ON NECESSARY CONDITIONS FOR EFFICIENCY IN DIRECTIONALLY DIFFERENTIABLE OPTIMIZATION PROBLEMS

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Abstract. This paper deals with multiobjective programming problems with inequality, equality and set constraints involving Dini or Hadamard differentiable functions. A theorem of the alternative of Tucker type is established, and from which Kuhn-Tucker necessary conditions for local Pareto minima with positive Lagrange multipliers associated with all the components of objective functions are derived.

1. Introduction

The key to identifying optimal solutions of constrained nonlinear optimization problems is the Lagrange multiplier conditions. One of the main approaches to establishing such multiplier condition for inequality constrained problems is based on the dual solvability characterizations of systems involving inequalities. Farkas initially established such a dual

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[0]To appear in *Nonlinear Functional Analysis and Applications.*

[0]2000 Mathematics Subject Classification: 90C46, 90C29.

[0]Keywords: Theorem of the alternative, Kuhn-Tucker necessary conditions, directionally differentiable functions.

[0]This research was partially supported by the Natural Science Council of Vietnam. M.H Nguyen acknowledges the financial support from the project *"ANR Jeunes Chercheuses et Jeunes Chercheurs"*, n^0 ANR-CEDEPTE-05-JCJC-0134-01.

characterization for nonlinear programming problems. This dual characterization is popularly known as Farkas's lemma which can also expressed as a so-called alternative theorem. Alternative theorems have played a crucial role in establishing necessary optimality conditions. Many works generalized classical theorems of the alternative such as Farkas' Theorem, Tucker's Theorem, Motzkin's Theorem and applied them to derive Fritz John and Kuhn-Tucker necessary conditions for optimality (see, e.g., [1], [2],[4]-[7],[9]-[14], and references therein). The classical Tucker theorem of the alternative and its generalizations play an important role in establishing Kuhn-Tucker necessary conditions for efficiency with Lagrange multipliers associated with all the components of objective functions to be positive. This attracts attention of mathematicians, since if a Lagrange multiplier corresponding to some component of the objective is equal to zero, then that component has no role in the considering necessary conditions.

Maeda [13] studies Fréchet differentiable multiobjective optimization problems with only inequality constraints and gives a Kuhn-Tucker necessary conditions for a Pareto minimum with positive Lagrange multipliers corresponding to all the components of the objective under a constraint qualification of Guignard type. Giorgi et al [4] study several constraint qualifications which generalize the constraint qualification introduced by Maeda [13] and the classical ones, and derive Kuhn-Tucker necessary conditions basing on establishing an alternative theorem for a system comprising sublinear inequalities and linear equalities. Ishizuka [6] gives an alternative theorem for a system containing only inequalities described by sup-functions, and derive Kuhn-Tucker necessary conditions for properly efficient solutions of multiobjective programs with inequality type constraints. Note that the aforementioned works are considered in finite dimensions. Recently, Luu and Nguyen[12] have developed Kuhn-Tucker necessary conditions for efficiency of Gâteaux differentiable multiobjective optimization problems in normed spaces involving inequality, equality and set constraints with Lagrange multipliers of the objective are all positive by proving a theorem of the alternative to a system comprising inequality, equality and an inclusion in normed spaces.

Motivated by the works mentioned above, this paper deals with the generalization of classical Tucker's theorem of the alternative to a system comprising inequalities described by sup-functions and an inclusion together with establishing Kunh-Tucker necessary conditions for efficiency with positive Lagrange multipliers associated with all the components of objective functions of directionally differentiable multiobjective optimization problems involving inequality, equality and set constraints in finite dimensions.

The paper is organized as follows. After Introduction and some preliminaries, Section 3 is devoted to present a theorem of the alternative to a system comprising inequalities described by sup-functions and an inclusion along with its consequences. From these results, section 4 gives Kuhn-Tucker necessary conditions for efficiency with positive Lagrange multipliers corresponding to all the components of the objective of the considering problem.

2. Preliminaries

Let f, g and h be mappings from $\mathbb{R}^n$ into $\mathbb{R}^p$, $\mathbb{R}^q$ and $\mathbb{R}^r$, respectively, and C be a nonempty subset of $\mathbb{R}^n$. Assume that f, g, h can be expressed as follows: $f = (f_1, \ldots, f_p)$, $g = (g_1, \ldots, g_q)$, $h = (h_1, \ldots, h_r)$, where $f_k, g_j, h_\ell : \mathbb{R}^n \to \mathbb{R}$ with $k \in I = \{1, \ldots, p\}; j \in J = \{1, \ldots, q\}; \ell \in L = \{1, \ldots r\}$.

We consider the following multiobjective programming problem (VP):

$$\begin{array}{ll} \min & f(x) \\ \text{s.t} & g_j(x) \leq 0, \quad j \in J; \\ & h_l(x) = 0, \quad l \in L; \\ & x \in C. \end{array}$$

Denote by M the feasible set of (VP)

$$M = \Big\{ x \in C : g_j(x) \leqslant 0, h_\ell(x) = 0, j \in J; \ell \in L \Big\}.$$

Recall that a point $\overline{x} \in M$ is said to be a local Pareto minimum of (VP) if there exists a number $\delta > 0$ such that

$$F \cap M \cap B(\overline{x}; \delta) = \emptyset,$$

where

$$F = \{x \in \mathbb{R}^n : f(x) \leq f(\overline{x}), f(x) \neq f(\overline{x})\},$$

and $B(\overline{x}; \delta)$ denotes the open ball of radius δ around $\overline{x}$.

Definition 2.1. *a)The tangent cone (or contingent cone) to C at $\overline{x} \in C$ is the following set:*

$$T(C; \overline{x}) = \Big\{ v \in \mathbb{R}^n : \exists\, v_n \to v, \exists\, t_n \downarrow 0^+ \text{ such that } \overline{x} + t_n v_n \in C, \forall\, n \Big\}.$$

b) The cone of sequential linear directions (or sequential radial cone) to C at $\overline{x} \in C$ is the following set:

$$Z(C;\overline{x}) = \big\{v \in \mathbb{R}^n : \exists\, t_n \downarrow 0^+ \text{ such that } \overline{x} + t_n v \in C, \forall\, n\big\}.$$

Note that both these cones are nonempty, $T(C;\overline{x})$ is closed and $Z(C;\overline{x}) \subset T(C;\overline{x})$.

Let K be a cone in $\mathbb{R}^n$. The polar cone to K is

$$K^* = \{\xi \in \mathbb{R}^n | < \xi, v > \leq 0 \ \forall v \in K\}.$$

If K is a subspace, then K^* is the orthogonal subspace $K^\perp$ to K. In case $K = T(C;\overline{x}), K^*$ is the normal cone $N(C;\overline{x})$ to C at $\overline{x}$.

Definition 2.2. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ and $\overline{x} \in \mathbb{R}^n$.*

a) The lower Dini derivative of f at $\overline{x}$ in a direction $v \in \mathbb{R}^n$ is

$$\underline{D}f(\overline{x};v) = \liminf_{t\downarrow 0^+} \frac{f(\overline{x}+tv) - f(\overline{x})}{t}\,;$$

b) The lower Hadamard derivative of f at $\overline{x}$ in the direction v is

$$\underline{d}f(\overline{x};v) = \liminf_{t\downarrow 0^+, u\to v} \frac{f(\overline{x}+tu) - f(\overline{x})}{t}\,.$$

Replacing "liminf" by "limsup" in a) or b), we get the upper Dini derivative $\overline{D}f(\overline{x};v)$ and the upper Hadamard derivative $\overline{d}f(\overline{x};v)$,respectively, of f at $\overline{x}$ in the direction v. In case $\underline{D}f(\overline{x};v) = \overline{D}f(\overline{x};v)$ (resp. $\underline{d}f(\overline{x};v) = \overline{d}f(\overline{x};v)$), we shall denote their common value by $Df(\overline{x};v)$(resp. $df(\overline{x};v)$), which is called the Dini derivative or directional derivative (resp. Hadamard derivative) of f at $\overline{x}$ in the direction v. The function f is said to be Dini differentiable or directionally differentiable (resp. Hadamard differentiable) at $\overline{x}$ if its Dini derivative(resp. Hadamard derivative) exists in all directions. Note that if $df(\overline{x};v)$ exists, then also $Df(\overline{x};v)$ exists and they are equal. In case f is Fréchet differentiable at $\overline{x}$ with Fréchet derivative $\nabla f(\overline{x})$, then

$$Df(\overline{x};v) = df(\overline{x};v) =< \nabla f(\overline{x}), v > .$$

Definition 2.3. *The Dini subdifferentiable of a Dini differentiable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ at $\overline{x}$ is*

$$\partial_D f(\overline{x}) = \{\xi \in \mathbb{R}^n | < \xi, v > \leq Df(\overline{x};v) \ \forall v \in \mathbb{R}^n\}.$$

In case the function $Df(\overline{x};.)$ is convex, there exists the subdifferentiable $\partial Df(\overline{x};.)(0)$ of this function at $v=0$ in Convex Analysis sense, and

$$\partial_D f(\overline{x}) = \partial Df(\overline{x};.)(0).$$

This set is nonempty, convex, compact , and

$$Df(\overline{x};v) = \max_{\xi\in\partial_D f(\overline{x})} \langle \xi, v\rangle.$$

Note that in case $Df(\overline{x};.)$ is convex, f was called quasidifferentiable at $\overline{x}$ by Pschenichnyi[16].

3. Theorem of the alternative

To derive necessary conditions for efficiency with all positive Lagrange multipliers of all the components of the objective, we establish the following theorem of the alternative.

Theorem 3.1. *Let $A_1, ..., A_p,, B_1, ..., B_q$ and C be nonempty subset of $\mathbb{R}^n$. Assume that*

a) K is an arbitrary nonempty closed convex subcone of $T(C;\overline{x})$ with vertex at the origin;

b) For each $i\in I$, the set

$$\cup\{cl\ co(A_i + \sum_{k=1,k\neq i}^{p} \alpha_k A_k + \sum_{j=1}^{q} \beta_j B_j) : \alpha_k \geq 0, k\neq i, \beta_j \geq 0\} + K^*$$

is closed in $\mathbb{R}^n$ where cl and co denote the closure and the convex hull, respectively.

Then the two following statements are equivalent:

i) For each $i\in\{1,\dots,p\}$, the system

$$\sup_{a_k\in A_k} \langle a_k, v\rangle \leqslant 0, \quad k\in I; k\neq i, \tag{3.1}$$

$$\sup_{a_i\in A_i} \langle a_i, v\rangle < 0, \tag{3.2}$$

$$\sup_{b_j\in B_j} \langle b_j, v\rangle \leqslant 0,\ j\in J, \tag{3.3}$$

$$v\in K \tag{3.4}$$

has no solution $v\in\mathbb{R}^n$.

ii) There exist $\overline{\lambda}_k > 0$ $(k \in I)$, $\overline{\mu}_j \geqslant 0$ $(j \in J)$ *such that*

$$0 \in cl\ co(\sum_{k=1}^{p} \overline{\lambda}_k A_k + \sum_{j=1}^{q} \overline{\mu}_j B_j) + K^*. \tag{3.5}$$

Proof. (i) $\Rightarrow$ (ii): Suppose that (i) holds which means that the system (3.1)-(3.4) has no solution. For each $i = 1, ..., p,$ we set

$$D_i = \cup\{\text{cl co}(A_i + \sum_{k=1,k\neq i}^{p} \alpha_k A_k + \sum_{j=1}^{q} \beta_j B_j) : \alpha_k \geq 0, k \neq i, \beta_j \geq 0\} + K^*.$$

Then by assumption, D_i is closed. Moreover,

$$D_i = \cup\{\text{cl}(\text{co}A_i + \sum_{k=1,k\neq i}^{p} \alpha_k \text{co}A_k + \sum_{j=1}^{q} \beta_j \text{co}B_j) : \alpha_k \geq 0, k \neq i, \beta_j \geq 0\} + K^*.$$

It is easy to check that the set

$$\cup\{cl(\text{co}A_i + \sum_{k=1,k\neq i}^{p} \alpha_k \text{co}A_k + \sum_{j=1}^{q} \beta_j \text{co}B_j) : \alpha_k \geq 0, k \neq i, \beta_j \geq 0\}$$

is convex,and hence, D_i is convex. We now show that $0 \in D_i (i = 1, ..., p)$.

If this were not so, there would exist $i_0 \in I$ such that $0 \notin D_{i_0}$. Applying strong separation theorem for a closed convex set and a point outside that set (see, e.g.,[15,Corollary 11.4.2]) yields the existence of a vector $\overline{v} \in \mathbb{R}^n \setminus \{0\}$ and a number $\alpha_0 \in \mathbb{R}$ such that

$$\langle \xi, \overline{v} \rangle < \alpha_0 < 0 (\forall \xi \in D_{i_0}),$$

which implies that

$$\langle a_{i_0}, \overline{v} \rangle + \sum_{k=1,k\neq i_0}^{p} \alpha_k \langle a_k, \overline{v} \rangle + \sum_{j=1}^{q} \beta_j \langle b_j, \overline{v} \rangle + \langle \zeta, \overline{v} \rangle < \alpha_0 < 0 \tag{3.6}$$

for all $a_k \in A_k, a_{i_0} \in A_{i_0}, b_j \in B_j, \zeta \in K^*, \alpha_k \geq 0, k \in I, k \neq i_0, \beta_j \geq 0, j \in J$.

Taking $\alpha_k = 0\ \forall\ k \in I, k \neq i_0, \beta_j = 0,\ j \in J, \zeta = 0$, we obtain that

$$\sup_{a_{i_0} \in A_{i_0}} \langle a_{i_0}, \overline{v} \rangle < 0. \tag{3.7}$$

We can show that

$$\sup_{b_j \in B_j} \langle b_j, \overline{v} \rangle \leq 0\ (\forall j \in J). \tag{3.8}$$

Assume the contrary, that there exists $j_0 \in J$ such that

$$\sup_{b_{j_0} \in B_{j_0}} \langle b_{j_0}, \overline{v} \rangle > 0,$$

then, by letting β_{j_0} be large enough, we arrive at a contradiction with (3.6). Similarly, we also get that

$$\sup_{a_k \in A_k} \langle a_k, \overline{v} \rangle \leq 0 \ \forall k \neq i_0. \tag{3.9}$$

Let us show that $\overline{v} \in K$. If this were false, there would exist $\zeta_0 \in K^*$ such that $\langle \zeta_0, \overline{v} \rangle > 0$. For $\alpha_k = 0 \ \forall \ k \neq i_0, \beta_j = 0, j \in J$, $\lambda \zeta_0 \in K^*$, for λ sufficiently large, we also get a contradiction with (3.6). Hence,

$$\langle \zeta, \overline{v} \rangle \leq 0, \forall \zeta \in K^*$$

which leads to the following

$$\overline{v} \in K^{**} = K. \tag{3.10}$$

It follows readily from (3.7)-(3.10) that the system (3.1)-(3.4) has a solution $\overline{v}$: a contradiction. Therefore, for each $i \in I, 0 \in D_i$. So there exists numbers $\overline{\alpha}_k^i \geq 0, \overline{\beta}_j^i \geq 0$ with $\overline{\alpha}_i^i = 1$ such that, for $i = 1, \ldots, p$,

$$0 \in \mathrm{cl}\ (\sum_{k=1}^{p} \overline{\alpha}_k^i \mathrm{co} A_k + \sum_{j=1}^{q} \overline{\beta}_j^i \mathrm{co} B_j) + K^*$$

as co$A$+co$B$ =co$(A + B)$.

Summing up these inclusions, we obtain

$$\begin{aligned} 0 \ &\in \ \sum_{i=1}^{p} \mathrm{cl}\ (\sum_{k=1}^{p} \overline{\alpha}_k^i \mathrm{co}\ A_k + \sum_{j=1}^{q} \overline{\beta}_j^i \mathrm{co}\ B_j) + K^* \\ &\subset \ \mathrm{cl} \sum_{i=1}^{p} (\sum_{k=1}^{p} \overline{\alpha}_k^i \mathrm{co} A_k + \sum_{j=1}^{q} \overline{\beta}_j^i \mathrm{co} B_j) + K^* \\ &= \ \mathrm{cl}\ [\sum_{k=1}^{p} \sum_{i=1}^{p} \overline{\alpha}_k^i \mathrm{co} A_k + \sum_{j=1}^{q} \sum_{i=1}^{p} \overline{\beta}_j^i \mathrm{co} B_j] + K^*, \end{aligned}$$

which implies that

$$0 \in \mathrm{cl}\ [\sum_{k=1}^{p} \overline{\lambda}_k \mathrm{co} A_k + \sum_{j=1}^{q} \overline{\mu}_j \mathrm{co} B_j] + K^*,$$

as cl$A$+cl$B$ $\subset$cl$(A + B)$, where $\overline{\lambda}_k = \sum_{k=1}^{p} \overline{\alpha}_k^i > 0, , \overline{\mu}_j = \sum_{i=1}^{p} \overline{\beta}_j^i \geq 0$.

Observing that $\sum_{k=1}^{p}\overline{\lambda}_k\text{co}A_k+\sum_{j=1}^{q}\overline{\mu}_j\text{co}B_j=\text{co}\,(\sum_{k=1}^{p}\overline{\lambda}_kA_k+\sum_{j=1}^{q}\overline{\mu}_jB_j)$ we arrive at (3.5).

(ii) $\Rightarrow$ (i):Suppose that (ii) holds. This implies that there exist $\overline{\lambda}_k>0,\overline{\mu}_j\geqslant 0$ $(k\in I, j\in J)$ such that (3.5) holds. We set

$$E=\text{cl co}[\sum_{k=1}^{p}\overline{\lambda}_kA_k+\sum_{j=1}^{q}\overline{\mu}_jB_j]+K^*.$$

By assumption, $0\in E$. It is obviously that E is convex.

We invoke Theorem 13.1 in [15] to deduce that

$$\sup_{\zeta\in E}\langle\zeta,v\rangle\geq 0\ \forall v.$$

It follows from this and Theorem 32.2 in [15] that, $\forall v$,

$$\sum_{k=1}^{p}\overline{\lambda}_k\sup_{a_k\in A_k}\langle a_k,v\rangle+\sum_{j=1}^{q}\overline{\mu}_j\sup_{b_j\in B_j}\langle b_j,v\rangle+\sup_{\xi\in K^*}\langle\xi,v\rangle\geq 0. \tag{3.11}$$

If (i) were false, there would exists $i\in I$ such that the system (3.1)-(3.4) has a solution $v_0\in\mathbb{R}^n$. It implies that

$$\sup_{\xi\in K^*}\langle\xi,v_0\rangle\leq 0, \tag{3.12}$$

and

$$\sum_{k=1}^{p}\overline{\lambda}_k\sup_{a_k\in A_k}\langle a_k,v_0\rangle+\sum_{j=1}^{s}\overline{\mu}_j\sup_{b_j\in B_j}\langle b_j,v_0\rangle<0\ . \tag{3.13}$$

Combining (3.12) and (3.13) yields that

$$\sum_{k=1}^{p}\overline{\lambda}_k\sup_{a_k\in A_k}\langle a_k,v_0\rangle+\sum_{j=1}^{q}\overline{\mu}_j\sup_{b_j\in B_j}\langle b_j,v_0\rangle+\sup_{\xi\in K^*}\langle\xi,v_0\rangle<0,$$

which contradicts (3.11). This completes the proof. $\square$

Remark 3.2. *a) If the cone $T(C;\overline{x})$ is replaced by the cone $Z(C;\overline{x})$, then Theorem 3.1 is still valid.*

b) Theorem 3.1 is a generalization of Proposition 2.2 in [6].

For each $i=1,...,p$, we set

$$T_i=\cup\{\text{cl co}(A_i+\sum_{k=1,k\neq i}^{p}\alpha_kA_k+\sum_{j=1}^{q}\beta_jB_j):\alpha_k\geq 0,k\neq i,\beta_j\geq 0\},$$

and denote by $coneT_i$ the cone generated by T_i.

In case the sets A_k and B_j are compact, we obtain the following consequence of Theorem 3.1.

Corollary 3.3. *Let $A_1, ..., A_p$, $B_1, ..., B_q$ be nonempty compact subset of $\mathbb{R}^n$. Assume that K is a nonempty closed convex subcone of $T(C;\overline{x})$ with the vertex at the origin. Suppose, in addition, that for each $i \in I$,*

$$0 \notin cl\ co(\bigcup_{k=1,k\neq i}^{p} A_k \cup \bigcup_{j=1}^{q} B_j),$$

and $(-coneT_i) \cap K^* = \{0\}$. Then the conclusions of Theorem 3.1 hold in which the cl in (3.5) is superfluous.

Proof. By assumption, for each $i = 1, ..., p$, the set $S_i = (\bigcup_{k=1,k\neq i}^{p} A_k \cup \bigcup_{j=1}^{q} B_j)$ is compact, and hence, coS_i is compact. Taking account of Proposition 1.4.7 [8] we deduce that the set cone(coS_i) is closed. This leads to the set

$$\cup\{\text{co}(\sum_{k=1,k\neq i}^{p} \alpha_k A_k + \sum_{j=1}^{q} \beta_j B_j) : \alpha_k \geq 0, k \neq i, \beta_j \geq 0\} \qquad (3.14)$$

is closed. On the other hand, in view of the compactness of coA_k and coB_j, it follows that

$$\text{cl}(\text{co}A_i + \sum_{k=1,k\neq i}^{p} \alpha_k \text{co}A_k + \sum_{j=1}^{q} \beta_j \text{co}B_j) =$$
$$\text{co}A_i + \sum_{k=1,k\neq i}^{p} \alpha_k \text{co}A_k + \sum_{j=1}^{q} \beta_j \text{co}B_j. \qquad (3.15)$$

for all $\alpha_k \geq 0, k \neq i, \beta_j \geq 0$. It follows readily from (3.15) that

$$\begin{aligned} T_i &= \cup\{\text{co}A_i + \sum_{k=1,k\neq i}^{p} \alpha_k \text{co}A_k + \sum_{j=1}^{q} \beta_j \text{co}B_j) : \alpha_k \geq 0, k \neq i, \beta_j \geq 0\} \\ &= \text{co}A_i + \cup\{\sum_{k=1,k\neq i}^{p} \alpha_k \text{co}A_k + \sum_{j=1}^{q} \beta_j \text{co}B_j) : \alpha_k \geq 0, k \neq i, \beta_j \geq 0\}. \end{aligned}$$

This along with (3.14) yields that T_i is closed. We invoke Corollary 9.1.1 [15] to deduce that $T_i + K^*$ is closed $(i = 1, ..., p)$. Applying Theorem 3.1 we obtain the desired conclusions. □

4. Kuhn-Tucker necessary conditions for efficiency

From the results obtained in the previous section, we shall establish Kuhn-Tucker necessary conditions for efficiency with Lagrange multipliers associated with all the components of the objective function to be positive.

We set

$$\begin{aligned} J(\overline{x}) &= \{j \in J : g_j(\overline{x}) = 0\}; \\ Q &= \{x \in C : f_k(x) \leqslant f_k(\overline{x}), g_j(x) \leqslant 0, h_\ell(x) = 0, k \in I, j \in J, \ell \in L\}; \\ Q^i &= \{x \in C : f_k(x) \leqslant f_k(\overline{x}), g_j(x) \leqslant 0, h_\ell(x) = 0, k \in I \backslash \{i\}, j \in J, \ell \in L\}. \end{aligned}$$

If for each $v \in Z(C; \overline{x}), Dh_l(\overline{x}; v)$ exists ($\ell \in L$), we put

$$\begin{aligned} C_D(Q; \overline{x}) &= \{v \in Z(C; \overline{x}) : \underline{D} f_k(\overline{x}; v) \leqslant 0, k \in I, \\ \underline{D} g_j(\overline{x}; v) &\leqslant 0, j \in J(\overline{x}), Dh_\ell(\overline{x}; v) = 0, \ell \in L\}. \end{aligned}$$

If for each $v \in T(C; \overline{x}), dh_l(\overline{x}; v)$ exists ($\ell \in L$), we put

$$\begin{aligned} C_d(Q; \overline{x}) &= \{v \in T(C; \overline{x}) : \underline{d} f_k(\overline{x}; v) \leqslant 0, k \in I, \\ \underline{d} g_j(\overline{x}; v) &\leqslant 0, j \in J(\overline{x}), dh_\ell(\overline{x}; v) = 0, \ell \in L\}. \end{aligned}$$

Note that $C_D(Q; \overline{x})$ and $C_d(Q; \overline{x})$ are cones with vertices at the origin. We recall some results in [12] which will be employed in the sequel.

Proposition 4.1. *[**12**]. Let $\overline{x} \in M$.*

a) If for each , $v \in T(C; \overline{x})$, the Hadamard directional derivatives

$$dh_1(\overline{x}; v), ..., dh_r(\overline{x}; v)$$

exist, then

$$\bigcap_{i=1}^{p} T(Q^i; \overline{x}) \subset C_d(Q; \overline{x}). \tag{4.1}$$

b) If for each $v \in Z(C; \overline{x})$, the Dini directional derivatives

$$Dh_1(\overline{x}; v), ..., Dh_r(\overline{x}; v)$$

exist, then

$$\bigcap_{i=1}^{p} Z(Q^i; \overline{x}) \subset C_D(Q; \overline{x}). \tag{4.2}$$

In general, the converse inclusions of (4.1) and (4.2) do not hold. As also in [12], we introduce the following constraint qualifications of Abadie type at $\overline{x}$

$$C_d(Q;\overline{x}) \subset \bigcap_{i=1}^{p} T(Q^i;\overline{x}), \tag{4.3}$$

$$C_D(Q;\overline{x}) \subset \bigcap_{i=1}^{p} Z(Q^i;\overline{x}). \tag{4.4}$$

If for each $v \in Z(C;\overline{x})$, the Dini directional derivatives $Df_k(\overline{x};v)$ and $Dh_l(\overline{x};v)$ exist , we set

$$\begin{aligned} L^i_D(f;\overline{x}) &= \{v \in Z(C;\overline{x}) : Df_i(\overline{x};v) < 0, \\ &\quad Df_k(\overline{x};v) \le 0, \forall k \in I, k \ne i\} \\ L_D(M;\overline{x}) &= \{v \in Z(C;\overline{x}) : \underline{D}g_j(\overline{x};v) \le 0, j \in J(\overline{x}), \\ &\quad Dh_\ell(\overline{x};v) = 0, \ell \in L\}. \end{aligned}$$

If for each $v \in T(C;\overline{x})$, the Hadamard directional derivatives $df_k(\overline{x};v)$ and $dh_l(\overline{x};v)$ exist , we set

$$\begin{aligned} L^i_d(f;\overline{x}) &= \{v \in T(C;\overline{x}) : df_i(\overline{x};v) < 0, \\ &\quad df_k(\overline{x};v) \le 0, \forall k \in I, k \ne i\}. \\ L_d(M;\overline{x}) &= \{v \in T(C;\overline{x}) : dg_j(\overline{x};v) \le 0, j \in J(\overline{x}), \\ &\quad dh_\ell(\overline{x};v) = 0, \ell \in L\}, \end{aligned}$$

where M indicates the feasible set of the Problem (VP).

Proposition 4.2. *[**12**]. Let $\overline{x}$ be a local efficient solution of Problem (VP). Assume that the functions g_j ($j \notin J(\overline{x})$) are continuous at $\overline{x}$, and for each $v \in T(C;\overline{x})$ (resp.$v \in Z(C;\overline{x})$), the Hadamard directional derivatives $df_k(\overline{x};v)$ and $dh_l(\overline{x};v)$ (resp. the Dini directional derivatives $Df_k(\overline{x};v)$ and $Dh_l(\overline{x};v)$), $k \in I, \ell \in L$, exist. Suppose, in addition, that the constraint qualification (4.3) (resp. (4.4)) holds at $\overline{x}$. Then, for each $i \in I$,*

$$\begin{aligned} L^i_d(f;\overline{x}) \cap L_d(M;\overline{x}) &= \emptyset \\ (\text{resp. } L^i_D(f;\overline{x}) \cap L_D(M;\overline{x}) &= \emptyset). \end{aligned}$$

To derive Kuhn-Tucker necessary conditions for efficiency of Problem (VP), we introduce the following assumption.

Assumption 4.3. *The functions f_k , g_j and h_l are Dini directionally differentiable at $\overline{x} \in \mathbb{R}^n$, and there are $c_l \in \mathbb{R}^n$ and the families $\mathcal{F}_k, \mathcal{G}_j$ of nonempty sets of $\mathbb{R}^n$ $(k \in I, j \in J(\overline{x}), \ell \in L)$ such that*

$$Df_k(\overline{x}; v) = \inf_{F_k \in \mathcal{F}_k} \sup_{z_k \in F_k} \langle z_k, v \rangle, \forall v \in \mathbb{R}^n, k \in I, \tag{4.5}$$

$$Dg_j(\overline{x}; v) = \inf_{G_j \in \mathcal{G}_j} \sup_{z_j \in G_j} \langle z_j, v \rangle, \forall v \in \mathbb{R}^n, j \in J(\overline{x}), \tag{4.6}$$

$$Dh_l(\overline{x}; v) = \langle c_l, v \rangle, \ell \in L. \tag{4.7}$$

Note that assumptions (4.5) and (4.6) were employed by Ishizuka [6]. The class of functions directionally differentiable whose directional derivatives have representations (4.5), (4.6) is rather wide, which contains all quasidifferentiable functions in the sense of Pschnichyi [16], Demyanov-Rubinov [3] and Ishizuka [5].

We are now in a position to formulate a Kuhn-Tucker necessary condition for efficiency of Problem (VP).

Theorem 4.4. *Let $\overline{x}$ be a local efficient solution of Problem (VP). Let f_k, g_j, h_l $(k \in I; j \in J(\overline{x}); \ell \in L)$ be Hadamard directionally differentiable (resp. Dini directionally differentiable) at $\overline{x}$. Let K be an arbitrary nonempty closed convex subcone of $T(C; \overline{x})$ (resp. $Z(C; \overline{x})$) with vertex at the origin and the functions g_j $(j \notin J(\overline{x}))$ continuous at $\overline{x}$. Assume that for each $i \in I$, the following set is closed*

$$\cup \{ cl\ co(F_i + \sum_{k=1, k \neq i}^{p} \alpha_k F_k + \sum_{j \in J(\overline{x})} \beta_j G_j) \ : \ \alpha_k \geq 0, k \neq i, \beta_j \geq 0 \} + lin\{c_l \ : \ l \in L\} + K^*.$$

Suppose, furthermore, that the constraint qualification (4.4) (resp (4.3)) and Assumption 4.3 are fulfilled. Then, there exist $\overline{\lambda}_k > 0, \overline{\mu}_j \geqslant 0, \overline{\nu}_\ell \in \mathbb{R}$ $(k \in I; j \in J; \ell \in L)$ and $\widetilde{F}_k \in \mathcal{F}_k, \widetilde{G}_j \in \mathcal{G}_j$ $(k \in I; j \in J; \ell \in L)$ such that

$$0 \in cl\ co(\sum_{k=1}^{p} \overline{\lambda}_k \widetilde{F}_k + \sum_{j=1}^{q} \overline{\mu}_j \widetilde{G_j}) + \sum_{l=1}^{r} \overline{v}_l c_l, \alpha_k + K^*, \tag{4.8}$$

$$\overline{\mu}_j g_j(\overline{x}) = 0, \forall j \in J. \tag{4.9}$$

Proof. We have only to prove this theorem in case f_k, g_j and h_l are Hadamard differentiable. In case f_k, g_j and h_l are Dini differentiable, the proof is analogous.

Taking account of Proposition 4.2, we get that for each $i \in I$,the following system has no solution $v \in \mathbb{R}^n$:

$$\begin{aligned} df_k(\overline{x}, v) &\leq 0, k \in I \backslash \{i\}, \\ df_i(\overline{x}, v) &< 0, \\ dg_j(\overline{x}, v) &\leq 0, j \in J(\overline{x}), \\ dh_l(\overline{x}, v) &= 0, l \in L, \\ v &\in K. \end{aligned}$$

Since $df_k(\overline{x}, v), dg_j(\overline{x}, v)$ and $dh_l(\overline{x}, v)$ exist, it follows that $Df_k(\overline{x}, v), Dg_j(\overline{x}, v)$ and $Dh_l(\overline{x}, v)$ also exist and they are equal. Hence, the following system has no solution solution $v \in \mathbb{R}^n$:

$$Df_k(\overline{x}, v) < 0, k \in I, \tag{4.10}$$

$$Dg_j(\overline{x}, v) < 0, j \in J(\overline{x}), \tag{4.11}$$

$$Dh_l(\overline{x}, v) = 0, l \in L, \tag{4.12}$$

$$v \in K. \tag{4.13}$$

By Assumption 4.3, the inequalities (4.10)-(4.12) imply that there exist $\widetilde{F_k} \in \mathcal{F}_k, \widetilde{G_j} \in \mathcal{G}_j$ and $c_l \in \mathbb{R}^n (k \in I; j \in J(\overline{x}); \ell \in L)$ such that

$$\begin{aligned} \sup_{z_k \in \widetilde{F_k}} \langle z_k, v \rangle &< 0, \ k \in I, \\ \sup_{z_j \in \widetilde{G_j}} \langle z_k, v \rangle &< 0, \ j \in J(\overline{x}), \\ \langle c_l, v \rangle &= 0, \ell \in L. \end{aligned}$$

The last equalities can be rewritten as follows

$$\langle c_l, v \rangle \leq 0, \langle -c_l, v \rangle \leq 0, \ell \in L. \tag{4.14}$$

It is obvious that

$$\cup \{ \text{cl co}(F_i + \sum_{k=1, k \neq i}^{p} \alpha_k F_k + \sum_{j \in J(\overline{x})} \beta_j G_j) : \alpha_k \geq 0, k \neq i, \beta_j \geq 0 \} + \text{lin}\{c_l : l \in L\} =$$

$$\cup \{ \text{cl co}(F_i + \sum_{k=1, k \neq i}^{p} \alpha_k F_k + \sum_{j \in J(\overline{x})} \beta_j G_j + \sum_{l=1}^{r} \gamma_l c_l) : \alpha_k \geq 0, k \neq i, \beta_j \geq 0, \gamma_l \in R \}.$$

Consequently, apply Theorem 3.1 to the system comprising (4.10),(4.11),(4.13),(4.14) and deduce that there exist $\overline{\lambda}_k > 0$, $\overline{\mu}_j \geqslant 0$,$\overline{\gamma}_\ell^+ \geq 0, \overline{\gamma}_\ell^- \geq 0$ $(k \in I; j \in$

$J(\overline{x}); \ell \in L)$ such that

$$0 \in \text{cl co}(\sum_{k=1}^{p} \overline{\lambda}_k \widetilde{F}_k + \sum_{j \in J(\overline{x})} \overline{\mu}_j \widetilde{G}_j + \sum_{l=1}^{r} (\overline{\gamma}_\ell^+ c_l + \overline{\gamma}_\ell^- (-c_l))) + K^*.$$

By setting $\overline{\gamma}_\ell = \overline{\gamma}_\ell^+ - \overline{\gamma}_\ell^-, \ell \in L$, one has

$$0 \in \text{cl co}(\sum_{k=1}^{p} \overline{\lambda}_k \widetilde{F}_k + \sum_{j \in J(\overline{x})} \overline{\mu}_j \widetilde{G}_j) + \sum_{l=1}^{r} \overline{\gamma}_\ell c_l + K^*. \tag{4.15}$$

By taking $\overline{\mu}_j = 0$ for $j \notin J(\overline{x})$,we get (4.9). Then, (4.15) implies that (4.8) holds, which completes the proof. □

In case directional derivatives are convex in directional variable, we get the following theorem.

Theorem 4.5. *Let $\overline{x}$ be a local efficient solution of Problem (VP). Let f_k, g_j, h_l $(k \in I; j \in J; \ell \in L)$ be Hadamard directionally differentiable (resp. Dini directionally differentiable) at $\overline{x}$, where $df_k(\overline{x}, .)$ and $dg_j(\overline{x}, .)$ (resp. $Df_k(\overline{x}, .), Dg_j(\overline{x}, .)$) are convex, $(k \in I; j \in J(\overline{x}))$ and $dh_l(\overline{x}, .)$ (resp. $Dh_l(\overline{x}, .)$), $l \in L$, be linear, which is given by $dh_l(\overline{x}, .) = \langle c_l, v\rangle$ (resp.$Dh_l(\overline{x}; v) = \langle c_l, v\rangle$), $\ell \in L$. Let K be an arbitrary nonempty closed convex subcone of $T(C; \overline{x})$ (resp. $Z(C; \overline{x})$) with vertex at the origin and the functions g_j $(j \notin J(\overline{x}))$ continuous at $\overline{x}$. Assume that for each $i \in I$, the following set is closed*

$$L_i = coneco(\bigcup_{k=1, k\neq i}^{p} \partial_D f_k(\overline{x})) + coneco(\bigcup_{j \in J(\overline{x})} \partial_D g_j(\overline{x})) + lin\{c_l : l \in L\} + K^*.$$

Suppose also that the constraint qualification (4.4) (resp (4.3)) is fulfilled. Then, there exist $\overline{\lambda}_k > 0$, $\overline{\mu}_j \geqslant 0$,$\overline{\nu}_\ell \in \mathbb{R}$ $(k \in I; j \in J; \ell \in L)$ such that

$$0 \in \sum_{k=1}^{p} \overline{\lambda}_k \partial_D f_k(\overline{x}) + \sum_{j=1}^{q} \overline{\mu}_j \partial_D g_j(\overline{x}) + \sum_{l=1}^{r} \overline{v}_l c_l + K^* \tag{4.16}$$

$$\overline{\mu}_j g_j(\overline{x}) = 0, j \in J. \tag{4.17}$$

Proof. As in the proof of Theorem 4.4, we have only to prove this theorem in case f_k, g_j, h_l $(k \in I; j \in J; \ell \in L)$ be Hadamard directionally differentiable, while the remainder is similarly proved.

Since $df_k(\overline{x}, v)$, $dg_j(\overline{x}, v)$ and $dh_l(\overline{x}, v)$ exist, $Df_k(\overline{x}, v), Dg_j(\overline{x}, v)$ and $Dh_l(\overline{x}, v)$ also exist and they are equal. In view of the convexity of

$Df_k(\overline{x},.)$ and $Dg_j(\overline{x},.)$, the Dini subdifferentials $\partial_D f_k(\overline{x})$ and $\partial_D g_j(\overline{x})$ are nonempty, convex , compact, and

$$\begin{aligned} Df_k(\overline{x},v) &= \max_{\xi\in\partial_D f_k(\overline{x})}\langle\xi,v\rangle, k\in I,\\ Dg_j(\overline{x},v) &= \max_{\xi\in\partial_D g_j(\overline{x})}\langle\xi,v\rangle, j\in J(\overline{x}). \end{aligned}$$

By assumption, L_i is closed, $i\in I$,and hence, $\partial_D f_i(\overline{x})+L_i$ is also closed, which means that for each $i\in I$, the set

$$\cup\{\text{cl }(\partial_D f_i(\overline{x})+\sum_{k=1,k\neq i}^{p}\alpha_k\partial_D f_k(\overline{x})+\sum_{j\in J(\overline{x})}\beta_j\partial_D g_j(\overline{x}))\ :\ \alpha_k\geq 0,k\neq i,\beta_j\geq 0\}$$
$$+\text{lin}\{c_l\ :\ l\in L\}+K^*$$

is closed. According to Theorem 4.4 , there exist $\overline{\lambda}_k>0$, $\overline{\mu}_j\geqslant 0$,$\overline{\nu}_\ell\in\mathbb{R}$ $(k\in I; j\in J;\ell\in L)$ such that (4.17) holds, and

$$0\in\text{cl co }(\sum_{k=1}^{p}\overline{\lambda}_k\partial_D f_k(\overline{x})+\sum_{j=1}^{q}\overline{\mu}_j\partial_D g_j(\overline{x}))+\sum_{l=1}^{r}\overline{v}_l c_l+K^*. \tag{4.18}$$

Since the set

$$\sum_{k=1}^{p}\overline{\lambda}_k\partial_D f_k(\overline{x})+\sum_{j=1}^{q}\overline{\mu}_j\partial_D g_j(\overline{x})$$

is closed convex, (4.18) implies (4.16). □

Remark 4.6. *In case $C=R^n$ and $h_1,...,h_r$ are Fréchet differentiable, from Theorem 4.5 above we obtain Theorem 4.3 [4] as a special case.*

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