Large Poisson game” or LPG refers to the asymptotic behavior of a sequence of Poisson games of expected size $n$ when $n$ is large enough.

In order to completely determine an election in a Poisson voting game, the voting rule remains to be specified. The focus of this paper is on scoring rules on the framework of three candidate elections. Following Myerson [10], we denote a generalized scoring rule in a three candidates election as a voting rule in which

the voter must assign: \[
\begin{align*}
\text{at most 1 point to one candidate,} \\
\text{at least 0 points to another candidate and} \\
\text{s points with } s \in [\underline{s}, \overline{s}] \text{ to the remaining candidate,}
\end{align*}
\]

with the convention that $0 \leq \underline{s} \leq \overline{s} \leq 1$. In the literature (for instance, see the axiomatization of Young [13]), most used ranked scoring rules satisfy $\underline{s} = \overline{s}$: for instance, Plurality voting (with $\underline{s} = \overline{s} = 0$), the Borda rule ($\underline{s} = \overline{s} = 1/2$) and Negative voting (with $\underline{s} = \overline{s} = 1$). There also exists non ranked scoring rules when $\underline{s} < \overline{s}$ such as Approval Voting in which $0 = \underline{s} < \overline{s} = 1$. The set of available ballots is denoted by $C$.

As shown by Myerson [8], assuming a Poisson population has two main advantages: common public information and independence of actions. This common public information property of Poisson Games entails that voters’ actions uniquely depend on their private information $t$ on this type of games in equilibrium. The second main advantage is usually referred as the independence of actions. The number of voters who choose a given ballot is independent from the number of voters who choose another ballot as a consequence of assuming that the number of voters is drawn from a Poisson random variable.

\(^2\)The distribution of types satisfies $r(t) \geq 0 \ \forall t \in T$ and $\sum_{t \in T} r(t) = 1$. 

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We represent voters’ actions by the strategy function $\sigma(c \mid t)$ \(^3\) which is a function from $T$ into $\Delta(C)$ the set of probability distributions over $C$. Formally, we will write

$$\sigma : \begin{cases} T &\longrightarrow \Delta(C) \\ t &\mapsto \sigma(\cdot \mid t). \end{cases}$$

A voter with type $t$ chooses ballot $c$ with probability $\sigma(c \mid t)$. Then, taking into account the distribution of types $r$ and the strategy function $\sigma(\cdot \mid t)$, the vote distribution $\tau = (\tau(c))_{c \in C}$ can be determined as follows. For each $c \in C$, we define

$$\tau(c) = \sum_{t \in T} r(t) \sigma(c \mid t).$$

The vote distribution $\tau$ represents the share of votes each ballot gets. We denote by $x(c)$ the Poisson random variable with parameter $n\tau(c)$ that describes the number of voters who choose ballot $c$. Furthermore the vote profile $x = (x(c))_{c \in C}$ is a vector of length $C$ of independent random variables (due to the independent actions property).

The set of electoral outcomes is denoted by $\mathcal{B}$, where

$$\mathcal{B} = \{ b \in \mathbb{R}^C \mid b(c) \text{ is a non-negative integer for all } c \in C \}$$

We denote by $b \in \mathcal{B}$ a vector of length $C$ of non-negative integer numbers. Each component $b(c)$ of vector $b$ accounts for the number of voters who vote for ballot $c$.

Given the vote profile $x$, the (common knowledge) probability that the outcome is equal to a vector $b \in \mathcal{B}$ is such that

$$P[x = b \mid n\tau] = \prod_{c \in C} \left( \frac{e^{-n\tau(c)}(n\tau(c))^{b(c)}}{b(c)!} \right).$$

Let $C_k$ denote the set of ballots in which candidate $k$ is approved. Given the vote profile $x$, the score distribution $\rho = (\rho(k))_{k \in K}$ describes the share of votes that each candidate gets. For each $k \in K$,

$$\rho(k) = \sum_{c \in C_k} \tau(c).$$

It follows that the number of voters that vote for a candidate $k$ is drawn from a Poisson random variable with mean $n\rho(k)$. Given the score distribution, we define the score profile $s = (s(k))_{k \in K}$ describes the number of voters who vote for each candidate $k$ with

$$s(k) = \sum_{c \in C_k} x(c) \sim \mathcal{P}(n\rho(k))$$

\(^3\)The strategy function satisfies $\sigma(c \mid t) \geq 0 \ \forall \ c \in C$ and $\sum_{c \in C} \sigma(d \mid t) = 1$. It plays the role of a strategy combination in a game with a constant number of players.
Given the vector $b \in B$, let $M(b) = \arg \max_{j \in K} \rho(j)$ denote the set of candidates with the most points. Assuming a fair toss of a coin, the probability of candidate $k$ winning the election given the vector $b \in B$ is

$$W[k \mid b] = \begin{cases} 1/\#(M(b)) & \text{if } k \in M(b) \\ \text{0} & \text{if } k \notin M(b). \end{cases}$$

Whenever the set $M(b)$ is single-valued, we refer to the candidate in this set as the Winner of the election as the probability of this candidate winning the election tends towards one as the number of voters tends towards infinity.

For any vector $b \subset B$ and any ballot $c \in C$, we let $b + \{c\}$ denote the vector such that one ballot $c$ is added. Thus, given the vote profile $x$, a voter with type $t$ casts the ballot $c$ that maximizes his expected utility

$$E_t[c \mid n] = \mathbb{E}I_{\{x \neq b\}} \sum_{k \in K} \frac{W[k \mid b + \{c\}] u_t(k)}{\mathbb{P}[x = b]}.$$

Again, for ease of notation, we write $E_t[c]$ for $E_t[c \mid n]$.  

**Definition 1.** We refer to $\sigma$ as an equilibrium of a finite Poisson voting game if for each $c \in C$ and each $t \in T$,

$$\sigma(c \mid t) > 0 \implies c \in \arg \max_{d \in C} E_t[d].$$

Nevertheless, as the focus of this work is on elections with a large number of voters, one shall look at the limits of equilibria as the expected number of voters $n$ tends to infinity. Thus, we refer to a *large equilibrium sequence* of to denote any equilibria sequence $\{\sigma_n\}_{n \to \infty}$ of the finite voting games of expected size $n$ such that the vectors $\sigma_n$ are convergent to some limit $\sigma$ as $n \to \infty$ in the sequence. We refer to this limit $\sigma$ as a *large equilibrium*. Furthermore, we refer to a sequence of outcomes in $B$ by $\{B_n\}_{n \to \infty}$. The limit $b$ of a sequence of vectors $\{b_n\}_{n \to \infty}$ in $B$ is a vector of $B$.

We assume that each voter determines which ballot he casts by maximizing his expected utility: a voter cares only about the influence of his own vote in determining the Winner’s identity. As we analyze elections with a large number of voters, a voter’s action has no impact in almost any possible outcome of the election. Indeed, it has some impact if and only if there is some set of candidates involved in a close race for first place where one ballot pivotally changes the result of the election: a pivot$^4$.

For some ballot $c$ and a pair of candidates $i$ and $j$, $\text{pivot}(c, i, j)$ denotes the event that adding one more ballot $c$ can change the winner from candidate $i$ to candidate $j$:

$$\text{pivot}(c, i, j) = \{b \in B \mid W[i \mid b] > W[i \mid b + \{c\}] \text{ and } W[j \mid b] < W[j \mid b + \{c\}]\}.$$  

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$^4$A discussion of the technical methods to compute such probabilities is out of the scope of this paper. The interested reader might find interesting the insights presented by Myerson [9], Myerson [10], Núñez [11] and Núñez [12].
Let $\text{pivot}(i, j)$ denote the event in which there is a close race such that one additional vote can pivotally change the winner of the election from one to the other of these two candidates,

$$P(i, j) = \bigcup_{c \in C} (P(c, i, j) \cup P(c, j, i)).$$

The event $\text{pivot}(i, j)$ is the union of the different outcomes in which one single ballot can change the outcome of the election. Voters take into account only the probabilities of these events in order to determine their best responses.

### 3 Sincerity of Approval Voting

Whereas the meaning of sincerity is clear in a rule like PV (a sincere voter simply votes for his preferred candidate), it is not that simple in a rule like AV where the voter can vote for different candidates.

**Definition 2 (Sincerity).** A scoring rule ballot is sincere for a voter if, given the score $s$ assigned to some candidate $k$, the scores assigned to candidates preferred by the voter to candidate $k$ are greater or equal than $s$.

With such a definition of sincerity, there need not be anymore a dichotomy between strategic voting and sincere voting. Indeed, this definition allows for several sincere AV ballots.

**Proposition 1.** Sincere behavior of voters is guaranteed under Approval Voting in an election with three candidates.

**Proof.** As in a Poisson game the voter knows that with strictly positive probability he will be the only voter showing up the day of the election, it is strictly dominated not to give one point to your preferred candidate under AV. Similarly, it is strictly dominated to vote for his least preferred candidate with AV as it negatively affects your expected utility (voting for such a candidate may only prevent your two preferred candidates of winning the election).

This result is not really “new” in the sense that elimination of weakly undominated strategies or equilibrium refinements such trembling-hand perfection lead to the same result: sincerity of AV in three-candidates election. However, as will be shown throughout, the current framework allows us to prove that the rest of the scoring rules are not sincere.

The remaining of this section presents a LPG with four candidates in which sincere behavior of the voters under AV is not satisfied. The equilibrium is quite stable as we do not specify the type distribution, only some general conditions for it to hold. Let us
consider a LPG $G$ in which four candidates $K = \{a, b, c, d\}$ are running for the election. Voters’ preferences are such that

$$u_{t_1} = (10, 2, 1, 3); \quad u_{t_2} = (1, 10, 2, 3) \text{ and } u_{t_3} = (1, 2, 10, 3).$$

Besides the distribution of types satisfies

$$r(t_1) < r(t_2) < r(t_3) < \frac{1}{2}.$$

It is not difficult to see that voters’ best responses are not sincere with the strategy function $\sigma_1$ depicted as follows:

$$\sigma_1 = (\{a, b\}, \{b\}, \{c\}).$$

**Proposition 2.** The strategy function $\sigma_1$ is a large equilibrium of the game $H$.

Prior to stating the proof of Proposition 2, next corollary illustrates a limit to the sincere behavior of strategic voters under AV.

**Corollary 1.** Sincere behavior of voters is not guaranteed under Approval Voting with more than three candidates.

**Proof.** In the equilibrium $\sigma_1$, the strategy function satisfies

$$\sigma_1(a, b \mid t_1) = \sigma_1(b \mid t_2) = \sigma_1(c \mid t_3) = 1.$$

Therefore, the vote distribution is such that

$$\tau(a, b) = r(t_1), \quad \tau(b) = r(t_2), \quad \tau(c) = r(t_3).$$

Given the vote distribution, the vote profile $x = (x(c))_{c \in C}$ is the following vector of random variables

$$x(a, b) \sim \mathcal{P}(nr(t_1)), \quad x(b) \sim \mathcal{P}(nr(t_2)) \text{ and } x(c) \sim \mathcal{P}(nr(t_3)).$$

In the equilibrium $\sigma_1$, the score distribution $\rho = (\rho(k))_{k \in K}$ is such that

$$\rho(a) = \tau(a, b) = r(t_1), \quad \rho(b) = \tau(a, b) + \tau(b) = r(t_1) + r(t_2) \text{ and } \rho(c) = r(t_3).$$

As the score distribution shows, the Winner of the election is candidate $b$. Finally, given the score distribution, the score profile $s = (s(k))_{k \in K}$ is such that

$$s(a) = x(a, b) \sim \mathcal{P}(r(t_1)n), \quad s(b) = x(a, b) + x(b) \sim \mathcal{P}((r(t_1) + r(t_2))n)$$

and

$$s(c) = x(c) \sim \mathcal{P}(r(t_3)n).$$
In order to show that $\sigma_1$ is an equilibrium, we need to prove that the vote distribution induces a probability distribution over the pivot outcomes in the election such $\sigma_1$ is a best response for all voters. As shown by the computations included in the appendix, the magnitudes of the pivot outcomes are ordered as follows:

$$\mu[pivot(b, c)] > \mu[pivot(a, b)] = \mu[pivot(a, c)]$$

Taking into account the ordering of the magnitudes, one can determine the ballot that each voter of a given type chooses. In particular, it is important to clarify why $t_1$-voters do not vote for candidate $d$, a candidate they prefer to candidate $b$. As we assume no one votes for candidate $d$, the only situation where voting for candidate $d$ is pivotal is when only a single voter votes. In this case, it is never optimal to vote for candidate $d$. Indeed, whenever voting for $d$ pivotally changes the outcome of the election it lowers the probability of winning of the best-ranked candidates ($a$, $b$ or $c$). Thus, no voter rationally votes for candidate $d$. Formally, the expected utility of casting ballot $\{a, b\}$ is strictly higher than the expected utility for $t_1$ voters of casting ballot $\{a, b, d\}$. Indeed, we can write

$$\Delta = E_{t_1}[a, b] - E_{t_1}[a, b, d]$$

However, the outcomes where adding candidate $d$ has an impact in the outcome are the ones where the score of each candidate is of at most equal to one (as no voter votes for candidate $d$). Among these ones, there are two outcomes with positive probability and where switching from ballot $\{a, b\}$ to ballot $\{a, b, d\}$ makes a change in the expected utility: $(0, 0, 0, 0)$ and $(0, 0, 1, 0)$ where each coordinate stands for the number of votes each candidate gets. Thus, we can rewrite the difference of expected utility as follows:

$$\Delta = P[x = (0, 0, 0, 0)]\left(\frac{u_{t_1}(a) + u_{t_1}(b)}{2} - \frac{u_{t_1}(a) + u_{t_1}(b) + u_{t_1}(d)}{3}\right) + P[x = (0, 0, 1, 0)]\left(\frac{u_{t_1}(a) + u_{t_1}(b) + u_{t_1}(c)}{3} - \frac{u_{t_1}(a) + u_{t_1}(b) + u_{t_1}(c) + u_{t_1}(d)}{4}\right)$$

Then, the effect of switching from ballot $\{a, b, d\}$ to ballot $\{a, b\}$ is located in two outcomes. Furthermore, in both outcomes, we have that

$$\left(\frac{u_{t_1}(a) + u_{t_1}(b)}{2} - \frac{u_{t_1}(a) + u_{t_1}(b) + u_{t_1}(d)}{3}\right) = \left(\frac{12}{2} - \frac{15}{3}\right) > 0,$$

and

$$\left(\frac{u_{t_1}(a) + u_{t_1}(b) + u_{t_1}(c)}{3} - \frac{u_{t_1}(a) + u_{t_1}(b) + u_{t_1}(c) + u_{t_1}(d)}{4}\right) = \left(\frac{13}{3} - \frac{16}{4}\right) > 0.$$
Repeating similar arguments for the different ballots yields to the conclusion that is a \textit{strict best response} not to vote for candidate \(d\) for \(t_1\)-voters. Furthermore, when a \(t_1\) voter decides between casting ballot \(\{a\}\) and ballot \(\{a, b\}\), he takes into account the influence of adding candidate \(b\). In order to do so, he cares about the most probable pivot outcome involving candidate \(b\); in this case, the one involving candidates \(b\) and \(c\). Therefore, as a \(t_1\)-voter prefers candidate \(b\) rather than candidate \(c\), he casts ballot \(\{a, b\}\). Similarly, one can show that the strategy function \(\sigma_1\) is a best response to the information and so this is an equilibrium. \(\square\)

\section{The lack of sincerity of the other scoring rules}

\subsection{The Bad Apple}

In this section, we reconsider a slightly modified version of the Bad Apple example of Myerson [10] in which a candidate is unanimously disliked (the Bad apple). Let us consider the LPG \(H\) in which there are three candidates \(K = \{a, b, c\}\) running for the election. Voters’ preferences can be described by the type distribution:

\[ u_{t_1} = (3, 2, 0) \text{ and } u_{t_2} = (2, 3, 0), \]

with \(r(t_1) = 0.5\) and \(r(t_2) = 0.5\). All voters prefer candidate \(a\) and \(b\) over candidate \(c\).

\textbf{Proposition 3.} Every scoring rule with \(0.5 > \bar{s} > 0\) may lead to insincere behavior at equilibrium.

\textit{Proof.} It is simple to check that under the strategy function \(\sigma_2\) with

\[ \sigma_2 = (\{1, 0, \bar{s}\}, \{0, 1, \bar{s}\}), \]

voters’ best responses are insincere as both \(t_1\) and \(t_2\) voters give \(\bar{s} > 0\) points to the least preferred candidate \(c\) and 0 points to their middle ranked candidate (\(b\) and \(a\) respectively). Besides the strategy function \(\sigma_2\) is a large equilibrium of the game \(G\) whenever \(\bar{s} < 0.5\). Indeed, as candidate \(c\) gets strictly less points than candidates \(a\) and \(b\), the pivot between candidates \(a\) and \(b\) is infinitely more probable than the other ones confirming that \(\sigma_2\) is a large equilibrium. \(\square\)

\textbf{Proposition 4.} Every scoring rule with \(\bar{s} \geq 0.5\) may lead to insincere behavior at equilibrium.

\textit{Proof.} With the strategy function \(\sigma_3\), both types of voters randomize as follows:

\[ \sigma_3(\{1, 0, \bar{s}\} | t_1) = \frac{1 + \bar{s}}{3\bar{s}}, \quad \sigma_3(\{1, \bar{s}, 0\} | t_1) = \frac{2\bar{s} - 1}{3\bar{s}} \]
and
\[
\sigma_3(\{0, 1, \bar{s}\} \mid t_2) = \frac{1 + \bar{s}}{3\bar{s}}, \quad \sigma_3(\{\bar{s}, 1, 0\} \mid t_2) = \frac{2\bar{s} - 1}{3\bar{s}}.
\]

Voters’ best responses are insincere as both \(t_1\) and \(t_2\) voters give \(\bar{s} > 0\) points to the least preferred candidate \(c\) and 0 points to their middle ranked candidate with strictly positive probability. If voters anticipate that the most probable pivot outcome occurs between candidates \(a\) and \(b\), they respectively vote \((1, 0, \bar{s})\) and \((0, 1, \bar{s})\). However, as \(\bar{s}\) is greater than 0.5, this would imply that any pivot outcome in which candidate \(a\) or \(b\) is included would also involve candidate \(c\). As voters prefer their middle ranked candidate to a lottery over the three candidates they deviate towards \(\{1, \bar{s}, 0\}\) and \(\{\bar{s}, 1, 0\}\). Given the strategy function \(\sigma_3\) (with \(\bar{s} \geq 0.5\)), the three candidates get the same expected score and the pivot probabilities are not too different (in the sense that their ratio tends towards a positive finite constant as the size of the population tends towards infinity). Therefore \(\sigma_3\) is a large equilibrium of the game \(H\). Myerson [10] analyzes the case of negative voting (in which \(\bar{s} = \bar{s} = 1\)) and shows that \(\sigma_3\) is a large equilibrium. \(\square\)

4.2 The Majority Preferred Candidate

It remains to be shown that the scoring rules in which \(\bar{s} = 0\) can lead strategic voters to insincere behavior. In this section, we analyze a modified version of an example by Nuñez [12] in which a candidate ranked first by the majority of the voters does not win the election under AV. Let us consider the LPG \(I\) in which there are three candidates \(K = \{a, b, c\}\) running for the election. Voters’ preferences can be described by the type distribution:
\[
u_{t_1} = (3, 0, 1), \quad u_{t_2} = (1, 3, 0) \quad \text{and} \quad u_{t_3} = (1, 0, 3),
\]
with \(r(t_1) = 0.1\), \(r(t_1) = 0.6\) and \(r(t_3) = 0.3\).

**Proposition 5.** Every scoring rule with \(\bar{s} = 0\) and \(\bar{s} < 1\) may lead to insincere behavior at equilibrium.

**Proof.** In the strategy function \(\sigma_4\) with
\[
\sigma_4 = (\{1, 0, 0\}, \{1, \bar{s}, 0\}, \{0, 0, 1\}),
\]

voters’ best responses are insincere: \(t_2\) voters give \(\bar{s} < 1\) points to their preferred candidate \(a\) and 1 point to their middle ranked candidate \(b\). Furthermore, the strategy function \(\sigma_4\) is a large equilibrium of the game \(G\) whenever \(\bar{s} \geq 0.5\). Indeed, voters anticipate that the pivot outcome between candidates \(a\) and \(c\) is the most likely one and then both \(t_1\) and \(t_2\) voters give 1 point to candidate \(a\) to prevent candidate \(c\) from winning the election and so do voters with type \(t_3\) by giving one point to candidate \(c\). Besides \(t_2\) voters give
\( \pi < 1 \) points to their preferred candidate \( b \) as it is dominated not to give any point to your preferred candidate. Hence, any pivot outcome in which candidate \( b \) is included would also include candidate \( a \) (as every voter that votes for candidate \( b \) also votes for candidate \( a \)). Then, the most probable pivot outcome occurs between candidates \( a \) and \( c \) which proves the claim.

5 Conclusion

This work presents a positive result concerning AV: it is shown that it is the unique scoring rule that leads strategic voters to sincere behavior in elections with three candidates in Poisson voting games. This result is in some way an extension of Brams and Fishburn [2]'s result under dichotomic preferences. Indeed, if preferences are dichotomic (either you like or dislike each candidate), AV leads to sincere behavior in natural way. However, the sincerity of AV has two main limits. As we have shown, sincerity is not ensured under AV in elections with more than three candidates. Furthermore, this sincerity does not ensure reasonable preference aggregation as shown by Nuñez [12] in Poisson games and De Sinopoli et al. [3] with traditional equilibrium refinements such as trembling-hand perfection and Mertens-stable sets.

An interesting extension of the present work would be to understand whether in a setting of information aggregation rather than preference aggregation (that is when voters know some signal about the true state of the world), AV may ensure sincere behavior for any number of candidates. Another interesting extension would be to set a comparison of scoring rules in the Score Uncertainty model (Laslier [7]) in which AV ensures sincere behavior of strategic voters under certain conditions.

References


6 Appendix: Lack of Sincerity of Approval voting with four candidates

We now provide the constrained minimization problems used in the proof of Proposition 2. The main theorems for the computation of magnitudes are included as supplementary material.

Magnitude of a pivot between candidates $a$ and $b$.

The Magnitude Equivalence Theorem or $MET$ (see Nuñez [12]) states that the magnitude of the pivot between candidates $a$ and $b$ coincides with the magnitude of the outcome $(s(a) = s(b) \geq s(c))$, i.e.

$$
\mu[pivot(a, b)] = \mu[(s(a) = s(b) \geq s(c))]
$$

Therefore, the Dual Magnitude Theorem or $DMT$ (see Myerson [10]) implies that this magnitude is equal to the solution of the following optimization problem.

$$
\mu[(s(a) = s(b) \geq s(c))] = \min_\lambda \tau(a, b) \exp[\lambda_3] + \tau(b) \exp[-\lambda_1 + \lambda_2] + \tau(c) \exp[-\lambda_3] - 1
$$
such that \( \lambda_i \geq 0 \forall i \). The solution of this constrained minimization problem entails that the magnitude of the pivot outcome between candidates \( a \) and \( b \) satisfies

\[
\mu[pivot(a, b)] = -r(t_2) - (\sqrt{r(t_1)} - \sqrt{r(t_3)})^2.
\]

**Magnitude of a pivot between candidates \( a \) and \( c \).**

Combining the \( MET \) and the \( DMT \), the magnitude of a pivot between candidates \( a \) and \( c \) is equal to

\[
\mu[pivot(a, c)] = \mu[(s(a) = s(c) \geq s(b))] = -r(t_2) - (\sqrt{r(t_1)} - \sqrt{r(t_3)})^2 = \mu[pivot(a, b)].
\]

**Magnitude of a pivot between candidates \( b \) and \( c \).**

Combining the \( MET \) and the \( DMT \), the magnitude of a pivot between candidates \( b \) and \( c \) is equal to

\[
\mu[pivot(b, c)] = \mu[(s(b) = s(c) \geq s(a))] = -(\sqrt{r(t_1)} + r(t_2) - \sqrt{r(t_3)})^2.
\]

Therefore, the magnitudes of the pivot outcomes are ordered as follows:

\[
\mu[pivot(b, c)] > \mu[pivot(a, b)] = \mu[pivot(a, c)].
\]