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Stabilizing through Poor Information

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Abstract

This paper studies the effect of asymmetric information on equilibrium stability in a class of linear models where the actual state depends on the forecasts about it. Stability is defined by the so-called eductive criterion which relies on common knowledge of rationality. The main result is that stability obtains when the proportion of uninformed agents is high enough. The expectational behavior of these agents indeed displays more inertia. This behavior, and then the actual outcome, are therefore easier to predict. This result is linked to the issue of informational efficiency. Extensions to cases with higher order uncertainty, additional agents' heterogeneity, and sunspots are also considered.

Keywords: Asymmetric information, eductive stability, rational expectations, stabilization.

JEL classification: C62, D82, D84.

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1 Introduction

The evidence suggests that there is considerable uncertainty regarding economic fundamentals. In addition, it is likely that decision makers have a more precise idea about fundamentals than the private sector. It is not clear whether decision makers whose purpose is to stabilize fluctuations should be transparent, i.e., should reveal their information about fundamentals. In the main strand of the literature, the stabilization purpose has been applied to equilibrium fluctuations, thus a priori assuming rational expectations (see, e.g., Cukierman and Meltzer, 1986). An alternative approach considers that agents are not a priori able to form rational expectations, and that they have to learn first the equilibrium law of motion of the economy. In this approach, the stabilization purpose can be understood as stabilizing the economy in an equilibrium situation (see, e.g., Bullard and Mitra, 2002, or Evans and Honkapohja, 2001, 2008). This paper studies the effect of informational asymmetries on equilibrium stability.

If the rational expectations hypothesis is relaxed, it becomes necessary to specify the forecasting behavior of economic agents. This paper focuses attention on the so-called eductive learning scheme: an equilibrium is stable whenever it is the only outcome surviving to the iterated process of elimination of non-best responses predictions triggered by the two assumptions of common knowledge of individual rationality and common knowledge of the model (Guesnerie, 2002).

In a class of self-referential linear models where the individual decision depends on the expectations of both the fundamentals and the average of others' forecasts, which encompasses Heinemann (2004) and Morris and Shin (2006), stability is shown to relate to (1) the sensitivity of the state of the economic system, interpreted as a price, to the average others' forecasts of this price, and (2) the proportion of agents informed about the true underlying economic fundamentals.

First, in accordance with the literature about learning under symmetric information in macroeconomic models, stability is favored whenever the influence of expectations onto the actual state of the economy is small enough. Indeed, when the actual price is not sensitive to individuals' forecasts, past history reflects fundamentals rather than the noise caused by agents' beliefs. The price then can be used as a valuable guide in adaptive learning processes (Grandmont, 1998). In the case of eductive learning, stability is favored since then it is no longer necessary to know precisely others' beliefs in order to predict the behavior of the economy (Guesnerie, 1992). Stabilizing policies should therefore be shaped so that the influence of forecasts is neutralized.

A specific feature of the paper with respect to the literature on learning in macroeconomic models is the presence of asymmetric information. Our main result shows that stability is negatively affected by the proportion of informed agents, i.e., there is a stabilizing effect of the lack of information. This effect is due to the fact that an agent's decision is more sensitive to his own price forecast when he is informed. This implies that the sensitivity of the actual price increases with the proportion of informed agents. Some inertia argument again prevails: it is easier to predict the forecast behavior of an agent who is not informed since his behavior does not sharply respond to his expectations. This suggests that stabilizing policies should not disclose information to a large proportion of agents.

This last property is reminiscent of the recent literature concerned with the stabilization problem from the equilibrium approach viewpoint (Woodford, 2003; Morris and Shin, 2006; Hellwig, 2008; and Nimark, 2008). This literature actually highlights that informational asymmetries may imply greater persistence of equilibrium fluctuations. In presence of informational asymmetries, an agent is not able to assess exactly how shocks on fundamentals influence others' decisions. Thus, as far as his optimal decision depends on others, his Bayesian Nash equilibrium behavior consists to adjust slowly to his private information. On the contrary, if all agents share the same information about fundamentals, they revise their decisions quickly, and the effects of a shock on the fundamentals is transitory (Woodford, 2003).

To summarize, informational asymmetries imply inertia, not only in the rational expectations equilibrium, but also in the learning process. This suggests that there is a close connection between stabilizing equilibrium fluctuations and stabilizing learning of this equilibrium: an equilibrium which displays greater inertia should be more learnable.

Such a connection is relevant only if fundamentals are not revealed by the observed price in the very short run, however. This raises the question of informational efficiency of the price (see Desgranges and Guesnerie (2000), Desgranges, Geoffard and Guesnerie (2003), and Desgranges and Heinemann (2008), for studying informational efficiency from the eductive viewpoint). With finitely many states of nature, the equilibrium price is fully revealing (Radner, 1979). If learning is taken into account, some efficiency criterion is required. Our criterion hinges on the number of steps of learning needed to assess with certainty that prices are fully revealing. The argument hinges on the fact that every step of learning defines a set of possible prices in each state of nature. When these sets do not intersect, the observation of the price allows agents to infer the true state of fundamentals. Our measure of informational efficiency appeals to the minimal number of steps for these sets to have no intersection: the smaller it is, the greater informational efficiency.

It is shown that informational efficiency improves whenever (1) the speed of convergence of learning is high, and (2) there is a large spread between equilibrium prices. Since the equilibrium belongs to the set of rationalizable solutions, a larger the spread between equilibrium prices makes more likely that the sets of possible prices at any given step of learning do not intersect. It follows that stabilizing equilibrium fluctuations, measured by the spread between equilibrium prices, is detrimental to informational efficiency. Stabilizing equilibrium fluctuations thus makes plausible that informational asymmetries persist over time, which improves stability of learning.

The paper is organized as follows. Section 2 describes the framework and the stability concept in the symmetric information case. Section 3 provides a characterization of stability in presence of asymmetric information. Section 4 examines the issue of informational efficiency. Extensions to higher order uncertainty, individual heterogeneity, and additional sunspot uncertainty are considered in Section 5.

2 Framework

There are Ω states of nature indexed by ω, $\omega = 1, \ldots, \Omega$. State ω occurs with probability $\pi(\omega)$, $0 \leq \pi(\omega) \leq 1$. In state ω, the actual price $p(\omega)$ is determined by

$$p(\omega) = \phi(\omega) \int_0^1 p_i^e di + \eta(\omega). \tag{1}$$

In state ω, the economic fundamentals are summarized by $(\phi(\omega), \eta(\omega))$. The forecast weight $\phi(\omega)$ represents the sensitivity of the economy to agents' forecasts and $\eta(\omega)$ is a scale factor. Eq. (1) can be thought of as a situation in which there is a continuum of infinitesimal agents $i \in [0, 1]$ whose individual forecasts p_i^e about the price matter through the aggregate forecast

$$P^e \equiv \int_0^1 p_i^e di. \tag{2}$$

In the sequel, we assume that the model exhibits strategic complementarity:

Hypothesis 1. For every $\omega = 1, \ldots, \Omega$, $\phi(\omega) > 0$.

Our analysis still applies when $\phi(\omega) < 0$ for every ω (i.e. there are strategic substituability). It does not straightforwardly extend, however, to the case where the signs of forecast weights differ across states of nature, i.e., the model exhibits strategic complementarity in some states and strategic substituability in others. The framework used by Morris and Shin (2006) fits (1), with $\phi(\omega)$ independent of ω.

2.1 The complete information case

One gets preliminary insights into the stability issue by focusing on the simple case where all the agents are perfectly informed of ω when they form their forecasts. A rational expectations equilibrium (REE) is then a vector of prices $(p^*(\omega), \omega = 1, ..., \Omega)$ such that $p^*(\omega) = \phi(\omega) p^*(\omega) + \eta(\omega)$, i.e., $p(\omega) = p^*(\omega)$ if $p_i^e(\omega) = p^*(\omega)$ for any i in (1). This equilibrium is unique if $\phi(\omega) \neq 1$. It can be interpreted as the Nash equilibrium of a strategic 'guessing' game. In this game, the strategies of agent i are the vectors of price forecasts $(p_i^e(1), ..., p_i^e(\Omega))$, and the ex-ante payoff of this agent is the opposite of his forecast error

$$- \sum_{1 \leq \omega \leq \Omega} \pi(\omega)(p(\omega) - p_i^e(\omega))^2,$$

where $p(\omega)$ is determined by (1).

In state ω, agent j's best-response to a profile $(p_i^e(\omega), i \in [0,1])$ of others' forecasts is

$$p_j^e(\omega) = \phi(\omega) P^e(\omega) + \eta(\omega), \tag{3}$$

where $P^e(\omega)$ is defined by (2), with $p_i^e = p_i^e(\omega)$. It is clear that the REE is the only Nash equilibrium of this game. In this equilibrium, each agent expects $p^*(\omega)$ because of the belief that all the others expect $p^*(\omega)$. Taking into account beliefs of higher order further implies that each agent believes that all the others believe that all the others expect $p^*(\omega)$. This process can be iterated ad infinitum, i.e., $p^*(\omega)$ is the only price in state ω to be consistent with common knowledge (CK) of every agent expecting it.

This observation suggests a stability criterion for the REE. To define it, we do not assume that the price forecast of every agent is CK. We make instead the weaker assumption of CK that $p_i^e(\omega) \in P^0(\omega)$ for all i and all ω, where $P^0(\omega) = [P^0_{\inf}(\omega), P^0_{\sup}(\omega)]$ and $p^*(\omega) \in P^0(\omega)$. This assumption triggers an iterative process.

At the first step of this process, it implies $P^e(\omega) \in P^0(\omega)$. Then, by (3), $p_j^e(\omega) \in P^1(\omega) = \left[P_{\inf}^1(\omega), P_{\sup}^1(\omega)\right]$, where

$$P_{\inf}^1(\omega) = \phi(\omega)P_{\inf}^0(\omega) + \eta(\omega),$$
$$P_{\sup}^1(\omega) = \phi(\omega)P_{\sup}^0(\omega) + \eta(\omega),$$

since $\phi(\omega) > 0$. Therefore, at the outcome of this first step, it is CK that every price forecast $p_i^e(\omega)$ of every agent i is in $P^1(\omega)$. More generally, at step $\tau \geq 1$, if $p_i^e(\omega) \in P^\tau(\omega)$ for every i, then $p_i^e(\omega) \in P^{\tau+1}(\omega)$ for every i, with $P^{\tau+1}(\omega) = \phi(\omega)P^\tau(\omega) + \eta(\omega)$.

The REE is stable when the sequence of intervals $(P^\tau(\omega), \tau \geq 0)$ converges to $\{p^*(\omega)\}$ for every ω. It is then the only rationalizable outcome in the game, once the strategy set of every agent has been initially restricted to $(P^0(1), ..., P^0(\Omega))$. Of course, the REE is stable if and only if $|\phi(\omega)| < 1$ for every ω. As advocated by Guesnerie (1992), stability thus obtains when the economic system is not too sensitive to forecasts in (1), or equivalently, agents' forecasts are not too sensitive to others' forecasts in (3).

2.2 The symmetric incomplete information case

Suppose now that uncertainty about fundamentals is no longer resolved when agents form their forecasts. The price forecast p_i^e can no longer depend on ω. Agents still know, however, that the actual price depends on ω. If agent i believes that the price in state ω is $p_i^e(\omega)$, his price forecast writes

$$p_i^e = \sum_{w=1}^{\Omega} \pi(w) p_i^e(w).$$

In this setting, a REE is a vector $(p^*(1), \ldots, p^*(\Omega))$ such that, for any ω,

$$p^*(\omega) = \phi(\omega) \sum_{w=1}^{\Omega} \pi(w) p^*(w) + \eta(\omega). \tag{4}$$

It coincides with the Nash equilibrium of the guessing game where all agents are uninformed of ω. In this game, the best-response of j to a profile $(p_i^e, i \in [0,1])$ of others' forecasts is

$$p_j^e = \sum_{w=1}^{\Omega} \pi(w) \left[\phi(w) P^e + \eta(w)\right] \equiv \bar{\phi} P^e + \bar{\eta},$$

where $\bar{\phi}$ and $\bar{\eta}$ represent the average forecast weight and scale factor, and the aggregate forecast is

$$P^e \equiv \sum_{w=1}^{\Omega} \pi(w) \int_0^1 p_i^e(w)\, di.$$

In the REE, therefore, it is CK that all the agents expect

$$p^* = \sum_{w=1}^{\Omega} \pi(w) p^*(w). \tag{5}$$

The stability criterion introduced above can be applied. If, at step τ, it is CK that $p_i^e \in P^\tau$ for every i (where P^τ is an interval including p^*), then it is CK that $p_i^e \in P^{\tau+1} = \bar{\phi} P^\tau + \bar{\eta}$ for every i. The sequence of intervals $(P^\tau, \tau \geq 0)$ converges to $\{p^*\}$ if and only if $\left|\bar{\phi}\right| < 1$, i.e., the average forecast weight is low enough.

3 Stability under Asymmetric Information

We now assume that there are α $(0 < \alpha < 1)$ informed agents who observe ω before they form their forecasts, and the $(1-\alpha)$ remaining agents have no information about ω at that time. The previous section suggests that stability should depend on the Ω values of $\phi(\omega)$ (because of the informed) and the value of $\bar{\phi}$ (because of the uninformed). The new issue is that the stability properties of prices $p^*(\omega)$ are now interdependent. Namely, in a given state ω, uninformed agents figure out what informed agents expect in every state. In order to form a correct forecast, informed agents have to guess the behavior of uninformed agents, and so they must take into account what uninformed believe about what they expect themselves in every state. Stability of a REE price in a given state therefore depends on the fundamentals in all the possible states.

The REE is a vector $(p^*(1), \ldots, p^*(\Omega))$ such that, for any ω,

$$p^*(\omega) = \phi(\omega) P^*(\omega) + \eta(\omega), \tag{6}$$

where

$$P^*(\omega) = \alpha p^*(\omega) + (1-\alpha) \sum_{w=1}^{\Omega} \pi(w) p^*(w). \tag{7}$$

It can be again interpreted as the Nash equilibrium of a guessing game. The timing of the game is:

1. The informed agents $i \in [0, \alpha]$ observe ω.

2. All the agents form their forecasts. A strategy of a player i consists of a price forecast, conditional to his information. It follows that, if i is informed, his strategy is a vector of price forecasts $(p_i^e(1), ..., p_i^e(\Omega))$, where $p_i^e(\omega)$ is the price expected by i to arise in state ω. If i is uninformed, then his strategy consists of a unique price forecast p_i^e.

3. In every state ω, the actual price $p(\omega)$ is determined by (1), $p(\omega) = \phi(\omega) P^e(\omega) + \eta(\omega)$, with the aggregate forecast

$$P^e(\omega) = \int_0^\alpha p_i^e(\omega)\, di + \int_\alpha^1 p_i^e di.$$

In this game, each agent chooses a forecast by minimizing his own squared forecast error. Hence, in state ω, when an informed agent i expects the aggregate forecast to be $P^e(\omega)$, his best-response is

$$p_i^e(\omega) = \phi(\omega) P^e(\omega) + \eta(\omega) \equiv R_\omega(P^e(\omega)). \tag{8}$$

If i is uninformed and expects the aggregate forecast to be $P^e(w)$ in every state w, then

$$p_i^e = \sum_{w=1}^{\Omega} \pi(w) R_w(P^e(w)). \tag{9}$$

Equations (8) and (9) allow us to define the stability criterion. Assume initially that the aggregate price forecast $P^e(\omega)$ in state ω belongs to some interval $P^0(\omega) = [P^0_{\text{inf}}(\omega), P^0_{\text{sup}}(\omega)]$, with $P^*(\omega) \in P^0(\omega)$. At the first step of the learning process, the best-response $p_i^e(\omega)$ of an informed agent i to an aggregate forecast $P^e(\omega)$ in $P^0(\omega)$ satisfies

$$p_i^e(\omega) \in \left[R_\omega(P^0_{\text{inf}}(\omega)), R_\omega(P^0_{\text{sup}}(\omega))\right]. \tag{10}$$

Similarly, for a uninformed agent,

$$p_i^e \in \left[\sum_{w=1}^{\Omega} \pi(w) R_w(P^0_{\text{inf}}(w)), \sum_{w=1}^{\Omega} \pi(w) R_w(P^0_{\text{sup}}(w))\right]. \tag{11}$$

As a consequence, $P^e(\omega) \in P^1(\omega) = \left[P^1_{\text{inf}}(\omega), P^1_{\text{sup}}(\omega)\right]$, with

$$P^1_{\text{inf}}(\omega) = \alpha R_\omega(P^0_{\text{inf}}(\omega)) + (1-\alpha)\sum_{w=1}^{\Omega}\pi(w)R_w(P^0_{\text{inf}}(w)),$$

$$P^1_{\text{sup}}(\omega) = \alpha R_\omega(P^0_{\text{sup}}(\omega)) + (1-\alpha)\sum_{w=1}^{\Omega}\pi(w)R_w(P^0_{\text{sup}}(w)).$$

More generally, if it is CK at step τ that $P^e(\omega) \in P^\tau(\omega)$ in state ω, then it is CK at step $(\tau+1)$ that $P^e(\omega) \in P^{\tau+1}(\omega)$ in state ω, with

$$P^{\tau+1}(\omega) = \alpha R_\omega(P^\tau(\omega)) + (1-\alpha)\sum_{w=1}^{\Omega}\pi(w)R_w(P^\tau(w)). \tag{12}$$

This relation defines a sequence of intervals $(P^\tau(\omega), \tau \geq 0)$. The REE is stable whenever this sequence converges toward $\{P^*(\omega)\}$, whatever ω is. The next result gives a necessary and sufficient condition for stability.

Proposition 1. *Assume that $\phi(\omega) > 0$ for any $\omega = 1, \ldots, \Omega$. Let $0 \leq \alpha \leq 1$. If $\alpha\phi(\omega) > 1$ for some ω, then the REE is unstable. If $\alpha\phi(\omega) < 1$ for every ω, then it is stable if and only if*

$$\sum_{w=1}^{\Omega}\pi(w)\frac{(1-\alpha)\phi(w)}{1-\alpha\phi(w)} < 1. \tag{13}$$

Proof. Consider, e.g., the Ω equations (12) corresponding to the lowest bounds $P^\tau_{\text{inf}}(\omega)$ of $P^\tau(\omega)$. Given (8), they can be rewritten in matrix form $\mathbf{p}^{\tau+1}_{\text{inf}} = \mathbf{M}\mathbf{p}^\tau_{\text{inf}} + \boldsymbol{\eta}$, where $\mathbf{p}^\tau_{\text{inf}}$ is the $\Omega \times 1$ vector $(P^\tau_{\text{inf}}(1), \ldots, P^\tau_{\text{inf}}(\Omega))$, $\boldsymbol{\eta}$ is the $\Omega \times 1$ vector $(\eta(1), ..., \eta(\Omega))$, and $\mathbf{M}$ is the $\Omega \times \Omega$ matrix $\alpha\mathbf{\Phi} + (1-\alpha)\mathbf{\Phi}\mathbf{\Pi}$ (with $\mathbf{\Phi}$ the diagonal $\Omega \times \Omega$ matrix whose $\omega\omega$th entry is $\phi(\omega)$, and $\mathbf{\Pi}$ the $\Omega \times \Omega$ stochastic matrix whose $\omega\omega'$th entry is $\pi(\omega')$). The REE is stable if and only if the spectral radius $\rho(\mathbf{M})$ of $\mathbf{M}$ is less than 1. The proof now hinges on the fact that for any $\Omega \times \Omega$ positive matrix $\mathbf{M}$, and any $\Omega \times 1$ vector $\mathbf{x} = (x_\omega)$ with every $x_\omega > 0$, we have

$$\min_\omega \frac{(\mathbf{M}\mathbf{x})_\omega}{x_\omega} \leq \rho(\mathbf{M}) \leq \max_\omega \frac{(\mathbf{M}\mathbf{x})_\omega}{x_\omega},$$

where $(\mathbf{Mx})_\omega$ stands for the ωth component of the $\Omega \times 1$ vector $\mathbf{Mx}$ (see Lemma 3.1.2. in Bapat and Raghavan (1997)). Let

$$Q(\mathbf{x}, \omega) = \frac{(\mathbf{Mx})_\omega}{x_\omega} = \phi(\omega) \left[\alpha + (1-\alpha) \frac{1}{x_\omega} \sum_{w=1}^{\Omega} \pi(w) x_w \right],$$

for any ω. Assume first that $\alpha\phi(\omega) > 1$ for some ω, e.g. $\omega = \Omega$. Then, consider the vector $\mathbf{x} = (\varepsilon, \ldots, \varepsilon, 1)'$ where $\varepsilon > 0$. When ε tends toward 0, $Q(\mathbf{x}, \omega)$ tends to $(+\infty)$ for every $\omega < \Omega$, and $Q(\mathbf{x}, \Omega) \geq \alpha\phi(\Omega) > 1$. Hence, $\min_\omega Q(\mathbf{x}, \omega) > 1$ for ε small enough, and so $\rho(\mathbf{M}) > 1$: the REE is unstable if $\alpha\phi(\omega) > 1$ for some ω. If, on the contrary, $\alpha\phi(\omega) < 1$ for any ω, then define

$$E = \sum_{w=1}^{\Omega} \pi(w) \frac{(1-\alpha)\phi(w)}{1-\alpha\phi(w)}.$$

Consider the $\Omega \times 1$ positive vector $\mathbf{x}$ whose ωth component is

$$x_\omega = \frac{1}{E} \frac{(1-\alpha)\phi(\omega)}{1-\alpha\phi(\omega)}.$$

If $E \geq 1$, then $Q(\mathbf{x}, \omega) > 1$ for any ω, so that $\min_\omega Q(\mathbf{x}, \omega) \geq 1$, and the REE is unstable. If, on the contrary, $E < 1$, then $Q(\mathbf{x}, \omega) < 1$ for any ω, so that $\max_\omega Q(\mathbf{x}, \omega) < 1$, and the REE is stable. □

When the REE is not stable, no price can be predicted. To see this, consider a state ω where $\phi(\omega) > 1$, which always exists when the REE is not stable. Uninformed agents cannot predict the price in such a state, and so they cannot select a unique price forecast in this state. Since their behavior does not depend on the actual state, this means that they cannot pick out a unique price forecast in any other state. As a result, expectations coordination is impossible in any state.

From Proposition 1, a small sensitivity of actual prices to forecasts favors stability, since the derivative of the LHS of (13) with respect to $\phi(\omega)$ is positive. However, the interaction between forecast weights and the information structure (summarized by α) is not clear. The next corollary focuses on this issue.

Corollary 2. *Assume that $\phi(\omega) > 0$ for any $\omega = 1, \ldots, \Omega$.*

If $\phi(\omega) < 1$ for any $\omega = 1, \ldots, \Omega$, then the REE is stable.

If $\inf_\omega \phi(\omega) < 1 < \sup_\omega \phi(\omega)$*, then provided that* $\bar{\phi} < 1$*, there exists a threshold proportion* α^**,* $0 < \alpha^* < 1$*, of informed agents such that stability of the REE obtains if and only if* $\alpha < \alpha^*$*; the threshold* α^* *is a decreasing function of each* $\phi(\omega)$*. If* $\bar{\phi} \geq 1$*, then stability never obtains.*

If $\phi(\omega) > 1$ *for any* $\omega = 1, \ldots, \Omega$*, then the REE is unstable.*

Proof. Assume first that $\phi(\omega) < 1$ for any $\omega = 1, \ldots, \Omega$. Then, $\alpha\phi(\omega) < 1$ and $(1-\alpha)\phi(\omega)/(1-\alpha\phi(\omega)) < 1$ for any ω. By Proposition 1, the REE is stable.

Let now $\inf_\omega \phi(\omega) < 1 < \sup_\omega \phi(\omega)$. If $\alpha > 1/\sup_\omega \phi(\omega)$, the REE is unstable, by Proposition 1. If $\alpha \leq 1/\sup_\omega \phi(\omega)$, then $\alpha\phi(\omega) < 1$ for every ω, and the REE is stable if and only if (13) is met. Let

$$F(\alpha) = \sum_{w=1}^{\Omega} \pi(w) \frac{\phi(w)}{1-\alpha\phi(w)} - \frac{1}{(1-\alpha)} \tag{14}$$

Since $F(\cdot)$ is a continuous and increasing function of α on the interval $[0, 1/\sup_\omega \phi(\omega)]$, with $F'(\alpha) > 0$ whatever α is, there is at most one value α such that $F(\alpha) = 0$ on this interval. Observe now that $F(0) = \bar{\phi} - 1$, and $F(\alpha)$ tends to $+\infty$ when α tends to $1/\sup_\omega \phi(\omega)$ from below. If, on the one hand, $\bar{\phi} \geq 1$, then $F(\alpha) \geq F(0) > 0$ for any $\alpha \in [0, 1/\sup_\omega \phi(\omega)]$, and the stability condition (13) is never satisfied. If, on the other hand, $\bar{\phi} < 1$, then there exists a unique solution α^* ($\alpha^* > 0$) to $F(\alpha) = 0$ in $[0, 1/\sup_\omega \phi(\omega)]$. The condition $F(\alpha) < 0$, i.e. the stability condition (13), is equivalent to $\alpha < \alpha^*$. Since $F(\alpha^*) = 0$ implicitly defines α^* as a function $(\phi(1), \ldots, \phi(\Omega))$, and since $F(\cdot)$ increases in every $\phi(\omega)$, α^* decreases in every $\phi(\omega)$.

Assume finally that $\phi(\omega) > 1$ for any ω. Then, $\bar{\phi} > 1$, and we have already seen that $F(\alpha) > 0$ for any $\alpha \in [0, 1/\sup_\omega \phi(\omega)]$. As a result, the stability condition (13) is never satisfied. □

If the REE is unstable, then an increase in the proportion of informed agents is never stabilizing. Instead, a decrease in the proportion of informed agents sometimes makes the equilibrium stable. In this sense, the presence of uninformed agents is stabilizing. The intuition for this result is simple. It stems from the sensitivity of individual forecasts to other's behavior. Indeed, when an informed agent i expects the aggregate forecast to undergo a change $dP^e(\omega) > 0$ in state ω, he will adjust his forecast for an amount $dp_i^e(\omega) = \phi(\omega)\, dP^e(\omega)$. In the same configuration, an uninformed agent will revise his forecast by $\pi(\omega)\phi(\omega)\, dP^e(\omega) \leq \phi(\omega)\, dP^e(\omega)$. The forecasting behavior of an uninformed agent is thus less sensitive to others' forecasts than the one of an informed agent. It is consequently easier to predict, which favors stability.

4 Informational Efficiency

If the actual price, once made public, reveals the underlying fundamentals, informational asymmetries do not persist over time. That is: the price is informationally efficient. When the REE is stable, informational efficiency obtains provided that $p^*(\omega) \neq p^*(\omega')$. On the other hand, when the REE is unstable, any price may arise in any state and the price is not revealing. These two statements suppose, however, an infinite number of steps of learning, while informational efficiency can obtain after a finite number of steps only.

After a finite number of steps τ, a price reveal ω if it belongs to $P^\tau(\omega) - P^\tau(\omega')$ for any $\omega' \neq \omega$. For any price in $P^\tau(\omega)$ to reveal ω at step τ, it is needed that $P^\tau(\omega)$ has no intersection with any other interval $P^\tau(\omega')$. In case of stability of the REE, there exists a threshold step $\underline{\tau}$ such that any price at step τ is revealing if and only if $\tau \geq \underline{\tau}$. The lower this threshold is, the more efficient the learning process. Hence, this threshold provides us a criterion for informational efficiency of the price.

Assume for convenience $\phi(\omega) = \phi$ and $P^0(\omega) = P^0$ for every ω (agents have no prior information about the state). By Proposition 1, the REE is stable if and only if $\phi < 1$.

Proposition 3. *Assume that the REE is stable ($\phi < 1$). Let $\underline{\tau}$ be the smallest integer τ satisfying*

$$p^0_{\sup} - p^0_{\inf} < \frac{\alpha}{\phi^\tau} \frac{1-(\alpha\phi)^\tau}{1-\alpha\phi} \inf_{\omega,\omega'} \left|\eta(\omega') - \eta(\omega)\right|. \tag{15}$$

Then, any price reveals the state ω at step τ of learning, i.e. the Ω sets $P^\tau(\omega)$ do not intersect, if and only if $\tau \geq \underline{\tau}$.

The threshold $\underline{\tau}$ increases with the proportion of uninformed agents $(1-\alpha)$, and with the forecast weight ϕ.

Proof. It follows from (12), with $\phi(\omega) = \phi$, that

$$\bar{p}^{\tau+1}_{\inf} \equiv \sum_{w=1}^{\Omega} \pi(w) p^\tau_{\inf}(w) = \phi^{\tau+1} p^0_{\inf} + \frac{1-\phi^{\tau+1}}{1-\phi}\bar{\eta},$$

$$\bar{p}_{\sup}^{\tau+1} \equiv \sum_{w=1}^{\Omega} \pi(w) p_{\sup}^{\tau}(w) = \phi^{\tau+1} p_{\sup}^{0} + \frac{1-\phi^{\tau+1}}{1-\phi}\bar{\eta}.$$

Let $dp^{\tau+1}(\omega',\omega) = p_{\sup}^{\tau+1}(\omega') - p_{\inf}^{\tau+1}(\omega)$, $d\eta(\omega',\omega) = \eta(\omega') - \eta(\omega)$, and $dp^0 = p_{\sup}^0 - p_{\inf}^0$. Then, from (12) and both previous equations, one gets

$$dp^{\tau+1}(\omega',\omega) = \alpha\phi dp^{\tau}(\omega',\omega) + \alpha d\eta(\omega',\omega) + (1-\alpha)\phi^{\tau+1} dp^0.$$

Hence, $dp^{\tau+1}(\omega',\omega)$ equals

$$(\alpha\phi)^{\tau+1} dp^0 + \alpha \frac{1-(\alpha\phi)^{\tau+1}}{1-\alpha\phi} d\eta(\omega',\omega) + (1-\alpha)\phi^{\tau+1} \frac{1-\alpha^{\tau+1}}{1-\alpha} dp^0.$$

The Ω sets $P^{\tau}(\omega)$ do not intersect if and only if $dp^{\tau}(\omega,\omega') < 0$ whenever $\eta(\omega') < \eta(\omega)$, which is equivalent to:

$$dp^0 < \frac{\alpha}{\phi^{\tau}} \frac{1-(\alpha\phi)^{\tau}}{1-\alpha\phi} \inf_{\omega,\omega'} |d\eta(\omega',\omega)|.$$

Since the LHS does not depend on τ and the RHS is increasing in τ, the threshold value $\underline{\tau}$ stated in the proposition is the smallest integer satisfying the above inequality. This shows the first part of the proposition. Observe now that

$$\frac{\alpha}{\phi^{\tau}} \frac{1-(\alpha\phi)^{\tau}}{1-\alpha\phi}$$

increases with α and τ (since $\phi < 1$). It decreases with ϕ, since

$$\frac{d}{d\phi}\left(\frac{\alpha}{\phi^{\tau}} \frac{1-(\alpha\phi)^{\tau}}{1-\alpha\phi}\right) = \frac{\alpha}{(1-\alpha\phi)\phi^{\tau+1}}\left(\alpha\phi\frac{1-\alpha^{\tau}\phi^{\tau}}{1-\alpha\phi} - \tau\right),$$

and

$$\alpha\phi\frac{1-\alpha^{\tau}\phi^{\tau}}{1-\alpha\phi} = \alpha\phi + (\alpha\phi)^2 + ... + (\alpha\phi)^{\tau} < \tau.$$

This concludes the proof. □

Informational efficiency improves, i.e., $\underline{\tau}$ is lower, when (1) the anchorage assumption is informative, i.e. $p_{\sup}^0 - p_{\inf}^0$ is small, and (2) the spread between equilibrium prices

$$|p^*(\omega') - p^*(\omega)| = \frac{|\eta(\omega') - \eta(\omega)|}{1-\alpha\phi}, \tag{16}$$

is large, i.e., the proportion of informed agents is high and $|\eta(\omega') - \eta(\omega)|$ is important. The influence of the forecast weight onto informational efficiency is a priori ambiguous, however. On the one hand, a higher ϕ increases the spread between equilibrium prices, which increases $\underline{\tau}$. On the other hand, it increases the speed of convergence to the REE, which lowers $\underline{\tau}$. The above proposition gives the net effect: a higher ϕ deteriorates informational efficiency.

5 Extensions

5.1 Higher Order Uncertainty

So far uninformed agents have used a common prior distribution of states, and this fact was CK. Our analysis actually holds if the probability $\pi_i(\omega)$ assigned by some uninformed agent i to state ω is private information, but

$$\pi(\omega) \equiv \frac{1}{(1-\alpha)} \int_\alpha^1 \pi_i(\omega) di$$

is CK. It may appear difficult to justify such an assumption in a framework which otherwise stipulates a high level of ignorance. We now consider the case of higher order uncertainty, where every agent is uncertain about others' beliefs over the different states of nature.

The aggregate forecast is

$$P^e(\omega) = \int_0^\alpha p_i^e(\omega) + \int_\alpha^1 \sum_{w=1}^{\Omega} \pi_i(w) p_i^e(w) di.$$

At step τ, the price forecasts $p_i^e(\omega)$ (for any i and any ω) belong to some interval $P^\tau(\omega) = \left[P_{\inf}^\tau(\omega), P_{\sup}^\tau(\omega)\right]$. We define higher order uncertainty as follows: every agent only knows that the aggregate price forecast

$$P^e(\omega) \in \left[\alpha P_{\inf}^\tau(\omega) + (1-\alpha) \inf_w P_{\inf}^\tau(w), \alpha P_{\sup}^\tau(\omega) + (1-\alpha) \sup_w P_{\sup}^\tau(w)\right]$$

for any ω.

When $\phi(\omega) = \phi$, the iterative learning process writes:

$$P_{\inf}^{\tau+1}(\omega) = \phi \left[\alpha P_{\inf}^\tau(\omega) + (1-\alpha) \inf_w P_{\inf}^\tau(w)\right] + \eta(\omega), \quad (17)$$

$$P_{\sup}^{\tau+1}(\omega) = \phi\left[\alpha P_{\sup}^{\tau}(\omega) + (1-\alpha)\sup_{w} P_{\sup}^{\tau}(w)\right] + \eta(\omega). \tag{18}$$

This shows that higher order uncertainty prevents agents to discover the REE. Indeed, $P_{\inf}^{\tau}(\omega) = P_{\sup}^{\tau}(\omega) = P_{\inf}^{\tau+1}(\omega) = P_{\sup}^{\tau+1}(\omega) = p^*(\omega)$ is not a solution of the system (17) and (18). Let $P_{\inf}(\omega)$ and $P_{\sup}(\omega)$ be the fixed points of this system ($P_{\inf}^{\tau}(\omega) = P_{\inf}^{\tau+1}(\omega) = P_{\inf}(\omega)$ for every ω in (17) and $P_{\sup}(\omega)$ is defined analogously from (18)). Stability corresponds to convergence of the sequence $\left[P_{\inf}^{\tau}(\omega), P_{\sup}^{\tau}(\omega)\right]$ toward $[P_{\inf}(\omega), P_{\sup}(\omega)]$ for every ω.

The next result shows that stability is not affected by higher order uncertainty, at least when $\phi(\omega) = \phi$.

Proposition 4. *The dynamics (17) and (18) is stable if and only if* $\phi < 1$.

Proof. The dynamics of lowest bounds $P_{\inf}^{\tau}(\omega)$ rewrites

$$\mathbf{p}_{\inf}^{\tau+1} = \phi\left[\alpha \mathbf{I}_{\Omega} + (1-\alpha)\,\mathbf{1}_{\Omega}\right]\mathbf{p}_{\inf}^{\tau},$$

where $\mathbf{1}_{\Omega}$ stands for the $\Omega \times \Omega$ stochastic matrix whose each entry in the $\underline{\omega}$th column is 1, where $\underline{\omega} = \arg\inf_w P_{\inf}(w)$, and any remaining entry is 0. The Ω eigenvalues of the matrix $\phi\left[\alpha \mathbf{I}_{\Omega} + (1-\alpha)\,\mathbf{1}_{\Omega}\right]$ are ϕ, $\alpha\phi$, ..., $\alpha\phi$. The same analysis applies to $P_{\sup}^{\tau}(\omega)$. □

Unlike the case examined in Section 4, stability does not necessarily imply efficiency of prices. The next result characterizes informational efficiency in presence of higher order uncertainty.

Proposition 5. *Let* $\phi < 1$. *Then, the price eventually reveals the state, i.e., no two intervals* $[P_{\inf}(\omega), P_{\sup}(\omega)]$ *and* $[P_{\inf}(\omega'), P_{\sup}(\omega')]$ *intersect, if and only if*

$$\alpha > 1 - \frac{\inf_{\omega \neq \omega'} |\eta(\omega) - \eta(\omega')|}{\sup_{\omega \neq \omega'} |\eta(\omega) - \eta(\omega')|}\left(\frac{1}{\phi} - 1\right). \tag{19}$$

Proof. Let $\phi < 1$. Let also $\eta(1) < \cdots < \eta(\Omega)$. Then, $\inf_w P_{\inf}(w) = P_{\inf}(1) = \eta(1)/(1-\phi)$,

$$P_{\inf}(\omega) = \frac{(1-\alpha)\,\phi}{1-\phi}\frac{\eta(1)}{1-\alpha\phi} + \frac{\eta(\omega)}{1-\alpha\phi} \text{ for } \omega > 1,$$

$$P_{\sup}(\omega) = \frac{(1-\alpha)\,\phi}{1-\phi}\frac{\eta(\Omega)}{1-\alpha\phi} + \frac{\eta(\omega)}{1-\alpha\phi} \text{ for } \omega < \Omega,$$

and $\sup_w P_{\sup}(w) = P_{\sup}(\Omega) = \eta(\Omega)/(1-\phi)$. Since $P_{\inf}(\omega) < P_{\inf}(\omega')$ and $P_{\sup}(\omega) < P_{\sup}(\omega')$ for $\omega < \omega'$, no two sets $P(\omega)$ and $P(\omega')$ intersect if and only if $P_{\sup}(\omega) - P_{\inf}(\omega') < 0$ whenever $\omega < \omega'$, or equivalently

$$\frac{\phi\,(1-\alpha)}{1-\phi}\frac{\eta(\Omega)-\eta(1)}{1-\alpha\phi} < \frac{\eta(\omega')-\eta(\omega)}{1-\alpha\phi}$$

for every $\omega < \omega'$. This inequality rewrites

$$\frac{\phi\,(1-\alpha)}{1-\phi} < \frac{\inf_{\omega,\omega'}|\eta(\omega)-\eta(\omega')|}{\eta(\Omega)-\eta(1)},$$

which leads to the result. □

The conditions for informational efficiency have the same flavor as in the absence of higher order uncertainty. Namely, informational efficiency is favored by a large proportion of informed agents α and a small forecast weight ϕ.

From Condition (19), informational efficiency becomes more likely when the term

$$\frac{\inf\limits_{\omega\neq\omega'}|\eta(\omega)-\eta(\omega')|}{\sup\limits_{\omega\neq\omega'}|\eta(\omega)-\eta(\omega')|}$$

is maximum. This happens when the spread between two successive $\eta(\omega)$ is constant. By (16), this corresponds to a situation where no two equilibrium prices are too close. The possibility of an equilibrium price that strongly differs from the others deteriorates informational efficiency.

5.2 Individual Heterogeneity

In the case of homogeneous agents, the influence of individual forecasts on the actual price is the same for every agent. This section analyzes stability in presence of some agents' heterogeneity. Let $\phi_I(\omega)$ and $\phi_U(\omega)$ be the forecast weights in state ω, respectively for an informed agent and an uninformed one. Hypothesis 1 becomes: $\phi_I(\omega) > 0$ and $\phi_U(\omega) > 0$ for every ω.

The actual price is then given by

$$p(\omega) = \phi_I(\omega)\int_0^{\alpha} p_i^e(\omega)\,di + \phi_U(\omega)\int_{\alpha}^{1}\sum_{w=1}^{\Omega}\pi(w)p_i^e(w)di + \eta(\omega)\,, \tag{20}$$

which replaces (1). A REE of (20) is a vector $(p^*(1), ..., p^*(\Omega))$ such that $p(\omega) = p_i^e(\omega) = p^*(\omega)$ in (20) for any ω and any i.

The learning process is defined as previously. An initial price restriction $P^0(\omega) = \left[P_{\text{inf}}^0(\omega), P_{\text{sup}}^0(\omega)\right]$ in state ω is postulated. At step τ, if it is CK that $p_i^e(\omega) \in P^\tau(\omega) = \left[P_{\text{inf}}^\tau(\omega), P_{\text{sup}}^\tau(\omega)\right]$, then it is CK that the actual price $p(\omega)$ belongs to $P^{\tau+1}(\omega) = \left[P_{\text{inf}}^{\tau+1}(\omega), P_{\text{sup}}^{\tau+1}(\omega)\right]$, where $P_{\text{inf}}^{\tau+1}(\omega)$ equals

$$\phi_I(\omega)\,\alpha P_{\text{inf}}^\tau(\omega) + \phi_U(\omega)(1-\alpha)\sum_{w=1}^{\Omega}\pi(w)P_{\text{inf}}^\tau(w) + \eta(\omega), \tag{21}$$

and $P_{\text{sup}}^{\tau+1}(\omega)$ equals

$$\phi_I(\omega)\,\alpha P_{\text{sup}}^\tau(\omega) + \phi_U(\omega)(1-\alpha)\sum_{w=1}^{\Omega}\pi(w)P_{\text{sup}}^\tau(w) + \eta(\omega). \tag{22}$$

It follows that it is also CK that $p_i^e(\omega) \in P^{\tau+1}(\omega)$ in state ω. The REE is the only limit point of this iterative process, i.e., $P_{\text{inf}}^\tau(\omega)$ and $P_{\text{sup}}^\tau(\omega)$ converges to $p^*(\omega)$ whenever these two sequences converge. The next result characterizes stability.

Proposition 6. *Let $\phi_I(\omega) > 0$ and $\phi_U(\omega) > 0$ for any $\omega = 1, \ldots, \Omega$. If $\alpha\phi_I(\omega) > 1$ for some ω, then the REE is unstable. If $\alpha\phi_I(\omega) < 1$ for every ω, then the REE is stable if and only if*

$$\sum_{w=1}^{\Omega}\pi(w)\frac{(1-\alpha)\phi_U(w)}{1-\alpha\phi_I(w)} < 1. \tag{23}$$

Proof. The REE is stable if and only if the spectral radius $\rho(\mathbf{M})$ of the $\Omega\times\Omega$ matrix $\mathbf{M} = \alpha\mathbf{\Phi}^I + (1-\alpha)\mathbf{\Phi}^U\mathbf{\Pi}$ is less than 1, where $\mathbf{\Phi}^I$ and $\mathbf{\Phi}^U$ are two $\Omega\times\Omega$ diagonal matrices whose $\omega\omega$th entry is $\phi^I(\omega)$ and $\phi^U(\omega)$, respectively. Let

$$Q(\mathbf{x}, \omega) = \alpha\phi_I(\omega) + (1-\alpha)\phi_U(\omega)\frac{1}{x_\omega}\sum_{w=1}^{\Omega}\pi(w)x_w.$$

As in Proposition 1, the first part of Proposition 6 follows by appealing to the $\Omega\times 1$ vector $\mathbf{x} = (\varepsilon, ..., \varepsilon, 1)$, where $\varepsilon > 0$ is small enough. The last part follows by appealing to the $\Omega\times 1$ vector $\mathbf{x}$ whose ωth component x_ω is

$$\frac{1}{E}\frac{(1-\alpha)\phi_U(\omega)}{1-\alpha\phi_I(\omega)},$$

with

$$E = \sum_{w=1}^{\Omega} \pi(w) \frac{(1-\alpha)\phi_U(w)}{1-\alpha\phi_I(w)}. \qquad \square$$

The map (20) shows that a change dp^e of every price forecast $p_i^e(\omega)$ implies a change $dp(\omega) = ((1-\alpha)\phi_U(\omega) + \alpha\phi_I(\omega))\, dp^e$. A sufficient condition for stability is that every $(1-\alpha)\phi_U(\omega) + \alpha\phi_I(\omega)$ is less than 1. In this case, $|dp(\omega)| < |dp^e|$, i.e., the actual price is not too sensitive to expectations. More generally, we have:

Corollary 7. *Assume that $\phi_I(\omega) > 0$ and $\phi_U(\omega) > 0$ for any $\omega = 1, \ldots, \Omega$.*

1. *If $\phi_I(\omega) < 1$ for any state ω, then there exists $\alpha^* < 1$ such that stability obtains if and only if $\alpha > \alpha^*$. Furthermore, $\alpha^* > 0$ if and only if $\bar{\phi}_U > 1$.*
2. *If, on the contrary, $\phi_I(\omega) > 1$ for any state ω, then there exists $\alpha^* < 1$ such that stability obtains if and only if $\alpha < \alpha^*$. Furthermore, $\alpha^* > 0$ if and only if $\bar{\phi}_U < 1$.*

Proof. If $\alpha \sup_\omega \phi_I(\omega) \geq 1$, the REE is unstable. If $\alpha \sup_\omega \phi_I(\omega) < 1$, the REE is stable if and only if (23) is met, i.e.

$$F(\alpha) \equiv \sum_{w=1}^{\Omega} \pi(w) \frac{(1-\alpha)\phi_U(w)}{1-\alpha\phi_I(w)} < 1.$$

The function $F(\cdot)$ is continuous in α and $F(0) = \bar{\phi}_U$. It is straightforward that, if $\phi_I(\omega) < 1$ for any ω, then $F'(\alpha) < 0$. It follows that $F(\alpha) < 1$ if and only if $\alpha > \alpha^*$. Since $F(1) = 0$, $\alpha^* < 1$. Lastly, $\bar{\phi}_U > 1$ if and only if $\alpha^* > 0$. If, on the contrary, $\phi_I(\omega) > 1$ for any ω, then $F'(\alpha) > 0$. It follows that $F(\alpha) < 1$ if and only if $\alpha < \alpha^*$. Since $F(1/\sup_\omega \phi_I(\omega)) = +\infty$, $\alpha^* < 1$. Again, $\bar{\phi}_U < 1$ if and only if $\alpha^* > 0$. $\square$

These properties are quite intuitive. Stability obtains when there are many informed agents if every $\phi_I(\omega)$ is less than 1. If, on the other hand, every $\phi_I(\omega)$ is greater than 1, stability obtains when there are many uninformed agents, provided that the actual price is not too sensitive to their forecasts ($\bar{\phi}_U < 1$). The intermediate situation where some $\phi_I(\omega)$ are less than 1, and others are greater than 1, is more intricate. It is studied Appendix 1. Stability is shown to be again favored by small forecast weights, and a low proportion of informed agents.

5.3 Sunspots

Consider a stochastic sunspot variable that can take Σ values ($S = 1, \ldots, \Sigma$) not correlated with fundamentals. Assume that its actual value is not known when agents form their forecasts. Namely, every agent i observes a private signal $s_i = 1, \ldots, \Sigma$ imperfectly correlated with S. Conditionally to S, private signals are independently and identically distributed across agents, and the probability $\Pr(s_i \mid S)$ that i observes s_i in sunspot event S is independent of i. Thus, in sunspot event S, there are $\Pr(s \mid S)$ agents who observe signal s ($s = 1, \ldots, \Sigma$).

Suppose that all the agents expect the price $p^e(\omega, S)$ to arise if the state of fundamentals is ω and the sunspot is S. In state (ω, S), there are $\alpha \Pr(s \mid S)$ informed agents whose price forecast is

$$\sum_{S'=1}^{\Sigma} \Pr(S' \mid s) p^e(\omega, S')$$

for any s. There are also $(1-\alpha)\Pr(s \mid S)$ uninformed agents who expect

$$\sum_{S'=1}^{\Sigma} \Pr(S' \mid s) \sum_{w=1}^{\Omega} \pi(w)\, p^e(w, S').$$

Let

$$\mu(S'|S) = \sum_{s=1}^{\Sigma} \Pr(s \mid S) \Pr(S' \mid s)$$

be the average probability (across agents) of sunspot S' if the actual sunspot is S. The aggregate price forecast $P^e(\omega, S)$ expresses as

$$\sum_{S'=1}^{\Sigma} \mu(S'|S) \left[\alpha p^e(\omega, S') + (1-\alpha) \sum_{w=1}^{\Omega} \pi(w)\, p^e(w, S') \right], \tag{24}$$

and the actual price $p(\omega, S)$, determined by (1) in state (ω, S), is such that

$$p(\omega, S) = \phi(\omega)\, P^e(\omega, S) + \eta(\omega). \tag{25}$$

A REE is a vector of $\Omega\Sigma$ prices $(p^*(1,1), \ldots, p^*(\Omega, \Sigma))$ such that $p^e(\omega, S) = p(\omega, S) = p^*(\omega, S)$ for every (ω, S) in (24) and (25). When $p^*(\omega, S)$ is independent of S, the REE is said to be 'fundamental'; otherwise, the REE is a sunspot REE.

The following result gives the conditions for existence of sunspot REE.

Proposition 8. *There exist sunspot REE if and only if the fundamental REE is unstable.*

Proof. Let us rewrite conditions (25) in matrix form. To this aim, let $\mathbf{p}(S)$ be the $\Omega \times 1$ vector whose ωth component is $p(\omega, S)$, and $\mathbf{p}$ be the $\Omega\Sigma \times 1$ vector $(\mathbf{p}(1), \ldots, \mathbf{p}(\Sigma))$. Let $\mathbf{S}$ be the $\Sigma \times \Sigma$ stochastic matrix whose SS'th entry is $\mu(S', S)$. Then, with $\mathbf{M}$ defined in Proposition 1, a REE is a vector $\mathbf{p}$ such that

$$\mathbf{p} = (\mathbf{M} \otimes \mathbf{S})\,\mathbf{p} + \mathbf{1}_\Sigma \otimes \boldsymbol{\eta}, \tag{26}$$

where the symbol $\otimes$ stands for the Kronecker product. Let $e(S)$ be the Sth eigenvalue of $\mathbf{S}$, with $e(S) \in [-1, 1]$ since $\mathbf{S}$ is a stochastic matrix. Let $\mu(\omega)$ be the ωth eigenvalue of $\mathbf{M}$. Then, the $\Omega\Sigma$ eigenvalues of $\mathbf{M} \otimes \mathbf{S}$ are $e(S)\mu(\omega)$ for any pair (ω, S). If $\rho(\mathbf{M}) < 1$, then all the eigenvalues of $\mathbf{M} \otimes \mathbf{S}$ have moduli less than 1, and so $\mathbf{M} \otimes \mathbf{S} - \mathbf{I}_{2\Omega}$ is invertible and there is a unique REE. If $\rho(\mathbf{M}) \geq 1$, there exist stochastic matrices such that $e(S) = 1/\rho(\mathbf{M})$ for some S. In this case, the matrix $\mathbf{M} \otimes \mathbf{S}$ has an eigenvalue equal to 1, and there are infinitely many $\mathbf{p}$ solution to (26), i.e. infinitely many sunspot REE and the fundamental REE. $\square$

It is natural to wonder whether a sunspot REE can be stable. In presence of the sunspot, the iterative process is modified as follows. An initial price restriction $P^0(\omega, S) = \left[P^0_{\inf}(\omega, S), P^0_{\sup}(\omega, S)\right]$ in state (ω, S) is assumed. At step τ, if it is CK that $p^e_i(\omega, S) \in P^\tau(\omega, S) = \left[P^\tau_{\inf}(\omega, S), P^\tau_{\sup}(\omega, S)\right]$, then it is CK that $p(\omega, S) \in P^{\tau+1}(\omega, S) = \left[P^{\tau+1}_{\inf}(\omega, S), P^{\tau+1}_{\sup}(\omega, S)\right]$, where $P^{\tau+1}_{\inf}(\omega, S)$ and $P^{\tau+1}_{\sup}(\omega, S)$ are respectively equal to

$$\phi(\omega) \sum_{S'=1}^{\Sigma} \mu(S', S) \left[\alpha P^\tau_{\inf}(\omega, S') + (1-\alpha) \sum_{w=1}^{\Omega} \pi(w)\, P^\tau_{\inf}(w, S')\right] + \eta(\omega)$$

and

$$\phi(\omega) \sum_{S'=1}^{\Sigma} \mu(S', S) \left[\alpha P^\tau_{\sup}(\omega, S') + (1-\alpha) \sum_{w=1}^{\Omega} \pi(w)\, P^\tau_{\sup}(w, S')\right] + \eta(\omega).$$

It follows that it is also CK that $p^e_i(\omega, S) \in P^{\tau+1}(\omega, S)$ in state (ω, S). The REE prices are the only limit points of this iterative process. The REE is stable when the two sequences $P^\tau_{\inf}(\omega, S)$ and $P^\tau_{\sup}(\omega, S)$ converge. A Corollary of Proposition 8 is

Corollary 9. *No sunspot REE is stable.*

Proof. The dynamics of the two sequences $P^{\tau}_{\inf}(\omega, S)$ and $P^{\tau}_{\sup}(\omega, S)$ is governed by the $\Omega\Sigma \times \Omega\Sigma$ matrix $\mathbf{M} \otimes \mathbf{S}$. Since the spectral radius of $\mathbf{M} \otimes \mathbf{S}$ is $\rho(\mathbf{M})$, a sunspot REE is stable if and only if $\rho(\mathbf{M}) < 1$. But then there is no sunspot REE (by Proposition 8). □

This result relies on the linear framework. In a nonlinear framework, locally stable sunspot REE may exist. When one of these REE exhibits revealing prices, local instability of the fundamental REE no longer prevents efficiency of the price. Extraneous uncertainty then ensures price efficiency.

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Appendix 1

We have:

Corollary 10. *Let* $\inf_\omega \phi_I(\omega) < 1 < \sup_\omega \phi_I(\omega)$.

1. *If* $E\phi_U < E\phi_I\phi_U$,[1] *then there exists* $\alpha^* < 1/\sup_\omega \phi_I(\omega) < 1$ *such that stability obtains if and only if* $\alpha < \alpha^*$. $\alpha^* > 0$ *if and only if* $E\phi_U < 1$.

2. *If* $E\phi_U \geq E\phi_I\phi_U$ *and* $E\phi_U \leq 1$, *then there exists* α^*, $0 < \alpha^* < 1/\sup_\omega \phi_I(\omega) < 1$, *such that stability obtains if and only if* $\alpha < \alpha^*$.

3. *If* $E\phi_U \geq E\phi_I\phi_U$ *and* $E\phi_U > 1$, *then,* (i) *either the equilibrium is unstable for every* α, (ii) *or there are two values* α_- *and* α_+ *with* $0 < \alpha_- < \alpha_+ < 1/\sup_\omega \phi_I(\omega) < 1$ *such that the equilibrium is stable if and only if* $\alpha \in [\alpha_-, \alpha_+]$. *Precisely, consider a vector* $(\phi_I(1), ..., \phi_I(\Omega))$. *In the space* $\mathbb{R}_+^\Omega$ *of the vectors* $(\phi_U(1), ..., \phi_U(\Omega))$, *there is a neighborhood of the hyperplane* $E\phi_U = 1$ *such that case* (i) *(resp.* (ii)*) obtains when* $(\phi_U(1), ..., \phi_U(\Omega))$ *is outside (resp. inside) this neighborhood. In particular, case* (i) *obtains when* $E\phi_U > \sup_\omega \phi_I(\omega)/(\sup_\omega \phi_I(\omega) - 1)$.

Proof. We write $F'(\alpha) = Q_+ - Q_-$ where[2]

$$\begin{aligned} Q_+ &= \sum_{\omega/\phi_I>1} \pi\phi_U \frac{\phi_I - 1}{(1-\alpha\phi_I)^2} \geq 0, \\ Q_- &= -\sum_{\omega/\phi_I<1} \pi\phi_U \frac{\phi_I - 1}{(1-\alpha\phi_I)^2} \geq 0. \end{aligned}$$

Q_+ and Q_- are both continuous, increasing and convex.

In the case $E\phi_U < E\phi_U\phi_I$, (that is $F'(0) > 0$) given that $(1-\alpha x)^{-2}$ is increasing in x for every given α, we have:

$$\begin{aligned} Q_+ &\geq \sum_{s/\phi_I>1} \pi\phi_U \frac{\phi_I - 1}{(1-\alpha)^2}, \\ Q_- &\leq -\sum_{s/\phi_I<1} \pi\phi_U \frac{\phi_I - 1}{(1-\alpha)^2}. \end{aligned}$$

[1] $E\phi_I\phi_U \overset{def}{=} \sum_{w=1}^{\Omega} \pi(w)\,\phi_I(w)\,\phi_U(w)$.

[2] We drop the index ω for simplicity.

It follows that $F'(\alpha) \geq F'(0)/(1-\alpha)^2 > 0$, F is increasing and $F\left(\frac{1}{\max \phi_I}\right) = +\infty$ so that stability obtains iff α is below a certain threshold α^*. Given that $F(0) = E\phi_U$, $\alpha^* > 0$ iff $E\phi_U < 1$. This proves the first point in the corollary.

In the case $E\phi_U > E\phi_U\phi_I$, (that is $F'(0) \leq 0$), at a point where $Q_+ = Q_-$, we have that

$$\begin{aligned}
\frac{dQ_+}{d\alpha} &\geq 2\sum_{s/\phi_I>1} \pi\phi_U \frac{1}{(1-\alpha)} \frac{\phi_I - 1}{(1-\alpha\phi_I)^2} = \frac{2Q_+}{(1-\alpha)}, \\
\frac{dQ_-}{d\alpha} &\leq -2\sum_{s/\phi_I<1} \pi\phi_U \frac{1}{(1-\alpha)} \frac{\phi_I - 1}{(1-\alpha\phi_I)^2} = \frac{2Q_-}{(1-\alpha)},
\end{aligned}$$

so that

$$\frac{dQ_-}{d\alpha} \leq \frac{2Q_-}{(1-\alpha)} = \frac{2Q_+}{(1-\alpha)} \leq \frac{dQ_+}{d\alpha},$$

i.e. Q_+ crosses Q_- from below at any intersection point. It follows that there is at most one intersection point. Notice now that

$$Q_+(0) < Q_-(0) \text{ and } Q_-\left(\frac{1}{\sup \phi_I}\right) < Q_+\left(\frac{1}{\sup \phi_I}\right) = +\infty,$$

implying that there is exactly one intersection point (denoted $\alpha_{\min} > 0$) between Q_+ and Q_-. It follows that $F(\alpha)$ is decreasing iff $\alpha \leq \alpha_{\min}$ and $F(\alpha)$ reaches a minimum at $\alpha_{\min}$. As a result, we have that, in the case $F(\alpha_{\min}) < 1$, there exists α_- and α_+ such that stability obtains iff $\alpha \in [\alpha_-, \alpha_+]$, while in the case $F(\alpha_{\min}) > 1$, stability never obtains.

To prove the second point in the corollary, notice that $\alpha_- = 0$ iff $E\phi_U < 1$. To prove the third point, notice first that, for α in $[0, 1/\sup \phi_I]$

$$\left(1 - \frac{1}{\sup \phi_I}\right) E\phi_U < (1-\alpha) E\phi_U < F(\alpha).$$

Fix a vector $(\phi_I(1), ..., \phi_I(\Omega))$. Consider a given vector $\boldsymbol{\phi}_U^1 = (\phi_U^1(1), ..., \phi_U^1(\Omega))$ such that $E\phi_U^1 = 1$, $E\phi_U^1 \geq E\phi_U^1\phi_I$. Define $\boldsymbol{\phi}_U = \lambda\boldsymbol{\phi}_U^1$ with $\lambda \geq 1$, and denote $F_\lambda = \lambda F_1$. The value $\alpha_{\min}$ such that $F'_\lambda(\alpha_{\min}) = 0$ does not depend on λ. $F_1(\alpha) < 1$ in a non empty interval. As $F_\lambda(\alpha)$ increases in λ and stability writes $F_\lambda(\alpha) < 1$, there is a value $\lambda_{\max}(\boldsymbol{\phi}_U^1)$ such that $F_\lambda(\alpha) < 1$ for some α iff $\lambda < \lambda_{\max}(\boldsymbol{\phi}_U^1)$. Consider now the set $I = \left\{\boldsymbol{\phi}_U / E\phi_U < \lambda_{\max}\left(\frac{1}{E\phi_U}\boldsymbol{\phi}_U\right)\right\}$. This a neighborhood of the hyperplane $E\phi_U = 1$ satisfying the third point. □