Optimal holding period
In Real Estate Portfolio

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Abstract

This paper considers the use of simulated cash flows to determine the optimal holding period in real estate portfolio to maximize its present value. The traditional DCF approach with an estimation of the resale value through a growth rate of the future cash flow does not let appear this optimum. However, if the terminal value is calculated from the trend of a diffusion process of the price, an optimum may appear under certain conditions. Finally we consider the sensitivity of the optimal holding period to the different parameters involved in the cash flow estimations. This methodology may be applied in commercial valuation and enables to get an optimal holding period for a given portfolio.

Key words: valuation, DCF, optimal holding period, commercial property

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Introduction

In a precedent paper (2005) we proposed to use dynamic cash flows for the rents inflows and for the terminal value in a real estate portfolio. These dynamics are supposed to be simple diffusion processes where the corresponding parameters are the trend and volatility, respectively for the rents and for the price. These parameters may be estimated from a rent index and a real estate index using Paris data, taking into account the empirical correlation between theses two indices. This approach lets appear an interesting rule played by the holding duration in the determination of the asset value.

Recent studies on real estate portfolios show the growing interest of the investors to determine an optimal holding period that optimizes the portfolio value. Obviously, it depends on a lot of factors: market conditions, costs of transactions, type of property, assets volatility, etc…

This issue has been studied for a long time in the literature relating to stocks (see Demsetz 1968, or Tinic 1972). Evidence suggests that transaction costs influence holding periods. More precisely, Amihud and Mendelson (1986) show that assets with high bid-ask spreads (which are usually a proxy for high transaction costs) would be held, in equilibrium, by investors who expect to hold the assets for a long time. Atkin and Dyl (1997) in an empirical research consider the effects of firm size, bid-ask spread and volatility of returns on the holding period of stocks for a sample of 2000 Nasdaq firms and 500-1100 Nyse firms over the period 1981-1993. They show a significant positive correlation between holding period and transaction costs and firm size, and a negative one between holding period and price variability.

Concerning real estate holding periods, research is more limited. For the US, Hendershott and Ling (1984), Gau and Wang (1994) or Fisher and Young (2000) argue the holding durations are principally conditioned by tax laws. For the UK market, the relationship between returns and holding period seems to be complex. Rowley, Gibson and Ward (1996) in a study realized from investors interviews show that investors or new property developers tend to have a holding period in mind from the start. Their conclusion is that for offices, the holding period decision is related with depreciation or obsolescence factors. However, for retail property, the decision is more empirical and would depend on active management as well as the state of the market.

In a more recent article, Collett, Lizieri and Ward (2003) underline how the knowledge of the holding period is important in the decision to invest in commercial real estate portfolios. Investment appraisal requires specifying an analysis period and the asset allocation depends on the variances and covariances of assets that are affected by the reference interval or analysis. Using the database of properties provided by IPD in the UK over an 18-year period they conclude from an empirical analysis that the median holding period is about seven years. But sales rates vary across the holding period (probably for rent cycles and lease structures reasons) and the holding period vary by property type. The larger or more expensive the properties are, the longer the holding periods. And if the return is greater, the holding period is lower. However, they were not able to propose conclusions about a possible link between volatility and holding period, because of an absence of proxy to measure this eventual relationship.

For small residential investment, Brown and Geurts (2005) gave an empirical response to the two following questions: how long does an investor own an apartment building and why do investors sell some property more frequently than others? Through a sample of apartment buildings of between 5 and 20 units over the period 1970-1990 in the city of San Diego, they found that the average holding period is around five years. They also deduced that investors sell property sooner when values rise faster than rent.

In an another article, Brown (2004) shows that considering the risk peculiar to real estate investments may explain the reasons for owning real estate by private investors and their buy-sell behavior. However, applying the CAPM for individuals to understand their portfolio management does not drive to relevant results as demonstrated by Geltner and Miller (2001).

In this paper, our purpose is to determine if there is an optimal holding duration of a real estate portfolio if the terminal value is computed using a growth rate for the prices. Firstly, we analyze the optimal holding period issue with the DCF method. Secondly, we determine the optimal holding duration for a terminal value trend. And finally we explicit the impact of other parameters on this optimal holding period through a sensitivity analysis.

1 Optimal holding period in DCF

In this first part, the holding period for a portfolio of real estate assets is considered in the traditional DCF framework.

Let us denote k as the weighted average cost of capital (WACC) used to discount the different free cash flows FCF_t, and P_T as the terminal value. We assume that the free cash flow of the terminal year T (the last year of the investment horizon) is used to estimate the terminal value and is supposed to continue growing at a constant rate forever after that date (g_∞). Consequently, the terminal value P_T of the asset is:

$$P_T = \frac{FCF_T\left(1+g_\infty\right)}{\left(k-g_\infty\right)\left(1+k\right)^T} \tag{1.1}$$

Then $P_{0,T}$ the present value of the asset sold at date T may be computed as:

$$P_{0,T} = \sum_{t=1}^{T} \frac{FCF_t}{\left(1+k\right)^t} + \frac{FCF_t\left(1+g_\infty\right)}{\left(k-g_\infty\right)\left(1+k\right)^T} \tag{1.2}$$

If g is the growth rate of the free cash flows, all the free cash flows in (1.2) may be calculated from FCF_1. Then the expression (1.2) becomes:

$$P_{0,T} = \sum_{t=1}^{T} \frac{FCF_1\left(1+g\right)^{t-1}}{\left(1+k\right)^t} + \frac{FCF_1\left(1+g\right)^{T-1}\left(1+g_\infty\right)}{\left(k-g_\infty\right)\left(1+k\right)^T} \tag{1.3}$$

In order to study the function $P_{0,T}$ as a function of T, we consider the function $P_{0,T+1} - P_{0,T}$. From (1.3) we deduce the following present value for the asset at date T+1:

$$P_{0,T+1} = \sum_{t=1}^{T+1} \frac{FCF_1\left(1+g\right)^{t-1}}{\left(1+k\right)^t} + \frac{FCF_1\left(1+g\right)^{T}\left(1+g_\infty\right)}{\left(k-g_\infty\right)\left(1+k\right)^{T+1}} \tag{1.4}$$

which can be expressed as a function of $P_{0,T}$ by considering separetely the last cash flow FCF_{T+1} and by substraying the discounted terminal value of $P_{0,T}$:

$$P_{0,T+1} = P_{0,T} + \frac{FCF_1\left(1+g\right)^T}{\left(1+k\right)^{T+1}} + \frac{FCF_1\left(1+g\right)^{T}\left(1+g_\infty\right)}{\left(k-g_\infty\right)\left(1+k\right)^{T+1}} - \frac{FCF_1\left(1+g\right)^{T-1}\left(1+g_\infty\right)}{\left(k-g_\infty\right)\left(1+k\right)^{T}} \tag{1.5}$$

After calculation (see appendix A.1), we obtain by discounting separately the first free cash flow:

$$P_{0,T+1} - P_{0,T} = FCF_1 \frac{\left(1+g\right)^{T-1}}{\left(1+k\right)^T}\left(\frac{g-g_\infty}{k-g_\infty}\right) \tag{1.6}$$

The sign of the right part of equation (1.6) corresponds to the sign of $g - g_\infty$.We have then the following situations:

- If $g > g_\infty$ then $P_{0,T+1} - P_{0,T} > 0$
- If $g = g_\infty$ then $P_{0,T+1} - P_{0,T} = 0$
- If $g < g_\infty$ then $P_{0,T+1} - P_{0,T} < 0$

Moreover, as $k > g$, the function $(1+g)/(1+k)$ is less than one and

$$\frac{(1+g)^{T-1}}{(1+k)^T} \tag{1.7}$$

is decreasing with *T*, which implies that

$$\lim_{T\to\infty}\left(P_{0,T+1} - P_{0,T}\right) = 0 \tag{1.8}$$

Hence, in the DCF approach, if we suppose a constant growth rate from the first period to infinity: $g = g_\infty$, and a constant discount rate, no optimal detention period can be detected. In fact, in this case, the terminal value which could seem to be separated from the cash flows is actually $P_{T+1,\infty}$ and the portfolio present value is in reality $P_{0,\infty}$:

$$P_{0,\infty} = \sum_{t=1}^{\infty}\frac{FCF_t}{(1+k)^t} = \frac{FCF_1}{1+k}\sum_{t=0}^{\infty}\left(\frac{1+g}{1+k}\right)^t = \frac{FCF_1}{1+k}\frac{1+k}{k-g} = \frac{FCF_1}{k-g} \tag{1.9}$$

which can be expressed by taking into account the break at date *T* in the valuation process:

$$\begin{aligned}
P_{0,\infty} &= \sum_{t=1}^{T}\frac{FCF_t}{(1+k)^t} + \sum_{t=T+1}^{\infty}\frac{FCF_t}{(1+k)^t} = \frac{FCF_1}{1+k}\sum_{t=0}^{\infty}\left(\frac{1+g}{1+k}\right)^{t-1} + \frac{1}{(1+k)^{T+1}}\sum_{t=0}^{\infty}FCF_{T+1}\left(\frac{1+g}{1+k}\right)^{t-1}\\
P_{0,\infty} &= \frac{FCF_1}{1+k}\sum_{t=0}^{T}\left(\frac{1+g}{1+k}\right)^{t-1} + \frac{P_{T+1,\infty}}{(1+k)^{T+1}}
\end{aligned} \tag{1.10}$$

This clearly illustrates the reason why the valuation is constant whatever is *T* in the case of a unique growth rate. With a constant discount rate, to get a non-constant present value according to the break at date *T,* it is necessary to consider two growth rates: one for the cash flow for the first *T* periods (g) and one after *T*+1 (g_∞). Then relation (1.10) becomes, by denoting $P_{0,\infty}^T$ the present value with a break at date T:

$$P_{0,\infty}^T = \frac{FCF_1}{1+k}\sum_{t=0}^{\infty}\left(\frac{1+g}{1+k}\right)^{t-1} + \frac{1}{(1+k)^{T+1}}\sum_{t=0}^{\infty}FCF_{T+1}\left(\frac{1+g_\infty}{1+k}\right)^{t-1} \tag{1.11}$$

The result in (1.6) shows that the function $P_{0,\infty}^T$ is:

- a concave and monotonic increasing function of *T* when $g > g_\infty$
- a constant function when $g = g_\infty$
- a convex and monotonic decreasing or increasing function of *T* when $g < g_\infty$

For instance, with $k = 8.40\%$, $g = 4\%$, $g_\infty = 3\%$ and $FCF_1 = 1$, Figure 1 illustrates the monotonous character of the function in the DCF approach. Figure 2 corresponds to a case where the loss in the terminal value is exactly balanced by the gain in cash flow: $g = g_\infty$. Figure 3 is obtained with $g_\infty = 4.5\%$.

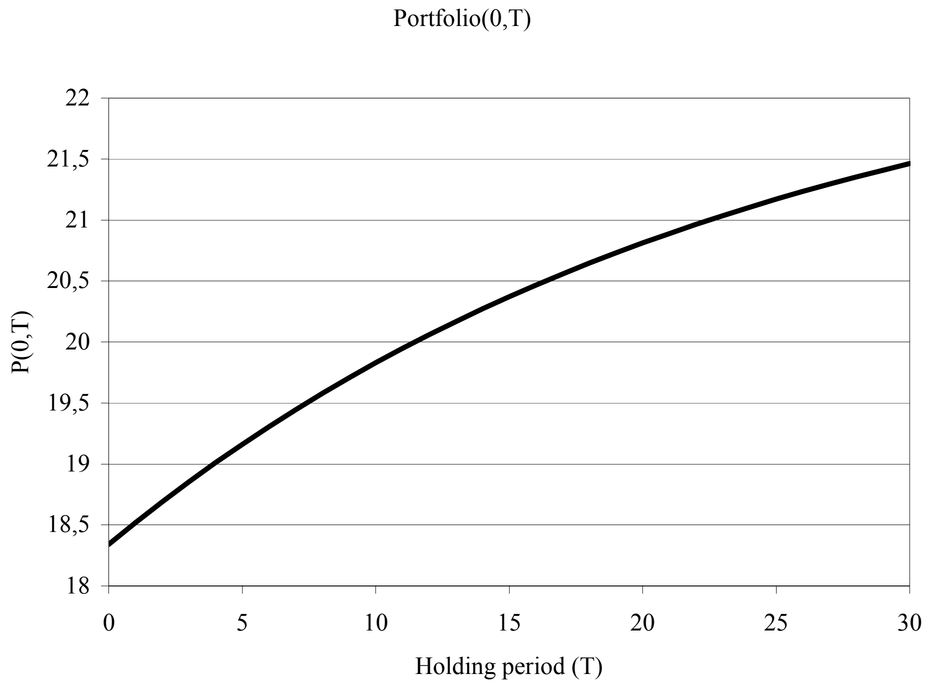

Figure 1: *Increase of the portfolio present value with the DCF approach ($g > g_\infty$)*

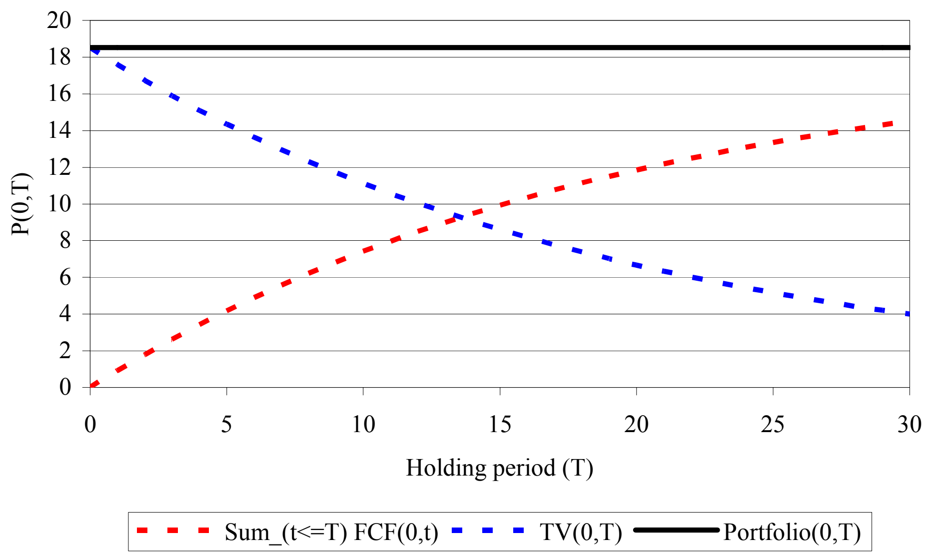

Figure 2: *Constant present value of a portfolio with the DCF approach ($g = g_\infty$)*

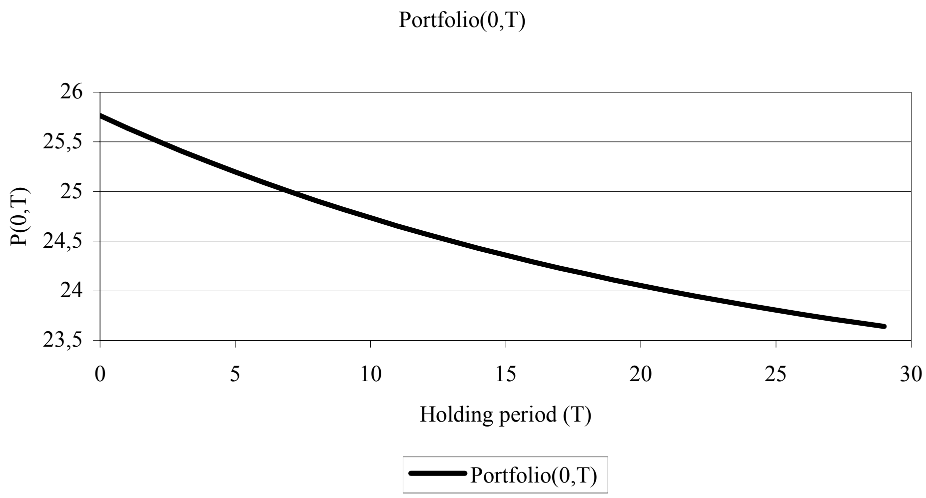

Figure 3: Decrease of the portfolio present value with the DCF approach ($g < g_\infty$)

We can conclude that the traditional DCF framework cannot let appear an optimal holding value for a portfolio, according to the asset present value, whatever the rates of expected growth are.

2 Determination of an optimal holding period using the terminal value trend

Baroni and al. (2005) have proposed the use of Monte Carlo simulation method in valuation and their main contribution is the modeling of the terminal value. They consider that the real estate price of the assets follows a geometric Brownian motion:

$$\frac{dP_t}{P_t} = \mu_P \, dt + \sigma_P \, dW_t \tag{1.12}$$

This equation assumes that real estate returns can be modeled as a simple diffusion process where parameters μ_P and σ_P are the trend and volatility. According to the modeling presented in (1.12), the expected return of the asset at time T is $e^{\mu T}$, which represents the trend. We propose this modeling to improve the DCF method in order to let appear an optimal detention period. To compare this new approach with the discrete case derived in section one, let us denote $\mu = e^{\mu}$-1.

First, we determine the optimal solution in this new approach and then we analyze it according the different parameters of interest.

2.1 Optimal solution

The expected present value of the asset sold at date T is:

$$\mathrm{E}\left(P_{0,T}\right) = \sum_{t=1}^{T} \frac{FCF_t}{(1+k)^t} + \frac{P_0 (1+\mu)^T}{(1+k)^T} \tag{1.13}$$

Then

$$\mathrm{E}\left(P_{0,T+1} - P_{0,T}\right) = \frac{FCF_{T+1}}{(1+k)^{T+1}} + \frac{P_0 (1+\mu)^{T+1}}{(1+k)^{T+1}} - \frac{P_0 (1+\mu)^T}{(1+k)^T} \tag{1.14}$$

$$\mathrm{E}\left(P_{0,T+1}-P_{0,T}\right)=\frac{FCF_1\left(1+g\right)^T}{\left(1+k\right)^{T+1}}+\frac{P_0\left(1+\mu\right)^{T+1}}{\left(1+k\right)^{T+1}}-\frac{P_0\left(1+\mu\right)^T}{\left(1+k\right)^T} \tag{1.15}$$

$$\left(1+k\right)^{T+1}\mathrm{E}\left(P_{0,T+1}-P_{0,T}\right)=FCF_1\left(1+g\right)^T+P_0\left(1+\mu\right)^T\left[\left(1+\mu\right)-\left(1+k\right)\right] \tag{1.16}$$

$\mathrm{E}\left(P_{0,T+1}-P_{0,T}\right)$ is composed of two different components, the first one associated to the free cash flows, and the second one associated to the terminal value:

$$\mathrm{E}\left(P_{0,T+1}-P_{0,T}\right)=\frac{1}{\left(1+k\right)^{T+1}}\left[FCF_1\left(1+g\right)^T+P_0\left(1+\mu\right)^T\left(\mu-k\right)\right] \tag{1.17}$$

Let us notice that if $\mu=k$, the expected difference does not depend on P_0, but only on the T+1[th] free cash flow:

$$\mathrm{E}\left(P_{0,T+1}-P_{0,T}\right)=FCF_0\left(\frac{1+g}{1+k}\right)^{T+1} \tag{1.18}$$

Hence, in this case, there is no optimal holding period.

We will now consider that $\mu\neq k$. We can deduce from (1.16) :

$$\mathrm{E}\left(P_{0,T+1}-P_{0,T}\right)>0\Leftrightarrow\frac{FCF_1}{P_0\left(k-\mu\right)}>\left(\frac{1+\mu}{1+g}\right)^T \tag{1.19}$$

$$\mathrm{E}\left(P_{0,T+1}-P_{0,T}\right)<0\Leftrightarrow\frac{FCF_1}{P_0\left(k-\mu\right)}<\left(\frac{1+\mu}{1+g}\right)^T \tag{1.20}$$

As long as $\mathrm{E}\left(P_{0,T+1}-P_{0,T}\right)>0$, see relation (1.19), there is no reason to sell the asset, the gain associated to the T+1[th] free cash flows is larger than the decrease of the discounted terminal value. When $\mathrm{E}\left(P_{0,T+1}-P_{0,T}\right)<0$, see relation (1.20), the situation is inversed. We deduct that an optimal sell date, when such a date exists, corresponds to[1]:

$$T^*=\frac{\ln\left(\dfrac{FCF_1}{P_0\left(k-\mu\right)}\right)}{\ln\left(\dfrac{1+\mu}{1+g}\right)} \tag{1.21}$$

in the cases where the logarithm function can be defined (the argument should be positive)

$$\frac{FCF_1}{P_0\left(k-\tilde{\mu}\right)}>0\Leftrightarrow\mu<k \tag{1.22}$$

$$\frac{1+\mu}{1+g}>0\Leftarrow\mu>-1\text{ and }g>-1 \tag{1.23}$$

and the optimum is a maximum (the function $P_{0,T}$ is concave ; see appendix A.2)

[1] If we consider the continuous solution (let us remind that T is an integer).

$$\begin{cases} \mathrm{E}\left(P_{0,T+2}-P_{0,T+1}\right)\text{-}\mathrm{E}\left(P_{0,T+1}-P_{0,T}\right)<0 \\ \qquad P_{0,T+2}-P_{0,T+1}>0 \qquad\qquad \Rightarrow g<\mu \text{ if } T^*>0 \\ \qquad P_{0,T+1}-P_{0,T}>0 \end{cases} \tag{1.24}$$

It is obvious that the mathematical restriction (1.23) does not constitute a restriction in practice. On the contrary, the restriction (1.22) is linked to the asset characteristics as shown in subsection 3.3.

2.2 Solution analysis

The existence of an optimal detention period comes from the fact the discounted portfolio value is the sum of two components, one increasing over time (the sum of the discounted free cash flows), and the other decreasing (the discounted terminal value). If the two components are increasing simultaneously no optimal detention period can be determined. The discounted terminal value is a positive function of the time if and only if $\mu > k$. Using the result in equation (1.18), our analysis focuses on the cases where $\mu < k$, as mentioned in (1.22) for a mathematical reason. Where $\mu < k$, the difference on a resale at time T or T+1, is simply a "loss" on the discounted terminal value (loss because $\mu < k$) balanced or not by a new free cash flow (the T+1$^{\text{th}}$).

If the previous condition allows convergence, there is no guarantee on the optimum sign. Indeed, if the gain in free cash flows is not high enough, the loss in the terminal value may never be compensated. This relation is valid in the cases where the free cash flow growth rate is smaller than the real estate return, see (1.24). This constraint corresponds to the fact that the numerator of T^* in (1.21) must be positive:

$$\ln\left(\frac{FCF_1}{P_0\left(k-\mu\right)}\right)>0 \Leftrightarrow \frac{FCF_1}{P_0\left(k-\mu\right)}>1 \Leftrightarrow \alpha>\left(k-\mu\right) \tag{1.25}$$

The higher this difference $(k-\mu)$ is, the larger the constraint (1.25). A large loss on the discounted terminal value may be balanced by a higher free cash flow.

Notice 1: if (1.25) is satisfied, the concavity issue brings the same constraint on the parameters than the positive character of the denominator of T^* in (1.21):

$$\ln\left(\frac{1+\mu}{1+g}\right)>0 \Leftrightarrow \mu>g \tag{1.26}$$

Then these constraints can be expressed as a function of μ:

a) $\mu < k$: necessary for an optimal holding period T^* (else T^* is infinite)

b) $\mu > \max\left(g, k-\alpha\right)$, from (1.24) and (1.25), which ensures a positive maximum T^*

These constraints are summed up as:

$$\max\left(g,k-\alpha\right)<\mu<k \tag{1.27}$$

The following example illustrates the optimal solution in the case where (1.27) is satisfied. With $k = 8.40\%$, $g = 3\%$, $\mu = 4.5\%$, $P_0 = 21$ and $FCF_1 = 1$, Figure 4 illustrates that the function $P_{0,T}$ is not monotonous. In this example, an optimal detention period of around 13.81 years appears (here, the free cash flow periodicity corresponds to one year).

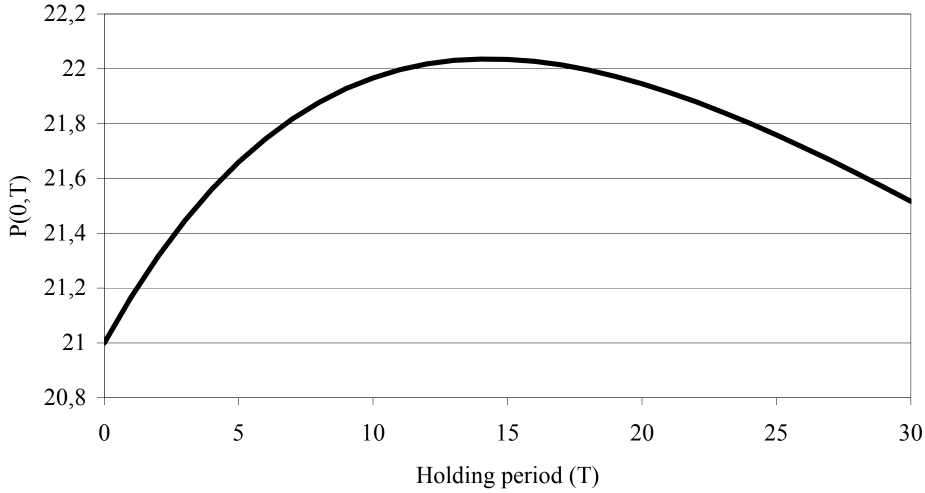

Figure 4: Optimal holding period

Notice 2: when $\mu > g$ and $\mu < k - \alpha$, the optimal detention period is negative. In fact, the value of T^* corresponds to a past time where the asset should have been sold. This could indicate an over evaluation of the asset at the beginning: the price P_0 seems too high and should have been smaller. A decrease of P_0 would imply a decrease in the constraint (1.25) in the sense that α would become higher with a smaller P_0. Figure 5 presents a negative optimal solution: $k = 7\%$, $g = 1\%$, $\mu = 2\%$, $P_0 = 21$ and $FCF_1 = 1$.

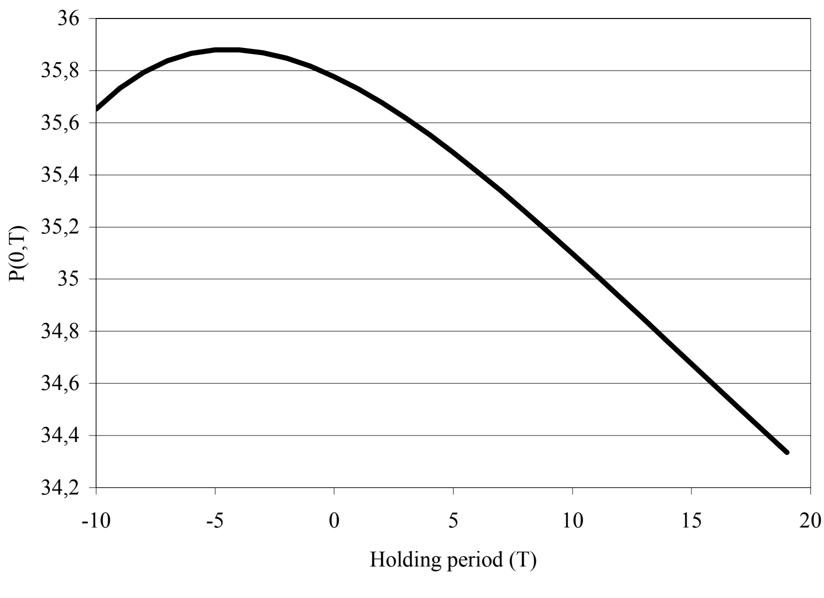

Figure 5: Optimal holding period (negative solution)

Notice 3: when the numerator and the denominator in (1.21) are negative, T^* is still positive. This occurs when (1.24) and (1.25) are simultaneously not satisfied, but as the function is convex, T^* corresponds to a minimum.

3 Sensitivity analysis

For a given price P_0 and a given WACC *k*, we are going to analyze the sensibility of the optimal detention period to:

- α (section 3.1)
- g (section 3.2)
- μ (section 3.1)

3.1 Sensitivity to α

The ratio between the first free cash flow and the initial price, denoted α, has an impact on the optimum value. To analyze the sensitivity of the optimal date T^* to α, let us compute the derivative of the function $T^*(\mu, g, \alpha)$ according to this variable. From (1.21) we compute:

$$\frac{\partial T^*(\mu, g, \alpha)}{\partial \alpha} = \frac{1}{\ln\left(\frac{1+\mu}{1+g}\right)} \times \frac{\frac{1}{k-\mu}}{\frac{\alpha}{k-\mu}} = \frac{1}{\alpha \times \ln\left(\frac{1+\mu}{1+g}\right)} \tag{1.28}$$

As this derivative is always positive, the higher is α, the higher the optimal detention period. If α increases, the sum of the discounted free cash flows increases as well. Then, the equilibrium between loss and gain arrives later. This effect is more pronounced when:

- The real estate return has a value close to the free cash flows growth rate ($\mu \xrightarrow[+]{} g$)
- The ratio between the first free cash flow and the initial price α is low ($\alpha \xrightarrow[+]{} 0$).

As an illustration, table Table *1* gives the optimal holding period evolution according to a variation of α. The initial price is constant, and only FCF_1 varies. In this example $k = 8.40\%$, $g = 3\%$, $\mu = 4.5\%$, $P_0 = 21$ and FCF_1 varies from 0.8 to 1.15. This is equivalent to a variation of α from 0.038 to 0.0548. $FCF_1 = 1$ corresponds to the example presented in Figure 4.

FCF_1	0,8	0,85	0,9	0,95	1	1,05	1,1	1,15
α	0,03809	0,0404	0,0428	0,0452	0,0476	0,05	0,0524	0,0548
T^*	-1,62	2,57	6,52	10,26	13,81	17,18	20,40	23,487

Table 1: optimal detention period according to α

The first optimal value is negative, as $\alpha < k - \mu$. The Figure 6 illustrates the modifications of the $P_{0,T}$ function and then the modification of the corresponding optimum. Figure 7 exhibits a more important effect than Figure 6, the real estate return being closer to the growth rate ($\mu = 3.5\%$). When α increases from 1.05 to 1.15, the optimum varies from 17.18 to 23.5 years in Figure 6. This variation becomes 4.17 to 22.96 years in Figure 7**.**

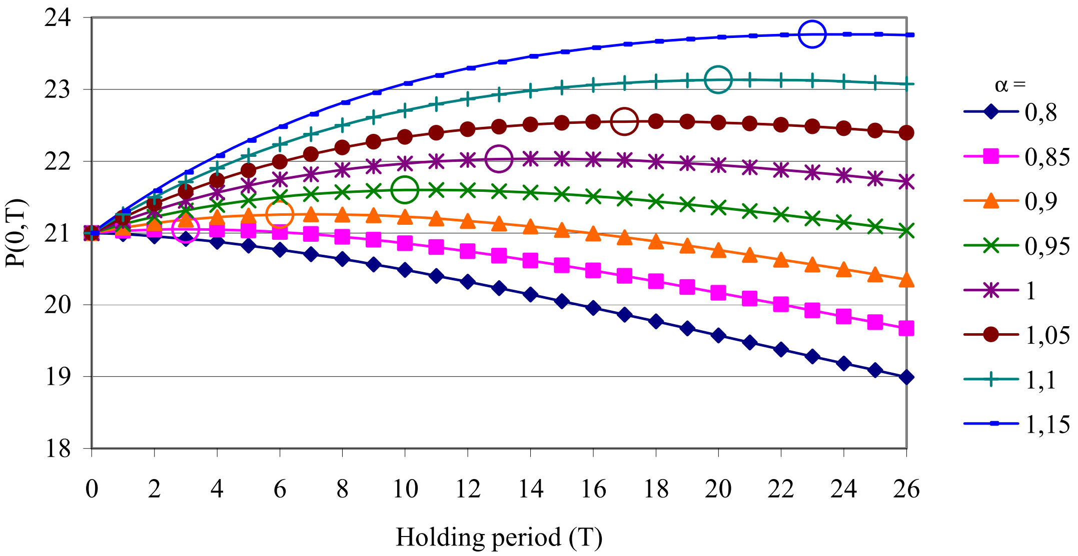

Figure 6: $P_{0,T}$ function according to α

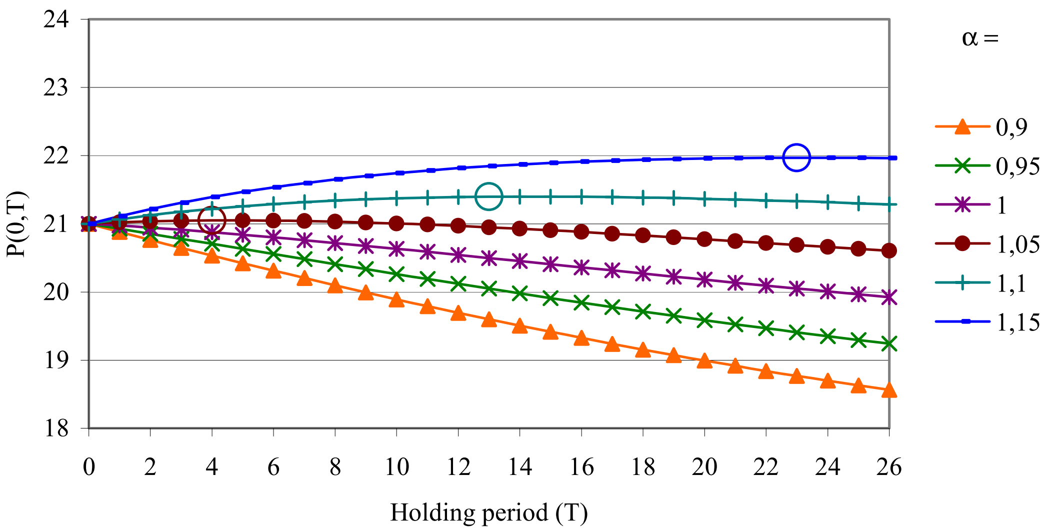

Figure 7: $P_{0,T}$ function according to α ($\mu \xrightarrow[+]{} g$)

3.2 Sensitivity to g

To analyze the sensitivity of the optimal date T^* to g, let us compute the derivative of the function $T^*(\mu,g,\alpha)$ according to this variable. From (1.21) we can compute:

$$\frac{\partial T^*(\mu,g,\alpha)}{\partial g} = -\ln\left(\frac{\alpha}{k-\mu}\right)\times\frac{-\dfrac{\dfrac{1+\mu}{(1+g)^2}}{\dfrac{1+\mu}{1+g}}}{\left[\ln\left(\dfrac{1+\mu}{1+g}\right)\right]^2} = -\ln\left(\frac{\alpha}{k-\mu}\right)\times\frac{-\dfrac{1}{1+g}}{\left[\ln\left(\dfrac{1+\mu}{1+g}\right)\right]^2} \quad (1.29)$$

$$\frac{\partial T^*(\mu,g,\alpha)}{\partial g} = \frac{\ln\left(\dfrac{\alpha}{k-\mu}\right)}{(1+g)\times\left[\ln\left(\dfrac{1+\mu}{1+g}\right)\right]^2} > 0 \quad (1.30)$$

This derivative is positive as soon as $\mu > k-\alpha$, which is verified (see section 2.1). An increase of the free cash flows growth rate implies that the gain obtained by the free cash flow will compensate longer the loss in the discounted terminal value.

With $k = 8.40\%$, $\mu = 4.5\%$, $P_0 = 21$ and $FCF_1 = 1$, *Figure 8* represents the function $P_{0,T}$ for different values of g. The optimal detention period are reported in *Table 2*.

g	0,005	0,01	0,015	0,02	0,025	0,03	0,035	0,04
T^*	5,12	5,86	6,85	8,25	10,33	13,81	20,77	41,63

Table 2: optimal detention period according to g

Instead of representing the functions $P_{0,T}$ as functions of g, the optimal solutions may be also represented for different values of μ, see *Figure 9*. The bold curve corresponds to the values reported in *Table 2* which are the optimum values of the different curves of *Figure 8*. The effect on the optimal value is more important when g is larger. Moreover, this effect disappears when the expected real estate return μ increases.

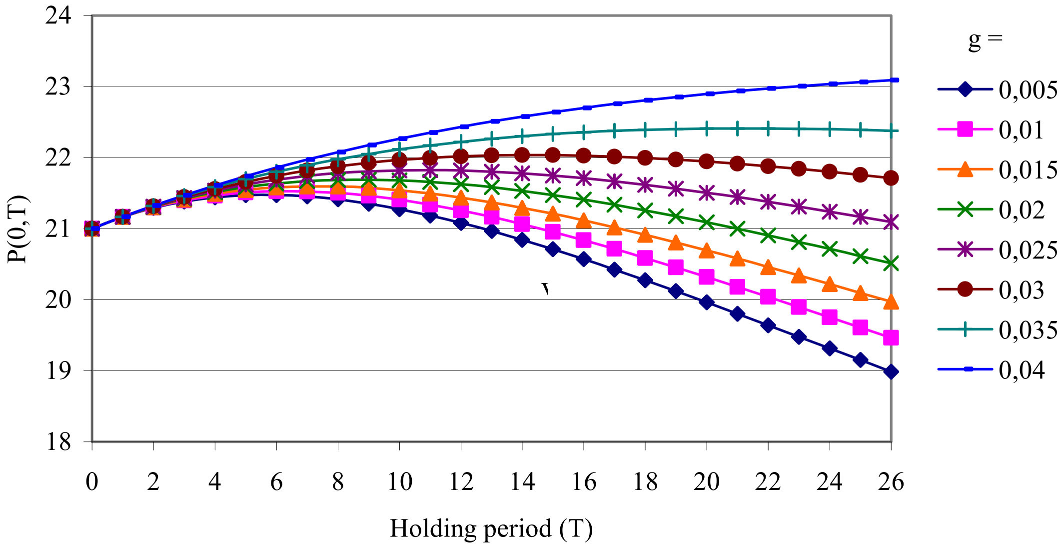

Figure 8: $P_{0,T}$ function according to g

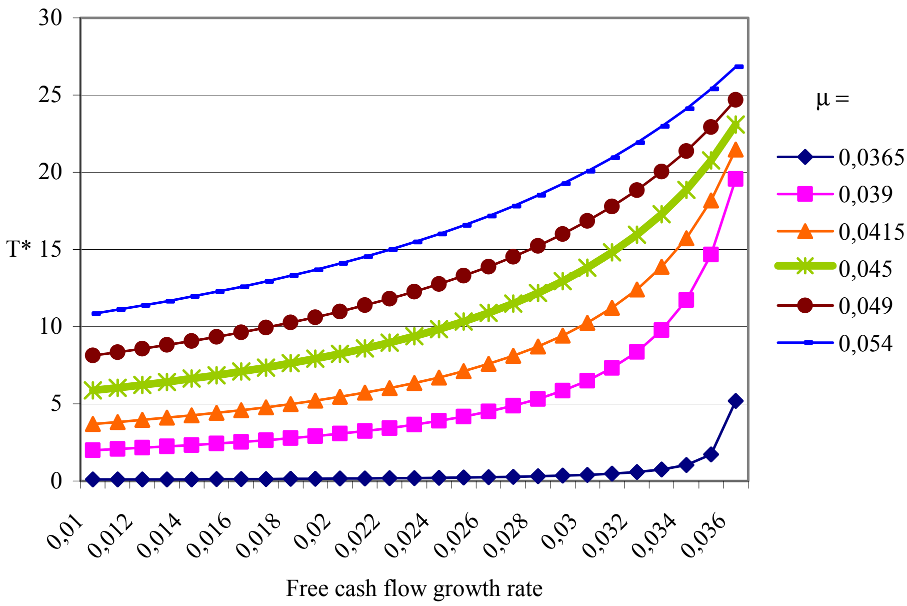

Figure 9: Optimal holding period as a function of g according to μ

3.3 Sensitivity to μ

If the expected real estate return increases, the terminal value has a more important weight in the valuation and then the discounted portfolio value is larger. But, is the optimal holding period shorter or longer? To answer, we have firstly to calculate analytically the derivative of the optimal function according to the real estate return/. Then, a numerical approach will be used to precise the effects with some examples.

3.3.1 Analytical approach

To determine how sensitive the optimal sell date is to μ, let us compute the derivative of the function $T^{*}(\mu,g,\alpha)$ according to μ. From (1.21) we obtain:

$$\frac{\partial T^{*}(\mu,g,\alpha)}{\partial \mu}=\frac{\left[\frac{\frac{-(-\alpha)}{(k-\mu)^{2}}}{\frac{\alpha}{k-\mu}}\times\ln\left(\frac{1+\mu}{1+g}\right)\right]-\ln\left(\frac{\alpha}{k-\mu}\right)\times\frac{\left(\frac{1}{1+g}\right)}{\left(\frac{1+\mu}{1+g}\right)}}{\left[\ln\left(\frac{1+\mu}{1+g}\right)\right]^{2}} \tag{1.31}$$

$$\frac{\partial T^{*}(\mu,g,\alpha)}{\partial \mu}=\frac{\frac{\ln\left(\frac{1+\mu}{1+g}\right)}{k-\mu}-\frac{\ln\left(\frac{\alpha}{k-\mu}\right)}{1+\mu}}{\left[\ln\left(\frac{1+\mu}{1+g}\right)\right]^{2}} \tag{1.32}$$

The sign of this function is the same as the sign of the numerator. Let us denote

$$S(\mu,g,\alpha)=\frac{\ln\left(\frac{1+\mu}{1+g}\right)}{k-\mu}-\frac{\ln\left(\frac{\alpha}{k-\mu}\right)}{1+\mu} \tag{1.33}$$

We then have
$$\operatorname{sign}\left[\frac{\partial T^{*}(\mu,g,\alpha)}{\partial \mu}\right]=\operatorname{sign}\left[S(\mu,g,\alpha)\right] \tag{1.34}$$

and to know this sign, the derivative of $S(\mu,g,\alpha)$ must be calculated (see A.3):

$$\frac{\partial S(\mu,g,\alpha)}{\partial \mu}=\frac{\ln\left(\frac{1+\mu}{1+g}\right)}{(k-\mu)^{2}}+\frac{\ln\left(\frac{\alpha}{k-\mu}\right)}{(1+\tilde{\mu})^{2}}>0 \tag{1.35}$$

As $\max(g,k-\alpha)<\mu<k$, this function is always positive. Then the function $S(\mu,g,\alpha)$ is monotonous in μ. To understand the behaviour of the derivative function (1.32), we may calculate its value in two limit cases when $\mu \xrightarrow[-]{} g$ or $\mu \xrightarrow[+]{} k-\alpha$ and $\mu \xrightarrow[-]{} k$.

For the first derivative, we deduce from (1.32) that

$$\text{a) sign}\left[\lim_{\varepsilon\to 0^+}\left.\frac{\partial T^*(\mu,g,\alpha)}{\partial \mu}\right|_{\tilde{\mu}=g+\varepsilon}\right]=\text{sign}\left[\frac{-\ln\left(\dfrac{\alpha}{k-g}\right)}{(1+g)}\right] \tag{1.36}$$

b)

$$\begin{aligned}\text{sign}\left[\lim_{\varepsilon\to 0^+}\left.\frac{\partial T^*(\mu,g,\alpha)}{\partial \mu}\right|_{\mu=k-\alpha+\varepsilon}\right] &= \text{sign}\left[\frac{\dfrac{1+(k-\alpha+\varepsilon)}{1+g}}{k-(k-\alpha+\varepsilon)}-\frac{\ln\left(\dfrac{\alpha}{k-(k-\alpha+\varepsilon)}\right)}{1+(k-\alpha+\varepsilon)}\right] \\ &= \text{sign}\left[\frac{\dfrac{1+k-\alpha+\varepsilon}{1+g}}{\alpha-\varepsilon}-\frac{\ln\left(\dfrac{\alpha}{\alpha-\varepsilon}\right)}{1+k-\alpha+\varepsilon}\right]\end{aligned} \tag{1.37}$$

$$\text{c) sign}\left[\lim_{\varepsilon\to 0^-}\left.\frac{\partial T^*(\mu,g,\alpha)}{\partial \mu}\right|_{\mu=k+\varepsilon}\right]=\text{sign}\left[\lim_{\varepsilon\to 0^-}\left(\frac{\ln\left(\dfrac{1+k}{1+g}\right)}{\varepsilon}-\frac{\ln\left(\dfrac{\alpha}{\varepsilon}\right)}{1+\mu}\right)\right] \tag{1.38}$$

$$\text{with }\lim_{\varepsilon\to 0^-}\left(\frac{\ln\left(\dfrac{1+k}{1+g}\right)}{\varepsilon}-\frac{\ln\left(\dfrac{\alpha}{\varepsilon}\right)}{1+\mu}\right)=+\infty .$$

Then

$$\begin{aligned}\text{sign}\left[\lim_{\varepsilon\to 0^+}\left.\frac{\partial T^*(\mu,g,\alpha)}{\partial \mu}\right|_{\tilde{\mu}=g+\varepsilon}\right] &< 0 \\ \text{sign}\left[\lim_{\varepsilon\to 0^+}\left.\frac{\partial T^*(\mu,g,\alpha)}{\partial \mu}\right|_{\mu=k-\alpha+\varepsilon}\right] &> 0 \\ \text{sign}\left[\lim_{\varepsilon\to 0^-}\left.\frac{\partial T^*(\mu,g,\alpha)}{\partial \mu}\right|_{\mu=k+\varepsilon}\right] &> 0\end{aligned} \tag{1.39}$$

From (1.35) and (1.39), two states of nature have to be considered:

State 1: if $k-\alpha > g$, $\dfrac{\partial T^*(\mu,g,\alpha)}{\partial \mu} > 0$ (1.40)

State 2: if $k-\alpha < g$, $\begin{cases} \dfrac{\partial T^*(\mu,g,\alpha)}{\partial \mu} < 0 & \mu < \bar{\mu} \\ \dfrac{\partial T^*(\mu,g,\alpha)}{\partial \mu} > 0 & \mu > \bar{\mu} \end{cases}$, with $\bar{\mu} \in \,]g;k[$ (1.41)

3.3.2 Numerical approach

State 1 is illustrated in *Figure 10* with $k = 8.40\%$, $g = 3\%$, $P_0 = 21$ and $FCF_1 = 1$. It exhibits how the function $P_{0,T}$ is changing over time and points out the augmentation of the optimum. Two situations are considered for state 2 (where $g = 3.75\%$): as while

Figure 11 focuses on values that leads to a diminution of the optimum, *Figure 12* underlines the changing in the evolution of the optimum. Moreover

Figure 11 shows the deformation of the curves $P_{0,T}$ that implies a reduction in the optimal holding period.

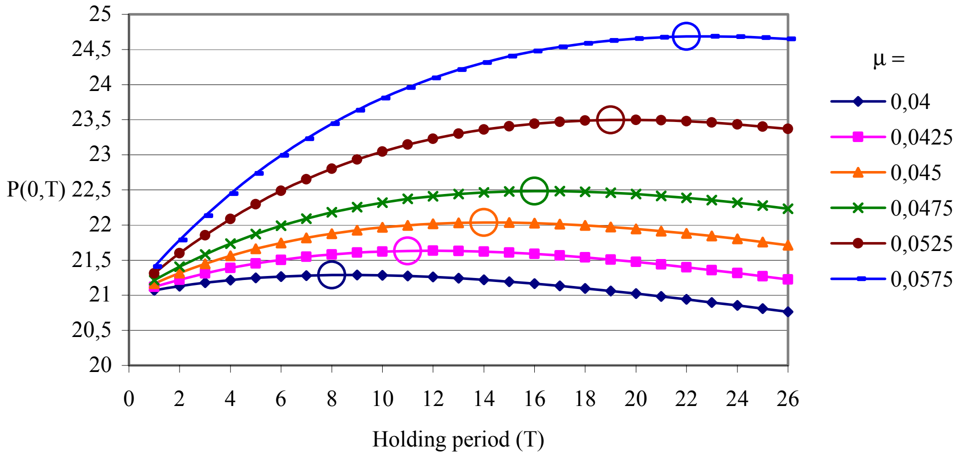

Figure 10 : $P_{0,T}$ function according to μ (state 1)

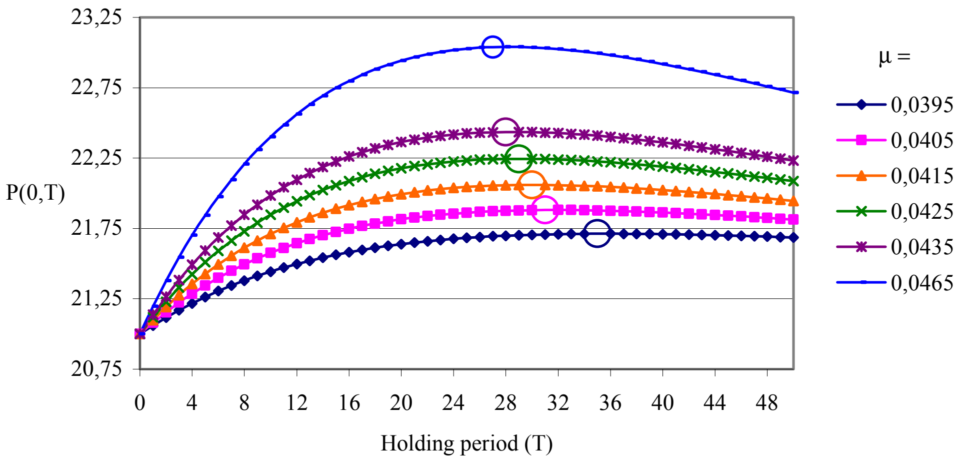

Figure 11 : $P_{0,T}$ function according to μ (state 2 where $\mu < \bar{\mu}$)

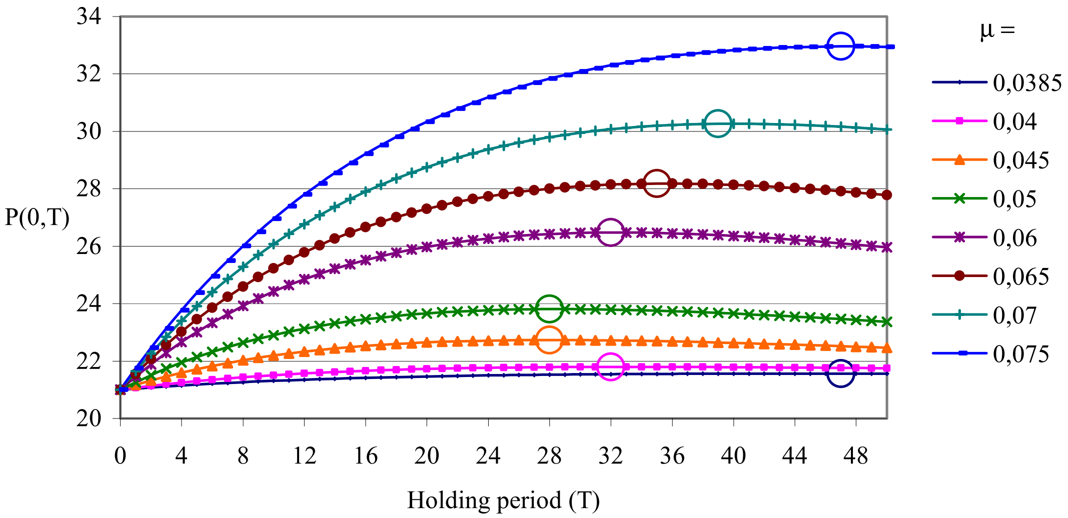

Figure 12 : $P_{0,T}$ function according to μ (state 2 where $\mu < \bar{\mu}$ and $\mu > \bar{\mu}$)

The different results on the optimal detention period are reported respectively in *Table 3*, *Table 4* and *Table 5*.

μ	0,035	0,0425	0,045	0,0475	0,05	0,0525	0,055	0,0575
T^*	8,18	11,40	13,81	15,78	17,52	19,12	20,68	22,24

Table 3: optimal detention period according to μ (case 1)

μ	0,0395	0,0405	0,0415	0,0425	0,0435	0,0445	0,0455	0,0465
T^*	35,18	31,33	29,56	28,61	28,08	27,80	27,67	27,66

Table 4: optimal detention period according to μ (case 2 where $\mu < \bar{\mu}$)

μ	0,0385	0,04	0,045	0,05	0,06	0,065	0,07	0,075
T^*	47,25	32,84	27,72	28,13	31,94	35,12	39,69	46,92

Table 5: optimal detention period according to μ (case 2 where $\mu < \bar{\mu}$ and $\mu > \bar{\mu}$)

The evolution of the optimum solution is represented in the following figures.

Figure 13 illustrates the evolution of the optimum $T^*(\mu, g, \alpha)$ as a function of μ in state 1.

As demonstrated in the previous section, the function is always increasing in this case. The range of values are $k-\alpha<\mu<k$. As μ tends to k, the function rises up as shown in the first derivative analysis (1.39). The derivative limit is infinity. *Figure 14* and *Figure 15* show the interaction between μ and g.

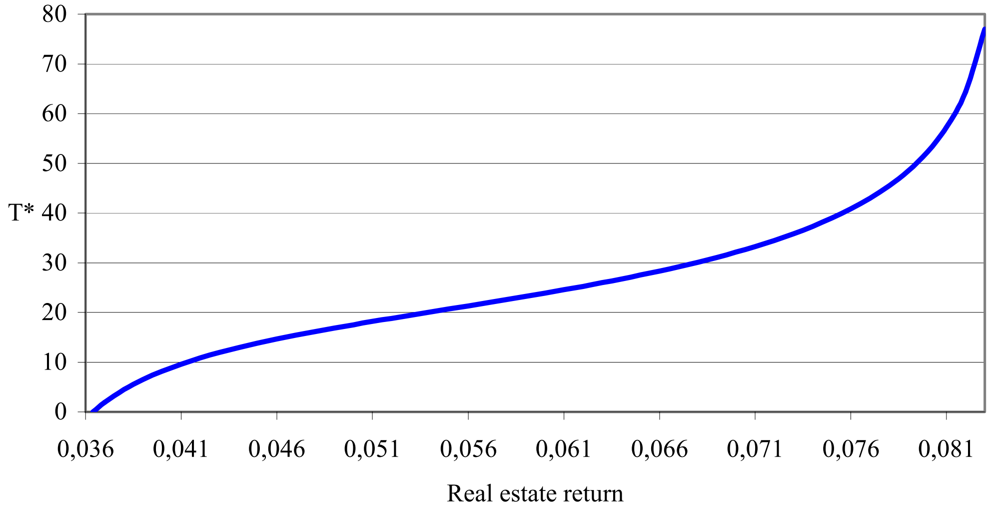

Figure 13 : $T^*(\mu, g, \alpha)$ *according to* μ *(state 1)*

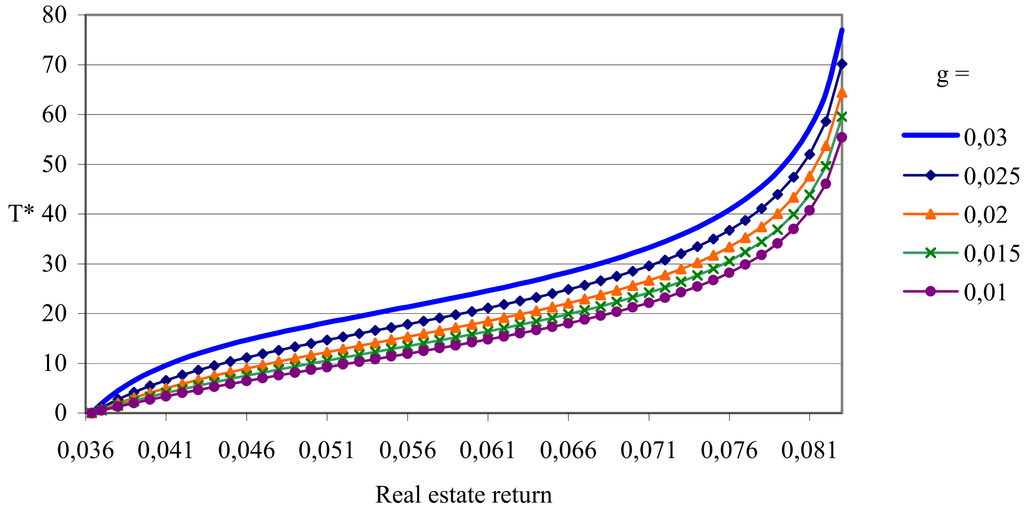

Figure 14 : $T^*(\mu, g, \alpha)$ *according to* μ *and* g *(state 1)*

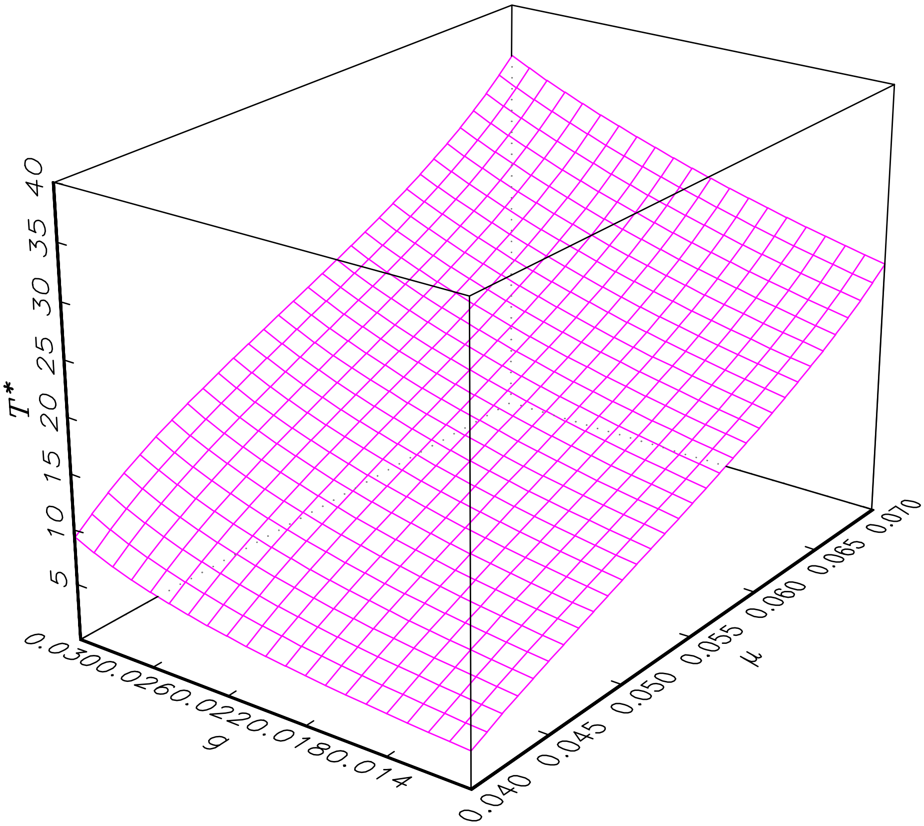

Figure 15 : $T^*(\mu, g, \alpha)$ *according to* μ *and* g *(state 1)*

Figure 17 illustrates the state 2. The interactions between μ *and* g are presented in, *Figure 18*, *Figure 19* and *Figure 19*

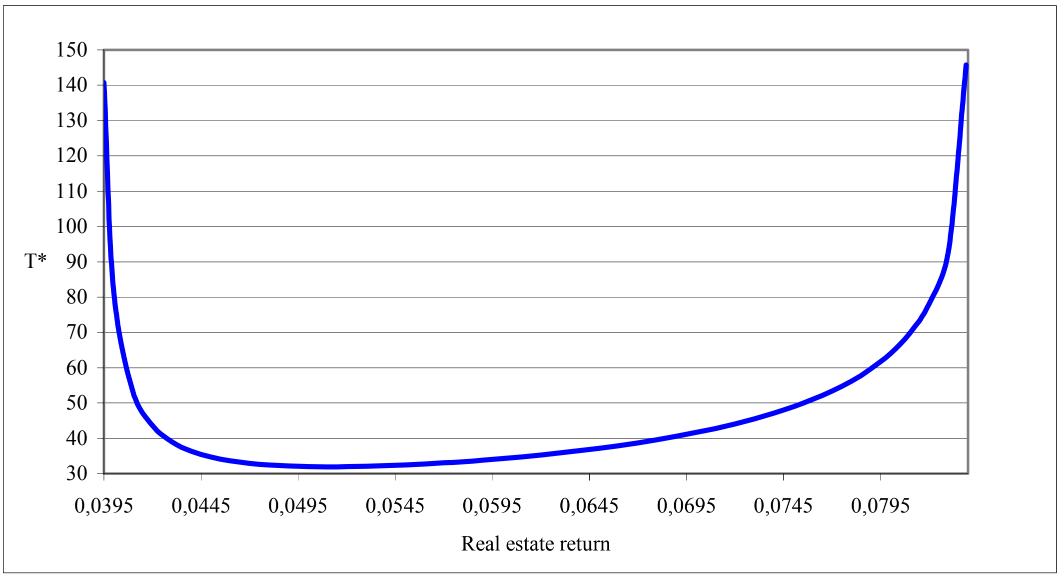

Figure 16 : $T^*(\mu, g, \alpha)$ *according to* . *(state 2 with* $\mu < \bar{\mu}$ *and* $\mu > \bar{\mu}$*)*

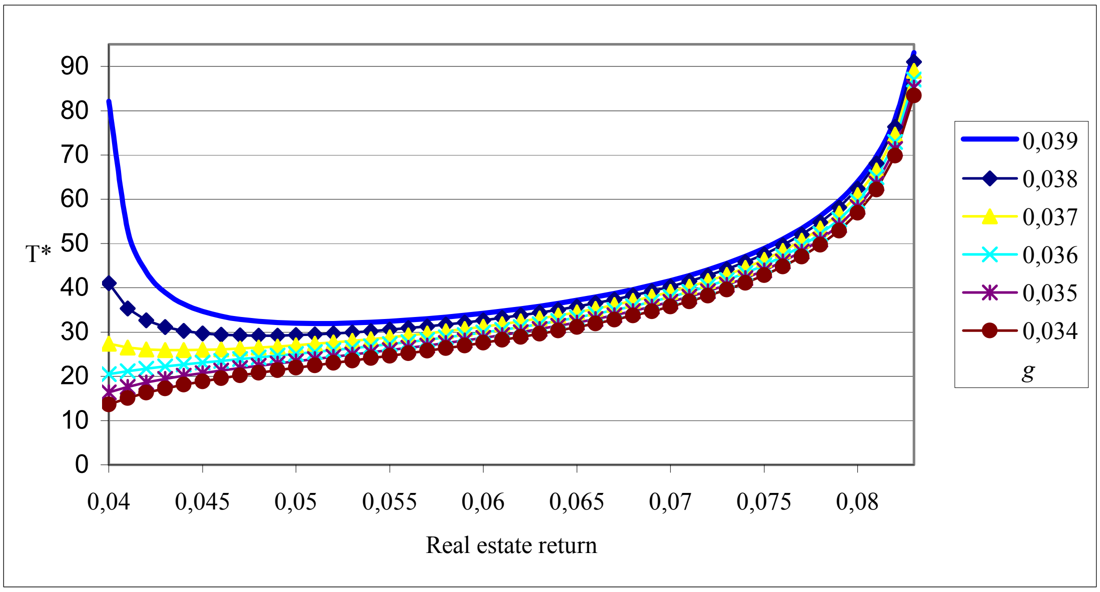

Figure 17 : $T^*(\mu, g, \alpha)$ *according to* μ *and* g *(state 2 with* $\mu < \bar{\mu}$ *and* $\mu > \bar{\mu}$*)*

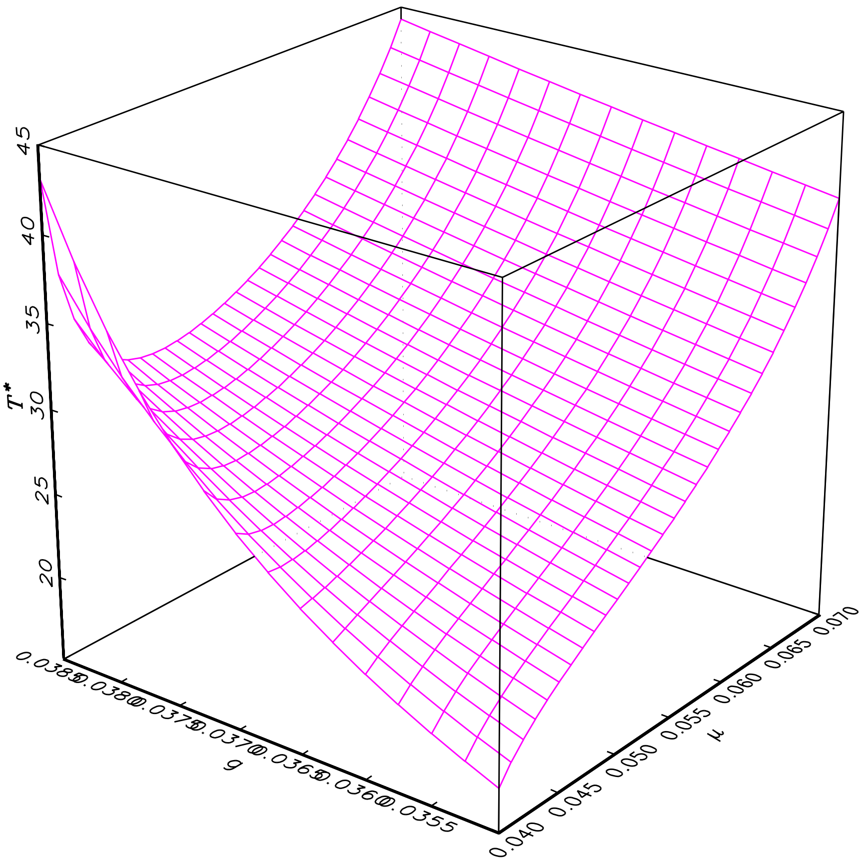

Figure 18 : $T^*(\mu, g, \alpha)$ *according to* μ *and* g *(state 2 with* $\mu < \bar{\mu}$ *and* $\mu > \bar{\mu}$*)*

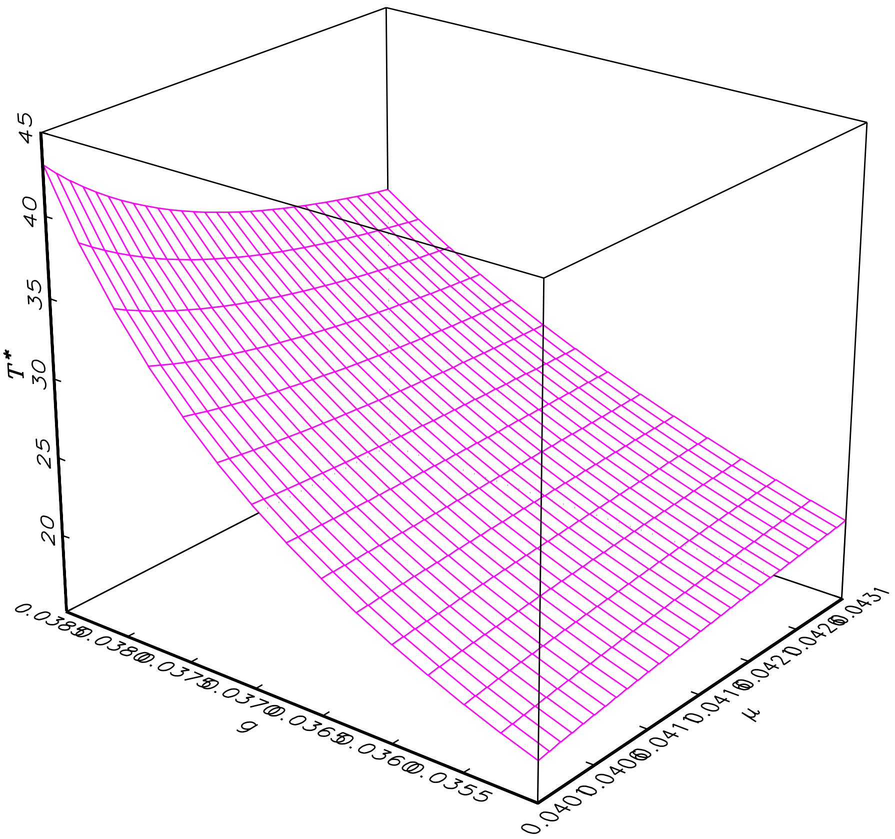

Figure 19 : $T^*(\mu, g, \alpha)$ *according to* μ *and* g *(state 1 and state 2 with* $\mu < \bar{\mu}$ *)*

In this last figures we can see the deformation of *T** in function of the expected price returns and the growth rate return of the cash flows and observe the various possible behaviors of this optimal holding period according to the assumptions concerning the discounting rate *k*.

Conclusion

As seen above, the consideration of all parameters that are influent to determine an optimal holding period reveals a certain complexity and the interaction between these different variables suggests difficulties to put into practice this approach. In fact the precise knowledge of the trend of the real estate price return and of the anticipated rent growth rate is the key issue. However, this approach which lies on the dynamics of price returns on a holding horizon may often be more realistic than assumptions on infinite growth rate of the cash flows. Moreover, on a diversified portfolio of assets, it is not necessarily too difficult to get a measure of the price returns trend from indexes. In this paper we essentially deal with the trends of cash flows and terminal value. The results we find to determine an optimal holding period may be changed if volatility parameters are introduced in the analysis, but that will be the object of a future research.

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Appendix

A.1

By multiplying the last term of (1.5) by $1+k$

$$P_{0,T+1}-P_{0,T}=\frac{FCF_1(1+g)^T}{(1+k)^{T+1}}+\frac{FCF_1(1+g)^{T-1}(1+g_\infty)(1+g-1-k)}{(k-g_\infty)(1+k)^{T+1}}$$

$$(1+k)^{T+1}(P_{0,T+1}-P_{0,T})=FCF_1(1+g)^T+\frac{FCF_1(1+g)^{T-1}(1+g_\infty)(g-k)}{k-g_\infty}$$

$$(1+k)^{T+1}(P_{0,T+1}-P_{0,T})=FCF_1(1+g)^T+FCF_1(1+g)^{T-1}(1+g_\infty)\left(\frac{g-k}{k-g_\infty}\right)$$

$$\frac{(k-g_\infty)(1+k)^{T+1}(P_{0,T+1}-P_{0,T})}{FCF_1}=(1+g)^{T-1}\left[(1+g)(k-g_\infty)+(1+g_\infty)(g-k)\right]$$

$$\frac{(k-g_\infty)(1+k)^{T+1}(P_{0,T+1}-P_{0,T})}{FCF_1(1+g)^{T-1}}=\left[(1+g)(k-g_\infty)+(1+g_\infty)(g-k)\right]$$

$$\frac{(k-g_\infty)(1+k)^{T+1}(P_{0,T+1}-P_{0,T})}{FCF_1(1+g)^{T-1}}=(1+k)(g-g_\infty)$$

Then the difference between the present values can be expressed as

$$P_{0,T+1}-P_{0,T}=FCF_1\left[\frac{(1+g)^{T-1}}{(1+k)^T}\right](g-g_\infty)$$

A.2

The function $P_{0,T}$ is concave if when the function is increasing, the second difference is negative. Let us denote $\Delta^2 P_{0,T}=(P_{0,T}-P_{0,T-1})\text{-}(P_{0,T-1}-P_{0,T-2})$

$$\begin{cases}\Delta^2 P_{0,T}<0\\ P_{0,T}-P_{0,T-1}>0\\ P_{0,T-1}-P_{0,T-2}>0\end{cases}\qquad \text{A2.1}$$

As $\forall T<T^*$, $P_{0,T}-P_{0,T-1}>0$ and $\forall T>T^*$, $P_{0,T}-P_{0,T-1}<0$, the previous constraints are equal to

$$\begin{cases}(P_{0,T}-P_{0,T-1})\text{-}(P_{0,T-1}-P_{0,T-2})<0\\ T<T^*\end{cases}$$

The second difference is equal to

$$\Delta^2 P_{0,T}=\frac{FCF_1(1+g)^{T-1}+P_0(1+\mu)^{T-1}(\mu-k)}{(1+k)^T}-\frac{FCF_1(1+g)^{T-2}+P_0(1+\mu)^{T-2}(\mu-k)}{(1+k)^{T-1}}$$

At time T we have

$$(1+k)^T\Delta^2 P_{0,T}=\alpha(1+g)^{T-1}+(1+\mu)^{T-1}(\mu-k)-\alpha(1+g)^{T-2}(1+k)+(1+\mu)^{T-2}(\mu-k)(1+k)$$

With the normalization $P_0 = 1$,

$$\Delta^2 P_{0,T} < 0 \Leftrightarrow \alpha(1+g)^{T-2}(g-k) + (1+\mu)^{T-2}(\mu-k)(\mu-k) < 0$$

$$\Delta^2 P_{0,T} < 0 \Leftrightarrow \alpha(1+g)^{T-2}(k-g) > (1+\mu)^{T-2}(\mu-k)(\mu-k)$$

$$\Delta^2 P_{0,T} < 0 \Leftrightarrow \alpha\frac{k-g}{(\mu-k)^2} > \left(\frac{1+\mu}{1+g}\right)^{T-2}$$

$$\Delta^2 P_{0,T} < 0 \Leftrightarrow \alpha\frac{k-g}{(\mu-k)^2}\left(\frac{1+\mu}{1+g}\right)^2 > \left(\frac{1+\mu}{1+g}\right)^T$$

$$\Delta^2 P_{0,T} < 0 \Leftrightarrow T \times \ln\left(\frac{1+\mu}{1+g}\right) < \ln\left(\frac{\alpha}{(k-\mu)} \times \frac{k-g}{(k-\mu)} \times \left(\frac{1+\mu}{1+g}\right)^2\right)$$

The constraint on T for the concavity is

$$\Delta^2 P_{0,T} < 0 \Leftrightarrow T < \frac{\ln\left(\frac{\alpha}{(k-\mu)} \times \frac{k-g}{(k-\mu)} \times \left(\frac{1+\mu}{1+g}\right)^2\right)}{\ln\left(\frac{1+\mu}{1+g}\right)} = T^* + \frac{\ln\left(\frac{k-g}{(k-\mu)} \times \left(\frac{1+\mu}{1+g}\right)^2\right)}{\ln\left(\frac{1+\mu}{1+g}\right)}$$

From A2.1, as $T < T^*$,

$$\Delta^2 P_{0,T} < 0 \Leftrightarrow \ln\left(\left(\frac{1+\mu}{1+g}\right)^2 \times \left(\frac{k-g}{k-\mu}\right)\right) > 0$$

$$\Delta^2 P_{0,T} < 0 \Leftrightarrow \left(\frac{1+\mu}{1+g}\right)^2 \times \left(\frac{k-g}{k-\mu}\right) > 1$$

Which leads to, $\Delta^2 P_{0,T} < 0 \Leftrightarrow \mu > g$;

A.3

$$\frac{\partial S(\mu,g,\alpha)}{\partial \mu} = \frac{\frac{\frac{1}{1+g}}{\frac{1+\mu}{1+g}} \times (k-\mu) - \ln\left(\frac{1+\mu}{1+g}\right) \times (-1)}{(k-\mu)^2} + \frac{-\frac{\frac{1}{(k-\mu)^2}}{\frac{\alpha}{k-\mu}} \times (1+\mu) + \ln\left(\frac{\alpha}{k-\mu}\right)}{(1+\mu)^2}$$

$$\frac{\partial S(\mu,g,\alpha)}{\partial \mu} = \frac{\frac{k-\mu}{1+g} + \ln\left(\frac{1+\mu}{1+g}\right)}{(k-\mu)^2} + \frac{-\left(\frac{1+\mu}{k-\mu}\right) + \ln\left(\frac{\alpha}{k-\mu}\right)}{(1+\mu)^2}$$

$$\frac{\partial S(\mu,g,\alpha)}{\partial \mu} = \frac{1}{(1+\mu)(k-\mu)} - \frac{1}{(k-\mu)(1+\mu)} + \frac{\ln\left(\frac{1+\mu}{1+g}\right)}{(k-\mu)^2} + \frac{\ln\left(\frac{\alpha}{k-\mu}\right)}{(1+\mu)^2}$$

$$\frac{\partial S(\mu,g,\alpha)}{\partial \mu} = \frac{\ln\left(\frac{1+\mu}{1+g}\right)}{(k-\mu)^2} + \frac{\ln\left(\frac{\alpha}{k-\mu}\right)}{(1+\mu)^2}$$