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Time, Bifurcations and Economic Applications

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Abstract

In this paper, we show how to recover discrete-time models from their continuous-time versions through Euler discretizations.

In the first part, we introduce general polynomial discretizations in backward and forward looking and we study the preservation of stability properties and local bifurcations under different discretizations.

In the second part, we apply these results to popular growth models. We show how to reconcile the traditional Solow models in discrete and continuous time through a backward-looking discretization. Discrete-time models of endogenous saving, such as Ramsey (1928), need hybrid discretizations of the continuous-time model because of the forward-looking nature of the Euler equation. The introduction of externalities allows us to illustrate the preservation of stability properties and local bifurcations.

Keywords: discretizations, bifurcations, growth models.
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1 Introduction

The issue of time representation, that is, the choice of a discrete or a continuous variable, is a fundamental concern in economic theory.

On the one side, most of theoretical models, especially in the growth literature, are built in continuous time and authors are forced to this option by no other reason than formal easiness, as Turnovsky (1977) recognizes. Gandolfo (1997) puts forward other arguments in favour of the continuous time: the common sense suggests that life unfolds continuously.

On the other side, economic transactions take place at given instants and data are available as discrete-time measurements: some authors argue that a discrete-time approach makes more sense from an empirical point of view.[1] From a methodological point of view, there is another difference between these representations which argues in favour of discrete time. A one-dimensional difference equation, such as the logistic map, can generate complex dynamics, while a higher-dimensional system is needed in continuous time (Guckenheimer and Holmes (1983)). As a consequence, one gains in simplicity by modeling complex dynamics in discrete time.[2] Finally, in discrete time, distinction between forward and backward-looking variables turns out to be more natural. For instance, introducing observed or expected inflation in a Taylor rule changes the dynamic properties of monetary policy.

These examples show that time modeling is neither trivial nor neutral and has economic consequences. The choice of time can determine the results independently of the underlying economic mechanisms. In the case of a logistic equation, the continuous time rules out in advance the occurrence of (a)periodic cycles.

In this paper, we don't address the question whether discrete or continuous-time models are more appropriate to represent the economic activity. We simply observe formal similarities and some differences of dynamic behavior and we want to contribute to understand the reasons.

A growing literature focuses on the dynamic effects of time representation. Theorists tackle the question in different ways.

On the one side, there are papers that consider specific models and compare the stability properties in discrete and continuous time. For instance, Carlstrom and Fuerst (2005) study the role of time specification on indeterminacy in models where the central bank implements an interest rate rule. Mino, Nishimura, Shimomura and Wang (2005) address the issue of stabilization policy in two-sector endogenous growth models with constant social returns. Time representation also matters under uncertainty: Leung (1995) shows that the consumption paths are different in discrete and continuous time when agents face an uncertain life-span.

On the other side, there are papers that address more general issues, such as the role of the period length, to reconcile discrete and continuous-time dynamics. Mercenier and Michel (1994) consider infinite-horizon optimizations and discretize continuous-time models as usually done in numerical simulation. Their goal is closely related to ours: the invariance property of the steady state can be achieved through an appropriate Euler discretization and simple restrictions on discounting. Anagnostopoulos and Giannitsarou (2008) play with the period length in a dynamic general equilibrium model: they recover popular models as particular cases of a general framework, compare the dynamic properties under both time specifications and conclude that the period length matters for indeterminacy. Hintermaier (2005) also shows that the existence of sunspot equilibria in discrete-time business cycle models depends on the period length. The length of the lag also plays a role: the literature on the time-to-build has been recently revisited in the light of time specification. Licandro and Puch (2006) compare continuous and discrete-time time-to-build models. Bambi (2008) makes an attempt to unify this literature, recovering multi-period investments in discrete-time from delay equations in continuous time. Cycles occur through Hopf bifurcations under both time representations.

In the first part of the paper, in the spirit of Krivine, Lesne and Treiner (2007), we bridge continuous and discrete-time dynamics through general polynomial discretizations. Then we study how the stability property of an invariant steady state are preserved under discretization. In the second part, we apply the theoretical results to popular growth models with or without market imperfections.

[1] Two main criticisms are addressed by Gandolfo (1997) to these apparently convincing arguments. First, although individual decisions are discrete, the fact that they are not synchronized and spread over time from a great number of agents, restores a theoretical justification for continuous-time models. In addition, statistical inference in continuous time knew consequent and satisfactory developments since the 1970s (see Bergstrom (1976), Bergstrom (1984), Gandolfo (1981) and Wymer (1972)).

[2] The logistic map exhibits stable fixed point, stable periodic cycles (of any order) and deterministic chaos. In addition, all these dynamic behaviors are sensitive to a single parameter value. Conversely, only monotonic orbits, either convergent or explosive, are generated by a single first-order differential equation.

A discretization is an approximation of the continuous-time system and the most common representation is the polynomial approximation. In this case, we distinguish discretizations according to the step, the order and the direction of discretization.

The step gives the length of the period in discrete time. Common discrete-time form are recovered under a unit step. The order is that of the Taylor expansion of the continuous-time model. A first-order approximation gives the classical Euler discretization. The direction depends on the backward or forward-looking nature of the Taylor expansion. A hybrid discretization mixes backward and forward-looking discretizations.

In the first part of the paper, we study the preservation of dynamic properties under different type of discretizations. We find that the steady state is invariant to the step, the order and the direction of discretization. In addition, the continuous-time stability properties of the steady state (sink, saddle, source) are preserved under a sufficiently small discretization step. This result holds in case of backward, forward or hybrid discretizations. Local bifurcations in continuous time such as the saddle node, the transcritical and the pitchfork are also preserved, while the Hopf bifurcation endures under a sufficient small discretization step. Flip and period-doubling bifurcations disappear in discrete time under a critical discretization step.

In the second part, we illustrate these properties with traditional growth models. The traditional Solow model in discrete time results from a backward-looking Euler discretization of the Solow model in continuous time. The traditional Ramsey model is recovered with a hybrid discretization of the Ramsey model in continuous time (we apply a Euler discretization in backward and forward looking to the law of motion and the Euler equation, respectively). Eventually, we introduce market imperfections (externalities) in both the models to obtain richer dynamics (cycles). Two-period cycles, arising in the Solow model with pollution, are ruled out when the discretization step becomes sufficiently small or the polynomial order sufficiently high (indeed quadratic forms are enough to exclude flip bifurcations). Limit cycles, emerging in the Ramsey model with positive externalities, are preserved under a critical discretization step.

The rest of the paper is organized as follows. In Section 2, we present the methodological issue of time discretization. Section 3 compares the stability properties in continuous and discrete time. Section 4 focuses on backward-looking discretizations of Solow models, while Section 5 apply hybrid discretizations to Ramsey models.

Part I
Theory

2 General methodology

The question we address concerns the typology of discretizations we need to recover some equivalence properties between discrete and continuous time models.

Discretizations based on polynomial representations were introduced by Euler and are today quite popular in computational science. From a theoretical point of view, the Euler approach can shed a light on the interplay between continuous and discrete-time dynamics and it proves to be pertinent to investigate and compare stability properties and bifurcations. In the spirit of Euler, we choose to apply a Taylor expansion to discretize a continuous-time system. We start by taking a general order expansion, then, we will consider linear and quadratic approximations.

2.1 Discretizations

Instead of considering a continuous variable t and the corresponding position $x(t)$ determined by an m-dimensional system of ordinary differential equations:

$$\dot{x} = f(x) \tag{1}$$

where $f \in C^0$, jointly with the initial condition $x_0 \equiv x(0)$, let us pick up a regular sequence of time values: $(t_n)_{n=0}^{\infty} = (nh)_{n=0}^{\infty}$, where h is a (possibly small) positive constant (discretization step), and the associated values: $x_n \equiv x(t_n) = x(nh)$.

The path from x_n to x_{n+1} can be reconstructed component by component through an appropriate integration of (1). More precisely, if we focus on the ith component of the vector $x \in \mathbb{R}^m$, we can integrate the time derivative on the right or on the left to obtain, respectively,

$$\begin{aligned} x_{in+1} - x_{in} &= x_i(nh+h) - x_i(nh) = \int_{nh}^{nh+\sigma} \dot{x}_i dt \bigg|_{\sigma=h} = \int_{nh}^{nh+\sigma} f_i(x(t))\, dt \bigg|_{\sigma=h} \\ x_{in+1} - x_{in} &= x_i(nh+h) - x_i(nh) = \int_{nh+\tau}^{nh+h} \dot{x}_i dt \bigg|_{\tau=0} = \int_{nh+\tau}^{nh+h} f_i(x(t))\, dt \bigg|_{\tau=0} \end{aligned}$$

with $i = 1, \ldots, m$.

Defining

$$\begin{aligned} \varphi_i(\sigma) &\equiv \int_{nh}^{nh+\sigma} f_i(x(t))\, dt \\ \psi_i(\tau) &\equiv \int_{nh+\tau}^{nh+h} f_i(x(t))\, dt \end{aligned}$$

we get $\varphi_i(h) = x_{in+1} - x_{in} = \psi_i(0)$. Clearly, $\varphi_i(h) = \psi_i(0)$.

Discretizing means approximating $\varphi_i(\sigma)$ $(\psi_i(\tau))$ with another (simple) function evaluated in $\sigma = h$ $(\tau = 0)$. The most popular approximation is the Euler-Taylor discretization: assuming that $f \in C^{q-1}$ and considering the qth order polynomial, we obtain a backward or a forward discretization, respectively:

$$x_{in+1} - x_{in} = \varphi_i(h) \approx \sum_{p=0}^{q} \frac{(h-0)^p}{p!} \varphi_i^{(p)}(0) = \sum_{p=1}^{q} \frac{(h-0)^p}{p!} \varphi_i^{(p)}(0) \tag{2}$$

$$x_{in+1} - x_{in} = \psi_i(0) \approx \sum_{p=0}^{q} \frac{(0-h)^p}{p!} \psi_i^{(p)}(h) = \sum_{p=1}^{q} \frac{(0-h)^p}{p!} \psi_i^{(p)}(h) \tag{3}$$

because $\varphi_i(0) = \psi_i(h) = 0$.

Let us call hybrid a discretization where (2) holds for some components of vector x and (3) holds for others. In economics, higher-dimensional models require often a hybrid discretization to recover the equivalence between discrete and continuous time. For instance, in the popular Ramsey model, a mix of discretization in backward looking (budget constraint) and forward looking (Euler equation) is required to recover the usual discrete-time form.

2.1.1 First-order discretizations

Setting $q = 1$, we obtain from (2) and (3):

$$x_{in+1} - x_{in} = \varphi_i(h) \approx (h-0)\, \varphi_i'(0) = h f_i(x(nh+0)) = h f_i(x_n) \tag{4}$$

$$x_{in+1} - x_{in} = \psi_i(0) \approx (0-h)\, \psi_i'(h) = (0-h)\left[-f_i(x(nh+h))\right] = h f_i(x_{n+1}) \tag{5}$$

This proves the following proposition.

Proposition 1 *The continuous-time dynamic system $\dot{x} = f(x)$ is discretized by linear forms. Using (2) and (3) we obtain in backward and forward looking, respectively:*

$$x_{in+1} - x_{in} \approx h f_i(x_n) \tag{6}$$

$$x_{in+1} - x_{in} \approx h f_i(x_{n+1}) \tag{7}$$

where the subscript i denotes the ith component of the vector.

Equation (6) (respectively (7)) constitutes a backward-looking (forward-looking) discretization, because the variation $x_{n+1} - x_n$ depends on the past value x_n (future value x_{n+1}) on the right-hand side. Equation

(6) is the classical Euler discretization.[3] In economics, forward-looking discretizations are of interest because agents behave according to their expectations.

The entire sequence $(x_n)_{n=0}^{\infty}$ can be computed forward (backward) from the initial condition x_0 (final condition x_n) by iterating the procedure: $x_1 \approx x_0 + hf(x_0)$, $x_2 \approx x_1 + hf(x_1) \approx x_1 + hf(x_0 + hf(x_0))$ and so on (respectively $x_{n-1} \approx x_n - hf(x_n)$, $x_{n-2} \approx x_{n-1} - hf(x_{n-1}) \approx x_n - hf(x_n) - hf(x_n - hf(x_n))$ and so on).

However, the sequences (x_n) are approximations of the true sequence $(x(nh))$, exact solution of system (1): the smaller h, the more accurate the representation. The easiness of the Euler's method makes it a popular technique to plot a phase diagram and find numerical solutions of a system of differential equations. In this paper, we are not interested in numerical simulations, but only in the change of dynamic properties, when one passes from continuous to discrete time: Euler's discretization is of great help to understand why some stability properties (dis)appear from a timing to another.

Conversely, given an ordinary m-dimensional discrete-time system: $x_{n+1} = g(x_n)$, we can define $f(x_n) \equiv [g(x_n) - x_n]/h$ and approximate the discrete-time system with $\dot{x} = f(x)$. As above, the smaller h, the more accurate the approximation. In the following, we will focus only on discretizations of continuous-time system.

2.1.2 Higher-order discretizations

As above, we define $x_n \equiv x(t_n) = x(nh)$, where $x \in \mathbb{R}^m$. We can approximate its ith component of x_{n+1} with a quadratic form.

Proposition 2 *The continuous-time dynamic system $\dot{x} = f(x)$ with $f \in C^1$ is discretized by second-order Taylor polynomials. Using (2) and (3), we obtain in backward and forward-looking, respectively:*

$$x_{in+1} \approx x_{in} + hf_i(x_n) + \frac{h^2}{2}\sum_{j=1}^{m} f_j(x_n)\frac{\partial f_i}{\partial x_{jn}}(x_n) \quad (8)$$

$$x_{in+1} \approx x_{in} + hf_i(x_{n+1}) - \frac{h^2}{2}\sum_{j=1}^{m} f_j(x_{n+1})\frac{\partial f_i}{\partial x_{jn+1}}(x_{n+1}) \quad (9)$$

where the subscript i denotes the ith component of the vector.

Proof. Focus on the ith component. Formulas (2) and (3) become, respectively,

$$x_{in+1} - x_{in} = \varphi_i(h) \approx (h-0)\varphi_i'(0) + \frac{(h-0)^2}{2!}\varphi_i''(0) \quad (10)$$

$$x_{in+1} - x_{in} = \psi_i(0) \approx (0-h)\psi_i'(h) + \frac{(0-h)^2}{2!}\psi_i''(h) \quad (11)$$

We know from (4) and (5) that $\varphi_i'(0) = f_i(x_n)$ and $\psi_i'(h) = -f_i(x_{n+1})$. In addition, noticing that $(dx_j/dt)(nh+t) = f_j(x(nh+t))$, we obtain[4]

$$\varphi_i''(\sigma) = \sum_{j=1}^{m}\frac{\partial f_i}{\partial x_j}(x(nh+\sigma))f_j(x(nh+\sigma))$$

$$\psi_i''(\tau) = -\sum_{j=1}^{m}\frac{\partial f_i}{\partial x_j}(x(nh+\tau))f_j(x(nh+\tau))$$

[3] An equivalent way of deriving (6) is the following. According to the definition of derivative, we can write $\dot{x}_i(t) \equiv \lim_{h\to 0}[x_i(t+h) - x_i(t)]/h$. If h is sufficiently small, we can set $\dot{x}(t) \approx [x(t+h) - x(t)]/h$ and, therefore, $[x(t+h) - x(t)]/h \approx f(x(t))$. We obtain $[x(t_n+h) - x(t_n)]/h \approx f(x(t_n))$, that is $[x(t_{n+1}) - x(t_n)]/h \approx f(x(t_n))$ where $t_{n+1} = t_n + h$, and, finally, $(x_{n+1} - x_n)/h \approx f(x_n)$, that is (6).

[4] f_i depends on all the components of the vector $x(t)$.

and, eventually,

$$\begin{aligned}\varphi_i''(0) &= \sum_{j=1}^{m} f_j(x_n)\frac{\partial f_i}{\partial x_{jn}}(x_n)\\ \psi_i''(h) &= -\sum_{j=1}^{m} f_j(x_{n+1})\frac{\partial f_i}{\partial x_{jn+1}}(x_{n+1})\end{aligned}$$

Replacing in (10) and (11) these results, we get (8) and (9). ■

In the case of a one-dimensional dynamics, discretizations (8) and (9) simplify to

$$\begin{aligned}x_{n+1} &\approx x_n + f(x_n)h + f(x_n)f'(x_n)h^2/2\\ x_{n+1} &\approx x_n + f(x_{n+1})h - f(x_{n+1})f'(x_{n+1})h^2/2\end{aligned}$$

while in the case of a two-dimensional dynamics, non-hybrid discretizations give in backward and forward-looking, respectively,

$$\begin{aligned}x_{n+1} &\approx x_n + \left[hI + \frac{h^2}{2}J_0(x_n)\right]f(x_n)\\ x_{n+1} &\approx x_n + \left[hI - \frac{h^2}{2}J_0(x_{n+1})\right]f(x_{n+1})\end{aligned}$$

where I is the identity matrix and J_0 is the Jacobian matrix of f.

Similarly, one derives higher-order discretizations. For instance, in the case of a one-dimensional dynamics in backward looking, one obtains

$$x_{n+1} \approx x_n + f(x_n)h + f(x_n)f'(x_n)h^2/2 + \left[f(x_n)f'(x_n)^2 + f(x_n)^2 f''(x_n)\right]h^3/6$$

If f is an analytic function, infinite-order backward or forward discretizations converges exactly to $x_{n+1} - x_n$ and the sign of approximation can be replaced by the equality:

$$x_{in+1} - x_{in} = \sum_{p=1}^{\infty}\frac{(h-0)^p}{p!}\varphi_i^{(p)}(0) = \sum_{p=1}^{\infty}\frac{(0-h)^p}{p!}\psi_i^{(p)}(h)$$

In this case, the Taylor polynomials become a convergent series and the discretized dynamics exactly represents the continuous time whatever the step h.

In general, a discretization is a closer approximation of a continuous-time system when the step h is smaller or the order of discretization q higher. As we will see below, the dynamic properties of a continuous-time system can be preserved lowering h or increasing q.

2.1.3 Dynamic optimization models

In economics, a large class of dynamic models are microfounded, that is based on rational individual behaviors. Agents are rational when they optimize their own objective under a system of constraints. Since intertemporal optimization is the starting point of any microfounded dynamic model in economics, it makes sense to compare optimization in continuous and discrete time and apply Euler discretizations in order to find some equivalence.

Some popular growth models, such those we will study at the end of the paper, are microfounded and can be derived as particular solutions of a general dynamic program: they rest on a common set of assumptions, namely intertemporal separability of the objective. In our approach, both the state and control variables enter the objective functional and the constraint. Instead of solving different specific models, we solve this general program. In the second part of the paper, the general solution will be applied to the specific models we are interested in.

After solving continuous and discrete-time programs of intertemporal optimization, we will discretize the first-order conditions in continuous time.

Let us maximize a general intertemporal functional

$$V \equiv \int_0^\infty \beta_t v(k_t, c_t)\, dt \tag{12}$$

where k_t and c_t denote the state and the control, subject to the law of motion

$$\dot{k}_t \leq s(k_t, c_t) \tag{13}$$

and a discounting process $\dot{\beta}_t = -\rho_t \beta_t$, where $\rho_t = \rho(t)$ is a given positive function of time. The initial conditions k_0 and $\beta_0 \equiv 1$ are also given.

Assumption 1 *$v : \mathbb{R}_+^2 \to \mathbb{R}$ and $s : \mathbb{R}_+^2 \to \mathbb{R}$ are C^2, strictly increasing in both the arguments ($\partial v/\partial k > 0$, $\partial v/\partial k > 0$) and strictly concave.*[5] *The Inada boundary conditions are also satisfied.*

The agent chooses the control in order to maximize the functional subject the law of motion. β_t is a general discounting which depends on the lapse of time. When the discount rate ρ_t is constant, discounting simplifies to $\beta_t = \beta_0 e^{-\rho t}$. In this case, it is equivalent to maximize (12) or $\int_0^\infty e^{-\rho t} v(k_t, c_t)\, dt$.

The Hamiltonian associated to the program is $H_t \equiv \beta_t v(k_t, c_t) + \lambda_t s(k_t, c_t)$. Maximizing H_t with respect to the costate, state and control variables, gives, respectively: $\partial H_t/\partial \lambda_t = s(k_t, c_t) = \dot{k}_t$, $\partial H_t/\partial k_t = \beta_t \partial v/\partial k_t + \lambda_t \partial s/\partial k_t = -\dot{\lambda}_t$, $\partial H_t/\partial c_t = 0$ (that is $\lambda_t/\beta_t = -(\partial v/\partial c_t)/(\partial s/\partial c_t)$) with $\dot{\beta}_t = -\rho_t \beta_t$ and transversality condition: $\lim_{t\to\infty} \lambda_t k_t = 0$. Setting $\mu_t \equiv \lambda_t/\beta_t$, the current-value shadow price, and noticing that

$$\dot{\lambda}_t = \beta_t \dot{\mu}_t + \mu_t \dot{\beta}_t \tag{14}$$

we obtain $\dot{\mu}_t = -\mu_t \left(\dot{\beta}_t/\beta_t + \partial s/\partial k_t\right) - \partial v/\partial k_t$ and

$$\mu_t = -\frac{\partial v/\partial c_t}{\partial s/\partial c_t} \tag{15}$$

We can apply the Implicit Function Theorem to equation (15) to obtain $c_t = c(k_t, \mu_t)$ with

$$\left(\frac{\partial c}{\partial k_t}, \frac{\partial c}{\partial \mu_t}\right) = \left(\frac{\frac{\partial v}{\partial c_t}\frac{\partial^2 s}{\partial c_t \partial k_t} - \frac{\partial s}{\partial c_t}\frac{\partial^2 v}{\partial c_t \partial k_t}}{\frac{\partial s}{\partial c_t}\frac{\partial^2 v}{\partial c_t^2} - \frac{\partial v}{\partial c_t}\frac{\partial^2 s}{\partial c_t^2}}, \frac{\left(\frac{\partial s}{\partial c_t}\right)^2}{\frac{\partial v}{\partial c_t}\frac{\partial^2 s}{\partial c_t^2} - \frac{\partial s}{\partial c_t}\frac{\partial^2 v}{\partial c_t^2}}\right) \tag{16}$$

Hence, we find a two-dimensional system in (k_t, μ_t):

$$\dot{k}_t = s(k_t, c(k_t, \mu_t)) \tag{17}$$

$$\dot{\mu}_t = -\mu_t \left[\frac{\dot{\beta}_t}{\beta_t} + \frac{\partial s}{\partial k_t}(k_t, c(k_t, \mu_t))\right] - \frac{\partial v}{\partial k_t}(k_t, c(k_t, \mu_t)) \tag{18}$$

Focus now on the corresponding program in discrete time. We maximize the utility series $\sum_{t=0}^\infty \beta_t v(k_t, c_t)$ under a sequence of constraints: $k_{t+1} - k_t \leq s(k_t, c_t)$ with $t = 0, 1 \ldots$ Under the assumptions $v_c > 0$ and $s_c < 0$, the Lagrangian multipliers are positive and the constraints is binding. The intertemporal smoothing is represented by a sequence of Euler equations. We obtain a two-dimensional system

$$k_{t+1} = k_t + s(k_t, c(k_t, \mu_t)) \tag{19}$$

$$\frac{\mu_t}{\mu_{t+1}} = \frac{\beta_{t+1}}{\beta_t}\left[1 + \frac{\partial s}{\partial k_{t+1}}\left(k_{t+1}, c\left(k_{t+1}, \mu_{t+1}\right)\right) + \frac{1}{\mu_{t+1}}\frac{\partial v}{\partial k_{t+1}}\left(k_{t+1}, c\left(k_{t+1}, \mu_{t+1}\right)\right)\right] \tag{20}$$

where μ_t is still given by (15). As above (15) allows us to define $c_t = c(k_t, \mu_t)$ with partial derivatives (16). The variables of system (19)-(20) are k_t and μ_t. We observe that μ_t is the current-value costate variable of the continuous-time program at time t, that is $\lambda_t = \beta_t \mu_t$.

[5] Let functions v and s satisfy the Arrow-Mangasarian sufficient conditions for maximization. The second-order restrictions are explicitly provided in Bosi and Ragot (2009).

The crucial question is whether the discrete-time system (19)-(20) can be recovered through a (first-order) Euler discretization. We mix a backward-looking discretization of constraint (17) and a forward-looking discretization of the Euler equation (18).

Discretizing the continuous-time constraint (17) gives:

$$k_{t+h} - k_t \approx hs\left(k_t, c\left(k_t, \mu_t\right)\right) \tag{21}$$

that is the discrete-time resource constraint (19) under a unit discretization step ($h = 1$). Because of the forward-looking nature of the Euler equation, we can not recover (20) in backward-looking. Using (14), equation (18) can be written in terms of $\lambda_t = \beta_t \mu_t$ instead of μ_t:

$$\dot{\lambda}_t = -\lambda_t \frac{\partial s}{\partial k_t}\left(k_t, c\left(k_t, \frac{\lambda_t}{\beta_t}\right)\right) - \beta_t \frac{\partial v}{\partial k_t}\left(k_t, c\left(k_t, \frac{\lambda_t}{\beta_t}\right)\right) \tag{22}$$

Let us call (22) the λ-type Euler equation and apply the forward-looking discretization (7) to (22):

$$\lambda_{t+h} - \lambda_t = -h\left[\lambda_{t+h}\frac{\partial s}{\partial k_{t+h}}\left(k_{t+h}, c\left(k_{t+h}, \frac{\lambda_{t+h}}{\beta_{t+h}}\right)\right) + \beta_{t+h}\frac{\partial v}{\partial k_{t+h}}\left(k_{t+h}, c\left(k_{t+h}, \frac{\lambda_{t+h}}{\beta_{t+h}}\right)\right)\right]$$

Replacing $\lambda_t = \beta_t \mu_t$, we obtain

$$\frac{\beta_t}{\beta_{t+h}}\frac{\mu_t}{\mu_{t+h}} = 1 + h\left[\frac{\partial s}{\partial k_{t+h}}\left(k_{t+h}, c\left(k_{t+h}, \mu_{t+h}\right)\right) + \frac{1}{\mu_{t+h}}\frac{\partial v}{\partial k_{t+h}}\left(k_{t+h}, c\left(k_{t+h}, \mu_{t+h}\right)\right)\right] \tag{23}$$

that is the discrete-time Euler equation (20) under a unit discretization step $h = 1$.

Hence, (1) a hybrid discretization of (2) a λ-type continuous-time system with (3) a unit discretization step gives exactly the traditional discrete time system.[6]

Traditional growth models in discrete time come from a unit-step hybrid approximation of the continuous-time system: backward-looking discretization of the constraint and a forward-looking discretization of the λ-type Euler equation.

3 Topological equivalence

In order to compare continuous-time and discrete-time system, we will study approximations in a neighborhood of the steady state and focus on the persistence of stability properties and elementary bifurcations.

3.1 Steady state

The system $\dot{x} = f(x)$ and its discrete-time approximation $x_{n+1} \approx x_n + hf(x_n)$ have the same steady state. Indeed, in both the cases we require $f(x) = 0$ (respectively, $\dot{x} = 0$ and $x_{n+1} = x_n$). We further notice that the system of m equations $f(x) = 0$ neither depend on the discretization degree h nor on the discretization method (forward or backward-looking).

[6] A quadratic approximation of (17)-(22) is also possible. As above, we focus on a backward-looking discretization of the constraint and a forward-looking discretization of the λ-type Euler equation. In the case the utility function no longer depends on the state variable ($v(k_t, c_t) = u(c_t)$ as in the Cass-Koopmans model (and *a fortiori* in Ramsey)), this quadratic approximation reduces to

$$\begin{aligned}
k_{t+h} - k_t &\approx hs_t + \frac{h^2}{2}\left(s_t - \mu_t \frac{\partial s}{\partial c_t}\frac{\partial c}{\partial \mu_t}\right)\frac{\partial s}{\partial k_t} \\
\frac{\beta_t}{\beta_{t+h}}\frac{\mu_t}{\mu_{t+h}} &\approx 1 + h\frac{\partial s}{\partial k_{t+h}} + \frac{h^2}{2}\frac{\partial s}{\partial k_{t+h}}\left(\frac{\partial s}{\partial k_{t+h}} + \mu_{t+h}\frac{\partial c}{\partial \mu_{t+h}}\frac{\partial^2 s}{\partial k_{t+h}\partial c_{t+h}}\right) \\
&\quad -\frac{h^2}{2}s_{t+h}\left(\frac{\partial^2 s}{\partial k_{t+h}^2} + \frac{\partial c}{\partial k_{t+h}}\frac{\partial^2 s}{\partial k_{t+h}\partial c_{t+h}}\right)
\end{aligned}$$

with $s_t \equiv s\left(k_t, c\left(k_t, \mu_t\right)\right)$.

3.2 Stability properties

The steady state is invariant to discretization. The subsequent question we raise is whether the stability properties are also preserved under discretization in a neighborhood of the steady state. On the one hand, we will prove a topological equivalence: a sink in continuous time remains a sink in discrete time under a sufficiently small discretization step; the same happens for a saddle point or a source.[7] On the other hand, we will see in the next section how the Euler discretization affects local bifurcations, that is how conditions for a specific bifurcation change under discretization.

At least, a two-dimensional system is required to study the three cases together (sink, saddle and source) and to consider hybrid discretizations. Without loss of generality, we linearize the following

$$\dot{x}_1 = f_1(x_1, x_2) \quad (24)$$

$$\dot{x}_2 = f_2(x_1, x_2) \quad (25)$$

Local dynamics around the steady state are represented by the Jacobian matrix

$$J_0 \equiv \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

evaluated at $f_1(x_1, x_2) = f_2(x_1, x_2) = 0$.

For simplicity, we will focus on first-order discretizations. Our equivalence results holds *a fortiori* for higher-order discretizations.[8]

3.2.1 Backward-looking discretization

We linearize the backward-looking discretization

$$x_{n+1} \approx x_n + hf(x_n) \quad (26)$$

of system (24)-(25) around the common steady state $f(x) = 0$ and we obtain $dx_{n+1} = J_1 dx_n = (I + hJ_0)\, dx_n$, where I and J_1 are the two-dimensional identity matrix and Jacobian matrix of system (26). We observe that J_0 depends on the steady state x which, in turn, does not depend on h: then,

$$J_1 = I + hJ_0 \quad (27)$$

depends only linearly on h.

Let us denote the trace and determinant of J_0 and J_1 by (T_0, D_0) and (T_1, D_1), respectively. The characteristic polynomial in discrete time is given by $P_1(\lambda) \equiv \lambda^2 - T_1\lambda + D_1$, where

$$T_1 = 2 + hT_0 \quad (28)$$

$$D_1 = 1 + hT_0 + h^2 D_0 = T_1 - 1 + h^2 D_0 \quad (29)$$

We can represent the stability properties in the plane of trace and determinant (see Samuelson (1941)).

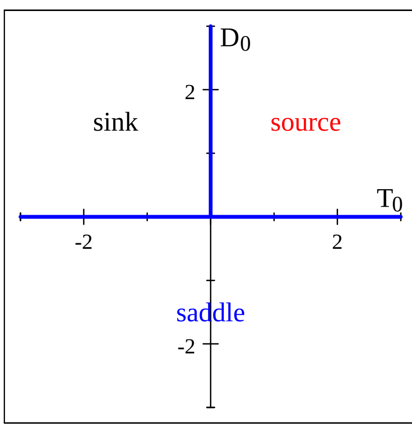

Fig. 1: Continuous time

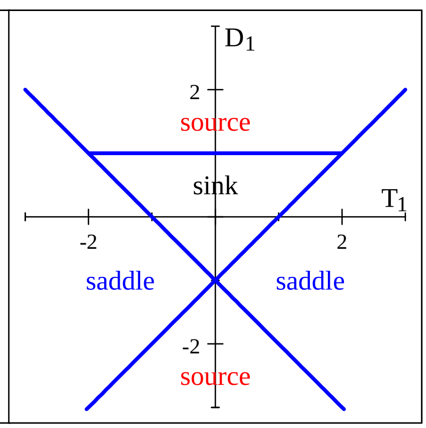

Fig. 2: Discrete time

[7] We would like to thank Jean-Michel Grandmont for his invaluable comments. Usual disclaimers apply.

[8] For second-order discretizations, the reader is referred to Bosi and Ragot (2009).

There are three critical values of the discretization step that determine the intervals of equivalence between the continuous and the discrete-time dynamics: $h_{H1} \equiv -T_0/D_0$,

$$h_{F1} \equiv -\frac{T_0}{D_0} - \sqrt{\left(\frac{T_0}{D_0}\right)^2 - \frac{4}{D_0}} \text{ and } h_{F2} \equiv -\frac{T_0}{D_0} + \sqrt{\left(\frac{T_0}{D_0}\right)^2 - \frac{4}{D_0}}$$

Proposition 3 *Consider $h > 0$.*

(1) Let the steady state be a sink in continuous time (Figure 3).

(1.1) If $T_0^2 < 4D_0$, then the steady state is a sink in discrete time if $h < h_{H1}$ and a source if $h_{H1} < h$.

(1.2) If $T_0^2 > 4D_0$, then the steady state is a sink if $0 < h < h_{F1}$, a saddle if $h_{F1} < h < h_{F2}$ and source if $h_{F2} < h$.

(2) If the steady state is a saddle in continuous time, then the steady state is a saddle in discrete time if $0 < h < h_{F2}$ and source if $h_{F2} < h$ (Figure 4).

(3) If the steady state is a source in continuous time, then the source property is preserved whatever $h > 0$ (Figure 5).

The system generically undergoes a Hopf bifurcation at h_{H1} and flip bifurcations at h_{Fi}, $i = 1, 2$.

Proof.

(1) Assume that the steady state is a sink in continuous time: $T_0 < 0 < D_0$. According to (29), $D_1 > T_1 - 1$. Focus on two cases: (1.1) $T_0^2 < 4D_0$ and (1.2) $T_0^2 > 4D_0$.

(1.1) If $T_0^2 < 4D_0$, then always $D_0h^2 + 2T_0h + 4 > 0$, that is $-T_1 - 1 < D_1$. So, the steady state is a sink if $D_1 < 1$, that is $h < h_{H1}$, and a source if $h > h_{H1}$. This case corresponds to the upper parabola in Figure 3. Increasing h away from zero means moving away from the point where $h = 0$, along the parabola.

(1.2) If $T_0^2 > 4D_0$, then $D_1 < -T_1 - 1$ iff $h_{F1} < h < h_{F2}$. In addition, $D_1 < 1$ iff $h < h_H$. We notice also that $0 < h_{F1} < h_{H1} < h_{F2}$. Then, the steady state is a sink if $0 < h < h_{F1}$, a saddle if $h_{F1} < h < h_{F2}$ and a source if $h_{F2} < h$. This case corresponds to the lower parabola in Figure 3.

(2) Assume now that the steady state is a saddle in continuous time: $D_0 < 0$. According to (29), $D_1 < T_1 - 1$. We observe that $h_{F1} < 0 < h_{F2}$ and that $D_1 > -T_1 - 1$ iff $h_{F1} < h < h_{F2}$. Thus, the steady state is a saddle if $0 < h < h_{F2}$ and a source if $h_{F2} < h$. If $T_0 < 0$ ($T_0 > 0$), the curve $\{(T_1(h), D_1(h)) : h > 0\}$ is represented by the leftward (rightward) branch of parabola in Figure 4.

(3) Assume now that the steady state is a source in continuous time: T_0 and $D_0 > 0$. (28) and (29) imply $T_1 > 2$ and $D_1 > T_1 - 1$ for every $h > 0$. Therefore the source property is preserved whatever $h > 0$. The branch of parabola in Figure 5 represents this case.

From (28) and (29), it is possible to plot a parametrized curve $(T_1(h), D_1(h))$ for each one of these different cases: $D_1 = T_1 - 1 + D_0 [(T_1 - 2)/T_0]^2$ given (T_0, D_0). ■

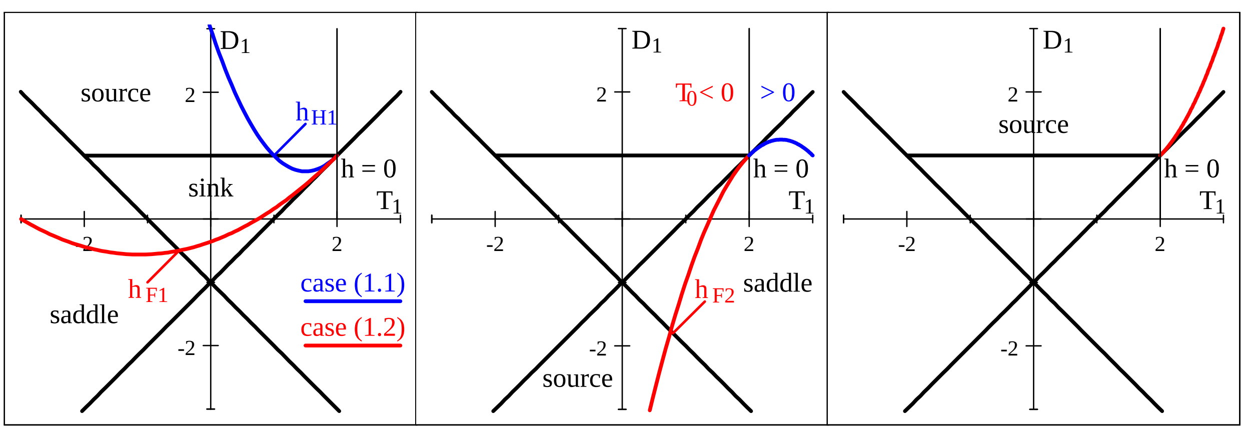

Fig. 3: Sink in continuous time   Fig. 4: Saddle in continuous time   Fig. 5: Source in continuous time

Corollary 4 *(topological equivalence in backward looking) In every case of Proposition 3, there exists a nonempty interval $(0, h^*)$ for the discretization step h where the stability properties of the continuous-time system are preserved.*

Proof. Straightforward. Simply observe that, in the case (3), $h^* = +\infty$. ■

3.2.2 Forward-looking discretization

We linearize now the forward-looking discretization

$$x_{n+1} \approx x_n + hf(x_{n+1}) \tag{30}$$

of system (24)-(25) around the common steady state $f(x) = 0$ to obtain $dx_{n+1} = J_1 dx_n = (I - hJ_0)^{-1} dx_n$.

Differently from the previous case, the Jacobian matrix of system (30) $J_1 = (I - hJ_0)^{-1}$ is no longer linear in h. The trace and the determinant of J_1 are now given by

$$T_1 = (2 - hT_0) D_1 \tag{31}$$

$$D_1 = \frac{1}{1 - hT_0 + h^2 D_0} = T_1 - 1 + h^2 D_0 D_1 \tag{32}$$

As above, we set three critical values: $h_{H2} \equiv T_0/D_0$,

$$h_{F3} \equiv \frac{T_0}{D_0} - \sqrt{\left(\frac{T_0}{D_0}\right)^2 - \frac{4}{D_0}} \text{ and } h_{F4} \equiv \frac{T_0}{D_0} + \sqrt{\left(\frac{T_0}{D_0}\right)^2 - \frac{4}{D_0}}$$

Proposition 5 *Consider $h > 0$.*

(1) If the steady state is a sink in continuous time, then the sink property is preserved in discrete time whatever $h > 0$.

(2) Let the steady state be a saddle in continuous time.

(2.1) If $D_1 > 0$, then the steady state is a saddle.

(2.2) If $D_1 < 0$, then the steady state is a saddle if $0 < h < h_{F4}$ and a sink if $h_{F4} < h$.

(3) Let the steady state be a source in continuous time.

(3.1) Let $D_1 < 0$. If $(T_0/D_0)^2 < 4/D_0$, then the source property is preserved whatever $h > 0$. If $(T_0/D_0)^2 > 4/D_0$, then the steady state is a source if $0 < h < h_{F3}$ or $h_{F4} < h$, and a saddle if $h_{F3} < h < h_{F4}$.

(3.2) Let $D_1 > 0$. If $(T_0/D_0)^2 < 4/D_0$, then the steady state is a source if $0 < h < h_{H1}$ and a sink if $h_{H2} < h$. If $(T_0/D_0)^2 > 4/D_0$, then the steady state is a source if $h < h_{F3}$, a saddle if $h_{F3}<h<h_{F4}$ and a sink if $h_{F4} < h$.

The system generically undergoes a Hopf bifurcation at h_{H2} and flip bifurcations at h_{Fi}, $i = 3, 4$.

Proof.

(1) Assume that the steady state is a sink in continuous time: $T_0 < 0 < D_0$. According to (32), $0 < D_1 < 1$ and, so, $D_1 > T_1 - 1$. We observe that $D_1 > -1/(1 + 2 - hT_0)$ or, equivalently, according to (31), $D_1 > -[(2 - hT_0) D_1] - 1 = -T_1 - 1$. Hence, the steady state is a sink in discrete time whatever $h > 0$.

(2) Assume now that the steady state is a saddle in continuous time: $D_0 < 0$. According to (32), $D_1 < T_1 - 1$ iff $D_1 > 0$. We notice that, according to (31) and (32), $D_1 < -T_1 - 1$ is equivalent to

$$\frac{1 - hT_0 + 2}{1 - hT_0 + h^2 D_0} < -1 \tag{33}$$

(2.1) If $D_1 > 0$, then $D_1 < T_1 - 1$. The steady state is a saddle.

(2.2) If $D_1 < 0$, then $1 - hT_0 + h^2 D_0 < 0$ and $D_1 < -T_1 - 1$, that is (33), is equivalent to $D_0 h^2 - 2T_0 h + 4 > 0$, that is to $h_{F3} < h < h_{F4}$. We observe that $h > 0$ and $h_{F3} < 0 < h_{F4}$. Thus, the steady state is a saddle if $0 < h < h_{F4}$ and a sink if $h_{F4} < h$.

(3) Assume now that the steady state is a source in continuous time: T_0 and $D_0 > 0$. According to (32), $D_1 < T_1 - 1$ iff $D_1 < 0$. We observe that $D_1 < -T_1 - 1$ is still equivalent to (33).

(3.1) If $D_1 < 0$ (that is $D_1 < T_1 - 1$), then $1 - hT_0 + h^2 D_0 < 0$ and $D_1 < -T_1 - 1$, that is (33), is equivalent to $D_0 h^2 - 2T_0 h + 4 > 0$.

If $(T_0/D_0)^2 < 4/D_0$, then $D_1 < -T_1 - 1$: the steady state is a source, whatever $h > 0$.

If $(T_0/D_0)^2 > 4/D_0$, then $D_1 > -T_1 - 1$ is equivalent to $h_{F3} < h < h_{F4}$ since $0 < h_{F3} < h_{F4}$. The steady state is a source if $0 < h < h_{F3}$ or $h_{F4} < h$, and a saddle if $h_{F3} < h < h_{F4}$.

(3.2) Consider the case $D_1 > 0$ (that is $D_1 > T_1 - 1$). Then $1 - hT_0 + h^2 D_0 > 0$ and $D_1 < -T_1 - 1$ is equivalent to $D_0 h^2 - 2T_0 h + 4 < 0$.

If $(T_0/D_0)^2 < 4/D_0$, then $D_1 > -T_1 - 1$. We have $D_1 > 1$ iff $h < T_0/D_0 \equiv h_{H2}$. Then, the steady state is a source if $0 < h < h_{H2}$ and a sink if $h_{H2} < h$.

If $(T_0/D_0)^2 > 4/D_0$, then $D_1 < -T_1 - 1$ iff h_{F3}<h<h_{F4}. We observe that $0 < h_{F3} < h_{H2} < h_{F4}$. Then, the steady state is a source if $h < h_{F3}$, saddle if h_{F3}<h<h_{F4} and sink if $h_{F4} < h$. ■

For brevity, we omit the figures corresponding to the cases of Proposition 5. Their construction is similar to that of Figures 3-5.

Corollary 6 *(topological equivalence in forward looking) In every case of Proposition 5, there exists a non-empty interval $(0, h^*)$ for the discretization step h where the stability properties of the continuous-time system are preserved.*

Proof. Straightforward. Simply observe that, in cases (1) and (2.1), $h^* = +\infty$. ■

3.2.3 Hybrid discretization

In economics, many higher-dimensional models require a hybrid discretization to recover the equivalence between discrete and continuous time, that is a mix of discretization in backward and forward looking. Without loss of generality, we consider a system where the first equation is discretized backward and the second one forward. Thus, the system of differential equations (24)-(25) becomes:

$$x_{1n+1} \approx x_{1n} + hf_1(x_{1n}, x_{2n}) \tag{34}$$

$$x_{2n+1} \approx x_{2n} + hf_2(x_{1n+1}, x_{2n+1}) \tag{35}$$

As we know, the steady state is invariant to the choice of time and to the type of discretization (backward/forward). The trace and the determinant of the Jacobian matrix J_1 of the hybrid system (34)-(35) become

$$T_1 = 2 + \frac{h(T_0 - hD_0)}{1 - h\partial f_2/\partial x_2} \tag{36}$$

$$D_1 = 1 + \frac{hT_0}{1 - h\partial f_2/\partial x_2} = T_1 - 1 + \frac{h^2 D_0}{1 - h\partial f_2/\partial x_2} \tag{37}$$

Notice that, in the particular case $\partial f_2/\partial x_2 = 0$, (36) and (37) write

$$T_1 = 1 + D_1 - h^2 D_0 \tag{38}$$

$$D_1 = 1 + hT_0 \tag{39}$$

Let

$$h_{F5} \equiv \frac{T_0 - 2f_{22}}{D_0} - \sqrt{\left(\frac{T_0 - 2f_{22}}{D_0}\right)^2 + \frac{4}{D_0}} \text{ and } h_{F6} \equiv \frac{T_0 - 2f_{22}}{D_0} + \sqrt{\left(\frac{T_0 - 2f_{22}}{D_0}\right)^2 + \frac{4}{D_0}}$$

where $f_{22} \equiv \partial f_2/\partial x_2$.

Proposition 7 *Consider $h > 0$.*

(1) Let $f_{22} \leq 0$.

(1.1) If the steady state is a sink in continuous time, then the steady state in discrete time is a sink if $0 < h < h_{F6}$, and a saddle if $h_{F6} < h$.

(1.2) Let the steady state be a saddle in continuous time.

(1.2.1) If $[(T_0 - 2f_{22})/D_0]^2 + 4/D_0 < 0$, or $[(T_0 - 2f_{22})/D_0]^2 + 4/D_0 > 0$ and $T_0 > 2f_{22}$, then the steady state is a saddle point.

(1.2.2) If $[(T_0 - 2f_{22})/D_0]^2 + 4/D_0 > 0$ and $T_0 < 2f_{22}$, then the steady state is a saddle if $0 < h < h_{F5}$, and a source if $h_{F5} < h$.

(1.3) If the steady state is a source in continuous time, then the steady state is a source if $0 < h < h_{F6}$ and a saddle if $h_{F6} < h$.

(2) Let $f_{22} > 0$ with $h < 1/f_{22}$. All the previous cases hold, provided we restrict the analysis to the interval $(0, 1/f_{22})$.

The system generically undergoes a Hopf bifurcation at h_{H2} and a flip bifurcation at h_{Fi}, $i = 5, 6$.

Proof.

(1) We consider the case $f_{22} \leq 0$ (the case $f_{22} = 0$, that is $T_1 = 1 + D_1 - h^2 D_0$ and $D_1 = 1 + hT_0$, is included).

(1.1) Assume that the steady state is a sink in continuous time: $T_0 < 0 < D_0$. Then from (37) we have $D_1 < 1$ and $D_1 > T_1 - 1$. We notice that, according to (36) and (37), $D_1 > -T_1 - 1$ is equivalent to

$$D_0 h^2 - 2(T_0 - 2f_{22})h - 4 < 0 \tag{40}$$

that is to $h_{F5} < h < h_{F6}$. We notice also that $h_{F5} < 0 < h_{F6}$. Thus, the steady state is a sink if $0 < h < h_{F6}$, and a saddle if $h_{F6} < h$.

(1.2) Assume now that the steady state is a saddle in continuous time: $D_0 < 0$. According to (37), $D_1 < T_1 - 1$. $D_1 > -T_1 - 1$ is equivalent to (40).

(1.2.1) If $[(T_0 - 2f_{22})/D_0]^2 + 4/D_0 < 0$, then $D_1 > -T_1 - 1$: the steady state is a saddle point. If $[(T_0 - 2f_{22})/D_0]^2 + 4/D_0 > 0$ and $T_0 > 2f_{22}$ we have $h_{F5} < h_{F6} < 0 < h$: the steady state is a saddle point.

(1.2.2) If $[(T_0 - 2f_{22})/D_0]^2 + 4/D_0 > 0$ and $T_0 < 2f_{22}$ we notice that $0 < h_{F5} < h_{F6}$. So, the steady state is a saddle if $0 < h < h_{F5}$, a source if $h_{F5} < h$.

(1.3) Assume now that the steady state is a source in continuous time: T_0 and $D_0 > 0$. According to (37), $D_1 > 1$ and $D_1 > T_1 - 1$. $D_1 > -T_1 - 1$ is equivalent to (40). We observe that $h_{F5} < 0 < h_{F6}$. Hence, source if $0 < h < h_{F6}$ and saddle if $h_{F6} < h$.

(2) The case $f_{22} > 0$ with $h < 1/f_{22}$ is similar to the previous one. More precisely, we have to consider the interval $(0, 1/f_{22})$ and only the bifurcation values in this interval.

(2.1) If the steady state is a sink in continuous time, then it is a sink if $0 < h < \min\{h_{F6}, 1/f_{22}\}$ and a saddle if $h_{F6} < 1/f_{22}$ and $h_{F6} < h < 1/f_{22}$.

(2.2) Let the steady state be a saddle in continuous time.

(2.2.1) If $[(T_0 - 2f_{22})/D_0]^2 + 4/D_0 > 0$ and $T_0 < 2f_{22}$, then the steady state is a saddle if $0 < h < \min\{h_{F5}, 1/f_{22}\}$ and a source if $h_{F5} < 1/f_{22}$ and $h_{F5} < h < 1/f_{22}$.

(2.2.2) If $[(T_0 - 2f_{22})/D_0]^2 + 4/D_0 < 0$ or $T_0 > 2f_{22}$, then the steady state is a saddle if $0 < h < 1/f_{22}$.

(2.3) Assume now that the steady state is a source in continuous time. Hence, the steady state is a source in discrete time if $0 < h < \min\{h_{F6}, 1/f_{22}\}$ and a saddle if $h_{F6} < 1/f_{22}$ and $h_{F6} < h < 1/f_{22}$.

We are interested in values of h lying in a right neighborhood of zero, where the stability properties are preserved. Therefore, the complicate case of a rough approximation with $h > 1/f_{22}$ and $f_{22} > 0$ is omitted. ■

As above, we omit the figures corresponding to the multiple cases of Proposition 7.

Corollary 8 *(topological equivalence in hybrid looking) In every case of Proposition 7, there exists a non-empty interval* $(0, h^*)$ *for the discretization step* h *where the stability properties of the continuous-time system are preserved.*

Proof. Straightforward. Simply observe that, in the case (1.2.1), $h^* = +\infty$. ■

3.2.4 Dynamic optimization models

The explicit structure of optimization models helps us to understand the (possible lack of) equivalence between bifurcations in continuous and discrete time. In the following, we reconsider the general program (12)-(13) and we linearize the first-order discretization.[9]

The existence of a steady state requires $\rho_t = \rho$ constant over time. In this case, the system writes

$$0 = s(k, c(k, \mu)) \tag{41}$$

$$\rho = \frac{\partial s}{\partial k}(k, c(k, \mu)) + \frac{1}{\mu}\frac{\partial v}{\partial k}(k, c(k, \mu)) \tag{42}$$

Local dynamics of continuous time system (17)-(18) are summarized by the following Jacobian matrix:[10]

$$J_0 \equiv \begin{bmatrix} s_k + s_c c_k & s_c c_\mu \\ -P & \rho - Q \end{bmatrix} \tag{43}$$

[9] For brevity, we omit the linearization of higher-order discretizations.

[10] In the following, given a generic function $z = z(x, y)$, $z_x \equiv \partial z/\partial x$ and $z_{xy} \equiv \partial^2 z/(\partial x \partial y)$ will denote the first and second-order (partial) derivatives.

where c_k and c_μ are given by (16), and

$$P \equiv v_{kk} + \mu s_{kk} + c_k (v_{kc} + \mu s_{kc}) \quad (44)$$

$$Q \equiv s_k + c_\mu (v_{kc} + \mu s_{kc}) \quad (45)$$

The trace and the determinant of the Jacobian matrix are given by

$$T_0 = \rho - Q + s_k + s_c c_k = \rho + s_c c_k - c_\mu (v_{kc} + \mu s_{kc}) \quad (46)$$

$$\begin{aligned} D_0 &= (\rho - Q)(s_k + s_c c_k) + P s_c c_\mu \quad (47) \\ &= (\rho - s_k)(s_k + s_c c_k) + c_\mu [s_c (v_{kk} + \mu s_{kk}) - s_k (v_{kc} + \mu s_{kc})] \end{aligned}$$

where $\rho - s_k = v_k/\mu$.

Discretizing $\dot{\beta}_t$ in forward-looking, we obtain

$$\beta_{t+h}/\beta_t \approx 1/\left(1 + h\rho_{t+h}\right) \quad (48)$$

We can replace (48) in (23) to get

$$\frac{\mu_t}{\mu_{t+h}} = \frac{1 + h\left[\frac{\partial s}{\partial k_{t+h}}\left(k_{t+h}, c\left(k_{t+h}, \mu_{t+h}\right)\right) + \frac{1}{\mu_{t+h}}\frac{\partial v}{\partial k_{t+h}}\left(k_{t+h}, c\left(k_{t+h}, \mu_{t+h}\right)\right)\right]}{1 + h\rho_{t+h}} \quad (49)$$

A constant discounting implies $\beta_{t+h}/\beta_t = \beta^h$, where $\beta = e^{-\rho}$. In this case, at the steady state, (49) gives (42). Assumption 1 on the fundamentals ensures the existence and the uniqueness of the steady state, solution of (41)-(42).

Focus now the local dynamics. Since, at the steady state, μ_t is stationary (while $\lambda_t = \beta_t \mu_t$ is not because β_t decreases over time), we linearize the system with a forward-looking μ-type Euler discretization.

The hybrid Euler discretization (21)-(23) becomes

$$k_{t+h} \approx k_t + hs\left(k_t, c\left(k_t, \mu_t\right)\right) \quad (50)$$

$$(1 + h\rho)\frac{\mu_t}{\mu_{t+h}} = 1 + h\left[\frac{\partial s}{\partial k_{t+h}}\left(k_{t+h}, c\left(k_{t+h}, \mu_{t+h}\right)\right) + \frac{1}{\mu_{t+h}}\frac{\partial v}{\partial k_{t+h}}\left(k_{t+h}, c\left(k_{t+h}, \mu_{t+h}\right)\right)\right] \quad (51)$$

Linearizing (50)-(51) gives

$$dk_{t+h} = [1 + h(s_k + s_c c_k)]\, dk_t + h s_c c_\mu d\mu_t$$

and

$$h[v_{kk} + \mu s_{kk} + c_k(v_{kc} + \mu s_{kc})]\, dk_{t+h} + (1 + h[s_k + c_\mu(v_{kc} + \mu s_{kc})])\, d\mu_{t+h} = (1 + h\rho)\, d\mu_t$$

(notice from (42) that $s_k = \rho - v_k/\mu$). Using (44) and (45), we find the associated Jacobian matrix J_1:

$$J_1 \equiv \begin{bmatrix} 1 + h(s_k + s_c c_k) & h s_c c_\mu \\ -[1 + h(s_k + s_c c_k)]\frac{hP}{1+hQ} & \frac{1+h\rho}{1+hQ} - \frac{hP}{1+hQ} h s_c c_\mu \end{bmatrix} \quad (52)$$

with the following trace and determinant

$$D_1 = [1 + h(s_k + s_c c_k)]\frac{1 + h\rho}{1 + hQ} \quad (53)$$

$$T_1 = 1 + h(s_k + s_c c_k) + \frac{1 + h\rho}{1 + hQ} - \frac{hP}{1 + hQ} h s_c c_\mu \quad (54)$$

The traces and determinants in (46) and (47), and in (53) and (54) will be reconsidered in Section 5 when a hybrid discretization will be applied to the most popular growth model pioneered by Ramsey (1928) and later refined by Cass (1965) and Koopmans (1965).

In the following, we study how conditions for elementary bifurcations change under a discretization of a continuous-time system. For brevity, we focus on two-dimensional backward-looking discretizations, but results can be easily extended to the case of hybrid or higher-dimensional dynamic systems.

3.3 Local bifurcations

We consider local bifurcations in stability of a simple attractor: the steady state, and we study the role of either the order or the discretization step h in the occurrence of these bifurcations.[11]

Two systems are topologically equivalent if they have similar trajectories.[12] Most of nonlinear system are topologically equivalent to their linearizations around a fixed point (steady state). The Großman-Hartman Theorem states that linearizations well behave around hyperbolic steady states[13], that is the stability properties are preserved.[14] In the following, we assume that the assumptions of the Großman-Hartman Theorem are satisfied and, namely, the steady states is hyperbolic.

In continuous time, a local bifurcation generically arises when the real part of an eigenvalue $\lambda(p)$ of the Jacobian matrix crosses zero in response to a change of parameter p. Without loss of generality, we normalize to zero the critical parameter value of bifurcation ($p = 0$) and we get generically two cases.

(1) Saddle-node bifurcation. A real eigenvalue crosses zero: $\lambda(0) = 0$.

(2) Hopf bifurcation. The real part of two complex and conjugate eigenvalues $\lambda(p) = a(p) \pm ib(p)$ crosses zero: $a(0) = 0$ and $b(p) \neq 0$ in a neighborhood of $p = 0$.

In discrete time, a local bifurcation generically occurs when one eigenvalue $\lambda(p)$ of the Jacobian matrix evaluated at the steady state, crosses the unit circle in response to a change of parameter p.[15] Normalizing as above to zero the critical parameter value of bifurcation ($p = 0$), we find generically three classes of elementary bifurcations.

(1) Saddle-node bifurcation: $\lambda(0) = +1$.

(2) Flip bifurcation: $\lambda(0) = -1$.

(3) Hopf bifurcation: $|\lambda(0)| = |a(0) \pm ib(0)| = 1$ with $b(0) \neq 0$.

Generically, only one eigenvalue is concerned with a saddle-node or a flip bifurcation and the bifurcation analysis can reduce to the study of a simple one-dimensional invariant manifold. Similarly, two complex (conjugated) eigenvalues are involved in the Hopf bifurcation and the bifurcation analysis simplifies to the study of a two-dimensional invariant manifold. When an eigenvalue (or a conjugated pair of eigenvalues in the case of Hopf) crosses the unit circle, generically, no other eigenvalue crosses simultaneously the circle. Then higher-dimensional dynamics reduces to a single equation or to a two-dimensional dynamics under a Hopf bifurcation (Central Manifold Theorem) and the movement of the other eigenvalues does not change the qualitative properties of dynamics. In other terms, only a one or two-dimensional central manifold is concerned with the bifurcation: the other manifolds preserve their qualitative properties.

For simplicity, we will study the occurrence of saddle-node bifurcations and flip bifurcations of one-dimensional dynamics and that of Hopf bifurcations of two-dimensional dynamics. Under the assumptions of the Central Manifold Theorem, there is no loss of generality with respect to higher-dimensional systems.

3.4 On the saddle-node equivalence

The continuous-time properties of the family of saddle-node bifurcations (saddle-node, transcritical and pitchfork) are preserved in discrete time. In a way, the saddle-node is the less sophisticated of the elementary bifurcations.

Focus for simplicity on the continuous-time one-dimensional dynamics $\dot{x} = f(x, p)$. The real eigenvalue $\lambda_0 = \partial f / \partial x$ depends on p, the bifurcation parameter. The first-order discretization is given by $x_{n+1} \approx$

[11] The bifurcation is local if the change of the orbit structure can be observed in an arbitrarily small neighborhood of the (normalized) steady state; the bifurcation is global otherwise. Good introductions to the theory of bifurcations are, among the others, Guckenheimer and Holmes (1983), Hale and Koçak (1991).

[12] Two dynamic systems f and g are topologically equivalent if there exists a homeomorphism (continuous function with continuous inverse) that maps f orbits into g orbits while preserving the sense of direction in time.

[13] A steady state x^* of a nonlinear system of differential equations $\dot{x} = f(x)$ (respectively, of a nonlinear system of difference equations $x_{t+1} = g(x_t)$) is said to be hyperbolic if the Jacobian matrix $J_0(x^*)$ of the system f evaluated at x^* has no eigenvalues with zero real parts (respectively, the Jacobian matrix $J_1(x^*)$ of the system g evaluated at x^* has no eigenvalues with moduli equal to one).

[14] If x^* is hyperbolic, there exists a neighborhood of x^* where $\dot{x} = f(x)$ is topologically equivalent to the linear system $\dot{x} = J_0(x^*)(x - x^*)$ (respectively, if $J_1(x^*)$ is invertible, $x_{t+1} = g(x_t)$ is topologically equivalent to the linear system $x_{t+1} = x^* + J_1(x^*)(x_t - x^*)$).

[15] We omit the case where the eigenvalue crosses zero. In this case, an orientation reversing map can locally become orientation preserving, without promoting the occurrence of cycles. In order to have a rigorous but concise introduction to bifurcations in discrete time, interested readers are highly recommended to see Grandmont (2008).

$x_n + hf(x_n, p)$ with eigenvalue $\lambda_1 = 1 + h\partial f/\partial x$ (evaluated at the steady state). A saddle node bifurcation arises in continuous time if $\lambda_0 = 0$, that is if $\partial f/\partial x = 0$ or, equivalently, $\lambda_1 = 1$. Since neither the steady state x nor f depend on h in the Euler discretization, this equivalence holds whatever the discretization step. Similarly, one proves the result in the case of forward-looking discretizations: $x_{n+1} \approx x_n + hf(x_{n+1}, p)$. The eigenvalue is given by $\lambda_1 = 1/(1 + h\partial f/\partial x)$ and $\partial f/\partial x = 0$ if and only if $\lambda_1 = 1$.

We conclude that under a first-order discretization (backward or forward-looking), a saddle-node bifurcation generically occurs in continuous time if and only if it arises in discrete time, whatever the discretization step h, that is even under an extremely rough approximation.

3.5 On the Hopf equivalence

As was the case for the stability properties in Section 3.2, conditions for Hopf bifurcation in discrete time tend to those in continuous time as the "distance" h between dynamics in continuous and discretized time tends to zero.

Proposition 9 *A Hopf bifurcation in continuous time generically arises when*

$$T_0 = 0 \tag{55}$$
$$D_0 > 0 \tag{56}$$

while, under a backward-looking discretization, it occurs when

$$T_0 = -hD_0 \tag{57}$$
$$D_0 \geq T_0^2/4 \tag{58}$$

where $h > 0$ is the discretization step. Under the assumption $f(x,p) \in C^2$ in a neighborhood of $(x(p_H), p_H)$ (where p_H is the Hopf bifurcation value in continuous time and $x(p_H)$ the corresponding steady state), the right-hand sides of (57) and (58) generically tend to zero as h goes to zero, and conditions (57)-(58) become closer to conditions (55)-(56).

Proof. The two roots of the continuous-time characteristic polynomial $P_0(\lambda) = \lambda^2 - T_0\lambda + D_0$ are: $\lambda = T_0/2 \pm \sqrt{T_0^2/4 - D_0}$. Roots are complex if and only if $D_0 > T_0^2/4$. In this case, the eigenvalues become $\lambda = \alpha \pm i\beta$ with $\alpha \equiv T_0/2$ and $\beta \equiv \sqrt{D_0 - T_0^2/4}$. Hopf bifurcation in continuous time generically requires: $\alpha = 0$ and $\beta \neq 0$, that is $T_0 = 0$ and $D_0 > T_0^2/4 = 0$.

Consider now the trace and determinant (28)-(29). It is known that a Hopf bifurcation generically arises in discrete time if and only if $D_1 = 1$ and $D_1 \geq T_1^2/4$ (complex and conjugated eigenvalues have the same modulus and cross together the unit circle if their product (determinant) is one). Equivalently, conditions to get a Hopf bifurcation become $T_1^2 \leq 4$ and $D_1 = 1$. Using (28)-(29), we get

$$T_1^2 = (2 + hT_0)^2 \leq 4 \tag{59}$$
$$D_1 = 1 + h(T_0 + hD_0) = 1 \tag{60}$$

(60) gives $T_0 + hD_0 = 0$ or, equivalently,

$$h = -T_0/D_0 \tag{61}$$

Replacing (61) in (59), we obtain $\left(2 - T_0^2/D_0\right)^2 \leq 4$ or, equivalently, $0 \leq T_0^2/D_0 \leq 4$. The left-hand inequality implies $D_0 > 0$. Therefore the right-hand inequality becomes $D_0 \geq T_0^2/4$.

Summing up, the necessary and sufficient conditions for a Hopf bifurcation in discrete time are, generically: $T_0 + hD_0 = 0$ and $D_0 \geq T_0^2/4$.

The derivatives appearing in J_0 and then in (T_0, D_0) depend directly and indirectly (through the steady state) on the parameter value p:

$$T_0(x(p), p) = -hD_0(x(p), p) \tag{62}$$
$$D_0(x(p), p) \geq T_0(x(p), p)^2/4 \tag{63}$$

where $x(p)$ is a stationary state corresponding to the parameter value p.

The Hopf bifurcation value p_H solves (62). Under the assumptions of the Implicit Function Theorem, equation (62) locally defines a continuous function[16] $p_H = p_H(h)$.

We compare (62)-(63) with conditions required in continuous time to obtain a Hopf bifurcation (55)-(56):

$$\begin{aligned} T_0(x(p), p) &= 0 \\ D_0(x(p), p) &> T_0(x(p), p)^2/4 \end{aligned}$$

Since $f(x,p) \in C^2 \subseteq C^1$, we apply the Implicit Function Theorem to $f_1(x_1, x_2, p) = 0$ and $f_2(x_1, x_2, p) = 0$ to obtain the continuity of $x(p)$ generically. Since $f(x,p) \in C^2$, we can apply the Implicit Function Theorem to (62), that is to

$$f_{11}(x(p), p) + f_{22}(x(p), p) + h[f_{11}(x(p), p) f_{22}(x(p), p) - f_{12}(x(p), p) f_{21}(x(p), p)] = 0$$

(where $f_{ij} \equiv \partial f_i / \partial x_j$), to obtain also the continuity of $p_H(h)$ generically.[17]

Generic continuity of x and p_H implies $\lim_{p \to p_H} x(p) = x(p_H)$ and $\lim_{h \to 0} p_H(h) = p_H(0)$. Thus,

$$\lim_{h \to 0} [h D_0(p_H(h), x(p_H(h)))] \to 0$$

(the continuity of D_0 ensues from $f(x,p) \in C^1$) and condition (62) converges to (55). (63) becomes closer to (56): indeed, when $T_0 \neq 0$ goes to zero, $D_0 \geq T_0^2/4 > 0$ remains strictly positive generically.[18] ■

In other terms, if a Hopf bifurcation arises in continuous time, it is (generically) possible to find a (sufficiently small) discretization step which preserves (by continuity) this bifurcation. Conditions for Hopf in discrete time can be made arbitrarily close to those in continuous time by simply reducing the period length h. Under mild continuity properties (namely, $f(x,p) \in C^2$), the discrete-time critical value $p_H(h)$ lies in a neighborhood of the continuous-time critical value $p_H(0)$.

We have considered a backward-looking discretization. Forward-looking and hybrid discretization are also of interest and similar conclusions hold. Just focus on the case of the hybrid discretization (34)-(35) which is of interest in endogenous saving models. Assuming for simplicity

$$\partial f / \partial x_2 = 0 \tag{64}$$

we obtain $T_1 = 2 + hT_0 - h^2 D_0$ and $D_1 = 1 + hT_0$. If $T_0 = 0$ and $D_0 > 0$ (conditions for Hopf bifurcation in continuous time, see (55)-(56)), we get also $D_1 = 1$ and $T_1 = 2 - h^2 D_0 < 2$, that is $T_1^2 \leq 4$ provided that

$$h^2 \leq 2/D_0 \tag{65}$$

Under condition (64) and inequality (65), the Hopf equivalence still holds between continuous and discrete time (see (59) and (61)).

One may question whether the equivalence holds in dynamic optimization models. We have discretized a λ-type Euler equation and transformed the resulting hybrid discretization in a μ-type system. Eventually, we have linearized the μ-type system around its stationary state (k, μ) (λ is not stationary).

Does the Hopf equivalence hold in general optimization models that satisfy (64) and suitable continuity properties? Focus on (53)-(54) and observe that

$$T_1 = 2 + \frac{h[T_0(1 + h\rho) - hD_0 - h\rho(\rho - Q)]}{1 + h\rho - h(\rho - Q)} \tag{66}$$

$$D_1 = 1 + \frac{h[T_0(1 + h\rho) - h\rho(\rho - Q)]}{1 + h\rho - h(\rho - Q)} \tag{67}$$

[16] The critical value for a Hopf bifurcation p_H depends on h, while, as seen above, the critical value for a saddle-node bifurcation p_S does not.

[17] More explicitly,

$$\begin{bmatrix} x_1'(p) \\ x_2'(p) \end{bmatrix} = -\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}^{-1} \begin{bmatrix} f_{1p} \\ f_{2p} \end{bmatrix}$$

where $f_{ip} \equiv \partial f_i / \partial p$, provided that $f_{11} f_{22} - f_{12} f_{21} \neq 0$. In addition, $p_H'(h) = -(f_{11} f_{22} - f_{12} f_{21})/\Delta$ with

$$\begin{aligned} &(1/h + f_{11})(f_{212} x_1' + f_{222} x_2' + f_{2p2}) + (1/h + f_{22})(f_{111} x_1' + f_{121} x_2' + f_{1p1}) \\ &- f_{12}(f_{211} x_1' + f_{221} x_2' + f_{2p1}) - f_{21}(f_{112} x_1' + f_{122} x_2' + f_{1p2}) \\ &\equiv \Delta \neq 0 \end{aligned}$$

where $f_{ijk} \equiv \partial f_i / (\partial x_j \partial x_k)$.

[18] The case where both the eigenvalues of J_0 are zero is non-generic.

where (T_0, D_0) and (T_1, D_1) are respectively given by (46)-(47) and (53)-(54). According to (42) and (45), when v no longer depends on k, we have $\rho = Q$. Then (66) and (67) reduce to[19]

$$T_1 = 1 + D_1 - \frac{h^2}{1 + h\rho} D_0 \tag{68}$$

$$D_1 = 1 + hT_0 \tag{69}$$

and $T_0 = 0$ iff $D_1 = 1$. Using (68) with $D_0 > 0$ and $D_1 = 1$, condition $T_1^2 \leq 4$ is equivalent to $h \leq 2\left[\rho/D_0 + \sqrt{1 + (\rho/D_0)^2}\right]$. Therefore, if a Hopf bifurcation arises in continuous time, under a sufficiently small discretization step, it occurs also in discrete time generically.

3.6 On the flip singularity

As seen above, the saddle-node bifurcation persists under a linear discretization, while the Hopf bifurcation is characterized by a continuity property (the smaller the step h, the closer the critical values in continuous and discrete time).

The main difference between these dynamics is the flip bifurcation: when the continuous-time eigenvalue is bounded from below, under linear and higher-order Euler approximations, the flip bifurcation disappears in discrete time when the discretization step h falls below a positive threshold h_F. The critical value h_F increases with the order q of Taylor discretization (see polynomial (2)-(3)).

In the following, we consider one-dimensional discretizations. There is no loss of generality under the assumptions of the Central Manifold Theorem.

A continuous-time scalar system: $\dot{x} = f(x, p)$, where p is the bifurcation parameter, can be approximated by a first-order Taylor polynomial: $x_{n+1} \approx x_n + hf(x_n, p) \equiv g(x_n, p)$. Consider a parametrized steady state: $f(x, p) = 0$. We introduce a simplified notation for the partial derivatives: $f_x \equiv \partial f/\partial x$, $f_p \equiv \partial f/\partial p$, $f_{xx} \equiv \partial^2 f/\partial x^2$, $f_{pp} \equiv \partial^2 f/\partial p^2$ and so on. As seen above, under the assumptions of the Implicit Function Theorem, the stationary state depends on the bifurcation parameter: $x = x(p)$.[20]

A flip bifurcation generically requires: $\lambda = g_x(x(p), p) = -1$ or, more explicitly:

$$\left.\frac{\partial [x_n + hf(x_n, p)]}{\partial x_n}\right|_{x_n = x(p)} = 1 + hf_x(x(p), p) = -1 \tag{70}$$

Applying the Implicit Function Theorem to (70), we get, locally, the critical value as a function of discretization degree: $p_F = p_F(h)$.[21]

Let us give now sufficient conditions to exclude flip bifurcations in discrete time. Without loss of generality, we set $h > 0$ and we call $X(p) \equiv \{(x, p) : f(x, p) = 0\}$ the set of stationary states x corresponding to a given parameter value p. $X(P) \equiv \cup_{p \in P} X(p)$ is the graph of the stationary states obtained by varying the (scalar) parameter p. $X(p)$ is empty, when the system admits no stationary states at p. In the sequel, we consider only the range of parameter values generating a nonempty set of stationary states: $P \equiv \{p : X(p) \neq \varnothing\}$. Let us also define the sets $Y \equiv f_x(X(P))$ and $Z \equiv f_x(S \times P)$ with $f \in C^1$ and S the domain of x. We provide sufficient conditions to exclude flip bifurcations.

Proposition 10 *(1) If* $\inf Y \geq 0$*, no flip bifurcation arises whatever* h*.*

(2) If $-\infty < \inf Y < 0$*, there exists a nonempty discretization range* $(0, h_F)$ *with* $h_F \equiv -2/\inf Y$*, where no flip bifurcation arises.*

Proof. (1) If $\inf Y \geq 0$, then $1 + hf_x(x(p), p) > 0 > -1$ whatever h and whatever the selection $(x(p), p) \in X(p)$. (2) If $\inf Y < 0$, solve $1 + h \inf Y > -1$ in order to exclude the flip bifurcation, that is, set $h < -2/\inf Y$. ∎

[19] Clearly, when $\rho = 0$, expressions (68)-(69) reduce to (38)-(39).
[20] If $f \in C^1$ and $f_x \neq 0$, we get $x'(p) = -f_p/f_x$.
[21] If $f \in C^2$, $f_x \neq 0$ and $f_{xx}/f_x \neq f_{px}/f_p$, then

$$p_F'(h) = \frac{1}{h} \frac{f_x/f_p}{f_{xx}/f_x - f_{xp}/f_p} \tag{71}$$

Corollary 11 *If* $-\infty < \inf Y$*, there exists a nonempty discretization range* $(0, h_F)$ *with* $h_F \equiv |-2/\inf Y|$*, where no flip bifurcation arises.*

Proof. Apply Proposition 10. ∎

Computing the graph $X(P)$ and its image with respect to f_x can be difficult. Let us provide another sufficient condition, less general than Corollary 11, but easier to check.

Corollary 12 *If* $-\infty < \inf Z$*, then there exists a discretization range* $(0, h_F)$ *with* $h_F \equiv |-2/\inf Z|$ *with no flip bifurcation.*

Proof. Simply notice that $X(P) \subseteq S \times P$ and apply Corollary 11. ∎

Bosi and Ragot (2009) provide explicit examples of Corollaries 11 and 12 with either bounded or unbounded parameter ranges.

In addition, they obtain the same qualitative results for higher-order and higher-dimensional discretizations under similar assumptions (namely boundedness of derivatives on $X(P)$).

On the one hand, in the case of two-dimensional dynamics, they prove that, if the derivatives of the Jacobian matrix are bounded on $X(P)$, there exists a critical step h_F such that $h \in (0, h_F)$ rules out the occurrence of flip bifurcations.

On the other hand, they show that, in the case of a qth-order discretization, if the qth derivatives of $f \in C^{q+1}$ are bounded over $X(P)$, then there exists a nonempty discretization range $(0, h_F)$, where generically no flip bifurcation arises.

In the rest of the paper, we focus on popular growth models to apply the equivalence results of our stability and bifurcation analysis.

Part II

Economic applications

Discrete-time version of popular dynamic models such as Solow (1956) can be derived through a backward-looking (Euler) discretization. To highlight the role of the discretization step in the occurrence of cycles of period two (flip bifurcation), we introduce negative externalities in the seminal Solow model (Day (1982)).

Hybrid discretizations are important in economic theory when agents' behavior results from a dynamic optimization. Households smooth consumption over time under a budget constraint with the wealth inherited from the past (backward-looking information), while considering the future interest rate in their intertemporal arbitrage (forward-looking information). The twofold nature of the dynamic system becomes more explicit when we discretize the continuous-time model. In order to recover the discrete-time model we need to discretize backward the budget constraint (as in Solow) and forward the Euler equation (intertemporal smoothing and endogenous saving). Influential examples of dynamic optimization is Ramsey (1928), later refined by Cass (1965) and Koopmans (1965), which is characterized by a saddle-path stability property. Introducing market imperfections can promote non-monotonic dynamics. Invariant closed curves (Hopf bifurcation) occur in Ramsey models with positive externalities (Zhang (2000)).

4 Backward-looking discretizations of Solow models

In this section, we compare the continuous-time and the discrete-time Solow models. We show that the discrete-time version ensues from a backward-looking discretization of the continuous-time one.

4.1 Solow models

The continuous-time version of Solow (1956) without technical progress is a two-dimensional dynamic system:

$$\dot{K}_t = sF(K_t, L_t) - \delta K_t \quad (72)$$

$$\dot{L}_t = gL_t \quad (73)$$

where K_t and L_t are the capital stock and the labor supply at time t. Parameters s, δ and g denote respectively the rates of saving, capital depreciation and demographic growth. Dynamics reduces to an intensive law of motion ($k = K/L$) under the assumption of a CRS technology:

$$\dot{k}_t = sf(k_t) - (\delta + g) k_t \tag{74}$$

Under the Inada conditions, the non-trivial steady state solves

$$f(k)/k = (\delta + g)/s \tag{75}$$

and is unique and locally stable: the eigenvalue of the intensive dynamics, evaluated at the steady state, is $\lambda_0 = -(1-a)(\delta+g) < 0$, where $a \equiv kf'(k)/f(k) \in (0,1)$ is the capital share. There is no room for (local) bifurcations.

In discrete time, the basic model writes:

$$K_{t+1} - K_t = sF(K_t, L_t) - \delta K_t \tag{76}$$

$$L_{t+1} - L_t = gL_t \tag{77}$$

and reduces to the intensive law:

$$k_{t+1} = [(1-\delta) k_t + sf(k_t)] / (1+g) \tag{78}$$

The positive steady state still solves $f(k)/k = (\delta+g)/s$ and is unique under the usual assumptions. Local stability is ensured by the eigenvalue in the unit circle:

$$\lambda_1 = 1 - (1-a)(\delta+g)/(1+g) \in (0,1) \tag{79}$$

As above, there is no room for local bifurcations.

A first-order discretization of system (72)-(73) gives

$$K_{n+1} \approx K_n + h[sF(K_n, L_n) - \delta K_n] \tag{80}$$

$$L_{n+1} \approx (1+hg) L_n \tag{81}$$

Normalizing (80) by L_n, we derive the intensive law: $k_{n+1} \approx [(1-h\delta) k_n + hsf(k_n)]/(1+hg)$. The discrete time dynamics (78) is recovered under a unit discretization step ($h = 1$). So, we can say that the discrete-time Solow model is actually the backward-looking Euler discretization of the continuous-time model.

The steady state does not depend on the discretization step and solves (75) as above, while the eigenvalue depends on h:

$$\lambda_1 = 1 - (1-a)(\delta+g)/(1/h+g) \tag{82}$$

Discretization introduces artificially the possibility of a flip bifurcation at $h_F \equiv 2/[(1-a)\delta - (1+a)g] > 2$ (under the assumption $g < \delta(1-a)/(1+a)$). However, the traditional discrete-time Solow model is characterized by monotonic stability because, as seen above, it corresponds to $h = 1 < h_F$: the unit discretization step rules out any flip bifurcation.

A continuity property holds: to recover the stability properties in continuous time we need to make our Euler-Taylor development "as close as possible" to the continuous-time system. Intuitively, we can either reduce the discretization step (as seen above: $h < h_F$) or increase the order of development. In the following, we prove that the second order is enough to exclude any artificial bifurcation in a Solow model.

Indeed, the second-order (backward-looking) discretization of (72)-(73) gives under constant returns to scale

$$k_{n+1} \approx \frac{k_n + [sf(k_n) - \delta k_n]\left(h + [sf'(k_n) - \delta] h^2/2\right) + sg[f(k_n) - k_n f'(k_n)] h^2/2}{1 + hg + (hg)^2/2} \tag{83}$$

with eigenvalue

$$\lambda_2 = \frac{1}{2} \frac{1 + (1 + h[ag - (1-a)\delta])^2}{1 + hg + (hg)^2/2} > 0 \tag{84}$$

which prevents the model from any flip bifurcation.[22]

[22] $h = 1$ implies $\lambda_2 \in (0,1)$: the saddle-node bifurcation is also excluded.

4.2 Externalities

A unit discretization step rules out any bifurcation in the Solow model. However, there is room for (flip) bifurcations in discrete-time Solow models with suitable market imperfections. The twofold question we raise is whether a smaller discretization step or a higher discretization order can remove the flip bifurcation observed in discrete time (under a unit discretization step).

We introduce in the Solow model negative productive externalities from a firm to another, by assuming that the environmental quality enhances factors' productivity and is, in turn, negatively affected by the average capital intensity. Formally, capital intensity k reduces the environmental quality to $m - k^{1-a}$, where $m > 0$ is the endowment of quality.

As in Day (1982), we assume a Cobb-Douglas production function and introduce an upper bound for the negative externality to ensure a positive TFP:

$$F(K_t, L_t) \equiv A\left(m - k_t^{1-a}\right) K_t^a L_t^{1-\alpha} \tag{85}$$

with

$$k_0 \in \left[0, m^{1/(1-a)}\right] \tag{86}$$

Replacing (85) in (72), we obtain the following law of motion:

$$\dot{k}_t \equiv sA\left(m - k_t^{1-a}\right) k_t^a - (\delta + g) k_t \tag{87}$$

Integrating (87) we find an explicit solution: $k_t = \left[k^{1-a} + \left(k_0^{1-a} - k^{1-a}\right) e^{-(1-a)(\delta+g+sA)t}\right]^{1/(1-a)}$. Restriction (86) ensures that the entire sequence of capital intensities lies in the interval $\left[0, m^{1/(1-a)}\right]$, the steady state k is asymptotically stable and the capital intensity converges monotonically towards its stationary value in the long run:

$$k = \lim_{t\to+\infty} k_t = \left[msA/(\delta + n + sA)\right]^{1/(1-a)} \in \left[0, m^{1/(1-a)}\right] \tag{88}$$

Therefore, in continuous time there is no room for bifurcations.

Conversely, in discrete time, persistent cycles and, possibly, chaos can arise. Introducing the externality (85) in the Solow model (76)-(77) gives:

$$k_{t+1} = \left[(1 - \delta - sA) k_t + sAmk_t^a\right] / (1 + g) \tag{89}$$

with steady state (88).

The eigenvalue of dynamics (89) is given by $\lambda_1 = a + (1-a)(1-\delta-sA)/(1+g) < 1$: only a flip bifurcation generically occurs at

$$A = A_F \equiv \frac{1}{s}\left[1 - \delta + (1+g)\frac{1+a}{1-a}\right] \tag{90}$$

Negative productive externalities generate cycles (when production increases, capital intensity goes up, productivity is lowered by the externalities and, eventually, production as well).

In the following, we prove two results: (1) on the one hand the discrete-time system still comes from a first-order backward-looking discretization of the original system (72)-(73) with (85), (2) a second-order discretization is enough to recover the continuous-time property and rule out the flip bifurcation.

(1) The intensive form of the first order discretization is given by

$$k_{n+1} \approx \left[(1 - h\delta - hsA) k_n + hsAmk_n^a\right] / (1 + hg)$$

Setting $h = 1$, we recover exactly the discrete-time Day model (equation (89)). In particular, we get the same flip bifurcation value as in (90).

(2) A second-order discretization constitutes a finer discretization of the continuous-time Day model and rules out the occurrence of flip bifurcations. Let us set $f_1(K_t, L_t) \equiv sA\left[m - (K_t/L_t)^{1-a}\right] K_t^a L_t^{1-\alpha} - \delta K_t$

and $f_2(K_t, L_t) \equiv gL_t$. Noticing that $L_{n+1}/L_n = 1 + hg + (hg)^2/2$ and that, under constant returns to scale, $F/\partial K_n = aAk_n^{a-1}m - A$ and $\partial F/\partial L_n = (1-a) Ak_n^a m$, we get the quadratic approximation

$$k_{n+1} = \frac{k_n + [sAk_n^a m - (\delta + sA) k_n] \left[h + \left(asAk_n^{a-1}m - \delta - sA\right) h^2/2\right] + (1-a) gsAk_n^a mh^2/2}{1 + hg + (hg)^2/2} \quad (91)$$

The steady state is still given by (88) and does not depend on h. The eigenvalue of the intensive law (91):

$$\lambda_2 = \frac{1}{2} \frac{1 + (1 + h[ag - (1-a)(\delta + sA)])^2}{1 + hg + (hg)^2/2}$$

is strictly positive and, therefore, there is no longer room for flip bifurcations, whatever the discretization step.

There are other possible extensions of the Solow (1956) model. The interested reader is referred to Bosi and Ragot (2008) for an application to the Keynesian Kaldor (1940) model where an exogenous aggregate saving function promotes the emergence of limit cycles through a Hopf bifurcation.

5 Hybrid discretizations of Ramsey models

The most popular optimal growth model is undoubtedly Ramsey (1928), later refined by Cass (1965) and Koopmans (1965). Ramsey argued against discounting utility of future generations as being "ethically indefensible". This "ethical" undiscounted utility functional in Ramsey (1928) is replaced in the Cass-Koopmans model (1965) by a weighted average of future felicities with decreasing weights over time (discounting).

5.1 Ramsey models

A benevolent planner determines the profile of capital accumulation in order to maximize the representative consumer's utility functional (12) subject to the resource constraint (13).[23] In Ramsey (1928), Cass (1965) and Koopmans (1965), the physical capital law of motion saving is specified as

$$s(k_t, c_t) = f(k_t) - \delta k_t - c_t \quad (92)$$

while the consumer's utility functional differs.

(1) In the Ramsey model, $\beta_t \equiv 1$ for every t and the felicity is defined as

$$v(k_t, c_t) = u(c_t) - u(c) \quad (93)$$

where c denotes the "bliss point". In order to ensure a bounded utility functional (a convergent integral), we fix a particular bliss point value: $c = f(k) - \delta k$ with $f'(k) = \delta$. This bliss point is the steady state value of consumption in the Ramsey model.[24]

(2) In the Cass-Koopmans model, $\dot{\beta}_t/\beta_t \equiv -\rho_t$ for every t and

$$v(k_t, c_t) = u(c_t) \quad (94)$$

Equation (15) reduces to $\mu = u'(c)$ and $c = c(k, \mu) = u'^{-1}(\mu) \equiv d(\mu)$ with $\partial c/\partial k = 0$ and $\partial c/\partial \mu = 1/u''$. System (17)-(18) simplifies:

$$\dot{k}_t = f(k_t) - \delta k_t - d(\mu_t) \quad (95)$$
$$\dot{\mu}_t = \mu_t [\rho_t + \delta - f'(k_t)] \quad (96)$$

(with $\rho_t = 0$ for every t in the Ramsey model).

[23] With no imperfections, a market economy decentralizes the planner's solution.
[24] The bliss point is the modified golden rule with a null discount rate.

In discrete time, the planner maximizes the utility series $\sum_{t=0}^{\infty} \beta_t u(c_t)$ (or $\sum_{t=0}^{\infty} [u(c_t) - u(c)]$ in the Ramsey model) subject to the sequence of resource constraints: $k_{t+1} - k_t + c_t \leq f(k_t) - \delta k_t$ with $t = 0, 1 \ldots$ System (19)-(20) writes:

$$k_{t+1} - k_t = f(k_t) - \delta k_t - c_t \tag{97}$$

$$\frac{\mu_t}{\mu_{t+1}} = \frac{\beta_{t+1}}{\beta_t} [1 + f'(k_{t+1}) - \delta] \tag{98}$$

(with $\beta_{t+1}/\beta_t = 1$ for every t in the Ramsey model).

Under (92) and (94), system (50)-(51) reduces to

$$k_{t+h} - k_t \approx h[f(k_t) - \delta k_t - d(\mu_t)] \tag{99}$$

$$\frac{\mu_t}{\mu_{t+h}} \approx \frac{\beta_{t+h}}{\beta_t} (1 + h[f'(k_{t+h}) - \delta]) \tag{100}$$

(with $\beta_{t+h}/\beta_t = 1$ for every t in the Ramsey model), and becomes the discrete-time system (97)-(98) under a unit discretization step ($h = 1$).

Dynamics generated by backward-looking approximations of the Euler equation work very differently from (98) because the productivity depends on k_t instead of k_{t+1}. The forward-looking component in the hybrid approximation not only allows us to recover the discrete-time model, but also makes economic sense because it captures saving decisions that depend on the future interest rate $f'(k_{t+1})$.

Focus now on the equivalence of stability properties.

System (99)-(100) comes from the discretization of the λ-type continuous-time system and a subsequent change of variable (μ_t instead of λ_t). As seen above, setting $h = 1$ gives (97)-(98). This proves that, in the Cass-Koopmans model, only a hybrid discretization of the continuous-time system expressed in the variables (k_t, λ_t) with a unit discretization step yields the traditional discrete-time system. The local analysis of the stability properties rests on an approximation around the steady state. However, the discretization variable λ_t cannot be the linearization variable, because the multiplier λ_t is non-stationary at the steady state. Conversely, μ_t becomes stationary at the steady state. Thus, we can linearize only the μ-type system (99)-(100). This question no longer matters in the Ramsey model where considering the multiplier λ_t or μ_t is indifferent (indeed, under no discounting, $\lambda_t = \mu_t$).

In the case of the Cass-Koopmans model (see (95)-(96)), the Jacobian matrix (43) of the continuous-time system simplifies:

$$J_0 = \begin{bmatrix} \rho & Ak/\mu \\ B\mu/k & 0 \end{bmatrix} \tag{101}$$

where $A \equiv \varepsilon[\rho/\alpha + \delta(1 - \alpha)/\alpha] > 0$ and $B \equiv (\rho + \delta)(1 - \alpha)/\sigma > 0$. α, σ and ε denote, respectively, the capital share in total income, the elasticity of capital-labor substitution and the elasticity of intertemporal substitution. The trace and determinant become $T_0 = \rho > 0$ and $D_0 = -AB < 0$. In the Ramsey model, $\rho = 0$: so, J_0 simplifies more with $T_0 = 0$. In both the cases, $D_0 < 0$ entails the saddle-path stability property.

In the Cass-Koopmans model, the Jacobian matrix (52) of the discretized system writes

$$J_1 \equiv \begin{bmatrix} 1 + h\rho & hAk/\mu \\ hB\mu/k & 1 + ABh^2/(1 + h\rho) \end{bmatrix} \tag{102}$$

The trace and determinant become $T_1 = 1 + D_1 + h^2 AB/(1 + h\rho)$ and $D_1 = 1 + h\rho$. In the Ramsey model ($\rho = 0$), J_1 also simplifies and we obtain $T_1 = 2 + h^2 AB$ and $D_1 = 1$. In both the cases, we have $1 \leq D < T_1 - 1$ and we recover the saddle-path stability property.

Summing up, saddle-path stability is a robust feature of the Ramsey-Cass-Koopmans framework and holds whatever the discretization step. However, Proposition 7 applies only to the Ramsey case ($\rho = 0$). Indeed, in the Ramsey model the discretization variable and the linearization variables are the same ($\lambda_t = \mu_t$), so, the general expressions (36) and (37) make sense. The Ramsey model corresponds to point (1.2.1) in Proposition 7 with $[(T_0 - 2f_{22})/D_0]^2 + 4/D_0 = -4/AB < 0$.

5.2 Externalities

We have seen that introducing market imperfections in the Solow models makes the discrete-time dynamics richer. There is room for cycles through a flip bifurcation in a Solow model with productive externalities. In the spirit of Proposition (10), reducing the step or increasing the order of discretization restores the monotonic stability property.

Similarly, we can introduce externalities in the Ramsey model to obtain cycles through a Hopf bifurcation. In order to illustrate Proposition (9), we show that reducing the step of discretization also restores the saddle-path stability property.

Externalities can affect either the production or the utility levels of economic agents. The public goods constitute a prominent class of externalities. Zhang (2000) introduces externalities of public spending in the Cass-Koopmans framework. As in Barro (1990), the public good plays the role of positive productive externality. However, Zhang (2000) considers also a public consumption good which enters households' utility functions. In his original model, Cobb-Douglas technology and preferences are considered and time is continuous.

We generalize Zhang in two directions: on the one side, we use more general production and utility functions; on the other side, we provide also the discrete-time version of Zhang and we compare bifurcations in continuous and discrete time. Exemplifying one of the simplest Hopf bifurcations in a Ramsey economy is the main asset of Zhang (2000) and the sense of revisiting his model in our work.

Zhang (2000) introduces two positive externalities in the Cass-Koopmans model:

(1) externalities of public capital (g) in a homogeneous production function as in Barro (1990): $Y \equiv F(K, L, g)$ or, in intensive terms, $y = f(k, g)$, where $y \equiv Y/L$ and $k \equiv K/L$;

(2) externality of public capital in the utility function: $u_t = u(c_t, g_t)$.

These functions satisfy suitable properties.

Assumption 2 *The production function $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ is CRS in (K_t, L_t). The intensive production function $f(k, g)$ is C^2, increasing in k and g and strictly concave in the private capital k ($\partial f/\partial k > 0$ and $\partial^2 f/\partial k^2 < 0$). In addition: $\partial^2 f/(\partial g \partial k) > 0$.*

Assumption 3 *The utility function $u : \mathbb{R}_+^2 \to \mathbb{R}$ is C^2, strictly increasing in c and g ($\partial u/\partial c > 0$, $\partial u/\partial g > 0$) and strictly concave in c ($\partial^2 u/\partial c^2 < 0$).*

According to Assumption 2, the impact of public capital on private production is positive ($\partial f/\partial g > 0$) and positively affects the marginal productivity of private capital ($\partial^2 f/(\partial g \partial k) > 0$).

For simplicity, we assume no population growth and no capital depreciation. The public budget is assumed to be balanced over time and the receipts to come from a homogenous tax on labor and capital earnings: $G_t = \tau Y_t = \tau F(K_t, L_t, g_t)$ (or, in per capita terms, $g_t = \tau y_t = \tau f(k_t, g_t)$). The implicit equation

$$g_t = \tau f(k_t, g_t) \tag{103}$$

locally determines the equilibrium public spending as a function of capital stock: $g_t = g(k_t)$.

Assumption 4

$$dg/dk = (\tau \partial f/\partial k)/(1 - \tau \partial f/\partial g) > 0 \tag{104}$$

In the following, we focus on the competitive dynamics which is different from the planner's solution because of the external effects. The representative household chooses the consumption path and the profile of capital accumulation in order to maximize the utility functional (12) subject to the resource constraint (13) with the following fundamentals:

$$s(k_t, c_t) = (1 - \tau)(r_t k_t + w_t l_t) - c_t \tag{105}$$

$$v(k_t, c_t) = u(c_t, g_t) \tag{106}$$

The initial endowment k_0 is given.

Under Assumption 2, profit maximization gives

$$(r_t, w_t) = (\partial f/\partial k_t, f(k_t, g_t) - k_t \partial f/\partial k_t) \tag{107}$$

while $g_t = g(k_t)$ solves the government budget constraint (103).

For simplicity, labor supply is inelastic: $l_t = 1$.

Under Assumptions 2 and 3 that replace Assumption 1, we can substitute (107) in the dynamic system (17)-(18) to obtain:[25]

$$\begin{aligned} \dot{k}_t &= (1-\tau) f(k_t, g(k_t)) - c(k_t, \mu_t) &(108)\\ \dot{\mu}_t &= \mu_t \left[\rho_t - (1-\tau) \frac{\partial f}{\partial k_t}(k_t, g(k_t))\right] &(109) \end{aligned}$$

In discrete time, the households maximize the utility series $\sum_{t=0}^{\infty} \beta_t u(c_t, g_t)$ subject to the sequence of budget constraints: $k_{t+1} - k_t + c_t \leq (1-\tau)(r_t k_t + w_t l_t)$ with $t = 0, 1 \ldots$

With fundamentals (105) and (106), system (19)-(20) reduces to

$$\begin{aligned} k_{t+1} - k_t &= (1-\tau)(r_t k_t + w_t l_t) - c_t &(110)\\ \frac{\mu_t}{\mu_{t+1}} &= \frac{\beta_{t+1}}{\beta_t}\left[1 + (1-\tau) r_{t+1}\right] &(111) \end{aligned}$$

Substituting $l_t = 1$ and (107) in (110)-(111), one gets

$$\begin{aligned} k_{t+1} - k_t &= (1-\tau) f(k_t, g(k_t)) - c(\mu_t, g_t(k_t)) &(112)\\ \frac{\mu_t}{\mu_{t+1}} &= \frac{\beta_{t+1}}{\beta_t}\left[1 + (1-\tau)\frac{\partial f}{\partial k_{t+1}}(k_{t+1}, g(k_{t+1}))\right] &(113) \end{aligned}$$

where $\mu_t = \partial u / \partial c_t$ is the current-value costate variable of the continuous-time program.

We raise the question whether the discrete-time dynamics can be obtained through an Euler discretization of the continuous-time system. As above, the answer is positive if we choose a hybrid discretization, that is, backward and forward-looking discretizations for the budget constraint and the Euler equation, respectively.

Under (105) and (106), system (50)-(51) simplifies to

$$\begin{aligned} k_{t+h} - k_t &\approx h\left[(1-\tau) f(k_t, g(k_t)) - c(\mu_t, g_t(k_t))\right] &(114)\\ \frac{\mu_t}{\mu_{t+h}} &\approx \frac{\beta_{t+h}}{\beta_t}\left[1 + h(1-\tau)\frac{\partial f}{\partial k_{t+h}}(k_{t+h}, g(k_{t+h}))\right] &(115) \end{aligned}$$

and, setting a unit discretization step ($h = 1$), we recover exactly the discrete-time system (112)-(113).

Under the forward-looking approximation $\beta_{t+h}/\beta_t \approx 1/\left(1 + h\rho_{t+h}\right)$, $\rho_t = \rho$ (that is $\beta_t = \beta_0 e^{-\rho t}$) and $h = 1$, (113) becomes:

$$\frac{\mu_t}{\mu_{t+h}} = \frac{1 + (1-\tau)\frac{\partial f}{\partial k_{t+1}}(k_{t+1}, g(k_{t+1}))}{1+\rho}$$

The existence of a steady state requires $\rho_t = \rho$ constant over time. In this case, equations (41)-(42) become:

$$\begin{aligned} c &= (1-\tau) f(k, g(k)) &(116)\\ \rho &= (1-\tau)\frac{\partial f}{\partial k}(k, g(k)) &(117) \end{aligned}$$

Solving (117) for k and replacing in (116) gives c.

Focus now on the steady state of the discretized time model (114)-(115) or, equivalently, when $h = 1$, of the discrete-time model (112)-(113).

Equation (114) evaluated at the steady state gives (116). In addition, we can replace β_{t+h}/β_t by $1/\left(1 + h\rho_{t+h}\right)$ in (115) to get

$$\frac{\mu_t}{\mu_{t+h}} \approx \frac{1 + h(1-\tau)\frac{\partial f}{\partial k_{t+h}}(k_{t+h}, g(k_{t+h}))}{1 + h\rho_{t+h}}$$

[25] The households maximizes the utility functional taking the externality g as given, and the (Arrow-Mangasarian) second-order conditions reduce to the partial concavity of u ($\partial^2 u/\partial c^2 < 0$) jointly with the partial concavity of s ($\partial^2 f/\partial k^2 < 0$).

Immediately, we obtain that $\mu_t = \mu_{t+h}$ and $\rho_{t+h} = \rho$ imply the steady state (117) of the continuous-time model.

The issues of existence and uniqueness rests on the solution of (116)-(117).

Proposition 13 *Let*

$$\varphi(k) \equiv \frac{\partial f}{\partial k}(k, g(k))$$

Under Assumptions 2, 3, 4 and the boundary conditions

$$\lim_{k\to 0^+} \varphi(k) < \rho/(1-\tau) \text{ and } \lim_{k\to +\infty} \varphi(k) > \rho/(1-\tau) \tag{118}$$

or

$$\lim_{k\to 0^+} \varphi(k) > \rho/(1-\tau) \text{ and } \lim_{k\to +\infty} \varphi(k) < \rho/(1-\tau) \tag{119}$$

a steady state exists.

Moreover, if, at the steady state: (1) $\varphi'(k) < 0$ in case (118), or (2) $\varphi'(k) > 0$ in case (119), then the steady state is unique.

Proof. Focus first on equation (117): $\varphi(k) = \rho/(1-\tau)$. The boundary conditions (118) and (119), jointly with the continuity of φ, are sufficient to ensure the existence of a strictly positive k.

Derivability of φ is entailed by Assumption 2 ($f(k,g)$ is twice continuously differentiable) and Assumption 4 (derivability of g):

$$\varphi'(k) \equiv \frac{\partial^2 f}{\partial k^2} + \frac{\partial^2 f}{\partial g \partial k} g'(k) \tag{120}$$

Derivability of φ implies continuity.

We notice that, under conditions (118) or (119), and continuity, the number of steady states is odd. In addition, given a strictly positive k, equation $g = \tau f(k,g)$ has a non-negative solution $g(k)$ because f is continuous, $f(k,0) \geq 0$ and $\lim_{g\to+\infty} \partial f/\partial g < 1/\tau$ (this inequality is entailed by Assumption 4). Thus $c = (1-\tau) f(k, g(k))$ is non-negative and μ is strictly positive (Assumption 3). If there are n steady states k_i with $k_i < k_{i+1}$ and $i = 1, \ldots, n$, the sign of φ' changes from steady state k_i to steady state k_{i+1}. In order to ensure the uniqueness, a sufficient condition is that, in case (118), always $\varphi'(k) < 0$ at the steady state, or, in case (119), always $\varphi'(k) > 0$ at the steady state. ■

(118) and (119) correspond to the cases of dominant increasing and dominant decreasing returns to scale, respectively. As we will see later (equation (128)), $\varphi'(k) > 0$ is a necessary condition to get a Hopf bifurcation. We conclude that increasing returns promote the uniqueness of the steady state and the occurrence of Hopf bifurcations (limit cycles). Conversely, this explains also why the Ramsey-Cass-Koopmans framework is characterized by saddle-path stability.

Define now the following elasticities to focus on the local dynamics:

$$\begin{aligned}
(\alpha_1, \alpha_2) &\equiv \left(\frac{\partial f}{\partial k}\frac{k}{f}, \frac{\partial f}{\partial g}\frac{g}{f}\right) \\
(\alpha_{11}, \alpha_{12}) &\equiv \left(\frac{\partial^2 f}{\partial k^2}\frac{k}{\partial f/\partial k}, \frac{\partial^2 f}{\partial g \partial k}\frac{g}{\partial f/\partial k}\right) \\
(\eta_{11}, \eta_{12}) &\equiv \left(\frac{\partial^2 u}{\partial c^2}\frac{c}{\partial u/\partial c}, \frac{\partial^2 u}{\partial g \partial c}\frac{g}{\partial u/\partial c}\right)
\end{aligned}$$

Notice that $\alpha_1 \equiv \alpha$ is the capital share in total income, while $\alpha_{11} \equiv \varepsilon_r$ is the elasticity of the interest rate with respect to the capital intensity and $\varepsilon = -1/\eta_{11} > 0$ is the elasticity of intertemporal substitution. Usual assumptions give $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_{11} < 0$, $\alpha_{12} > 0$, $\eta_{11} < 0$, $\eta_{12} \lesseqqgtr 0$.

At the steady state, the discounting is constant over time: $\beta_{t+h}/\beta_t = \beta^h$, where $\beta = e^{-\rho}$. Using $(c,g) = (1-\tau, \tau) f$, $(\partial f/\partial k, \partial u/\partial c) = (\rho/(1-\tau), \mu)$ and $c/k = \rho/\alpha_1$, the Jacobian matrix (43) simplifies:

$$J_0 = \begin{bmatrix} \rho\left[1 + \frac{1}{\alpha_1}\left(\alpha_2 - \frac{\partial c}{\partial g}\frac{g}{c}\right)\frac{kg'(k)}{g}\right] & -\frac{\partial c}{\partial \mu} \\ -\rho\left[\alpha_{11} + \alpha_{12}\frac{kg'(k)}{g}\right]\frac{\mu}{k} & 0 \end{bmatrix} \tag{121}$$

Differentiating (103) gives

$$\frac{kg'(k)}{g} = \frac{\alpha_1}{1-\alpha_2} \tag{122}$$

We observe that Assumption 4 implies $kg'(k)/g > 0$, that is $\alpha_2 < 1$. We get also

$$\left(\frac{dc}{d\mu}\frac{\mu}{c}, \frac{\partial c}{\partial g}\frac{g}{c}\right) = \left(\frac{1}{\eta_{11}}, -\frac{\eta_{12}}{\eta_{11}}\right) \tag{123}$$

Replacing (122) and (123) in (121), we find

$$J_0 = \begin{bmatrix} \rho\frac{1+\eta_{12}/\eta_{11}}{1-\alpha_2} & -\frac{1}{\eta_{11}}\frac{\rho}{\alpha_1}\frac{k}{\mu} \\ -\rho\left(\alpha_{11}+\frac{\alpha_1\alpha_{12}}{1-\alpha_2}\right)\frac{\mu}{k} & 0 \end{bmatrix} = \begin{bmatrix} \rho\frac{1-\varepsilon\eta_{12}}{1-\alpha_2} & \rho\frac{\varepsilon}{\alpha}\frac{k}{\mu} \\ -\rho\left(\varepsilon_r+\frac{\alpha\alpha_{12}}{1-\alpha_2}\right)\frac{\mu}{k} & 0 \end{bmatrix} \tag{124}$$

The trace and the determinant in continuous time (46)-(47) become:

$$\begin{aligned} D_0 &= \varepsilon\rho^2\left(\frac{\varepsilon_r}{\alpha}+\frac{\alpha_{12}}{1-\alpha_2}\right) \\ T_0 &= \rho\frac{1-\varepsilon\eta_{12}}{1-\alpha_2} \end{aligned}$$

but now, in contrast with the Cass-Koopmans framework ($T_0 > 0$ and $D_0 < 0$), saddle-path stability is no longer ensured. Indeed, since $\alpha > 0$, $\varepsilon > 0$, $\varepsilon_r < 0$ and $\alpha_2 \in (0,1)$, we have

$$T_0 \geq 0 \Leftrightarrow \eta_{12} \leq \frac{1}{\varepsilon} \tag{125}$$

$$D_0 > 0 \Leftrightarrow \alpha_{12} > -\frac{1-\alpha_2}{\alpha}\varepsilon_r \tag{126}$$

k_t is a predetermined variable, while μ_t is a jump variable. So, local indeterminacy requires that both the eigenvalues have negative real parts, that is $T_0 < 0$ and $D_0 > 0$, or more explicitly:

$$\eta_{12} > \frac{1}{\varepsilon} \text{ and } \alpha_{12} > -\frac{1-\alpha_2}{\alpha}\varepsilon_r \tag{127}$$

As seen above, increasing returns promotes the occurrence of Hopf bifurcations.

We remark also that, using (120) and computing the elasticity of φ at the steady state, we get

$$\frac{k\varphi'(k)}{\varphi(k)} = \varepsilon_r + \frac{\alpha\alpha_{12}}{1-\alpha_2} \tag{128}$$

Thus, increasing returns ($\varphi'(k) > 0$) require sufficiently large positive externalities ($\alpha_{12} > -(1-\alpha_2)\varepsilon_r/\alpha$) that imply in turn, according to (126), a necessary condition to the occurrence of Hopf bifurcations ($D_0 > 0$).

Focus now on the hybrid discretization (114)-(115). At the steady state, (114) becomes $c = (1-\tau) f$, while, under a forward-looking approximation with a constant ρ ($\beta_{t+h}/\beta_t \approx 1/(1+h\rho)$), (115) gives $\partial f/\partial k = \rho/(1-\tau)$. Moreover, the government budget constraint becomes $g = \tau f$. Finally, $\mu = \partial u/\partial c$. Thus, unsurprisingly, we recover (116)-(117).

Differentiating (114)-(115) around this steady state or, equivalently, applying (52) with (105) and (106), and eventually replacing (122) and (123), gives the system $\left(dk_{t+h}, d\mu_{t+h}\right)^T = J_1 \left(dk_t, d\mu_t\right)^T$, where

$$J_1 = \begin{bmatrix} 1 & 0 \\ \frac{h\rho}{1+h\rho}\left(\varepsilon_r+\frac{\alpha\alpha_{12}}{1-\alpha_2}\right)\frac{\mu}{k} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1+h\rho\frac{1-\varepsilon\eta_{12}}{1-\alpha_2} & h\rho\frac{\varepsilon}{\alpha}\frac{k}{\mu} \\ 0 & 1 \end{bmatrix} \tag{129}$$

and $\alpha \equiv \alpha_1$, $\varepsilon_r \equiv \alpha_{11}$ and $\varepsilon = -1/\eta_{11}$.

The determinant and the trace (equations (53)-(54)) are given by:

$$D_1 = 1 + h\rho\frac{1-\varepsilon\eta_{12}}{1-\alpha_2} \tag{130}$$

$$T_1 = 1 + D_1 - h\rho\frac{\varepsilon}{\alpha}\frac{h\rho}{1+h\rho}\frac{(1-\alpha_2)\varepsilon_r+\alpha\alpha_{12}}{1-\alpha_2} \tag{131}$$

We observe that the Zhang model generalizes the Ramsey-Cass-Koopmans benchmark. Under no externalities in production and utility ($\alpha_2 = \alpha_{12} = \eta_{12} = 0$), we recover exactly the Jacobians of the Cass-Koopmans model (with $\delta = 0$). Indeed, the continuous-time matrix (124) collapses in (101), while the hybrid matrix (129) becomes (102).

Local indeterminacy occurs if the steady state is a sink, that is if $D_1 < 1$, $D_1 > T_1 - 1$ and $D_1 > -T_1 - 1$. Using (130)-(131), $D_1 < 1$, $D_1 > T_1 - 1$ are respectively equivalent to inequalities (127) whatever the discretization step h, while $D_1 > -T_1 - 1$ becomes

$$2 > \frac{h\rho}{1-\alpha_2}\left(\frac{h\rho}{1+h\rho}\frac{\varepsilon}{2\alpha}\left[\left(1-\alpha_2\right)\varepsilon_r + \alpha\alpha_{12}\right] + \varepsilon\eta_{12} - 1\right)$$

which is satisfied for a sufficiently small h. Then, we find that, under a sufficiently small discretization step, multiple equilibria arise in discrete time around a sink if and only if they occur in continuous time according to conditions (127).

One of the assets of the Zhang model (2000) is the occurrence of a Hopf bifurcation, which generically requires $T_0 = 0$ and $D_0 > T_0^2/4 = 0$ (see Section 3.5), that is, according to (125) and (126):

$$\eta_{12} = 1/\varepsilon\,(>0) \tag{132}$$

$$\alpha_{12} > -\frac{1-\alpha_2}{\alpha}\varepsilon_r\,(>0) \tag{133}$$

In other terms, cycles require the synergy of external effects on production ($\alpha_{12} > 0$) and on consumption ($\eta_{12} > 0$). Both the externalities are necessary: for instance, in the Barro model (1990), even if $\alpha_{12} > 0$, saddle-path stability prevails because $\eta_{12} = 0$.

It is known that a Hopf bifurcation generically arises in discrete time if and only if $D_1 = 1$ and $T_1^2 \leq 4$ (see Section 3.5). Replacing (130) in $D_1 = 1$, we get

$$\eta_{12} = 1/\varepsilon \tag{134}$$

as in the continuous-time case (whatever the discretization step), while, replacing (131) in $T_1^2 \leq 4$ with $D = 1$ and using $\eta_{12} = 1/\varepsilon$, we need

$$\alpha_{12} \geq -\frac{1-\alpha_2}{\alpha}\varepsilon_r \tag{135}$$

$$\frac{(h\rho)^2}{1+h\rho} \leq 4\frac{\alpha}{\varepsilon}\frac{1-\alpha_2}{\left(1-\alpha_2\right)\varepsilon_r + \alpha\alpha_{12}} \tag{136}$$

(136) is equivalent to

$$h \leq h_H \equiv \frac{1}{\rho}\left[\omega + \sqrt{\omega^2 + 2\omega}\right]\left(> \frac{\omega}{\rho} > 0\right)$$

where

$$\omega \equiv 2\frac{\alpha}{\varepsilon}\frac{1-\alpha_2}{\left(1-\alpha_2\right)\varepsilon_r + \alpha\alpha_{12}}\,(>0)$$

Conditions (134) and (135) are respectively equivalent to conditions (132) and (133). Since the RHS of (136) is positive under condition (135), inequality (136) is satisfied for $h < h_H$.

We have shown in Section 3.5 that, under a sufficiently small discretization step, a Hopf bifurcation occurs in discrete time if and only if it arises in continuous-time. More precisely, Proposition 9 applies to the Zhang model in the case $\rho = 0$, that is in the Ramsey version of Zhang (2000). Indeed, as seen above, our Proposition 9 holds if the discretization and the linearization variables are the same. In Zhang (as in Cass-Koopmans) the discretization variable is λ_t, while the linearization variable is μ_t. However, when $\rho = 0$, $\lambda_t = \mu_t$ and Proposition 9 works. When $\rho > 0$, a Hopf equivalence still holds for small discretization steps between the continuous-time system and the hybrid λ-type discretization, but the critical condition is different from $h < h_H$.

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