Hedging global environment risks: An option based portfolio insurance*

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Abstract

This paper introduces a financial hedging model for global environment risks. Our approach is based on portfolio insurance under hedging constraints. Investors are assumed to maximize their expected utilities defined on financial and environmental asset values. The optimal investment is determined for quite general utility functions and hedging constraints. In particular, our results suggest how to introduce derivative assets written on the environmental asset.

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1 Introduction

The purpose of the environmental economics is to examine the relations between the (local or entire) economy and the environment. From the early 1950s, the Washington DC environmental and resource economics organization (called "Resources for the Future") has examined many economic problems linked to environmental protection: regulation of water pollution illustrated for the Delaware River system by Kneese and Bower (1968), natural resources control by Barnett and Morse (1968), the concept of value beyond simple use introduced by Krutilla (1967). Safirova, Hourde, Lipman, Harrington and Baglino (2006) also analyze the regional long-term impacts (at the city level) of congestion pricing as a way to alleviate congestion and its environmental impacts. This paper takes a long term and general perspective ignoring the more focus issues (in particular those related to consumer behavior with respect to the environment - see, e.g. the discussion of de Palma and Pahaut, 1996 on this issue).

As mentioned in Kolstad (2000), many important questions arise.

How should the economy activity be regulated in order to reach specific environmental goals? What is the right balance between the industrial production, which is necessary for the economic development and the reduction of pollution, which has a social value (but also to some extend, an economic value) ?

Among several such questions, several financial problems arise: How could one evaluate the costs of pollution controls? For example, what is the fair value of carbon emission fees? How a given firm can allocate its wealth between production capacity (for instance, energy) and environmental investments, such as pollution fees? Such questions are difficult to handle since they are embedded in a risky or uncertain environment. Moreover, market are often incomplete.

We propose here a model which take into account both the benefit (or utility) of consumption as well as the utility derived from the quality of the environment. The quality of the environment can influence indirectly the utility of individual consumption (for example, the enjoyment of leisure good can be reduced if the air is polluted). However, environmental quality has also an direct effect on individual utility (one could be concerned by the quality of the environment per se, with respect to the future generation, etc.).

Using a standard microeconomic approach of the environment, we can consider a given society of individual consumers i having preferences on both a material good g and an environmental quality level e. In the Von Neumann and Morgenstern framework, these preferences are represented by utility functions $U_i(g, e)$. The trade-off between g and e is determined by expected utility maximization. Based on Pareto criterion, the choice of the whole society might be represented by a social welfare function analogous to an individual utility function. However, if the environment is not sufficiently valued by individuals who are rather selfish materialists than biocentrists, policy makers can provide right incentives to control environmental risks such as global warming. This problem is due to the market failure of the public provision of non-market goods. It is in line with the criticism of the utilitarian criterion suggested by Sagoff (1994): Should public policy be only determined from individual preferences, without introducing a concept of "what is right"?

To solve such dilemma, we propose in this paper to introduce specific *hedging constraints.*

At a given horizon, the environmental quality e will have to be above a minimal level. The production of the material good also may satisfy such constraint: for a developed economy, many individuals will not accept to reduce their consumption beyond some level, and for developing country, natural resources for example will have to be exploited to guarantee sufficient wealth.

The production of the material good and the control of environmental quality can be based on investment on both financial and environmental assets. Then, this management problem is similar to a portfolio optimization problem with or without insurance conditions. Portfolio insurance payoff provides for a benefit payable at maturity. It gives the investor the ability to limit downside risk while allowing some participation in upside markets. The portfolio payoff is a function of the value at maturity of some specified portfolio of common assets. Such portfolio management has been introduced by Leland and Rubinstein (1976) who considered a portfolio invested in a risky asset S, usually a financial index such as the $S\&P$, covered by a listed put written on it. The comparison between main insurance methods with various criteria has been examined for example in Black and Perold (1992), Bookstaber and Langsam (2000), Bertrand and Prigent (2001). Most portfolio optimization models consider an investor who maximizes the expected utility of his terminal wealth, by trading in continuous time (see for example Cox and Huang (1989) or Cvitanic and Karatzas (1996)). The continuous-time setup is also usually introduced to study portfolio insurance (see for example, Grossman and Vila (1989), Basak (1995) or Grossman and Zhou (1996)). The key assumption is that markets are complete: all portfolio profiles at maturity can be perfectly hedged. The optimal positioning problem has also been addressed in the partial equilibrium framework by Leland (1980) and Brennan and Solanki (1981). The value of the portfolio is a function of the common assets and crucially depends on the investor's risk aversion. Following this approach, Carr and Madan (1998) consider markets in which exist out-of-the-money European puts and calls of all strikes. As they mentioned, this assumption allows to examine the optimal positioning in a complete market and is the counterpart of the assumption of continuous trading. This approximation is justified when there is a large number of option strikes (e.g. for the S&P500, for example).

In this paper, we analyze the optimal hedging investment among financial and environmental assets. They are determined for two main cases. It can be applied, for example, to model the choice of a firm which must allocate its capital between energy investment and pollution fees. It can also be used to model the social trade-off between material good production and environmental investment.

In the first case, it is assumed that the two bundles can be substituted: they are considered as purely monetary amounts. At any time, the "investor" can buy or sell one of these assets to buy or sell the other. For example, a firm can simultaneously trade on energy and pollution fees such as carbon emission fees.

In the second case, the investment problem is split into two parts. The "investor" trades on two separate funds: one fund for the production of material good which can be for instance a purely financial investment, and the other fund is devoted to the environmental quality. For example, a social planner may have to choose between investment for global production and a environmental non-market good.

In Section 2, a review of basic results about insurance portfolio optimization is provided . In Section 3, fundamental examples are detailed. In particular, we examine optimal portfolios for the Cobb-Douglas utility functions. Concluding comments are presented in Section 4.

2 Optimal Insured Portfolio

In what follows, some of the main investment optimization results with hedging constraints are recalled (see Prigent (2007) for an overview of the portfolio optimization problems).

2.1 The financial market and the environmental asset

Assume that the financial market is complete, arbitrage free and frictionless. Asset values are supposed to follow continuous-time diffusion processes. Financial markets can be assumed to be complete by introducing two sources of risk. As shown by Duffie and Huang (1985), continuous-time rebalancing allows such assumption of dynamic market completeness.

Asset prices dynamics are defined by:

(1) Cash:

$$\frac{dC_t}{C_t} = r_t dt, \tag{1}$$

where the instantaneous interest rate r follows a diffusion process. For example, r_t may be an Ornstein-Uhlenbeck process given by:

$$dr_t = a_r(b_r - r_t)dt - \sigma_r dW_{1,t}, \tag{2}$$

where a_r, b_r and σ_r are positive constants and W_1 is a standard Brownian motion. The market price of interest rate risk is assumed to be constant (see Vasicek (1977)).

(2) Stock index:

$$\frac{dS_t}{S_t} = (r_t + \theta_S(t, S_t))dt + \sigma_1(t, S_t)dW_{1,t} + \sigma_2(t, S_t)dW_{2,t}. \tag{3}$$

(3) Environmental investment:

$$\frac{dI_t}{I_t} = (r_t + \theta_I(t, I_t))dt + \delta_1(t, I_t)dW_{1,t} + \delta_2(t, I_t)dW_{2,t}, \tag{4}$$

where W_1 and W_2 are two independent standard Brownian motion. The diffusion coefficients σ_1, σ_2, δ_1, δ_2 are assumed to be positive functions which satisfy usual assumptions so that the above stochastic differential equations have one and only one solution[1]. The parameter θ_S is the risk premium of the stock, and θ_I is the risk premium of the environmental investment.

The volatility matrix $\begin{pmatrix} \sigma_1(t,S_t) & \sigma_2(t,S_t) \\ \delta_1(t,I_t) & \delta_2(t,I_t) \end{pmatrix}$ is assumed to be invertible.

Therefore, the market is complete and there exists a unique risk-neutral probability $\mathbb{Q}$ associated to two market risk premia, λ and λ_r, with density η with respect to the initial probability $\mathbb{P}$ given by:

$$\eta_t = \exp\left[-\int_0^t (\lambda_{1,t} dW_{1,t} + \lambda_{2,t} dW_{2,t}) - 1/2 \int_0^t (\lambda_{1,t}^2 + \lambda_{2,t}^2) dt\right].$$

The premia λ and λ_r are determined from the relation:

$$\begin{aligned} \theta_S(t,S_t) &= \sigma_1(t,S_t)\lambda_{1,t} + \sigma_2(t,S_t)\lambda_{2,t}, \\ \theta_I(t,I_t) &= \delta_1(t,I_t)\lambda_{1,t} + \delta_2(t,I_t)\lambda_{2,t}. \end{aligned}$$

2.2 Optimal payoffs as functions of a benchmark

We suppose that the investor determines an optimal payoff h which is a function defined on all possible values of the assets (C,S,I) at maturity. Since the market is complete, this payoff can be achieved by the investor.

Remark 1 *The market can be complete for example if the financial market evolves in continuous-time and all options can be dynamically replicated by a perfect hedging strategy. This is the previous assumption on the three assets. It can still be complete if for example, in one period setting, European options of all strikes are available on the market. In this setting, the inability to trade continuously potentially induces investment in cash, asset I, asset S and all European options with underlying I and S (if cash and environmental asset are non stochastic, only European options on S are required). The market can be also incomplete. In that case, the solution given in this section is only "theoretical" but still interesting to know since the optimal payoff can be approximated by investing on traded assets (in practice, the investor defines an approximation method, which may take transaction costs or liquidity problems into account).*

According to the investor's risk aversion and horizon, the investor chooses the weights to invest on financial and environmental assets. The resulting portfolio value $(V_t)_t$ is self-financing. It means that the process $(V_t \exp(-\int_0^t r_s ds))_t$ is a $\mathbb{Q}$-martingale where $\mathbb{Q}$ is the risk-neutral probability.
Denote by $\eta = \frac{d\mathbb{Q}}{d\mathbb{P}}$ the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to the

[1] See for example Jacod and Shiryaev (2003) for sufficient conditions such as Lipshitz and linear growth.

historical probability $\mathbb{P}$. Denote also by M_T the process $\eta_T \exp(-\int_0^T r_s ds)$. Due to the no-arbitrage condition, the budget constraint corresponds to the following relation:

$$V_0 = \mathbb{E}_{\mathbb{Q}}[V_T \exp(-\int_0^T r_s ds)] = \mathbb{E}_{\mathbb{P}}[V_T M_T].$$

Assume that the investor wants to maximize an expected utility under the statistical probability $\mathbb{P}$. As usual, the utility U of the investor is supposed to be increasing, concave and twice-differentiable. Suppose as in Karatzas, Lehoczky, Sethi and Shreve (1986) that the marginal utility U' satisfies:

$$\lim_{o+} U' = +\infty \text{ and } \lim_{+\infty} U' = 0.$$

Denote by J the inverse of the marginal utility U'.

2.2.1 The non insured portfolio

This subsection is an extension of the results in Brennan and Solanki (1981) or Carr and Madan (1997) to more general markets. It provides an overview of results in Prigent (2006). Consider an investor who wants to maximize the expected utility of his terminal wealth. Under the standard condition of no-arbitrage, the assets prices are calculated under risk neutral probabilities. If markets exist for out-of-the-money European puts and calls of all strikes, then it implies the existence of an unique risk-neutral probability that may be identified from option prices (see Breeden and Litzenberger (1978)). Otherwise, if there is no continuous-time trading, generally the market is incomplete and a one particular risk-neutral probability $\mathbb{Q}$ is used to price the options. It is also possible that stock prices change continuously but the market may be still dynamically incomplete. Again, it is assumed that one risk-neutral probability is selected. Assume that prices are determined under such measure $\mathbb{Q}$. Denote by $\frac{d\mathbb{Q}}{d\mathbb{P}}$ the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to the historical probability $\mathbb{P}$. Denote by η_T the discount factor and by M_T the product $\eta_T \frac{d\mathbb{Q}}{d\mathbb{P}}$.

Due to the no-arbitrage condition, the budget constraint corresponds to the following relation :

$$V_0 = \mathbb{E}_{\mathbb{Q}}[h(C_T, S_T, I_T)\eta_T] = \mathbb{E}_{\mathbb{P}}[h(C_T, S_T, I_T)M_T].$$

The investor has to solve the following optimization problem:

$$Max_h \mathbb{E}_{\mathbb{P}}[U(h(C_T, S_T, I_T)] \text{ under } V_0 = \mathbb{E}_{\mathbb{P}}[h(C_T, S_T, I_T)M_T]. \tag{5}$$

To simplify the presentation of the main results, we suppose as usual that the function h fulfils:

$$\int_{\mathbb{R}^{+3}} h^2(c, b, s)\mathbb{P}_{(C_T, S_T, I_T)}(dc, ds, di) < \infty \ .$$

It means that $h \in \mathbb{L}^2(\mathbb{R}^{+3}, \mathbb{P}_{X_T}(dx))$ where $X_T = (C_T, S_T, I_T)$ which is the set of the measurable functions with squares that are integrable on $\mathbb{R}^{+3}$ with respect to the distribution $\mathbb{P}_{X_T}(dx)$.

With the utility function U is associated a new functional Φ_U which is defined on the space $\mathbb{L}^2(\mathbb{R}^{+3}, \mathbb{P}_{X_T}(dx))$ by:

$$\text{For any } Y \in \mathbb{L}^2(\mathbb{R}^{+3}, \mathbb{P}_{X_T}(dx)),\ \Phi_U(Y) = \mathbb{E}_{\mathbb{P}_{X_T}}[U(Y)].$$

Φ_U is usually called the Nemitski functional associated with U (see for example Ekeland and Turnbull (1983) for definition and basic properties). From the properties of the utility function U, the Nemitski functional Φ_U is concave and differentiable on $\mathbb{L}^2(\mathbb{R}^{+3}, \mathbb{P}_{X_T})$. Besides, the budget constraint is a linear function of h. So there exists exactly one solution h^e. The function h^e is the solution of $\frac{\partial L}{\partial h} = 0$ where the Lagrangian L is defined by :

$$L(h, \lambda) = \int_{\mathbb{R}^{+3}} [U(h(x))] \mathbb{P}_{X_T}(dx) + \lambda \left(V_0 - \int_{\mathbb{R}^{+3}} h(x) g(x) \mathbb{P}_{X_T}(dx) \right).$$

The parameter λ is the Lagrange multiplier associated to the budget constraint. Thus, h^e satisfies: $U'(h^e) = \lambda g$. Therefore, $h^e = J(\lambda g)$.

Proposition 2 *Introduce the conditional expectation of M_T under the σ-algebra generated by (C_T, S_T, I_T). Denote it by g. Assume that g is a function defined on the set of the values of $X_T = (C_T, S_T, I_T)$. and $g \in \mathbb{L}^2(\mathbb{R}^{+3}, \mathbb{P}_{X_T})$.*
Then, the optimization problem is reduced to:

$$Max_{h \in \mathbb{L}^2(\mathbb{R}^{+3}, \mathbb{P}_{X_T})} \int_{\mathbb{R}^{+3}} [U(h(x))] \mathbb{P}_{X_T}(dx) \tag{6}$$

$$under: V_0 = \int_{\mathbb{R}^{+3}} h(x) g(x) \mathbb{P}_{X_T}(dx).$$

We deduce that the optimal payoff h^e is given by:

$$h^e = J(\lambda g), \tag{7}$$

where λ is the scalar Lagrange multiplier such that

$$V_0 = \int_{I\!R^{+3}} J(\lambda g(x)) g(x) \mathbb{P}_{X_T}(dx).$$

Suppose for example that there exist only cash and environmental assets. Then, the properties of the optimal payoff h^* as function of the benchmark I can be analyzed. Since the utility function U is concave, the marginal utility U' is decreasing, then J is also decreasing, from which we deduce:

Corollary 3 *The function h^* is an increasing function of the environmental I_T if and only f the conditional expectation g of $\frac{d\mathbb{Q}}{d\mathbb{P}}$ under the σ-algebra generated by I_T is a decreasing function of I_T. More precisely: assume that g is differentiable. From the optimality conditions, the derivative of the optimal payoff is given by:*

$$h'(i) = \left(-\frac{U'(h(i))}{U''(h(i))} \right) \times \left(-\frac{g(i)'}{g(i)} \right).$$

Remark 4 *In most cases, g is decreasing. Introduce the tolerance of risk $T_o(h(i))$ equal to the inverse of the absolute risk-aversion:*

$$T_o(h(i)) = -\frac{U'(h(i))}{U''(h(i))}$$

As it can be seen, $h'(i)$ depends on the tolerance of risk. The design of the optimal payoff can also be specified. Denote $Y(i) = -\frac{g'(i)}{g(i)}$. Assuming that g is twice-differentiable, we have:

$$h''(i) = [X'(h(i)) + \frac{Y'(i)}{Y(i)^2}] \times [X(h(i))Y^2(i)]. \tag{8}$$

Differentiating twice with respect to i, and from previous corollary, we deduce the result. Therefore, usually, higher is the tolerance of risk, higher is h"(i).

2.2.2 The insured portfolio

This section presents a generalization of Prigent (2006) to the case of two stochastic assets. We assume now that the investor wants a specific guarantee. This one can be required to get an additional insurance against risk or to reach a sufficiently high environmental quality level. Such guarantee can be modelled by letting a function h_0 defined on the possible values of the assets S_T and I_T. Whatever the value of $X_T = (S_T,\ I_T)$, the investor wants to get a final portfolio value above the floor $h_0(X_T)$. For instance, if h_0 is linear, with $h_0(s,i) = (a_s s + b_s, a_i i + b_i,)$, then, when the asset Y falls, the investor is sure of getting at least b (equal to a fixed percentage of his initial investment) and if the asset Y rises, he make profits out of the rises at a percentage a.

The general solution The optimal payoff with insurance constraints on the terminal wealth is solution of the following problem:

$$\begin{aligned} & Max_h \mathbb{E}_{\mathbb{P}}[U(h(X_T)] && (9)\\ V_0 &= \mathbb{E}_{\mathbb{P}}[h(X_T)M_T] \\ h(X_T) &\geq h_0(X_T) \end{aligned}$$

As it can be seen, the initial investment V_0 must be higher than $\mathbb{E}_{\mathbb{P}}[h_0(X_T)M_T]$ if the insurance constraint must be satisfy. To solve this optimization problem, introduce the sets

$$K_1 = \{h \in \mathbb{L}^2(\mathbb{R}^{+3}, \mathbb{P}_{X_T}) | V_0 = \mathbb{E}_{\mathbb{P}}[h(X_T)M_T]\},$$

and

$$K_2 = \{h \in \mathbb{L}^2(\mathbb{R}^{+3}, \mathbb{P}_{X_T}) | h \geq h_0\}.$$

The set $K = K_1 \cap K_2$ is a convex set of $\mathbb{L}^2(\mathbb{R}^{+3}, \mathbb{P}_{X_T})$. Consider the following indicator function of K, denoted by δ_K and defined by:

$$\delta_K(h) = \begin{cases} 0 & if \quad h \in K, \\ +\infty & if \quad h \notin K. \end{cases}$$

Since K is closed and convex, δ_K is lower semi-continuous and convex. Denote by $\partial\delta_K$ the subdifferential of δ_K (see for example Ekeland and Turnbull(1983) for definition and properties of subdifferentials)). The optimization problem is equivalent to:

$$Max_h \left(\mathbb{E}[U(h(X_T)] - \delta_K(h)\right) \tag{10}$$

The optimality conditions leads to (similar proof as in Prigent (2006)):

Proposition 5 *There exists a scalar λ_c and a function h_c defined on $L^2(I\!R^{+3}, P_{X_T})$ such that:*

$$h^* = J(\lambda_c g + h_c), \tag{11}$$

where λ_c is solution of :

$$y?, V_0 = \int_0^\infty J[yg(x) + h_c(x)]g(x)f(x)dx. \tag{12}$$

and $h_c \in \partial\delta_{K_2}(h^*)$

The optimal payoff h^* can be determined by introducing the unconstrained optimal payoff h^e associated to the modified coefficient λ_c (i.e. $h^e = J(\lambda_c g)$). λ_c can also be considered as a Lagrange multiplier associated to a non insured optimal portfolio but with a modified initial wealth. When h^e is greater than the hedging floor h_0, then $h^* = h^e$. Otherwise, $h^* = h_0$. Indeed, the optimal payoff is given by:

$$h^* = Max(h_0, h^e). \tag{13}$$

Generally, as mentioned previously, h_0 is increasing and h^e also. Therefore, the optimal payoff is an increasing function of the components of $X_T = (S_T,\, I_T)$.

To illustrate such results in the one-dimensional case, assume for instance that the utility function of the individual is a CRRA utility and there exist only two assets: the cash and the environmental asset.

$$U(i) = \frac{i^\alpha}{\alpha},$$

with $0 < \alpha < 1$ from which we deduce $J(i) = i^{\frac{1}{\alpha-1}}$.

Suppose that the interest rate r is constant and that the environmental asset value $(I_t)_t$ follows a geometric Brownian motion. Then, $(I_t)_t$ is given by:

$$I_t = I_0 \exp\left[(\mu - 1/2\sigma^2)t + \sigma W_t\right].$$

Introduce the following parameters:

$$\theta = \frac{\mu - r}{\sigma}, \ A = -\frac{1}{2}\theta^2 T + \frac{\theta}{\sigma}(\mu - \frac{1}{2}\sigma^2)T, \psi = e^A (I_0)^{\frac{\theta}{\sigma}}, \text{ and } \kappa = \frac{\theta}{\sigma}.$$

Recall that in this framework, the conditional expectation g of $\frac{d\mathbb{Q}}{d\mathbb{P}}$ under the σ-algebra generated by I_T is given by:

$$g(i) = \psi i^{-\kappa}.$$

Therefore, $h^e(i)$ satisfies:

$$h^e(i) = d \times i^m \text{ with } \quad d = c\psi^{\frac{1}{\alpha-1}} \text{ and } \ m = \frac{\kappa}{1-\alpha} > 0. \tag{14}$$

Then, if there is no insurance constraint, the optimal payoff is given by:

$$h^e(i) = \frac{V_0 e^{rT}}{\int_0^\infty g(i)^{\frac{\alpha}{\alpha-1}} f(i)di} \times g(i)^{\frac{1}{\alpha-1}}. \tag{15}$$

If the insurance constraint is required then the optimal payoff must be solution of

$$\begin{aligned} & Max_h \mathbb{E}\left[\frac{(h(I_T))^\alpha}{\alpha}\right] \\ V_0 \quad = \quad & e^{-rT}\mathbb{E}[h(I_T)] \text{ and } h(I_T) \geq h_0(I_T) \end{aligned} \tag{16}$$

Then:

Proposition 6 *The optimal payoff with guarantee is given by:*

$$h^* = (\lambda_c g + h_c)^{\frac{1}{\alpha-1}}, \tag{17}$$

where λ_c is solution of:

$$y?, V_0 e^{rT} = \int_0^\infty [yg(i) + h_c(i)]^{\frac{1}{\alpha-1}} g(i) f(i)di, \tag{18}$$

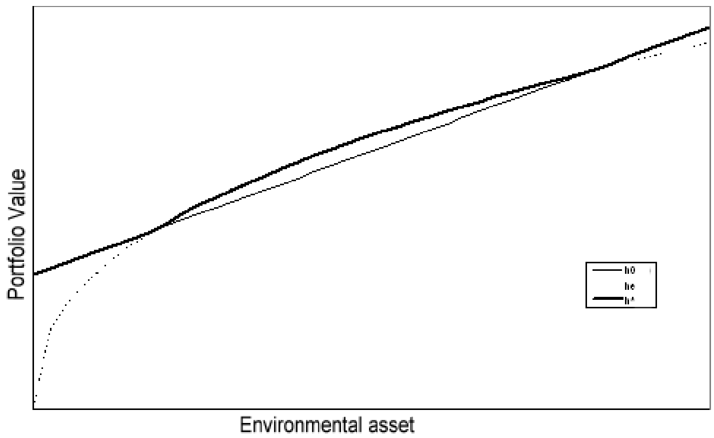
and h_c is a negative function satisfying the property of previous corollary. Assume as usual that h_0 is increasing and continuous, then the optimal payoff is an increasing continuous function of the benchmark at maturity.

Corollary 7 *If there is no insurance constraint, the concavity/convexity of the optimal payoff is determined by the comparison between the risk-aversion and the ratio* $\kappa = \frac{\mu - r}{\sigma^2}$ *which is the Sharpe ratio divided by the volatility* σ*.*
i) h^e *is concave if* $\kappa < 1 - \alpha$*.*
ii) h^e *is linear if* $\kappa = 1 - \alpha$*.*
iii) h^e *is convex if* $\kappa > 1 - \alpha$*.*

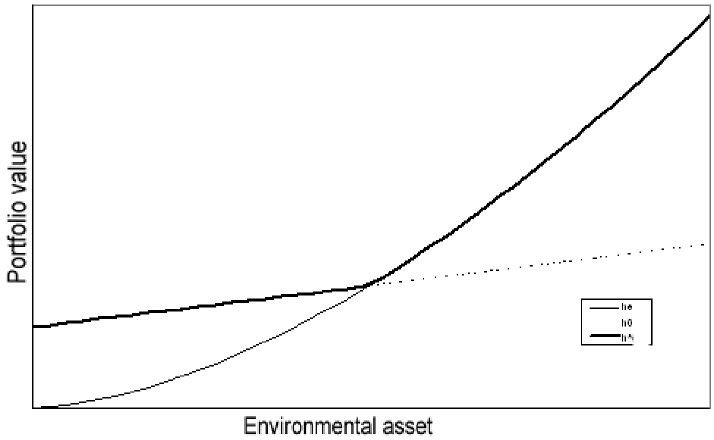
Remark 8 *As it can be seen, the graph of the optimal payoff changes from concavity to convexity according to the increase of the risk-aversion of the individual. If for example, the insurance constraint is linear* ($h_0(i) = ai + b$)*, it looks like the unconstrained case's one, except when* h^* *is equal to the constraint* h_0*.*

Consider the dynamically complete case where the environmental asset value follows the usual geometric Brownian motion. The concavity/convexity of the optimal payoff depends on the comparison between the Sharpe type ratio $\frac{\mu - r}{\sigma^2}$ and the risk aversion ($1 - \alpha$).

If $\kappa < 1 - \alpha$, h^{**} is concave.



If $\kappa > 1 - \alpha$, h^{**} is convex.



Remark 9 *We can also consider a restricted case: a portfolio with cash and an environmental asset, and only a finite number of options written on it. In that case, if the guaranteed payoff is linear, the optimal (polygonal) payoff is still convave/convex according to the degree of risk aversion.*

3 Basic examples

All the previous properties are illustrated in the following examples.

In what follows, we assume that the diffusion coefficients of the asset prices are linear functions.

- The cash value is defined by:

$$\frac{dC_t}{C_t} = rdt, \tag{19}$$

where the instantaneous riskless interest rate r is constant.

- The stock index and the environmental investment are given by:

$$\frac{dS_t}{S_t} = (r + \theta_S)dt + \sigma_1 dW_{1,t} + \sigma_2 dW_{2,t}. \tag{20}$$

$$\frac{dI_t}{I_t} = (r + \theta_I)dt + \delta_1 dW_{1,t} + \delta_2 dW_{2,t}, \tag{21}$$

where W_1 and W_2 are two independent standard Brownian motions, and where σ_1, σ_2, δ_1, δ_2 are positive constants. The parameter θ_S is the constant risk premium of the stock, and θ_I is the constant risk premium of the environmental investment.

From Equation (30), the stock price is equal to:

$$S_t = S_0 \exp\left[\left(r + \theta_S - \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2\right)\right)t + \sigma_1 W_{1,t} + \sigma_2 W_{2,t}\right],$$

and, from Equation (21), the environmental asset has the following value:

$$I_t = I_0 \exp\left[\left(r + \theta_I - \frac{1}{2}\left(\delta_1^2 + \delta_2^2\right)\right) t + \delta_1 W_{1,t} + \delta_2 W_{2,t}\right].$$

Since the market is complete, there exists a unique risk-neutral probability $\mathbb{Q}$ associated to two market risk premia, λ_1 and λ_2, with density η with respect to the initial probability $\mathbb{P}$ given by:

$$\eta_t = \exp\left[-\left(\lambda_1 W_{1,t} + \lambda W_{2,t}\right) - 1/2(\lambda_1^2 + \lambda_2^2)t\right].$$

The premia λ_1 and λ_2 are determined from the relation:

$$\begin{aligned} \theta_S &= \sigma_1\lambda_1 + \sigma_2\lambda_2, \\ \theta_I &= \delta_1\lambda_1 + \delta_2\lambda_2. \end{aligned}$$

Denote by d the determinant of the previous linear system:

$$d = \sigma_1\delta_2 - \sigma_2\delta_1.$$

Lemma 10 *Introduce the functions A_t and B_t defined by:*

$$\begin{aligned} A_t &= \left(r + \theta_S - \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2\right)\right) t, \\ B_t &= \left(r + \theta_I - \frac{1}{2}\left(\delta_1^2 + \delta_2^2\right)\right) t. \end{aligned}$$

Therefore, we have:

$$\begin{aligned} \ln\left[\frac{S_t}{S_0}\right] - A_t &= \sigma_1 W_{1,t} + \sigma_2 W_{2,t}, \\ \ln\left[\frac{I_t}{I_0}\right] - B_t &= \delta_1 W_{1,t} + \delta_2 W_{2,t}. \end{aligned}$$

We deduce:

$$\begin{aligned} W_{1,t} &= \frac{1}{d}\left[\delta_2\left(\ln\left[\frac{S_t}{S_0}\right] - A_t\right) - \sigma_2\left(\ln\left[\frac{I_t}{I_0}\right] - B_t\right)\right], \\ W_{2,t} &= \frac{1}{d}\left[-\delta_1\left(\ln\left[\frac{S_t}{S_0}\right] - A_t\right) + \sigma_1\left(\ln\left[\frac{I_t}{I_0}\right] - B_t\right)\right]. \end{aligned}$$

Thus, the process η is given by:

$$\begin{aligned} \eta_t &= \exp\left[-\left(\lambda_1 W_{1,t} + \lambda_2 W_{2,t}\right) - 1/2(\lambda_1^2 + \lambda_2^2)t\right] \\ &= \left[\frac{S_t}{S_0}\right]^{\left(\frac{-\lambda_1\delta_2 + \lambda_2\delta_1}{d}\right)} \left[\frac{I_t}{I_0}\right]^{\left(\frac{\lambda_1\sigma_2 - \lambda_2\sigma_1}{d}\right)} \\ &\quad \exp\left[A_t\left(\frac{\lambda_1\delta_2 - \lambda_2\delta_1}{d}\right) + B_t\left(\frac{-\lambda_1\sigma_2 + \lambda_2\sigma_1}{d}\right)\right] \\ &\quad \exp\left[-1/2(\lambda_1^2 + \lambda_2^2)t\right]. \end{aligned}$$

3.1 Example 1 (global optimization)

Consider an investor having a utility function U defined on the amount $A^{C,S}$ invested on the financial assets and on the environmental amount A^I.

The investor's utility U is supposed to be increasing, concave and twice-differentiable with respect to its two arguments. Assume also that both marginal utilities $\frac{\partial U(.,y)}{\partial x}$ and $\frac{\partial U(x,.)}{\partial y}$ satisfy: $\forall(x,y)$,

$$\begin{aligned} \lim_{x\to o+}\frac{\partial U\left(.,y\right)}{\partial x} &= +\infty \text{ and } \lim_{x\to+\infty}\frac{\partial U\left(.,y\right)}{\partial x}=0, \\ \lim_{y\to o+}\frac{\partial U\left(x,.\right)}{\partial y} &= +\infty \text{ and } \lim_{y\to+\infty}\frac{\partial U\left(x,.\right)}{\partial y}=0. \end{aligned} \tag{22}$$

Denote by $J_x(.,A_T^{I*})$ and $J_y(A_T^{C,S*},.)$ the inverse of the marginal utilities $\frac{\partial U\left(x,A_T^{I*}\right)}{\partial x}$ and $\frac{\partial U\left(A_T^{C,S*},y\right)}{\partial y}$, respectively.

Proposition 11 *Under previous assumptions on the utility function and the stock and environmental assets, the optimal portfolio solution* $\left(A_T^{C,S},A_T^I\right)^*$ *satisfies the two following relations:*

$$A_T^{C,S}=J_x(a\eta_T,A_T^{I*}) \text{ and } A_T^I=J_y\left(A_T^{C,S*},a\eta_T\right).$$

If $U(x,y)=\frac{x^\alpha}{\alpha}\frac{y^\beta}{\beta}$ *with* $0<\alpha<1$, $0<\beta<1$ *and* $0<\alpha+\beta<1$, *then* V_T^* *is a function of asset values* S_T *and* I_T *given by:*

$$V_T^*=V_0\times S_T^{\left(\frac{-\lambda_1\delta_2+\lambda_2\delta_1}{d(\alpha+\beta-1)}\right)}I_T^{\left(\frac{\lambda_1\sigma_2-\lambda_2\sigma_1}{d(\alpha+\beta-1)}\right)}\times f(T),$$

where $f(.)$ *is a deterministic function.*

Proof. We have to solve the following optimization problem:

$$\mathrm{Max}_\theta\ \mathbb{E}_{\mathbb{P}}[U(A_T^{C,S},A_T^I)] \text{ under } A_T^{C,S}\geq a_T^{C,S}\text{and } A_T^I\geq a_T^I.$$

The global investment value is given by:

$$V_T=A_T^{C,S}+A_T^I.$$

Due to market completeness, this problem is equivalent to:

$$\mathrm{Max}_{\left(A_T^{C,S},A_T^I\right)}\ \mathbb{E}_{\mathbb{P}}[U(A_T^{C,S},A_T^I)] \tag{23}$$

$$\text{under } V_0 = e^{-rT}\mathbb{E}_{\mathbb{Q}}[V_T]. \tag{24}$$

1) Consider the solution $\left(A_T^{C,S**},A_T^{I**}\right)$ of the free problem (without terminal constraint). This solution must satisfy:

$$\frac{\partial U\left(A_T^{C,S**},A_T^{I*}\right)}{\partial x}=a\eta_T=\frac{\partial U\left(A_T^{C,S*},A_T^{I*}\right)}{\partial y},$$

where the Lagrangian parameter a is such that $V_0^* = e^{-rT}\mathbb{E}_{\mathbb{Q}}[V_T^*]$.

Then, $\left(A_T^{C,S*}, A_T^{I*}\right)$ is solution of the following system:

$$\begin{aligned} A_T^{C,S*} &= J_x(a\eta_T, A_T^{I*}), \\ A_T^{I*} &= J_y\left(A_T^{C,S*}, a\eta_T\right). \end{aligned}$$

For the special case $U(x,y) = \frac{x^\alpha}{\alpha}\frac{y^\beta}{\beta}$, we deduce:

$$\begin{aligned} \frac{\partial U(x,y)}{\partial x} &= x^{\alpha-1}\frac{y^\beta}{\beta}, \ \frac{\partial U(x,y)}{\partial y} = \frac{x^\alpha}{\alpha}y^{\beta-1}, \\ & \text{and} \\ J_x(u,y) &= \left(\frac{\beta u}{y^\beta}\right)^{\frac{1}{\alpha-1}}, \ J_y(x,v) = \left(\frac{\alpha v}{x^\alpha}\right)^{\frac{1}{\beta-1}}. \end{aligned}$$

Therefore, we have:

$$A_T^{C,S*} = (a\eta_T)^{\frac{1}{\alpha+\beta-1}}\,\alpha^{\frac{\beta}{\alpha+\beta-1}}\beta^{\frac{1-\beta}{\alpha+\beta-1}}, \tag{25}$$

$$A_T^{I*} = (a\eta_T)^{\frac{1}{\alpha+\beta-1}}\,\alpha^{\frac{1-\alpha}{\alpha+\beta-1}}\beta^{\frac{\alpha}{\alpha+\beta-1}}. \tag{26}$$

The initial investment condition is equivalent to:

$$V_0 = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[(a\eta_T)^{\frac{1}{\alpha+\beta-1}}\left(\alpha^{\frac{\beta}{\alpha+\beta-1}}\beta^{\frac{1-\beta}{\alpha+\beta-1}} + \alpha^{\frac{1-\alpha}{\alpha+\beta-1}}\beta^{\frac{\alpha}{\alpha+\beta-1}}\right)\right],$$

which implies that the Lagrangian multiplier b is given by:

$$a = \left(\frac{V_0 e^{rT}}{\left(\alpha^{\frac{\beta}{\alpha+\beta-1}}\beta^{\frac{1-\beta}{\alpha+\beta-1}} + \alpha^{\frac{1-\alpha}{\alpha+\beta-1}}\beta^{\frac{\alpha}{\alpha+\beta-1}}\right)\mathbb{E}_{\mathbb{P}}\left[\eta_T^{1+\frac{1}{\alpha+\beta-1}}\right]}\right)^{(\alpha+\beta-1)},$$

with

$$\mathbb{E}_{\mathbb{P}}\left[\eta_T^{1+\frac{1}{\alpha+\beta-1}}\right] = \exp\left[\frac{1}{2}\frac{\alpha+\beta}{(\alpha+\beta-1)^2}\left(\lambda_1^2+\lambda_2^2\right)T\right].$$

Therefore, the portfolio value is equal to

$$V_T^* = (a\eta_T)^{\frac{1}{\alpha+\beta-1}}\left(\alpha^{\frac{\beta}{\alpha+\beta-1}}\beta^{\frac{1-\beta}{\alpha+\beta-1}} + \alpha^{\frac{1-\alpha}{\alpha+\beta-1}}\beta^{\frac{\alpha}{\alpha+\beta-1}}\right).$$

Consequently,

$$V_T^* = V_0 \times \left(\frac{S_T}{S_0}\right)^{\left(\frac{-\lambda_1\delta_2+\lambda_2\delta_1}{d(\alpha+\beta-1)}\right)}\left(\frac{I_T}{I_0}\right)^{\left(\frac{\lambda_1\sigma_2-\lambda_2\sigma_1}{d(\alpha+\beta-1)}\right)} \times f(T),$$

with

$$f(T) = \exp\left[rT + A_T\left(\frac{\lambda_1\delta_2 - \lambda_2\delta_1}{d(\alpha+\beta-1)}\right) + B_T\left(\frac{-\lambda_1\sigma_2 + \lambda_2\sigma_1}{d(\alpha+\beta-1)}\right)\right] \times \exp\left[-\frac{(2\alpha+2\beta-1)(\lambda_1^2+\lambda_2^2)}{2(\alpha+\beta-1)^2}T\right]. \quad (27)$$

■

Remark 12 *Both optimal investments* $A_T^{C,S*}$ *and* A_T^{I*} *(respectively on* C, S *and* I*) are functions of* S_T *and* I_T*. This is due to the dynamic strategy which is based on a simultaneous trade on both the financial and environmental assets. At any time, the investor can rebalance his portfolio by transferring any amount from one asset to the other one.*

Remark 13 *The optimal initial amounts are given by*

$$A_0^{C,S*} = e^{-rT}\mathbb{E}_{\mathbb{Q}}[A_T^{C,S*}] = e^{-rT}\left(\alpha^{\frac{\beta}{\alpha+\beta-1}}\beta^{\frac{1-\beta}{\alpha+\beta-1}}\right)\left(a^{\frac{1}{\alpha+\beta-1}}\mathbb{E}_{\mathbb{P}}[\eta_T^{\frac{\alpha+\beta}{\alpha+\beta-1}}]\right),$$

$$A_0^{I*} = e^{-rT}\mathbb{E}_{\mathbb{Q}}[A_T^{I*}] = e^{-rT}\left(\alpha^{\frac{1-\alpha}{\alpha+\beta-1}}\beta^{\frac{\alpha}{\alpha+\beta-1}}\right)\left(a^{\frac{1}{\alpha+\beta-1}}\mathbb{E}_{\mathbb{P}}[\eta_T^{\frac{\alpha+\beta}{\alpha+\beta-1}}]\right),$$

with

$$a^{\frac{1}{\alpha+\beta-1}} = \frac{V_0 e^{rT}\mathbb{E}_{\mathbb{P}}[\eta_T^{\frac{\alpha+\beta}{\alpha+\beta-1}}]}{\left(\alpha^{\frac{\beta}{\alpha+\beta-1}}\beta^{\frac{1-\beta}{\alpha+\beta-1}} + \alpha^{\frac{1-\alpha}{\alpha+\beta-1}}\beta^{\frac{\alpha}{\alpha+\beta-1}}\right)\mathbb{E}_{\mathbb{P}}[\eta_T^{\frac{\alpha+\beta}{\alpha+\beta-1}}]},$$

Then, we have:

$$A_0^{C,S*} = V_0\frac{\alpha}{\alpha+\beta} \text{ and } A_0^{I*} = V_0\frac{\beta}{\alpha+\beta}. \quad (28)$$

Therefore, both optimal weights $A_0^{C,S*}/V_0$ *and* A_0^{I*}/V_0 *are respectively equal to the ratios defined from the exponents of the Cobb-Douglas utility* $\frac{\alpha}{\alpha+\beta}$ *and* $\frac{\beta}{\alpha+\beta}$*. Additionally, the ratio* $A_0^{I*}/A_0^{C,S*}$ *is equal to* $\frac{\beta}{\alpha}$*.*

- The ratio of the optimal amounts at maturity $A_T^{C,S*}$ *and* A_T^{I*} *also is equal to* $\frac{\beta}{\alpha}$*.*

Remark 14 *Assume that the utility function of the individual is only defined on the portfolio value* V_T *(selfish materialist). Under the previous assumptions on the stock and environmental assets, the optimal portfolio* V_T^* *is solution is given by:*

$$V_T^* = J(b\eta_T),$$

where J *is the inverse of the marginal utility and* b *is the Lagrangian parameter associated to the budget condition:*

$$V_0 = e^{-rT}\mathbb{E}_{\mathbb{Q}}[V_T^*].$$

If $U(x) = \frac{x^\alpha}{\alpha}$ with $0 < \alpha < 1$ (CRRA utility), then V_T^* is a function of asset values S_T and I_T given by:

$$V_T^* = V_0 \times \left(\frac{S_T}{S_0}\right)^{\left(\frac{-\lambda_1\delta_2+\lambda_2\delta_1}{d(\alpha-1)}\right)} \left(\frac{I_T}{I_0}\right)^{\left(\frac{\lambda_1\sigma_2-\lambda_2\sigma_1}{d(\alpha-1)}\right)} \times g(T),$$

where $g(.)$ is a deterministic function defined by:

$$\begin{aligned} g(T) &= \exp\left[rT + A_T\left(\frac{\lambda_1\delta_2-\lambda_2\delta_1}{d(\alpha-1)}\right) + B_T\left(\frac{-\lambda_1\sigma_2+\lambda_2\sigma_1}{d(\alpha-1)}\right)\right] \\ &\quad \times \exp\left[-1/2\frac{(2\alpha-1)(\lambda_1^2+\lambda_2^2)}{(\alpha-1)^2}T\right]. \end{aligned}$$

If hedging constraints are introduced, the optimization problem is:

$$\begin{aligned} & \operatorname{Max}_{(A_T^{C,S},A_T^I)} \mathbb{E}_{\mathbb{P}}[U(A_T^{C,S}, A_T^I)] \qquad (29) \\ \text{under } A_T^{C,S} &\geq a_T^{C,S} \text{ and } A_T^I \geq a_T^I, \\ & \text{with} \\ V_0 &= e^{-rT}\mathbb{E}_{\mathbb{Q}}[V_T] \geq e^{-rT}\mathbb{E}_{\mathbb{Q}}[a_T^{C,S} + a_T^I]. \end{aligned}$$

Proposition 15 *The solution of Problem (29) is given by (see Relation (13)):*

$$\begin{aligned} A_T^{C,S*c} &= Max\left[(a\eta_T)^{\frac{1}{\alpha+\beta-1}}\alpha^{\frac{\beta}{\alpha+\beta-1}}\beta^{\frac{1-\beta}{\alpha+\beta-1}}, a_T^{C,S}\right], \\ & \textit{and} \\ A_T^{I*c} &= Max\left[(a\eta_T)^{\frac{1}{\alpha+\beta-1}}\alpha^{\frac{1-\alpha}{\alpha+\beta-1}}\beta^{\frac{\alpha}{\alpha+\beta-1}}, a_T^I\right]. \end{aligned}$$

Remark 16 *The two previous optimal amounts are call options written on the same underlying asset which is the product of two powers of both the financial and environmental assets $S_T^{\left(\frac{-\lambda_1\delta_2+\lambda_2\delta_1}{d(\alpha+\beta-1)}\right)} I_T^{\left(\frac{\lambda_1\sigma_2-\lambda_2\sigma_1}{d(\alpha+\beta-1)}\right)}$. The "strikes" are respectively the hedge amounts $a_T^{C,S}$ and a_T^I. More precisely, we have:*

$$\begin{aligned} A_T^{C,S*c} &= a_T^{C,S} + Max\left[S_T^{\left(\frac{-\lambda_1\delta_2+\lambda_2\delta_1}{d(\alpha+\beta-1)}\right)} I_T^{\left(\frac{\lambda_1\sigma_2-\lambda_2\sigma_1}{d(\alpha+\beta-1)}\right)} f(T)\frac{\alpha}{\alpha+\beta} - a_T^{C,S}, 0\right], \\ & \textit{and} \\ A_T^{I*c} &= a_T^I + Max\left[S_T^{\left(\frac{-\lambda_1\delta_2+\lambda_2\delta_1}{d(\alpha+\beta-1)}\right)} I_T^{\left(\frac{\lambda_1\sigma_2-\lambda_2\sigma_1}{d(\alpha+\beta-1)}\right)} f(T)\frac{\beta}{\alpha+\beta} - a_T^I, 0\right]. \end{aligned}$$

3.2 Example 2 (separate optimization)

Assume now that investments on the risky financial asset S and on the environmental asset I cannot be substituted. At initial date, the investor splits his endowment V_0 into two parts: the first one is invested into a financial portfolio; the second one is used to trade on the environmental asset.

Assets S and I are assumed to be driven by two independent Brownian motions W_1 and W_2:

$$\frac{dS_t}{S_t} = (r + \theta_S)dt + \sigma_1 dW_{1,t}. \tag{30}$$

$$\frac{dI_t}{I_t} = (r + \theta_I)dt + \delta_2 dW_{2,t}, \tag{31}$$

where σ_1 and δ_2 are non negative constants.

Since the two markets are complete, there exist two unique risk-neutral probability $\mathbb{Q}_1$ and $\mathbb{Q}_2$ associated to two market risk premia, $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$, with density η_1 and η_2 with respect to the initial probability $\mathbb{P}$ given by:

$$\begin{aligned} \eta_{1,t} &= \exp\left[-\widetilde{\lambda}_1 W_{1,t} - 1/2\widetilde{\lambda}_1^2 t\right], \\ \eta_{2,t} &= \exp\left[-\widetilde{\lambda}_2 W_{2,t} - 1/2\widetilde{\lambda}_2^2 t\right], \end{aligned}$$

where the premia λ_1 and λ_2 are determined from the relation:

$$\theta_S = \sigma_1 \widetilde{\lambda}_1 \text{ and } \theta_I = \delta_2 \widetilde{\lambda}_2.$$

Denote by $\mathbb{E}_i\left[X(W_1, W_2)\right]$ the expectation with respect to the Brownian motion W_i :

$$\begin{aligned} \mathbb{E}_1\left[X(W_1, W_2)\right] &= \int X(W_1, W_2)dW_1, \\ \mathbb{E}_2\left[X(W_1, W_2)\right] &= \int X(W_1, W_2)dW_2. \end{aligned}$$

Denote respectively by $\widetilde{J}_x(., A_T^{I**})$ and $\widetilde{J}_y\left(A_T^{C,S**}, .\right)$ the inverse of the expected marginal utilities $\mathbb{E}_2\left[\frac{\partial U\left(x, A_T^{I**}\right)}{\partial x}\right]$ and $\mathbb{E}_1\left[\frac{\partial U\left(A_T^{C,S**}, y\right)}{\partial y}\right]$.

Proposition 17 *(no hedging constraint) Under previous assumptions on the stock and environmental assets, the optimal portfolio solution is a function of asset values S and I given by:*

$$A_T^{C,S**} = \widetilde{J}_x(a_1\eta_{1,T}, A_T^{I**}) \text{ and } A_T^{I**} = \widetilde{J}_y\left(A_T^{C,S**}, a_2\eta_{2,T}\right).$$

If $U(x,y)=\frac{x^\alpha}{\alpha}\frac{y^\beta}{\beta}$ with $0<\alpha<1$ and $0<\beta<1$, then V_T^{**} is a function of asset values S_T and I_T given by:

$$V_T^{**}=S_T^{\left(\frac{\widetilde{\lambda}_1}{\sigma_1(1-\alpha)}\right)}f_{A_1}(T)+I_T^{\left(\frac{\widetilde{\lambda}_2}{\delta_2(1-\beta)}\right)}f_{A_2}(T),$$

where $f_1(.)$ and $f_2(.)$ are two deterministic functions.

Proof. The global investment value is given by:

$$V_T=A_T^{C,S}+A_T^I.$$

Due to market completeness, this problem is equivalent to:

$$\begin{aligned}
& \mathrm{Max}_{(A_0^I,A_T^{C,S},A_T^I)}\ \mathbb{E}_{\mathbb{P}}[U(A_T^{C,S},A_T^I)] && (32)\\
& \text{under} && (33)\\
A_0^{C,S} \quad = \quad & e^{-rT}\mathbb{E}_{\mathbb{Q}_1}[A_T^{C,S}] \text{ and } A_0^I=e^{-rT}\mathbb{E}_{\mathbb{Q}_2}[A_T^I]. && (34)
\end{aligned}$$

1) First step: For fixed initial investment A_0^I, consider the solution $\left(A_T^{C,S**},A_T^{I**}\right)$ of the free problem (without guarantee constraint). This solution must satisfy:

$$\begin{aligned}
\mathbb{E}_2\left[\frac{\partial U\left(A_T^{C,S**},A_T^{I**}\right)}{\partial x}\right] &= a_1\eta_{1,T},\\
\mathbb{E}_1\left[\frac{\partial U\left(A_T^{C,S**},A_T^{I**}\right)}{\partial y}\right] &= a_2\eta_{2,T},
\end{aligned}$$

where a_1 and a_2 are the Lagrangian parameters associated to the two budget constraints.

Then, $\left(A_T^{C,S},A_T^I\right)^{**}$ is solution of the following system:

$$\begin{aligned}
A_T^{C,S**} &= \widetilde{J}_x(a_1\eta_{1,T},A_T^{I**}),\\
A_T^{I**} &= \widetilde{J}_y\left(A_T^{C,S**},a_2\eta_{2,T}\right).
\end{aligned}$$

For the special case $U(x,y)=\frac{x^\alpha}{\alpha}\frac{y^\beta}{\beta}$, note that, by independence,

$$\mathbb{E}\left[U\left(A_T^{C,S},A_T^I\right)\right]=\mathbb{E}_1\left[\frac{\left(A_T^{C,S}\right)^\alpha}{\alpha}\right]\mathbb{E}_2\left[\frac{\left(A_T^I\right)^\beta}{\beta}\right].$$

Thus, the functions $\widetilde{J}_x$ and $\widetilde{J}_y$ satisfy:

$$\widetilde{J}_x(u)=\left(\frac{u}{\mathbb{E}_2\left[\frac{(A_T^I)^\beta}{\beta}\right]}\right)^{\frac{1}{\alpha-1}},\ \widetilde{J}_y(x)=\left(\frac{v}{\mathbb{E}_1\left[\frac{(A_T^{C,S})^\alpha}{\alpha}\right]}\right)^{\frac{1}{\beta-1}}.$$

Note also that, for this special case, we can separately maximize $\mathbb{E}_1\left[\frac{\left(A_T^{C,S}\right)^\alpha}{\alpha}\right]$ and $\mathbb{E}_2\left[\frac{\left(A_T^{I}\right)^\beta}{\beta}\right]$ for given initial investments (respectively $V_0 - A_0^I$ and A_0^I).

Therefore, we deduce:

$$A_T^{C,S**} = \left(a_1 \eta_{1,T}\right)^{\frac{1}{\alpha-1}} \text{ and } A_T^{I**} = \left(a_2 \eta_{2,T}\right)^{\frac{1}{\beta-1}},$$

where the two Lagrangian parameters are determined from conditions:

$$V_0 - A_0^I = e^{-rT}\mathbb{E}_{\mathbb{Q}_1}\left[A_T^{C,S**}\right] \text{ and } A_0^I = e^{-rT}\mathbb{E}_{\mathbb{Q}_2}\left[A_T^{I**}\right].$$

We have:

$$a_1 = \left(\frac{\left(V_0 - A_0^I\right)e^{rT}}{\exp(\frac{1}{2}\frac{\alpha}{(\alpha-1)^2}T)}\right)^{\alpha-1} \text{ and } a_2 = \left(\frac{A_0^I e^{rT}}{\exp(\frac{1}{2}\frac{\beta}{(\beta-1)^2}T)}\right)^{\beta-1}$$

Introduce the functions $\widetilde{A}_t$ and $\widetilde{B}_t$ defined by:

$$\begin{aligned} \widetilde{A}_t &= \left(r + \theta_S - \frac{1}{2}\sigma_1^2\right)t, \\ \widetilde{B}_t &= \left(r + \theta_I - \frac{1}{2}\delta_2^2\right)t. \end{aligned}$$

Therefore, we have:

$$\begin{aligned} \ln\left[\frac{S_t}{S_0}\right] - \widetilde{A}_t &= \sigma_1 W_{1,t}, \\ \ln\left[\frac{I_t}{I_0}\right] - \widetilde{B}_t &= \delta_2 W_{2,t}. \end{aligned}$$

We deduce:

$$\begin{aligned} W_{1,t} &= \frac{1}{\sigma_1}\left(\ln\left[\frac{S_t}{S_0}\right] - \widetilde{A}_t\right), \\ W_{2,t} &= \frac{1}{\delta_2}\left(\ln\left[\frac{I_t}{I_0}\right] - \widetilde{B}_t\right). \end{aligned}$$

Thus, the process η_1 and η_2 are given by:

$$\begin{aligned} \eta_{1,t} &= \exp\left[-\widetilde{\lambda}_1 W_{1,t} - 1/2\widetilde{\lambda}_1^2 t\right] = \left[\frac{S_t}{S_0}\right]^{\left(\frac{-\widetilde{\lambda}_1}{\sigma_1}\right)} \exp\left[\widetilde{A}_t \frac{\widetilde{\lambda}_1}{\sigma_1} - 1/2\widetilde{\lambda}_1^2 t\right], \\ \eta_{2,t} &= \exp\left[-\widetilde{\lambda}_2 W_{2,t} - 1/2\widetilde{\lambda}_2^2 t\right] = \left[\frac{I_t}{I_0}\right]^{\left(\frac{-\widetilde{\lambda}_2}{\delta_2}\right)} \exp\left[\widetilde{B}_t \frac{\widetilde{\lambda}_2}{\delta_2} - 1/2\widetilde{\lambda}_2^2 t\right]. \end{aligned}$$

These relations imply:

$$A_T^{C,S**} = \left[\frac{S_t}{S_0}\right]^{\left(\frac{\tilde{\lambda}_1}{\sigma_1(1-\alpha)}\right)} f_{A_1}(T) \text{ and } A_T^{I**} = \left[\frac{I_t}{I_0}\right]^{\left(\frac{\tilde{\lambda}_2}{\delta_2(1-\beta)}\right)} f_{A_2}(T), \quad (35)$$

with

$$f_{A_1}(T) = a_1^{\frac{1}{\alpha-1}} \exp\left[(\alpha-1)\left(\widetilde{A}_t \frac{\widetilde{\lambda}_1}{\sigma_1} - 1/2\widetilde{\lambda}_1^2 t\right)\right], \quad (36)$$

$$f_{A_2}(T) = a_2^{\frac{1}{\beta-1}} \exp\left[(\beta-1)\left(\widetilde{B}_t \frac{\widetilde{\lambda}_2}{\delta_2} - 1/2\widetilde{\lambda}_2^2 t\right)\right]. \quad (37)$$

Therefore, V_T^{**} is a function of the asset values S_T and I_T which is given by:

$$\begin{aligned} V_T^{**} &= A_T^{C,S**} + A_T^{I**}, \\ &= S_T^{\left(\frac{\tilde{\lambda}_1}{\sigma_1(1-\alpha)}\right)} f_{A_1}(T) + I_T^{\left(\frac{\tilde{\lambda}_2}{\delta_2(1-\beta)}\right)} f_{A_2}(T), \end{aligned}$$

2) Second step: we must choose A_0^I such that $\mathbb{E}_{\mathbb{P}}[U(A_T^{C,S}, A_T^I)^{**}]$ is maximal. Since we have

$$A_T^{C,S**} = \left(a_1\eta_{1,T}\right)^{\frac{1}{\alpha-1}} \text{ and } A_T^{I**} = \left(a_2\eta_{2,T}\right)^{\frac{1}{\beta-1}},$$

we deduce that the indirect utility function is equal to:

$$\mathbb{E}\left[U\left(A_T^{C,S}, A_T^I\right)^{**}\right] = a_1^{\frac{\alpha}{\alpha-1}} a_2^{\frac{\beta}{\beta-1}} \mathbb{E}_1\left[\frac{\eta_{1,T}^{\frac{\alpha}{\alpha-1}}}{\alpha}\right] \mathbb{E}_2\left[\frac{\eta_{2,T}^{\frac{\beta}{\beta-1}}}{\beta}\right].$$

Since $\mathbb{E}_1\left[\frac{\eta_{1,T}^{\frac{\alpha}{\alpha-1}}}{\alpha}\right]$ and $\mathbb{E}_2\left[\frac{\eta_{2,T}^{\frac{\beta}{\beta-1}}}{\beta}\right]$ are positive and do not depend on A_0^I, we have to maximize $a_1^{\frac{\alpha}{\alpha-1}} a_2^{\frac{\beta}{\beta-1}}$ with respect to A_0^I.

Note that:

$$a_1^{\frac{\alpha}{\alpha-1}} = \left(\frac{\left(V_0 - A_0^I\right)e^{rT}}{\exp\left(\frac{1}{2}\frac{\alpha}{(\alpha-1)^2}T\right)}\right)^{\alpha} \text{ and } a_2^{\frac{\beta}{\beta-1}} = \left(\frac{A_0^I e^{rT}}{\exp\left(\frac{1}{2}\frac{\beta}{(\beta-1)^2}T\right)}\right)^{\beta}.$$

Thus, the optimization problem is equivalent to the maximization of the function $\left(V_0 - A_0^I\right) f^{\alpha}\left(A_0^I\right)^{\beta}$. Then, the optimal solution $\left(V_0 - A_0^I, A_0^I\right)^{**}$ is given by:

$$A_0^{C,S**} = \left(V_0 - A_0^I\right)^{**} = \frac{\alpha}{\alpha+\beta}V_0 \text{ and } A_0^{I**} = \frac{\beta}{\alpha+\beta}V_0. \quad (38)$$

■

Remark 18 *- For the Cobb-Douglas utility function, both optimal initial weights $A_0^{C,S**}/V_0$ and A_0^{I**}/V_0 are respectively equal to the ratios $A_0^{C,S*}/V_0$ and A_0^{I*}/V_0 of Example 1.*

- However, contrary to Example 1, the ratio of the optimal amounts at maturity $A_T^{C,S**}$ and A_T^{I**} is no longer constant. It is a random variable involving a ratio of powers of financial and environmental asset values.

Remark 19 *The optimal portfolio value is an increasing function of both asset values S_T and I_T. It is the sum of two power functions with exponents respectively equal to $\frac{\theta_S}{(\sigma_1)^2}\frac{1}{(1-\alpha)}$ and $\frac{\theta_I}{(\delta_2)^2}\frac{1}{(1-\beta)}$. Thus, the concavity/convexity with respect to these values is determined from the comparison of the Sharpe type ratios $\frac{\theta_S}{(\sigma_1)^2}$ and $\frac{\theta_I}{(\delta_2)^2}$ with the relative risk aversions $(1-\alpha)$ and $(1-\beta)$.*

Remark 20 *Examine Example 1 when the correlation between the financial and the environmental assets is equal to 0. We have $\sigma_2 = 0$ and $\delta_1 = 0$.*

The optimal solution is given by:

$$V_T^{**} = V_0 \times \left(\frac{S_T}{S_0}\right)^{\left(\frac{\lambda_1}{\sigma_1(1-\alpha-\beta)}\right)} \left(\frac{I_T}{I_0}\right)^{\left(\frac{\lambda_2}{\delta_2(1-\alpha-\beta)}\right)} f(T),$$

where $f(.)$ is the deterministic function defined in (27):

$$\begin{aligned} f(T) &= \exp\left[rT + A_T\left(\frac{-\lambda_1}{\sigma_1(1-\alpha-\beta)}\right) + B_T\left(\frac{-\lambda_2}{\delta_2(1-\alpha-\beta)}\right)\right] \\ &\quad \times \exp\left[-(1/2)\frac{(2\alpha+2\beta-1)(\lambda_1^2+\lambda_2^2)}{(\alpha+\beta-1)^2}T\right]. \end{aligned}$$

Note that $\lambda_1 = \widetilde{\lambda}_1$ and $\lambda_2 = \widetilde{\lambda}_2$. We also have $A_T = \widetilde{A}_T$ and $B_T = \widetilde{B}_T$.

Therefore, V_T^{**} is a product of powers of S_T and I_T contrary to the separate case where it is a weighted sum of powers of these two variables.

This difference is due to the constraint in Example 2 where arbitrage between the financial and the environmental assets is forbidden. Due to this restriction, the indirect utility function $\mathbb{E}_{\mathbb{P}}[U(A_T^{C,S}, A_T^I)^{**}]$ (second case) is smaller than $\mathbb{E}_{\mathbb{P}}[U(A_T^{C,S}, A_T^I)^{*}]$ (first case).

When hedging constraints are introduced, we have to solve the following optimization problem:

$$\begin{aligned} & \mathrm{Max}_{(A_0^I, A_T^{C,S}, A_T^I)}\ \mathbb{E}_{\mathbb{P}}[U(A_T^{C,S}, A_T^I)] && (39) \\ \text{under } A_T^{C,S} &\geq a_T^{C,S} \text{and } A_T^I \geq a_T^I, \\ & \text{and} && (40) \\ A_0^{C,S} &= e^{-rT}\mathbb{E}_{\mathbb{Q}_1}[A_T^{C,S}] \geq e^{-rT}\mathbb{E}_{\mathbb{Q}_1}[a_T^{C,S}], && (41) \\ A_0^I &= e^{-rT}\mathbb{E}_{\mathbb{Q}_2}[A_T^I] \geq e^{-rT}\mathbb{E}_{\mathbb{Q}_2}[a_T^I]. && (42) \end{aligned}$$

Using results such as Relation (13) and Relation (35), we deduce:

Proposition 21 *The solution of Problem (39) is given by:*

$$\begin{aligned} A_T^{C,S**c} &= Max\left[\left[\frac{S_t}{S_0}\right]^{\left(\frac{\widetilde{\lambda}_1}{\sigma_1(1-\alpha)}\right)} f_{A_1}(T), a_T^{C,S}\right], \\ &\quad and \\ A_T^{I**c} &= Max\left[\left[\frac{I_t}{I_0}\right]^{\left(\frac{\widetilde{\lambda}_2}{\delta_2(1-\beta)}\right)} f_{A_2}(T), a_T^{I}\right], \end{aligned}$$

where a_1 and a_2 are Lagrangian parameters associated to budget constraints, taking account of hedging constraints.

Remark 22 *The optimal solutions are call options with "strikes" $a_T^{C,S}$ and a_T^I since we have:*

$$A_T^{C,S**c} = a_T^{C,S} + Max\left[\left[\frac{S_t}{S_0}\right]^{\left(\frac{\widetilde{\lambda}_1}{\sigma_1(1-\alpha)}\right)} f_{A_1}(T) - a_T^{C,S}, 0\right],$$

and

$$A_T^{I**c} = a_T^{I} + Max\left[\left[\frac{I_t}{I_0}\right]^{\left(\frac{\widetilde{\lambda}_2}{\delta_2(1-\beta)}\right)} f_{A_2}(T) - a_T^{I}, 0\right].$$

For constant guaranteed amounts $a_T^{C,S}$ and a_T^I, they are usual call power options, respectively written on the financial and environmental assets.

To illustrate the previous results, consider the case of non correlated financial and environmental assets. This excludes, for example, climate changes insofar they are a consequence of economic activity.

For the global optimization, the optimal amounts respectively invested on the financial and environmental assets are given by:

$$A_T^{C,S*} = V_0 \times \left(\frac{\alpha}{\alpha+\beta}\right) \times \left[\frac{S_T}{S_0}\right]^{\left(\frac{\widetilde{\lambda}_1}{\sigma_1(1-\alpha-\beta)}\right)} \left[\frac{I_T}{I_0}\right]^{\left(\frac{\widetilde{\lambda}_2}{\delta_2(1-\alpha-\beta)}\right)} f_A(T), \tag{43}$$

$$A_T^{I*} = V_0 \times \left(\frac{\beta}{\alpha+\beta}\right) \times \left[\frac{S_T}{S_0}\right]^{\left(\frac{\widetilde{\lambda}_1}{\sigma_1(1-\alpha-\beta)}\right)} \left[\frac{I_T}{I_0}\right]^{\left(\frac{\widetilde{\lambda}_2}{\delta_2(1-\alpha-\beta)}\right)} f_A(T), \tag{44}$$

$$\begin{aligned} \text{with} \quad f_A(T) &= \exp\left[rT + \left(\frac{1}{\alpha+\beta-1}\right)\left(\widetilde{A}_T\left(\frac{\widetilde{\lambda}_1}{\sigma_1}\right) + \widetilde{B}_T\left(\frac{\widetilde{\lambda}_2}{\delta_2}\right)\right)\right] \\ &\quad \times \exp\left[-(1/2)\frac{(2\alpha+2\beta-1)\,(\widetilde{\lambda}_1^2 + \widetilde{\lambda}_2^2)}{(\alpha+\beta-1)^2}T\right]. \end{aligned}$$

For the separate optimization, they are given by:

$$
\begin{aligned}
A_T^{C,S**} &= V_0 \times \left(\frac{\alpha}{\alpha+\beta}\right) \times \left[\frac{S_T}{S_0}\right]^{\left(\frac{\widetilde{\lambda}_1}{\sigma_1(1-\alpha)}\right)} f_{A_1}(T) && (45)\\
&\text{and}\\
A_T^{I**} &= V_0 \times \left(\frac{\beta}{\alpha+\beta}\right) \times \left[\frac{I_T}{I_0}\right]^{\left(\frac{\widetilde{\lambda}_2}{\delta_2(1-\beta)}\right)} f_{A_2}(T), && (46)\\
&\text{with}\\
f_{A_1}(T) &= \exp\left[rT + \widetilde{A}_T \frac{\widetilde{\lambda}_1}{\sigma_1(\alpha-1)} - 1/2 \frac{(2\alpha-1)\,\widetilde{\lambda}_1^2}{(\alpha-1)^2} T\right],\\
f_{A_2}(T) &= \exp\left[rT + \widetilde{B}_T \frac{\widetilde{\lambda}_2}{\delta_2(\beta-1)} - 1/2 \frac{(2\beta-1)\,\widetilde{\lambda}_2^2}{(\beta-1)^2} T\right].
\end{aligned}
$$

As seen in Relations (43,44) and (45,46), the powers of financial and environmental assets $\frac{\widetilde{\lambda}_1}{\sigma_1(1-\alpha-\beta)}$ and $\frac{\widetilde{\lambda}_2}{\delta_2(1-\alpha-\beta)}$ for the global optimization are higher than those for the separate optimization $\frac{\widetilde{\lambda}_1}{\sigma_1(1-\alpha)}$ and $\frac{\widetilde{\lambda}_2}{\delta_2(1-\beta)}$, since we have $\frac{1}{(1-\alpha)} < \frac{1}{(1-\alpha-\beta)}$ and $\frac{1}{(1-\beta)} < \frac{1}{(1-\alpha-\beta)}$.

This implies that the first solution is more sensitive to both asset fluctuations around their initial values S_0 and I_0. For high asset returns, the optimal amounts $A_T^{C,S*}$ and A_T^{I*} (global optimization) are higher than the corresponding ones, $A_T^{C,S**}$ and A_T^{I**} for the separate optimization. For small asset returns, it is the converse. Such results hold when same hedging constraints are introduced for both optimization constraints such as predetermined constant amounts at a given horizon.

Consider a numerical base case with the following parameter values:

$$
\begin{aligned}
V_0 &= 100, C_0 = 1, S_0 = 10, I_0 = 10, r = 2\%, \theta_S = 5\%, \theta_I = 2\%,\\
\sigma_1 &= 20\%,\ \delta_2 = 20\%.
\end{aligned}
$$

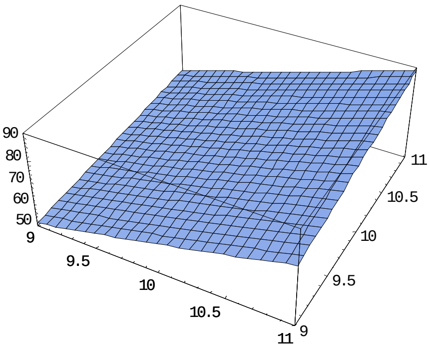
Note that these values are standard for financial markets, and, for the environmental asset, they correspond for example to emission fees.

We also assume that the parameters of the Cobb-Douglas utility function are given by:

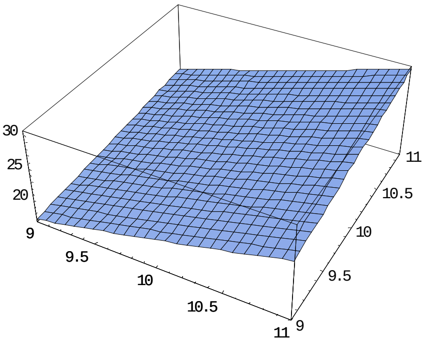
$$\alpha = 0.3 \text{ and } \beta = 0.1$$

They satisfy the two conditions: $0 < \alpha < 1, 0 < \beta < 1, \alpha + \beta < 1$. Since $\beta \neq 0$, the individual is sensitive to environmental risk. However, $\alpha = 3\beta$, which means that the individual is more sensitive to the purely financial asset.

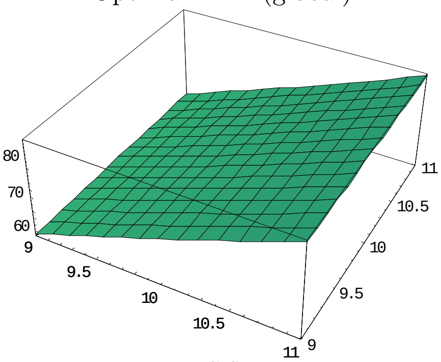
We illustrate the comparison for asset variations of $+/-10\%$.



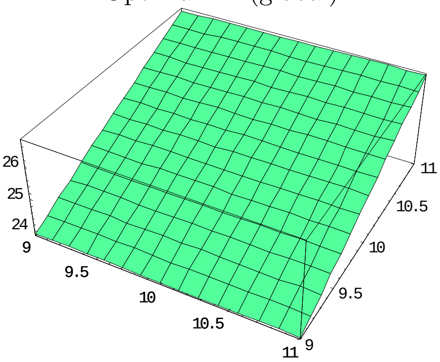
Optimal $A^{C,S}$ (global)



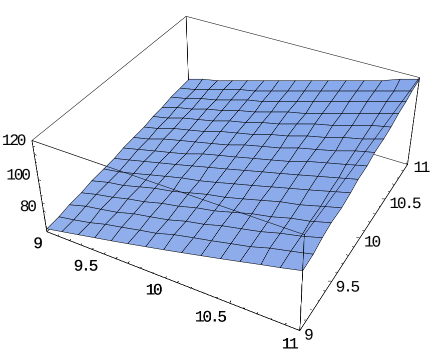
Optimal A^{I} (global)



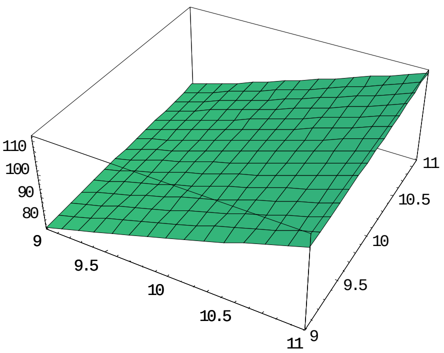
Optimal $A^{C,S}$ (separate)



Optimal A^{I} (separate)



Whole investment (global)



Whole investment (separate)

4 Conclusion

Using the expected utility theory, the optimal investment under financial and environmental hedging constraints have been determined for a large class of models. In particular, the results suggest that environmental derivative assets have to be introduced in order to maximize the expected utility of individuals. The optimal solution clearly depends on the risk aversion and on the hedging condition. The optimal portfolio is determined for quite general utility functions and insurance constraints. It has been also illustrated with the Cobb-Douglas utility functions. The concavity/convexity of the investment profile is determined from the level of risk aversion and from asset performances, for example a Sharpe type ratio. In this example, we have assumed the case of non correlated financial and environmental assets. The set-up that we have developed could allow to take such (important) correlation into account, and consider the choice of economic development in conjunction with a choice of the current and future quality of the environment.

As a matter of fact, the hedging constraints envisaged on the terminal invested amounts are quite general and can be applied to a large variety of practical cases. This will be a topic for future research.

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