Risk and the Endogenous Formation of Risk-sharing Coalitions.*

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Abstract

In this paper we analyse the segmentation of society into risk-sharing coalitions voluntarily formed by agents differing with respect to risk under a unanimous acceptance rule, in the absence of financial markets. We obtain a partition belonging to the core of the membership game. It is homophily-based: the less risky agents congregate together and reject more risky ones into other coalitions, etc. There is perfect risk sharing within a coalition but not at the society level. The distribution of risk affects the number and the size of these coalitions. A more risky society is not necessarily more segmented than a less risky one. Finally, the empirical evidence on imperfect risk-sharing when agents rely on informal insurance schemes can be understood when the endogenous partition of society with respect to risk is taken into account.

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1 Introduction.

In many developing economies, people face large fluctuations of their income (see, in particular, Townsend, 1994, for ICRISAT villages in India or Dubois, Jullien and Magnac, 2008, for Pakistan villages). Nonetheless, the idiosyncratic part of income risk is relatively large suggesting that insurance against shocks is feasible (see Townsend 1995 or Dercon 2004 for reviews of empirical evidence about the idiosyncratic nature of income shocks). We should thus expect risk-averse households to group together to mutualize risk. If risk is fully insured, the theory tells us that individual consumption is determined by aggregate consumption (see Borch 1962, Arrow, 1964, Wilson, 1968). However, this proposition has been subject to many empirical rebuttals. In developing economies, it has been found that households are able to protect consumption against adverse income shocks but full insurance is not achieved (see, among many others, Townsend, 1994, Kazianga and Udry 2005)[1][2]. Moreover, empirical works emphasize the local nature of risk sharing (see for instance Dercon *et alii* 2006, Arcand and Fafchamps 2006, Barr and Genicot 2008, and the survey of Fafchamps, 2008, on the role of families and kinship networks in sharing risk)[3]. Therefore the issue with respect to risk sharing is to understand why there are limits to insurance and not perfect risk sharing within a given society. Why risk-bearing arrangements do not encompass the whole society?

The standard explanation is that complete risk sharing is prevented due to limited commitment. Risk-sharing arrangements are seen as self-enforcing contracts immune to either individual deviations (Kocherlakota, 1996, Ligon, Thomas and Worrall, 2002) or group deviations (see Dubois 2002, Genicot and Ray, 2003). Since individuals can defect on these contracts, it makes sense not to cover a too risky agent through a common risk-sharing agreement.

In the present paper, we offer an alternative explanation based on the endogenous formation of risk-sharing groups. There is no full risk sharing because individuals who differ with respect to their exposure to risk select themselves into groups in which they mutualize risk. We stress that society segmentation into various risk-sharing groups plays a crucial role in providing differentiated insurance schemes to agents unequally affected by risk. The limits of communities themselves cannot be considered exogenous to risk bearing.

Formally, we study a society comprised of many individuals characterized by idiosyncratic risks.

[1] Ray (1998), Dubois (2002), Dercon (2004) or Townsend (2005) are excellent surveys of the literature.

[2] For developped economies also, empirical evidence does not support the full insurance hypothesis (Mace, 1991, Cochrane 1991, Altonji, Hayashi and Kotlikoff, 1996, Attanasio and Davis 1996).

[3] In particular, Arcand and Fafchamps (2006) studying a dataset from Senegal and Burkina Faso, emphasize that membership in community-based organizations is very common in these countries. Essentially, a community-based organization is created by producers in order to provide services to the members of the group. Among these services, a community-based organization provides insurance.

The level of risk associated with an individual (measured by the variance of the distribution from which is drawn the idiosyncratic shock) is specific to this individual. Individuals have the possibility to form a group in order to mutualize risk. We consider that individuals commit to share the random component of their income equally with members of their risk-sharing group.[4] We then examine the segmentation of society into such risk-sharing groups. We show that the resulting "core partition" exists and is unique (under mild assumption). It turns out that the key dimension of the coalition formation process is risk heterogeneity measured by ratios of variance between individuals. This leads the core partition to be homophily-based: coalitions gather together agents similar with respect to the variance of the idiosyncratic shock. Within each coalition belonging to the core partition, there is complete risk sharing but not within the whole society: risk-premiums paid by individuals differ between coalitions.

We study the impact of specific variance schedules on the core partition and show thanks to these cases how the number and the size of coalitions belonging to the core partition are affected by the distribution of risk within society

Turning then to the comparison of two societies identical with respect to the number of individuals (and thus risks), one being more "risky" than the other, we compare their core partitions and prove that the more risky one is not necessarily more segmented than the less risky one, nor that the aggregate risk premium associated with risk-sharing in the more risky society is always higher than the one associated with the less risky society.

Finally, we discuss the empirical implications of this partitioning of society and prove how the empirical evidence can be explained when the grand coalition does not form, that is when society is segmented into more than one risk-sharing coalition.

The relationship between risk and group formation has already been studied by various authors. In particular, Genicot and Ray (2003) develop a group formation approach where one risk-sharing coalition forms in the presence of possible self defection. However they do not study the partition of society into possibly multiple coalitions. Our work is also closely related to Taub and Chade (2002) who study in a dynamic setup whether the current core partition is immune to future individual defections. Our focus is different as we build a setup that allows us to characterize a relationship between the risks characteristics of a society and the membership, size of risk-sharing groups and the extent of risk coping. Recent works have developed network formation models where informal insurance is essentially characterized by bilateral relations (see Bramoullé and Kranton 2007, and

[4] Given our setup, it turns out that, *conditional on group membership*, such an insurance scheme leads to consumption allocations consistent with Pareto optimal allocations which can be obtained by effective state-contingent markets.

Bloch, Genicot and Ray, 2008). Our coalition formation approach allows for multilateral transfers between individuals which are supported by empirical evidence (see Dercon *et alii* 2006 for the case of group-based funeral insurance or Arcand and Fafchamps 2008 for insurance services rendered by community-based organizations). Finally, our paper is also related to Henriet and Rochet (1987) who build a model of endogenous formation of mutuals using a cooperative game theoretic approach. The modelling strategy is different from ours as they assume a continuum of agents, the existence of some congestion costs and a binomial distribution of shock. Further, they focus on formal insurance activity and do not address the issue of mutualization of risk under informal insurance schemes.

2 The Model.

2.1 Society.

We consider a society $\mathbf{I}$ formed of N agents, indexed by $i = 1, ..., N$. There is no production in this society and agents are endowed with quantities of a consumer good. The individual endowment y_i allotted to individual i has a deterministic component w_i and is affected by an idiosyncratic risk ε_i :

$$y_i = w_i + \varepsilon_i + \nu$$

ε_i is normally distributed with variance σ_i^2: $\varepsilon \rightsquigarrow \mathcal{N}(0, \sigma_i^2)$. ν denotes a common shock normally distributed with variance σ_ν^2. Without loss of generality, we index individuals as follows: for i and $i' = 1, .., N$ with $i < i'$ then $\sigma_i^2 < \sigma_{i'}^2$. We will thus say that a lower indexed individual is a "less risky agent" (strictly speaking, individual risk is associated with the law of motion of ε_i).

Given these differences among individuals, we define $\lambda_i \equiv \frac{\sigma_i^2}{\sigma_{i-1}^2}$ for $i \in \mathbf{I}$. λ_i is called the "risk ratio" between agents $i-1$ and i. Then it will be useful in the sequel to use the following

Definition 1 *Any society* $\mathbf{I}$ *can be characterized by a risk-ratio schedule* $\Lambda = \{\lambda_2, \lambda_3, ..., \lambda_N\}$ *with* $\lambda_i \equiv \frac{\sigma_i^2}{\sigma_{i-1}^2}$ *for* $i \in \mathbf{I}$.

Each individual has the same CARA utility function

$$U_i = -\frac{1}{\alpha} e^{-\alpha c_i} \tag{1}$$

where c_i denotes private consumption and α a positive parameter measuring the absolute risk aversion. It is identical for all individuals. c_i may differ from y_i if agent i enters some risk-sharing group. We will provide more detail in the next subsection. It is assumed that there are no financial markets allowing any agent to insure himself against his idiosyncratic risks. Finally there is perfect information in the following sense: a priori the various idiosyncratic variances are common knowledge and the realised individual shocks are also perfectly and universally observed when they occur.

2.2 Risk-Sharing Groups.

Individuals have an incentive to belong to a risk-sharing group: a group provides insurance by pooling some resources and redistributing these resources among its members.

Here we consider the following insurance scheme: agent i pays the full value of his realized idiosyncratic shock ε_i in a common fund. The common fund is then equally redistributed among members of the group. Formally, consider a group $S \subseteq I$, formed of a finite number $n \leq N$ of agents. S is a subset of I whose membership is left undefined at this stage. The consumption of agent i belonging to S is equal to:

$$c_i = w_i + \frac{\sum_{k \in S} \varepsilon_k}{n} + \nu \tag{2}$$

where n is the cardinal of S.

This insurance scheme deserves attention because it leads to allocations that are Pareto optimal and consistent with the existence of complete markets. Precisely, according to the First Welfare Theorem, allocations obtained with complete markets are Pareto optimal when, for any individual i, the lagrange multiplier associated with her budget constraint equals the inverse of her weight in the social value function of the Pareto program (see Ljungqvist and Sargent, 2004, p. 216). Given that we assume a CARA utility function and a risk aversion parameter identical among individuals, we can check that our insurance rule leads to allocations that are optimal *and* identical to the one obtained with complete markets.[5]

This insurance rule is different from credit as transfers paid at a date t do not depend on history, for instance agents who had suffered from bad shocks in the past and who received transfers do not have to reimburse them while they benefit from a good shock.

The expected utility of individual i in group S, $V_i(S)$, applying this insurance rule is:

$$V_i(S) = -E\left[\frac{1}{\alpha} e^{-\alpha w_i - \alpha \frac{\sum_{k \in S} \varepsilon_k}{n} - \alpha \nu}\right].$$

As we assume a CARA utility function and normal distribution for each idiosyncratic shock, the Arrow-Pratt approximation is exact:

$$V_i(S) = -\frac{1}{\alpha} e^{-\alpha\left[w_i - \frac{\alpha}{2n^2}\sum_{k \in S} \sigma_k^2 - \frac{\alpha}{2}\sigma_\nu^2\right]} \tag{3}$$

[5] Let us stress that our risk-sharing groups are not equivalent to mutuals studied by Henriet and Rochet (1987). They consider a mutual as an insurance company that provides full insurance, the premium is the same for all clients and expected profits are nil. With respect to our insurance rule, an agent i in group S pays the premium ε_i, he receives t such that the *ex post* budget constraint for the group is binding: $nt = \sum_{k \in S} \varepsilon_k$.

We define the certainty-equivalent income for individual i in group S, denoted by $\omega_i(S)$, as:

$$\omega_i(S) = w_i - \frac{\alpha}{2}\sum_{k\in S}\frac{\sigma_k^2}{n^2} - \frac{\alpha}{2}\sigma_\nu^2 \tag{4}$$

The risk premium for any individual i in group S, denoted by $\pi(S)$, is equal to $\frac{\alpha}{2}\sum_{k\in S}\frac{\sigma_k^2}{n^2} + \frac{\alpha}{2}\sigma_\nu^2$. It is immediate to remark that it is the same for every member of S.

Hence, the formation of a group relies on the following trade-off. Accepting a new member has two opposite effects: on the one hand, everything else equal, the higher its size, the lower the risk premium; on the other hand, accepting an individual increases the sum of individual risks leading members to pay a higher risk premium. Therefore when assessing the net benefit of accepting a given individual, characterized by a certain variance, an insider has to weigh these two effects. But clearly, given the choice between two agents, any insider prefers the one with the lesser variance. Remark that the risk premium is a non linear function of the size n_j.

We define the individual gain for agent i from membership to group S rather than to group S' as follows:

$$\pi(S') - \pi(S)$$

It amounts to the reduction in the risk premium represented by being a member of S rather than a member of S'. In other words, an agent prefers joining a group (provided she is accepted in this group) in which her certainty-equivalent income is higher.

Suppose an agent i forming its own risk sharing group, i.e. $S' = \{i\}$, this difference becomes:

$$\pi(\{i\}) - \pi(S).$$

Considering two individuals i and $i' > i$, $i, i' \in S$, i' benefits more than i from being in S rather than being alone, as we get:

$$\pi(\{i'\}) - \pi(S) = \frac{\alpha}{2}\left(\sigma_{i'}^2 - \sum_{k\in S}\frac{\sigma_k^2}{n^2}\right) > \pi(\{i\}) - \pi(S) = \frac{\alpha}{2}\left(\sigma_i^2 - \sum_{k\in S}\frac{\sigma_k^2}{n^2}\right) \tag{5}$$

In other words, the riskier an agent, the more he benefits from belonging to a given group (rather than remaining alone): individual gains from a group are differentiated and actually increasing with the riskiness of the agent. This is the core characteristics of a group functioning under our insurance rule.

More generally, given two different risk-sharing groups differing by their membership, and therefore, the exposure to risk of their members, the gains for joining either of them for a given agent differ. The desire of each agent is to join the group which generates for him the highest gain.

3 Risk-sharing and the segmentation of society.

From above, it is immediate that the characteristics and in particular the size of a group matters for its members. In particular, as agents have different needs for risk sharing, and expose the members of the group to which they may belong to idiosyncratic risks, the membership of a group is a matter of concern. This leads to the question of the endogenous segmentation of society into risk-sharing groups.

We consider that a group is a coalition or club of individuals and a partition of the society is a set of coalitions. More formally,

Definition 2 *A non-empty subset S_j of $\mathbf{I}$ is called a coalition and $\mathcal{P} = \{S_1, ..., S_j, ..., S_J\}$ for $j = 1, ..., J$ is called a partition of $\mathbf{I}$ if (i) $\bigcup_{j=1}^{J} S_j = \mathbf{I}$ and (ii) $S_j \bigcap S_{j'} = \emptyset$ for $j \neq j'$.*

According to this definition, any individual belongs to one and only one coalition. The size of the $j-th$ coalition, $S_j \subseteq \mathbf{I}$, is denoted by n_j. We will use interchangeably the terms "coalition" and "risk-sharing coalition".

To address the issue of segmentation of society into risk-sharing coalitions, we consider the following sequence of events:

1. Agents form risk-sharing coalitions and a partition of society is obtained.
2. Individuals commit to pay transfers according to the insurance rule of Equation (2) in each coalition.
3. Idiosyncratic shocks are realized. Agents then consume their after-transfer income.

We solve this coalition-formation game by looking at a core partition defined as follows:

Definition 3 *A partition $\mathcal{P}^* = \left\{S_1^*, ..., S_j^*, ..., S_J^*\right\}$ belongs to the core of the coalition-formation game if:*

$$\nexists \pounds \subseteq \mathbf{I} \text{ such that } \forall i \in \pounds,\ V_i(\pounds) > V_i(\mathcal{P}^*)$$

where $V_i(\mathcal{P}^*)$ denotes the utility for agent i associated with partition $\mathcal{P}^*$.

According to this definition, a core partition is such that no subset of agents is willing to secede. It amounts to say that coalitions are formed according to a unanimity rule: (i) no one can be compelled to stay in a given group and (ii) to be accepted in a group, there must be unanimous consent by all existing members of this group.

Two remarks are important at this stage. First, we assume full commitment. That is, no agent is able to renege on her chosen coalition once the state of nature is realized. This is a key difference with for instance Genicot and Ray (2003). Second, if at each date there was a draw in the distribution law of shocks and no storage technology, the problem at the coalition formation stage would be the same, and the solution to the game would remain identical over time[6].

3.1 The core partition.

Then we are able to offer the following:

Proposition 1 *A core partition* $\mathcal{P}^* = \left\{S_1^*, ..., S_j^*, ..., S_J^*\right\}$ *exists and is characterized as follows:*

i/ *It is unique if*

$$\forall z = 2, ..., N-1, \quad \frac{\lambda_{z+1} - \lambda_z}{\lambda_{z+1} - 1} \geq -\frac{1}{z+1}. \tag{6}$$

ii/ *It is consecutive, that is, if* i *and* $\widetilde{i}$ *both belong to* S_j^* *then* $\forall i'$, $i > i' > \widetilde{i}$, $i' \in S_j^*$.

iii/ *For any two individuals* $i \in S_j^*$ *and* $i' \in S_{j'}^*$ *such that* $\sigma_i^2 < \sigma_{i'}^2$*, the risk premium* $\pi(S_j^*) < (=)\pi(S_{j'}^*)$ *if* $j < (=)j'$.

Proof. See Appendix. ■

We first prove the existence of a core partition of **I**. Based on voluntary agreements, agents are able by themselves to form coalitions so as to share risk. No institutional constraint is involved. The fact that there is no financial markets does not mean that there is no way to get insured against risk.[7] Remark that the partition is Pareto-optimal as we are focusing on the core. But we hasten to add that this result depends on the risk-sharing rule we consider.

The first result, *i/*, provides a sufficient condition for the core partition to be unique. The condition on uniqueness depends on the rank of individuals. If the risk ratios are increasing with the index z, this condition is always met. The condition may appear stringent when $\lambda_z > \lambda_{z+1}, \forall z = 2, ..., N-1$. The expression $\frac{-1}{z+1}$ is an increasing function of z which equals $\frac{-1}{3}$ when $z = 2$, $\frac{-1}{N}$ when $z = N-1$, and tending to 0 when N is sufficiently large.[8]

Turning to the characteristics of the core partition, the second result, *ii/*, is about consecutivity. Coalitions belonging to the core partition are homogeneous in the following sense: they include

[6] This is congruent with the formal setting of the risk sharing issue developed in Townsend (1994).

[7] Admittedly, the result does not rule out the existence of singletons within the core partition. Singletons are degenerate risk-sharing coalitions.

[8] Let us stress that the core partition is generically unique (see for instance Farrell and Schotchmer, 1985) but we need to provide a sufficient condition for uniqueness in order to proceed our comparative static exercises.

agents who are "close" in terms of exposure to risk. Take an individual who has to choose between two individuals in order to form a risk-sharing coalition. It is easy to check that he always prefers the less risky of the pair. This implies that if an agent i is willing to form a coalition with some other agent i', then all agents with a lower risk than i' are also accepted by i in the coalition.[9]

The third result, *iii/*, is in line with consecutivity. Take individual characterized by σ_1^2. He is accepted by any possible coalition and chooses the group that incurs the lowest risk premium. More risky individuals may not be accepted by agents characterized by low risks to pool resources in a same group and pay higher risk premium in other coalitions.

Given the consecutivity property, from now on, we adopt the following convention that for any S_j^* and $S_{j'}^*$, $j' > j$ when $\sigma_i^2 < \sigma_{i'}^2$, $\forall i \in S_j^*, \forall i' \in S_{j'}^*$. Another way to express consecutivity is to say that a core partition can be characterized by a series of "pivotal agents", that is agents who are the most risky agents of the coalition they belong to:

Definition 4 *Given the coalition S_j^* of size n_j in the core-partition, the pivotal agent, defined by the integer $p_j \in \{1, ..., N\}$, associated with S_j^* and the next agent $p_j + 1$ are characterized by variances $\sigma_{p_j}^2$ and $\sigma_{p_j+1}^2$, respectively, such that:*

$$\pi(S_j^* \backslash \{p_j\}) \geq \pi(S_j^*) \text{ and } \pi(S_j^* \cup \{p_j + 1\}) > \pi(S_j^*)$$

Hence,

$$\sigma_{p_j}^2 \leq [2n_j - 1] \sum_{k \in S_j \backslash \{p_j\}} \frac{\sigma_k^2}{(n_j - 1)^2} \tag{7}$$

and

$$\sigma_{p_j+1}^2 > [2n_j + 1] \sum_{k \in S_j} \frac{\sigma_k^2}{{n_j}^2}. \tag{8}$$

A pivotal agent, associated with the $j - th$ coalition S_j^*, is by the consecutivity property, the most risky agent belonging to this club. He is the ultimate agent for which the net effect of his inclusion in the club is beneficial for all other (less risky) agents belonging to the club. Even though he increases the numerator of risk premium paid by all agents in the club (as he is more risky than any of them), thus inflicting a loss to their welfare, his addition also increases its denominator. Actually, his inclusion decreases the risk premium paid by each member of the coalition S_j^*. But if this coalition were to include the next agent, $p_j + 1$, as he is more risky than p_j, the net effect of his inclusion would be negative for all other agent of S_j^*. Therefore they prefer not to let him in. On the whole, the pivotal agent p_j generates the lowest possible risk premium paid by each member of the coalition S_j^*.

[9] The consecutivity property is also obtained in Henriet and Rochet (1987) but it is with respect to probability of accident and not with respect to the variance of shocks.

Let us remark that the definition of a pivotal agent depends neither on the level of the variance nor on the degree of risk aversion. The conditions (7) and (8) can be rewritten as:

$$1 \leq \frac{[2n_j - 1]}{(n_j - 1)^2} \sum_{k \in S_j \setminus \{p_j\}} \prod_{z=k+1}^{p_j - 1} \frac{1}{\lambda_z} \tag{9}$$

and

$$1 > \frac{[2n_j + 1]}{n_j^2} \sum_{k \in S_j} \prod_{z=k+1}^{p_j} \frac{1}{\lambda_z} \tag{10}$$

respectively. What matters in the formation of a coalition, is the heterogeneity of the exposure to risk measured by risk ratios. Consider the less risky agent, characterized by σ_1^2. If he forms a coalition, it is necessarily with a more risky agent. The best choice for him is agent 2 who adds the lesser increase in the common risk premium:

$$\begin{aligned} \pi(\{1,2\}) &= \frac{\alpha}{8}(\sigma_1^2 + \sigma_2^2) = \frac{\alpha \sigma_1^2}{8}(1 + \lambda_2) \\ &< \pi(\{1,i\}) = \frac{\alpha}{8}(\sigma_1^2 + \sigma_i^2) = \frac{\alpha \sigma_1^2}{8}(1 + \prod_{k=2}^{i} \lambda_k), \forall i > 2. \end{aligned}$$

This formula makes clear that agent 1 prefers to form a coalition with agent 2 than with any other agent in society, because he is relatively closer to him in terms of risk. Eventually, what matters for agent 1, is the sequence of risk ratios, that is the individual variances relative to his own. This reasoning can then be generalized to any n-agent coalition so as to obtain the core partition.

Given the consecutivity property of the core partition, the coalition S_j^* is fully defined by the two agents whose indices are $p_{j-1} + 1$ and p_j. In other words, the core partition is defined by the set of pivotal agents. Then we are able to offer the following:

Proposition 2 *The core partition is characterized by a set of J pivotal agents indexed by p_j satisfying (7) - (8) for $j = 1, ..., J-1$ and $\sigma_{p_J}^2 = \sigma_N^2$.*

Remark that the last coalition is peculiar. Its pivotal agent is *per force* agent N who satisfies condition (7) and not condition (8). We refer to this ultimate coalition as the "residual" risk sharing coalition.

Finally, Proposition 2 highlights that, depending on the risk-ratio schedule, our insurance rule may lead to various risk-sharing groups. We could obtain the grand coalition belonging to the core if the risk heterogeneity was sufficiently small. Further, the core partition depends on the assumed insurance rule. Remark that, building on Theorem 2 of Baton and Lemaire (1981), the insurance rule that gives to individual i the following level of consumption:

$$c_i = w_i + \frac{\sum_{k \in S} \varepsilon_k}{n} + \nu + \frac{\alpha}{2n}\left(\frac{\sum_{k \in S} \sigma_k^2}{n} - \sigma_i^2\right)$$

would lead to the grand coalition in the core of the coalition formation game for any risk-ratio schedule. Indeed, this rule yields the following certainty-equivalent income

$$\omega_i(S) = w_i - \frac{\alpha}{2}\frac{\sigma_i^2}{n} - \frac{\alpha}{2}\sigma_\nu^2$$

which monotonously decreases with the size of the risk sharing group. Hence, every individual i wishes to form a club encompassing the whole society.

3.2 Particular risk-ratio schedules.

We have just emphasized the importance of the risk-ratio schedule Λ characterizing a society $\mathbf{I}$ in the endogenous determination of the core partition of this society in differentiated risk-sharing coalitions. In this subsection, we explore the link between patterns of the risk-ratio schedule and the characteristics of the core partition. This allows us to better understand how heterogeneity affects the way individuals congregate so as to share risk. Formally, we want to assess the impact of Λ on the series of pivotal agents, i.e. on the number and size of risk-sharing coalitions.

We restrict the analysis to risk-ratio schedules with simple monotonicity properties: either the series of λ_i increases, decreases or remains constant. We then offer the following

Proposition 3 *If the risk-ratio schedule* $\Lambda = \{\lambda_2, \lambda_3, ..., \lambda_N\}$ *is such that:*

- $\lambda_i = \lambda$, $\forall i \in \mathbf{I}$, *then* $n_j^* = n, \forall j = 1, ..., J-1$;
- $\lambda_i \leq \lambda_{i+1}$, $\forall i \in \mathbf{I}$, *then* $n_j^* \geq n_{j+1}^*, \forall j = 1, ..., J-1$;
- $\lambda_i \geq \lambda_{i+1}$, $\forall i \in \mathbf{I}$, *then* $n_j^* \leq n_{j+1}^*, \forall j = 1, ..., J-1$.

Proof. See Appendix. ■

This proposition makes clear that risk heterogeneity affects the core partition, that is the way agents collectively cope with risk. To understand this proposition, we have to keep in mind that each individual makes his decision about membership with several principles in mind that we have previously uncovered. First, he wants to join the least risky coalition, that is a coalition whose members have a lower variance than himself; second, he wants to be joined by the less risky agents among those who are more risky than himself; third, when selecting (approving the admission of) members in his coalition, he cares about the risk ratios. Consecutivity, the ordering of coalition-risk premia, and the impact of risk ratios in determining the pivotal agent of any coalition are the key elements for understanding how a core partition relates to the risk ratio schedule.

First, consider that the risk ratios are constant and equal to λ. From (9) and (10), we see that inequalities determining the pivotal agent are identical for any club S_j. It turns out that coalitions in the core partition have the same size. In fact, it amounts to say that with constant risk ratios individuals, while deciding to form a risk-sharing group, individuals face the same trade-off whatever the level of their exposure to risk.

Second, consider that the risk ratios are increasing with the rank of individuals. Again, agent 1 selects the pivotal agent of the first club p_1. Consider now the problem facing the agent following the pivotal agent, $p_1 + 1$. He has to select the pivotal agent p_2. The condition determining this agent implies higher values of the risk ratios than the condition determining p_1 (remember that the absolute values of variances of the first agents do not matter). Hence, pondering the benefit of increasing size and cost of higher risk bearing potential members, he chooses a pivotal agent corresponding to a lower size. Repeating the argument, we find that the succeeding club sizes decrease.

Third, the case where the risk ratios are decreasing with the rank of individuals is easily understood by using a similar argument. The less risky agent of the second club has to form the coalition he wants to enter among less (relatively) risky candidates than agent 1. Hence, pondering the benefit of increasing size and cost of higher risk bearing potential members, he chooses a pivotal agent corresponding to a higher size.

3.3 Comparing stochastic distributions and risk sharing groups.

Are more risky societies more segmented in smaller / more numerous risk sharing coalitions? Is it possible to compare the extent of insurance among two societies differing with respect to their risk? These are the issues our coalition formation setup allows us to address. Consider two societies, $\mathbf{I}$ and $\mathbf{I}'$ with a similar number of agents, and characterized by different risk ratio schedules, Λ and Λ'. Hence the core partitions differ. Formally, we shall compare the two core partitions of two different societies using the following

Definition 5 *Let us consider* $\mathcal{P}^* = \left\{S_1^*, ..., S_j^*, ..., S_J^*\right\}$ *the core partition in society* $\mathbf{I}$ *and* $\mathcal{P}'^* = \left\{S_1'^*, ..., S_k'^*, ..., S_{J'}'^*\right\}$ *the core partition in society* $\mathbf{I}'$*. We say that* $\mathbf{I}$ *is weakly less segmented than* $\mathbf{I}'$ *if for any* $i = 1, ..., N$, $i \in S_j$ *in* $\mathcal{P}^*$ *and* $i \in S_k'$ *in* $\mathcal{P}'^*$ *we have* $k \geq j$*.*

This definition is based on the risk-premium ordering property stressed in Proposition 1. It amounts to say that the number of clubs where the risk premium is lower than the risk premium any individual i currently pays in his club is higher in the more segmented society. This definition captures the idea that in a more segmented society any individual i has less possibility to mutualize risk with less risky individuals. It turns out that when society $\mathbf{I}$ is weakly less segmented than $\mathbf{I}'$ the number of risk-sharing coalitions is lower in $\mathbf{I}$.

Proposition 4 *For two societies* $\mathbf{I}$ *and* $\mathbf{I}'$*,* $\mathbf{I}$ *being characterized by* $\Lambda = \{\lambda_2, \lambda_3, ..., \lambda_N\}$ *and* $\mathbf{I}'$ *being characterized by* $\Lambda' = \{\lambda'_2, \lambda'_3, ..., \lambda'_N\}$*, such that* $\lambda_i < \lambda'_i$ *whatever* $i = 2, ..., N$*, then society* $\mathbf{I}$ *is weakly less segmented than* $\mathbf{I}'$*.*

Proof. See Appendix. ■

This Proposition highlights the crucial impact of risk heterogeneity on the allocation of risk in any society. The higher is risk heterogeneity the lower is the chance for any individual to mutualize risk with less risky individuals.

Let us provide an intuition for the proof of Proposition 4 by taking the special case where $\lambda_i = \lambda_{i+1} = \lambda, \lambda'_i = \lambda'_{i+1} = \lambda'$ and $\lambda < \lambda'$ whatever $i = 2, ..., N$. Consider agent 1 in society $\mathbf{I}$. Taking into account that the decision for membership only depends on the risk ratios, and pondering the trade-off between the benefit of size and the cost of higher marginal relative risk, agent 1 is willing to be included in a larger risk sharing coalition in society $\mathbf{I}$ than in society $\mathbf{I}'$. As we have seen, the agent following the first pivotal agent faces the same trade-off as agent 1. Hence the second club is of the same size than the first club, and consequently is of a larger size in society $\mathbf{I}$ than in society $\mathbf{I}'$. Repeating the argument, we find that the number of non-residual clubs is (weakly) reduced in the core partition of society $\mathbf{I}$ compared to the core partition of society $\mathbf{I}'$. The case of decreasing and increasing λs can similarly be dealt with.

Moreover, Proposition 4 stresses the fact that it is impossible to determine a non-ambiguous relationship between risk and social segmentation. We shall use the second-order stochastic dominance (SS-Dominance hereafter) criterion. Taking two societies $\mathbf{I}$ and $\mathbf{I}'$ such that for any agent, ε_i SS-Dominates ε'_i, $\mathbf{I}$ will be considered as more risky than $\mathbf{I}'$. Take two societies $\mathbf{I}$ and $\mathbf{I}'$ satisfying condition on risk ratio schedules of Proposition 4. Proposition 4 is valid eventhough $\sigma_i^2 > \sigma_i'^2$, $\forall i = 1, ..., N$ or $\sigma_i^2 < \sigma_i'^2$, $\forall i = 1, ..., N$. In other words, $\mathbf{I}$ is weakly less segmented than $\mathbf{I}'$ either when $\mathbf{I}$ is more risky than $\mathbf{I}'$ or when $\mathbf{I}'$ is more risky than $\mathbf{I}$.

3.4 Stochastic distributions, social segmentation and risk premiums.

There is another dimension for the assessment of the impact of higher risk: it is the resource cost of dealing with risk. More precisely, even if an increase in risk does not lead to a more segmented society, it may still lead to a higher insurance cost paid by society as a whole.

To get a better understanding of this issue, we first define the aggregate risk premium.

Definition 6 *The aggregate risk premium associated with the core partition $\mathcal{P}$ is defined as:*

$$\overline{\pi}(\mathcal{P}) = \frac{1}{N}\sum_{i=1}^{N}\pi_i = \frac{1}{N}\left(\sum_{j=1}^{J+1} n_j \pi(S_j)\right) \tag{11}$$
$$= \frac{1}{N}\frac{\alpha}{2}\left(\sum_{j=1}^{J+1}\frac{1}{n_j}\sum_{k\in S_j}\sigma_k^2\right) + \frac{\alpha}{2}\sigma_\nu^2$$

The aggregate risk premium is an indicator of the willingness to pay for risk coping, at the society level. Given the partition of society in risk sharing coalition, it is affected by the partition since it shapes the individual risk premia (see above).

Intuitively, more individual risk should lead to a higher aggregate risk premium. An increase in risk heterogeneity, by means of an increase in someone's variance leads to higher individual risk premia, hence higher average risk premium. This is obviously true if the coalition formation is taken as given. Then it is true that if for each agent, her variance increases, then the individual risk premia increase as well as the average risk premium. However this is not necessarily true when agents form their risk-sharing coalitions. It may happen that the change in the whole core-partition leads to different risk-sharing arrangements, the outcome of which is to decrease the average risk premium.

This counter-intuitive result is proven in the following

Proposition 5 *For two societies $\mathbf{I}$ and $\mathbf{I}'$ such that any ε_i SS-Dominates ε_i' for every $i = 1, ..., N$, then society $\mathbf{I}$ may be characterized by a higher average risk premium than $\mathbf{I}'$*

$$\overline{\pi}(\mathcal{P}') < \overline{\pi}(\mathcal{P}). \tag{12}$$

where $\mathcal{P}$ (resp. $\mathcal{P}'$) is the core partition associated with $\mathbf{I}$ ($\mathbf{I}'$).

Proof. See Appendix. ■

Proposition 5 highlights the fact that if endogenous formation of risk sharing group is taken into consideration than we cannot claim that all individuals pay a higher risk premium in a riskier society. We get such a result using Proposition 4 that stresses an ambiguous relationship between social segmentation and second-order stochastic dominance. We consider the case where society $\mathbf{I}'$ is less segmented than $\mathbf{I}$ eventhough agents face more risk (higher idiosyncratic variances) in $\mathbf{I}'$ than in $\mathbf{I}$. Hence, in society $\mathbf{I}'$, risk may be allocated in larger coalitions. In other words, in society $\mathbf{I}'$, individuals have the possibility to mutualize risk on a larger scale. This leads that the sum of these risk premia may be lower in the more risky society and some individuals will pay lower risk premium in this society.

4 Segmentation and empirical issues.

The aim of this section is to study empirical implications of endogenous group formation. In particular, considering the regression usually run to test for optimal risk sharing (Townsend, 1994), we want to examine the influence of group formation on the values of estimated coefficients.

If there is optimal risk sharing then individual consumption of any agent i is determined by aggregate consumption and does not depend on the individual income of agent i. Formally, given a society $\mathbf{I}$, consider the regression of individual consumptions on individual incomes and aggregate consumption expressed as follows:

$$c_{it} = \alpha_i + \beta_i \bar{c}_{It} + \zeta_i y_{it} + u_{it}, \tag{13}$$

where c_{it} is consumption of individual i, $\bar{c}_{It}$ is society $\mathbf{I}$ average consumption, y_{it} is income of individual i, and u_{it} is an error term.

Optimal risk sharing implies $\beta_i = 1$ and $\zeta_i = 0$, for all i. As was said in the introduction, this theory is rejected for village economies.[10] Can this rejection be related to the fact that the grand coalition does not form? In other words, does the partition of the population in multiple risk-sharing groups play a role? Suppose the village is a closed society (which is more or less the case as far as insurance is concerned). It may not be the right cluster of agents with respect to risk. In other words, the grand coalition may not form in this society because it is too diverse in terms of risk. Within the village, there may be such an heterogeneity with respect to risk that agents willingly form smaller risk-sharing coalitions. Villagers may group into smaller "neighborhoods" or districts within the village, according to their particular exposure to risk.

Let us take into account endogenous group formation and consider the core partition $\mathcal{P}^* = \left\{S_1^*, ..., S_j^*, ..., S_J^*\right\}$ of a society $\mathbf{I}$. Hence for an agent i belonging to S_j^* we have $c_i = y_i + \frac{\sum_{z \in S_j^*} \varepsilon_z}{n_j^*} + \nu$.

The OLS estimates of $\widehat{\beta}_i$ and $\widehat{\zeta}_i$ are given by the following formulas:

$$\widehat{\beta}_i = \frac{cov\left(\bar{c}_t, c_{it}\right) var\left(y_{it}\right) - cov\left(y_{it}, c_{it}\right) cov\left(\bar{c}_t, y_{it}\right)}{var\left(\bar{c}_t\right) var\left(y_{it}\right) - \left[cov\left(\bar{c}_t, y_{it}\right)\right]^2}$$

$$\widehat{\zeta}_i = \frac{cov\left(y_{it}, c_{it}\right) var\left(\bar{c}_t\right) - cov\left(\bar{c}_t, c_{it}\right) cov\left(\bar{c}_t, y_{it}\right)}{var\left(y_{it}\right) var\left(\bar{c}_t\right) - \left[cov\left(\bar{c}_t, y_{it}\right)\right]^2}$$

Therefore, in our setup when the core partition forms, these estimates for individual i belonging

[10] The evidence provided by Townsend on three Indian villages has been the subject of close scrutiny. Grimard (1997) rejects perfect risk-sharing using panel data for the Ivory Coast. Ogaki and Zhang (2001), based on Indian and Pakistanese data, reject perfect risk-sharing across villages but do not reject it within villages.

to S_j^* are equal to:

$$\widehat{\beta}_i = \frac{\left(\sigma_\nu^2 + \frac{\sum_{k\in S_j}\sigma_k^2}{n_j N}\right)\left(\sigma_\nu^2+\sigma_i^2\right) - \left(\sigma_\nu^2 + \frac{\sigma_i^2}{n_j}\right)\left(\sigma_\nu^2 + \frac{\sigma_i^2}{N}\right)}{\left(\sigma_\nu^2+\sigma_i^2\right)\left(\sigma_\nu^2 + \frac{\sum_{m\in I}\sigma_m^2}{N^2}\right) - \left[\sigma_\nu^2 + \frac{\sigma_i^2}{N}\right]^2}$$

$$\widehat{\zeta}_i = \frac{\left(\sigma_\nu^2 + \frac{\sigma_i^2}{n_j}\right)\left(\sigma_\nu^2 + \frac{\sum_{m\in I}\sigma_m^2}{N^2}\right) - \left(\sigma_\nu^2 + \frac{\sum_{k\in S_j}\sigma_k^2}{n_j N}\right)\left(\sigma_\nu^2 + \frac{\sigma_i^2}{N}\right)}{\left(\sigma_\nu^2+\sigma_i^2\right)\left(\sigma_\nu^2 + \frac{\sum_{m\in I}\sigma_m^2}{N^2}\right) - \left[\sigma_\nu^2 + \frac{\sigma_i^2}{N}\right]^2}$$

Membership has thus a direct impact on values of $\widehat{\beta}_i$ and $\widehat{\zeta}_i$. If the grand coalition forms, that is if $S_j^* = I$, we have $\widehat{\beta}_i = 1$ and $\widehat{\zeta}_i = 0$. However if society is partitioned into different risk sharing groups, then we have $\widehat{\beta}_i \neq 1$ and $\widehat{\zeta}_i \neq 0$. Running regression (13) without taking into account social segmentation would lead to a dismissal of the theory of optimal risk sharing eventhough individuals are fully insured within their coalitions. Having information on the right groupings of individuals and considering the following regression

$$c_{it} = \alpha_i + \beta_i \overline{c}_{jt}^* + \zeta_i y_{it} + u_{it}$$

with $\overline{c}_{jt}^*$ the average consumption within S_j^* instead of regression (13) would lead to support optimal risk sharing. Indeed, it is easy to check that $\widehat{\beta}_i = 1$ and $\widehat{\zeta}_i = 0$.

However, running regression (13) is not without merits. Indeed in some cases, it allows us to get some information about the core partition, that is the segmentation of society into risk-sharing groups, provided agents are left free to set their risk sharing arrangements as they wish.

If we consider a large population and assume that $\lim_{N\to\infty} \frac{\sigma_N^2}{\sigma_1^2} < \infty$, we obtain (see Appendix for a proof of this approximation)

$$\widehat{\zeta}_i \simeq \frac{1}{n_j} \text{ and } \widehat{\beta}_i \simeq 1 - \frac{1}{n_j}$$

It is therefore obvious that all agents in the same club S_j^* are characterized by the same value of $\widehat{\zeta}_i$ and $\widehat{\beta}_i$. Then, the larger is a club, the lower (higher) is the value of $\widehat{\zeta}_i$ ($\widehat{\beta}_i$) for its members. This implies that individuals in a larger club are able to share risk more efficiently as their individual consumptions depend less from their individual income.

Moreover, for a given income distribution, we show that individual consumption is more dependent of individual income, the deeper is risk heterogeneity between two succeeding individuals, i.e. higher λ_i. This comes from Proposition 3: when $\lambda_i \lesseqgtr \lambda_{i+1}$ for $i = 1, ..., N-1$, then $\widehat{\zeta}_i \gtreqless \widehat{\zeta}_{i'}$ for $i \in S_j^*$ and $i' \in S_{j'}^*$, $j' > j$ and $\widehat{\beta}_i \lesseqgtr \widehat{\beta}_{i'}$ for $i \in S_j^*$ and $i' \in S_{j'}^*$, $j' > j$. This is due to the result that the more heterogenous individuals are with respect to idiosyncratic shocks, the less they are willing to share risk within the same coalition.

Finally, a more segmented society weakly leads to a higher $\widehat{\zeta}_i$ and a lower $\widehat{\beta}_i$ reflecting the fact that the extent of insurance is done on a lower scale. Moreover, computing $\sum_{i=1}^{N} \zeta_i$ gives us the number of clubs of the core partition.

5 Conclusion

Non-financial risk sharing arrangements are widely-used in developing economies. In the absence of proper and well-functioning financial markets, agents rely on informal insurance schemes, often based on a social or geographical (the "village") proximity. Hence it is legitimate to ask how are designed the risk sharing mechanisms in a society and what are their properties and consequences.

In the present paper, considering a society without financial markets and relying on a particular insurance rule, we have studied the endogenous formation of risk-sharing coalitions. Agents can form any possible group but commit to stay in the group once they choose to belong to.

We have obtained a characterization of the optimal segmentation of society with respect to risk, depending on the differentiated idiosyncratic risks born by individuals. It is unique (under mild assumptions), and consecutive: a coalition integrates agents of relatively similar risks. There is perfect risk sharing within a coalition. However, there is no full insurance across society. In other words, the amplitude of risk sharing cannot be studied without precisely taking into account the memberships of risk-sharing groups and their differences.

Then we have discussed the role of risk heterogeneity on the segmentation of society. Focusing on three special cases, we characterize the relationship between the number, the size and the membership of risk-sharing coalitions (i.e. the properties of the segmentation) and the distribution of risk across society.

Turning to the comparison of different societies, we prove some counter-intuitive results. In particular, a more risky society may turn out to be less segmented than a less risky one. This gives additional support to the claim that the way risk-sharing groups are formed is crucial for the understanding of the extent of collective risk-sharing.

Finally, we provided a discussion of the empirical evidence on imperfect risk-sharing in informal societies. We prove how the segmentation of society into multiple risk-sharing coalitions can provide an explanation of some empirical puzzles. This has practical implications for empirical researchers working on the subject: the relevant borders of risk-sharing groups for obtaining an exact picture of risk-sharing schemes must be precisely and carefully assessed.

The present research proves how coalition theory tools can be applied to study the functioning of an economy in the presence of uncertainty when agents are risk-averse. It can be extended along several lines, where these tools are also of potential interest.

First, other insurance rules than the one we have studied here could be analyzed, in particular when the whole income is taxed so as to fund the insurance scheme. It is likely to generate more complex arrangements and no so clear-cut results as in the present setting. More generally, various studies have proven that many devices are used to collectively attain some insurance: cooperative networks (Fafchamps and Lund, 2003), altruism (Foster and Rosenzweig, 2001), marriages (Rosenzweig and Stark, 1989), state contingent loans (Udry, 1994).

Second, the assumption of full-commitment (the impossibility of individual defection) could be relaxed so as to assess the impact of defection on the number and the size of risk-sharing coalitions.

Third, the introduction of risk-sharing coalitions, maybe enacted by an entrepreneur, in the context of financial markets, is of clear interest. Even in economies with financial markets, some form of coalitions exist: they may take the form of insurance companies, informal risk sharing arrangements, or even publicly organized welfare institutions. We think that coalition theory is a relevant tool to investigate these issues.

Finally, the link between risk-sharing and growth could be studied when agents are both able to accumulate some production factors and voluntarily form coalitions. This could shed some light on the relationship between risk-sharing and risk-taking over the long term. [11]

[11] Jaramillo, Kempf and Moizeau (2005) study the relationship between coalition formation and growth, when agents are unequally endowed in a primitive stage, in an endogenous growth model, but not taking into account uncertainty.

References

[1] Arcand, J.-L. and M. Fafchamps, 2008, "Matching in Community-Based Organizations", mimeo.

[2] Arrow, K., 1964, "The Role of Securities in the Optimal Allocation of Risk Bearing", *Review of Economic Studies*, 31, 91-96.

[3] Attanasio, O., and S. Davis, 1996, "Relative Wage Movements and the Distribution of Consumption", *Journal of Political Economy*, 104, 1227-1262.

[4] Banerjee, S., H. Konishi and T. Sonmez, 2001, "Core in a Simple Coalition Formation Game", *Social Choice and Welfare*, 18, 135-153.

[5] Barr, A., and G. Genicot, 2008, "Risk Sharing, Commitment and Information: An Experimental Analysis", *Journal of the European Economic Association*, forthcoming.

[6] Baton, B., and J. Lemaire, 1981, "The Core of a Reinsurance Market", *Astin Bulletin*, 12, 57-71.

[7] Bloch, F., Genicot, G. and D. Ray, 2008, "Informal Insurance in Social Networks", *Journal of Economic Theory*, 143, 36-58.

[8] Borch, R., 1962, "Equilibrium in a Reinsurance Market", *Econometrica*, 30, 424-444.

[9] Bramoullé, Y., and R. Kranton, 2007, "Risk-Sharing Networks", *Journal of Economic Behavior and Organization*, 64, 275-294.

[10] Dercon, S., 2004, "Income Risk, Coping Strategies and Safety Nets", in *Insurance against Poverty*, Oxford University Press, Oxford: U.K..

[11] Dercon, S., De Weerdt, J., Bold, T., and Pankhurst, A., 2006, "Group-Based Funeral Insurance in Ethiopia and Tanzania", *World Development*, 34, 685-703.

[12] Dubois, P. 2002, "Consommation, partage de risque et assurance informelle : développements théoriques et tests empiriques récents", *L'Actualité économique*, 116-149.

[13] Dubois, P. 2006, "Heterogeneity of Preferences, Limited Commitment and Coalitions: Empirical Evidence on the Limits to Risk Sharing in Rural Pakistan", http://gremaq.univ-tlse1.fr/perso/dubois/duboisen.htm.

[14] Dubois, P., Jullien, B. and Magnac Th., 2008, “Formal and Informal Risk Sharing in LDCs: Theory and Empirical Evidence”, *Econometrica*, 76, 679-725.

[15] Fafchamps, M., 2008, “Risk Sharing Between Households”, in J. Benhabib, A. Bisin and M. Jackson eds, *Handbook of Social Economics*, Elsevier.

[16] Fafchamps, M., and Lund, 2003, “Risk-Sharing Networks in Rural Philippines”, *Journal of Development Economics*, 71, 261-287.

[17] Foster and Rosenzweig, 2001, “Imperfect commitment, altruism and the Family: Evidence from Transfer Behavior in Low-Income Rural Areas”, *Review of Economics and Statistics*, 83, 389–407.

[18] Genicot, G., and D. Ray, 2003, “Group Formation in Risk-Sharing Arrangements”, *Review of Economic Studies*, 40, 87-113.

[19] Grimard, F., 1997,“Household Consumption Smooting through Ethnic Ties: Evidence from Côte d’Ivoire”, *Journal of Development Economics*, 53, 391-422.

[20] Hayashi, F., J. Altonji and L. Kotlikoff, 1996, “Risk Sharing between and within Families”, *Econometrica*, 64, 261-294.

[21] Henriet, D. and J.-Ch. Rochet, 1987, “Some Reflections on Insurance Pricing”, *European Economic Review*, 31, 863-885.

[22] Jaramillo, F., H. Kempf and F. Moizeau, 2003, “Inequality and Club Formation”, *Journal of Public Economics*, 931-955.

[23] Jaramillo, F., H. Kempf and F. Moizeau, 2003, “Inequality and Growth Clubs”, in G. Demange and M. Wooders, *Group Formation in Economics*, Cambridge University Press.

[24] Kazianga, H. and Ch. Udry, 2005, “Consumption Smoothing ? Livestock, Insurance and Drought in Rural Burkina Faso”, mimeo.

[25] Kimball, M., 1988, “Farmer Cooperatives as Behavior Toward Risk”, *American Economic Review*, 78, 224-232.

[26] Kocherlakota, N., 1996, “Implications of Efficient Risk Sharing without Commitment”, *Review of Economic Studies*, 63, 595-609.

[27] Ligon, E., Thomas, J. and T. Worral, 2002, “Mutual Insurance and Limited Commitment: Theory and Evidence in Village Economies”, *Review of Economic Studies*, 69, 115-139.

[28] Ljungqvist, L. and Th. Sargent, *Recursive Macroeconomic Theory*, MIT Press, 2004.

[29] Mace, B., 1991,“Full insurance in the presence of aggregate uncertainty”, *Journal of Political Economy*, 99, 928-956.

[30] Ogaki, M. and Q. Zhang, 2001, “Decreasing Relative Risk Aversion and Tests of Risk-Sharing”, *Econometrica*, 69, 515-526.

[31] Ray, D., *Development Economics*, Princeton University Press, 1998.

[32] Rosenzweig and Stark, 1989, “Consumption Smoothing, Migration and Marriage,: Evidence from Rural India”, *Journal of Political Economy*, 97, 905-926.

[33] Taub, B. and Chade, H., 2002, “Segmented Risk in a Continuous-Time Setting”, *Economic Theory*, 20, 645-675.

[34] Townsend, R., 1994, “Risk and Insurance in Village India”, *Econometrica*, 62, 539-591.

[35] Townsend, R., 1995, “Consumption Insurance: An Evaluation of Risk-Bearing Systems in Low-Income Economies”, *Journal of Economic Perspectives*, 9, 83-102.

[36] Udry, 1994, “Risk and Insurance in a Rural Credit Market: An Empirical Investigation of Northern Nigeria”, *Review of Economic Studies*, 61, 495-526.

[37] Wilson, R., 1968, “The Theory of Syndicates”, *Econometrica*, 36, 119-132.

6 Appendix.

6.1 Proof of Proposition 1.

Existence. Given the value of $V_i(S_j)$, if for the two groups S_j and $S_{j'}$, we have:

$$V_i(S_j) \geq V_i(S_{j'}) \Longleftrightarrow \frac{\alpha^2}{2\binom{n_j^2}{}} \sum_{k\in S_j} \sigma_k^2 \leq \frac{\alpha^2}{2\binom{n_{j'}^2}{}} \sum_{k\in S_{j'}} \sigma_k^2$$

then we have:

$$V_{i'}(S_j) \geq V_{i'}(S_{j'}), \forall i' \in I.$$

This implies that the common ranking property is satisfied, that is:

$$\forall i,k \in \mathbf{I},\ V_i(S_j) \geq V_i(S_{j'}) \Leftrightarrow V_k(S_j) \geq V_k(S_{j'}).$$

According to Banerjee *et al.* (2001), the common ranking property implies that a core partition exists.

Proof of (i): Uniqueness.

Here we assume consecutivity. Let us define p_j the most risky agent of the consecutive group $S_j \backslash \{p_j\}$ with size n_j satisfying the two following inequalities:

$$\sigma_{p_j}^2 \leq [2n_j+1] \sum_{k\in S_j\backslash\{p_j\}} \frac{\sigma_k^2}{n_j^2}$$

and

$$\sigma_{p_j+1}^2 > [2n_j+3] \sum_{k\in S_j} \frac{\sigma_k^2}{(n_j+1)^2}.$$

Let us consider the consecutive group S_j whose lowest-individual-risk agent is i. Given the definition of the most risky agent, we can introduce the two following functions: $\Gamma(n) = \frac{n}{2n+1}$ and $\Theta(i,n) = \frac{1}{n}\frac{\sum_{k=i}^{i+n-1}\sigma_k^2}{\sigma_{i+n}^2}$ with $n = 1, ..., N-i+1$. Let us denote $n^*(i)+1$ the size of group S_j such that:

$$\Gamma(n^*(i)) \leq \Theta(i, n^*(i))$$

and

$$\Gamma(n^*(i)+1) > \Theta(i, n^*(i)+1)$$

It is easy to check that $\Gamma(n)$ is an increasing function of n and $\Gamma(1) = \frac{1}{3}$. Given $\Theta(i,1) = 1 > \Gamma(1)$, if $\Theta(i,n)$ is decreasing with respect to n whatever $i \in I$ and $n \leq N-i$, then $n^*(i)$ is unique as $\Gamma(n) \leq \Theta(i,n)$ for $n \leq n^*(i)$ and $\Gamma(n) > \Theta(i,n)$ for $n > n^*(i)$.

The function $\Theta(i,n(i))$ is decreasing if and only if:

$$\Delta\Theta(i,n) \equiv \Theta(i,n(i)+1) - \Theta(i,n(i)) = \frac{1}{n+1}\frac{\sigma_{i+n}^2 + \sum\limits_{k=i}^{i+n-1}\sigma_k^2}{\sigma_{i+n+1}^2} - \frac{1}{n}\frac{\sum\limits_{k=i}^{i+n-1}\sigma_k^2}{\sigma_{i+n}^2} < 0 \Longleftrightarrow$$

$$\psi\left(i,n\right) = n\sigma_{i+n}^2 - \left((n+1)\frac{\sigma_{i+n+1}^2}{\sigma_{i+n}^2} - n\right)\left(\sum_{k=i}^{i+n-1}\sigma_k^2\right) < 0.$$

Let us consider the function $\psi\left(i,n\right)$. It is negative for all i, $n \leq N-i$ if

$$\psi\left(i,1\right) = \sigma_{i+1}^2 - \left(2\frac{\sigma_{i+2}^2}{\sigma_{i+1}^2} - 1\right)\left(\sigma_i^2\right) \leq 0 \text{ and } \Delta\psi\left(i,n\right) \equiv \psi\left(i,n+1\right) - \psi\left(i,n\right) \leq 0.$$

Defining $\lambda_{i+1} = \frac{\sigma_{i+1}^2}{\sigma_i^2}$, the inequality $\psi\left(i,1\right) \leq 0$ is equivalent to

$$\frac{\left(\frac{\sigma_{i+1}^2-\sigma_i^2}{\sigma_i^2}\right) - \left(\frac{\sigma_{i+2}^2-\sigma_{i+1}^2}{\sigma_{i+1}^2}\right)}{\left(\frac{\sigma_{i+2}^2-\sigma_{i+1}^2}{\sigma_{i+1}^2}\right)} = \frac{\lambda_{i+1} - \lambda_{i+2}}{\lambda_{i+2} - 1} \leq 1 \tag{14}$$

Moreover, $\forall n \geq 1$, $\Delta\psi\left(i,n\right) \leq 0$ is equivalent to

$$\Delta\psi\left(i,n\right) = ((n+1)\lambda_{i+n+1} - \left(n+2\right)\lambda_{i+n+2} + 1)(\sigma_{i+n}^2 + \left(\sum_{k=i}^{i+n-1}\sigma_k^2\right)) \leq 0 \Leftrightarrow$$

$$\frac{\lambda_{i+n+1} - \lambda_{i+n+2}}{\lambda_{i+n+2} - 1}(n+1) \leq 1.$$

Defining $z \equiv i+n+1$, we can rewrite this inequality as follows:

$$\frac{\lambda_z - \lambda_{z+1}}{\lambda_{z+1} - 1}(z+1)(\frac{z-i}{z+1}) \leq 1$$

As $0 \leq \frac{(z-i)}{(z+1)} \leq 1$, we deduce that if for all $z = 3,...,N-1$, $\frac{\lambda_z - \lambda_{z+1}}{\lambda_{z+1}-1}(z+1) \leq 1$, then $\triangle\psi\left(i,n\right) \leq 0$.

Given equation (14), we deduce that if for all $z = 2,...,N-1$, $\frac{\lambda_z - \lambda_{z+1}}{\lambda_{z+1}-1}\left(z+1\right) \leq 1$ then $\triangle\psi\left(i,n\right) \leq 0$ and $\psi\left(i,1\right) \leq 0$, $\forall i = 1,...,N$.

Hence, when for all $z = 2,...,N-1$, $\frac{\lambda_z - \lambda_{z+1}}{\lambda_{z+1}-1}\left(z+1\right) \leq 1$, we deduce that there is a unique size n_j for the club S_j.

Proof of (ii): Consecutivity.

By contradiction, let us consider a core-partition $\mathcal{P}^*$ characterized by some non consecutive groups, that is, there exist individual $i, \widetilde{i} \in S_j^*$ and $i' \in S_{j'}^*$ with $i < i' < \widetilde{i}$.

Suppose first that $\pi(S_j^*) \geq \pi(S_{j'}^*)$. As $i < i' < \widetilde{i} \Longleftrightarrow \sigma_i^2 < \sigma_{i'}^2 < \sigma_{\widetilde{i}}^2$, we have $\pi(S_{j'}^*) > \pi((S_{j'}^* \backslash \{i'\}) \cup \{i\})$, which leads to

$$\forall z \in (S_{j'}^* \backslash \{i'\}) \cup \{i\},\ V_z((S_{j'}^* \backslash \{i'\}) \cup \{i\}) > V_z(\mathcal{P}^*).$$

Second, assume that $\pi(S^*_{j'}) \geq \pi(S^*_j)$. We have $\pi(S^*_j) > \pi((S^*_j \backslash \{\widetilde{i}\}) \cup \{i'\})$, which leads to

$$\forall z \in (S^*_j \backslash \{\widetilde{i}\}) \cup \{i'\},\ V_z((S^*_j \backslash \{\widetilde{i}\}) \cup \{i'\}) > V_z(\mathcal{P}^*).$$

Hence a contradiction with the fact that $\mathcal{P}^*$ is assumed to be a core-partition.

Proof of (iii): Risk premium ordering.

Consider the first group S^*_1. Let us define the group $\pounds_j = \left\{1, ..., n^*_j\right\}$ which is consecutive, comprised of the lowest-individual-risk agents and has the same size as group S^*_j. From the definition of the core-partition, we know that, $\forall \pounds \subset I$, $\forall z \in S^*_1$ and $\pounds$, $V_z(S^*_1) \geq V_z(\pounds)$ and in particular $\forall z \in S^*_1$ and $\pounds_j$, $\forall j = 2, ..., J$, $V_z(S^*_1) > V_z(\pounds_j)$ which means that $\forall \pounds_j$, $\pi(S^*_1) < \pi(\pounds_j)$. Moreover, given the consecutivity property, it is easy to show that $\pi(\pounds_j) < \pi(S^*_j)$, $\forall j > 1$. Hence, $\pi(S^*_1) < \pi(S^*_j)$. Considering the population $I \backslash (S^*_1 \cup S^*_2 \cup ... \cup S^*_j)$, the same argument can be applied for S^*_{j+1} leading to the result $\pi(S^*_1) < \pi(S^*_2) < \pi(S^*_3) < ... < \pi(S^*_j) < ... < \pi(S^*_{J-1})$.

This completes the proof.

6.2 Proof of Proposition 3.

Let us first denote $S^c(i)$ any consecutive group whose less risky individual is i. We will denote by $\widehat{n}(i)$ the size of $S^c(i)$ such that $\widehat{n}(i) = \arg\max V_i(S^c(i))$ in the subset $\mathbf{I} \backslash \{1, 2, ..., i-1\}$, for a risk ratio schedule Λ. Hence, $\widehat{n}(i)$ satisfies inequalities characterizing a pivotal agent:

$$\Gamma(\widehat{n}(i) - 1) \leq \Theta(i, \widehat{n}(i) - 1) \tag{15}$$

and

$$\Gamma(\widehat{n}(i)) > \Theta(i, \widehat{n}(i)) \tag{16}$$

From Proof of Proposition 1, we know that $\Gamma(n)$ is an increasing function of n and, under some condition, $\Theta(i, n)$ decreases with respect to n. We can rewrite $\Theta(i, n)$ as follows: $\frac{1}{n} \frac{\sum_{k=i}^{i+n-1} \sigma^2_k}{\sigma^2_{i+n}}$

$$\Theta(i, n) = \frac{1}{n} \sum_{v=i}^{i+n-1} \prod_{z=v+1}^{i-1+n} \frac{1}{\lambda_z}$$

$\Theta(i, n)$ is a function of i such that:

(i) When $\lambda_z = \lambda$, $\forall z \in I$, then $\Theta(i, n) = \Theta(i', n)$ $\forall i, i'$.

(ii) When $\lambda_z \leq \lambda_{z+1}$, $\forall z \in I$, then $\Theta(i, n) \geq \Theta(i', n)$ for $i < i'$.

(iii) When $\lambda_z \geq \lambda_{z+1}$, $\forall z \in I$, then $\Theta(i, n) \leq \Theta(i', n)$ for $i < i'$.

Hence, items (i), (ii), (iii) and inequalities (15) and (16) lead to Proposition 3.

6.3 Proof of Proposition 4.

We will denote $S^c(i)$ the consecutive club whose lowest risky agent is individual i. Let us denote by $\widehat{n}(i|\,\Lambda)$ the size of $S^c(i)$ such that $\widehat{n}(i|\,\Lambda) = \arg\max V_i(S^c(i))$, for a risk ratio schedule Λ.

We first offer the following Lemma

Lemma 1 *For two societies I and I' characterized respectively by $\Lambda = \{\lambda_2, \lambda_3, ..., \lambda_N\}$ and $\Lambda' = \{\lambda'_2, \lambda'_3, ..., \lambda'_N\}$ with $\lambda_z < \lambda'_z$ for $z = 2, ..., N$, we have $\widehat{n}(i|\,\Lambda) \geq \widehat{n}(i|\,\Lambda')$.*

Proof. Let us denote $\Theta(\overrightarrow{\lambda}_{i,n}) \equiv \Theta(i,n) = \frac{1}{n}\sum\limits_{v=i}^{i+n-1}\prod\limits_{z=v+1}^{i-1+n}\frac{1}{\lambda_z}$ with $\overrightarrow{\lambda}_{i,n} = (\lambda_{i+1}, \lambda_{i+2}, ..., \lambda_{i+n-1})$. Hence for two vectors $\overrightarrow{\lambda}_{i,n}$ and $\overrightarrow{\lambda'}_{i,n}$ where $\lambda'_z > \lambda_z$, $\forall z = i+1, ..., i+n-1$, we have $\Theta(\overrightarrow{\lambda}_{i,n}) > \Theta(\overrightarrow{\lambda'}_{i,n})$, $\forall i \in I$ and $\forall n = 1, ..., N-i+1$. Given inequalities (15) and (16) and that $\Theta(\overrightarrow{\lambda}_{i,n}) > \Theta(\overrightarrow{\lambda'}_{i,n})$, it is thus easy to deduce that the optimal size of the consecutive group beginning with agent i is larger under $\Lambda = \{\lambda_2, \lambda_3, ..., \lambda_N\}$ than under $\Lambda' = \{\lambda'_2, \lambda'_3, ..., \lambda'_N\}$. Hence, Lemma 1. ■

Lemma 2 *Let us denote $p_{S^c(i)}$ the pivotal agent of any consecutive club $S^c(i)$. For any society I, any $i' < i$ we have $p_{S^c(i)} > p_{S^c(i')}$.*

Proof. We know that $\sigma^2_{p_{S^c(i)}}$ satisfies

$$\sigma^2_{p_{S^c(i)}} \leq \left[2n^c_j - 1\right]\sum_{k=i}^{p_{S^c(i)}-1}\frac{\sigma^2_k}{(n^c_j-1)^2} \tag{17}$$

and

$$\sigma^2_{p_{S^c(i)}+1} > \left[2n^c_j + 1\right]\sum_{k=i}^{p_{S^c(i)}}\frac{\sigma^2_k}{n^{c2}_j}. \tag{18}$$

Let us consider the consecutive club $S^c(i') = \{i', ..., p_{S^c(i)}+1\}$. By assumption on the individuals ordering, we have

$$\sum_{k=i}^{p_{S^c(i)}}\frac{\sigma^2_k}{n^c_j} > \sum_{k=i'}^{p_{S^c(i)}}\frac{\sigma^2_k}{n'^c_j} \text{ for any } i' < i$$

Hence as $\frac{2n'+1}{n'} < \frac{2n+1}{n}$ for any $n' > n$, we thus have

$$\sigma^2_{p_{S^c(i)}+1} > \left[2n^c_j + 1\right]\sum_{k=i}^{p_{S^c(i)}}\frac{\sigma^2_k}{n^{c2}_j} > \left[2n'^c_j + 1\right]\sum_{k=i'}^{p_{S^c(i)}}\frac{\sigma^2_k}{n'^{c2}_j}, \text{ for any } i' < i.$$

We easily deduce that $p_{S^c(i)} > p_{S^c(i')}$ for any $i' < i$. ■

Let us now define $p^*_j(\Lambda)$ the pivotal agent of club S_j in the core partition associated to Λ. Let us consider individual 1. Using Lemma 1, for $\Lambda = \{\lambda_2, \lambda_3, ..., \lambda_N\}$ and $\Lambda' = \{\lambda'_2, \lambda'_3, ..., \lambda'_N\}$

with $\lambda_z < \lambda'_z$ for $z = 2, ..., N$, we deduce that $p_1^*(\Lambda) \geq p_1^*(\Lambda')$. Using Lemma 2, we thus deduce that $p_2^*(\Lambda) \equiv p^*_{S^c(p_1(\Lambda)+1)}(\Lambda) > p_{S^c(p_1(\Lambda')+1)}(\Lambda)$. Using again Lemma 1 allows us to say that $p_{S^c(p_1(\Lambda')+1)}(\Lambda) \geq p^*_{S^c(p_1(\Lambda')+1)}(\Lambda') \equiv p_2^*(\Lambda')$. Hence $p_2^*(\Lambda) \geq p_2^*(\Lambda')$. Iterating this reasoning until $j = J$ allows us to say that $p_j^*(\Lambda) \geq p_j^*(\Lambda')$ for any $j = 1, ..., J$. Hence for any $i = 1, ..., N$ we thus deduce that the number of pivotal agents associated with Λ such that $p_j^*(\Lambda) \leq i$ compared to the number of pivotal agents associated with Λ' such $p_j(\Lambda') \leq i$ is higher for Λ than Λ'. This ends proof of Proposition 4.

6.4 Proof of Proposition 5.

Let us consider the two following societies. In society $\mathbf{I'}$, there are N individuals characterized with $\sigma_i'^2 = 1$. Hence, $\mathcal{P}' = \{I'\}$. In society $\mathbf{I}$, n_1 individuals are characterized with σ_1^2 and n_2 individuals are characterized with σ_2^2 such that $1 > \sigma_2^2 > \sigma_1^2$. Let us choose σ_1^2 and σ_2^2 such that $\mathcal{P} = \{S_1^*, S_2^*\}$ with S_1^* (respectively S_2^*) comprised of the n_1 (respectively n_2) individuals with σ_1^2 (respectively σ_2^2). Hence, σ_1^2, σ_2^2, n_1, n_2 and x are such that

$$\frac{n_1\sigma_1^2 + x\sigma_2^2}{(n_1 + x)^2} > \frac{n_1\sigma_1^2}{(n_1)^2} \text{ for all } x \in \{1, ..., n_2\}$$

which is equivalent to

$$\sigma_2^2 > \sigma_1^2 \frac{2n_1 + x}{n_1}$$

As the RHS is an increasing function of x, a sufficient condition for this inequality to hold is

$$\sigma_2^2 > \sigma_1^2 \frac{2n_1 + n_2}{n_1}.$$

Thus, given both core partitions, we deduce that

$$\overline{\pi}(\mathcal{P}) = \frac{\alpha}{2}\frac{1}{N}\left(n_1\frac{n_1\sigma_1^2}{(n_1)^2} + n_2\frac{n_2\sigma_2^2}{(n_2)^2}\right) + \frac{\alpha}{2}\sigma_\nu^2 \text{ and } \overline{\pi}(\mathcal{P}') = \frac{\alpha}{2}\frac{1}{N} + \frac{\alpha}{2}\sigma_\nu^2$$

In order to have $\overline{\pi}(\mathcal{P}) > \overline{\pi}(\mathcal{P}')$, σ_1^2 and σ_2^2 must be such that:

$$\sigma_1^2 + \sigma_2^2 > 1$$

Clearly there exist σ_1^2 and σ_2^2 that satisfy the following inequalities:

$$1 > \sigma_1^2;\ 1 > \sigma_2^2;\ \sigma_1^2 + \sigma_2^2 > 1;\ \sigma_2^2 > \sigma_1^2 \frac{2n_1 + n_2}{n_1}.$$

For example, take $\sigma_1^2 < \frac{n_1}{3n_1+n_2}$ which satisfies $1 > \sigma_1^2$. As $\sigma_1^2 > 0$, we have $1 > 1 - \sigma_1^2$. Notice that $\sigma_1^2 < \frac{n_1}{3n_1+n_2}$ is equivalent to $1 - \sigma_1^2 > \sigma_1^2 \frac{2n_1+n_2}{n_1}$. So that for any σ_2^2 such that $1 > \sigma_2^2 > 1 - \sigma_1^2$, the four inequalities are satisfied.

6.5 Regression coefficients β_i and ζ_i.

If our model is true, we have obtained that

$$\beta_i = \frac{\left(\sigma_\nu^2 + \frac{\sum_{k\in S_j}\sigma_k^2}{n_j N}\right)\left(\sigma_\nu^2+\sigma_i^2\right) - \left(\sigma_\nu^2+\frac{\sigma_i^2}{n_j}\right)\left(\sigma_\nu^2+\frac{\sigma_i^2}{N}\right)}{\left(\sigma_\nu^2+\sigma_i^2\right)\left(\sigma_\nu^2+\frac{\sum_{m\in I}\sigma_m^2}{N^2}\right) - \left[\sigma_\nu^2+\frac{\sigma_i^2}{N}\right]^2}$$

$$\zeta_i = \frac{\left(\sigma_\nu^2+\frac{\sigma_i^2}{n_j}\right)\left(\sigma_\nu^2+\frac{\sum_{m\in I}\sigma_m^2}{N^2}\right) - \left(\sigma_\nu^2 + \frac{\sum_{k\in S_j}\sigma_k^2}{n_j N}\right)\left(\sigma_\nu^2+\frac{\sigma_i^2}{N}\right)}{\left(\sigma_\nu^2+\sigma_i^2\right)\left(\sigma_\nu^2+\frac{\sum_{m\in I}\sigma_m^2}{N^2}\right) - \left[\sigma_\nu^2+\frac{\sigma_i^2}{N}\right]^2}$$

Let us suppose that $\lim_{N\to\infty}\frac{\sigma_N^2}{\sigma_1^2}<\infty$. This implies that

$$\lim_{N\to\infty}\frac{\sigma_N^2}{N\sigma_1^2}=0$$

As $\frac{\sum_{m\in I}\sigma_m^2}{N^2\sigma_i^2}\leq\frac{\sigma_N^2}{\sigma_1^2 N}$ $\forall i=1,...,N$, we thus easily deduce that when $\lim_{N\to\infty}\frac{\sigma_N^2}{\sigma_1^2}<\infty$, then

$$\lim_{N\to\infty}\frac{\sum_{m\in I}\sigma_m^2}{N^2\sigma_i^2}=0, \forall i=1,...,N$$

Let us start with ζ_i. Dividing by σ_i^2, it can be expressed as follows

$$\zeta_i\left(S_j\right) = \frac{\left(\frac{\sigma_\nu^2}{\sigma_i^2}+\frac{1}{n_j}\right)\left(\frac{\sigma_\nu^2}{\sigma_i^2}+\frac{\sum_{m\in I}\sigma_m^2}{\sigma_i^2 N^2}\right) - \left(\frac{\sigma_\nu^2}{\sigma_i^2}+\frac{\sum_{k\in S_j}\sigma_k^2}{\sigma_i^2 n_j N}\right)\left(\frac{\sigma_\nu^2}{\sigma_i^2}+\frac{1}{N}\right)}{\left(\frac{\sigma_\nu^2}{\sigma_i^2}+1\right)\left(\frac{\sigma_\nu^2}{\sigma_i^2}+\frac{\sum_{m\in I}\sigma_m^2}{\sigma_i^2 N^2}\right) - \left[\frac{\sigma_\nu^2}{\sigma_i^2}+\frac{1}{N}\right]^2}$$

As $\lim_{N\to\infty}\frac{\sum_{m\in I}\sigma_m^2}{N^2\sigma_i^2}=0, \forall i=1,...,N$,

$$\zeta_i \simeq \frac{\left(\frac{\sigma_\nu^2}{\sigma_i^2}+\frac{1}{n_j}\right)\left(\frac{\sigma_\nu^2}{\sigma_i^2}\right) - \left(\frac{\sigma_\nu^2}{\sigma_i^2}+\frac{\sum_{k\in S_j}\sigma_k^2}{\sigma_i^2 n_j N}\right)\left(\frac{\sigma_\nu^2}{\sigma_i^2}\right)}{\left(\frac{\sigma_\nu^2}{\sigma_i^2}+1\right)\left(\frac{\sigma_\nu^2}{\sigma_i^2}\right) - \left[\frac{\sigma_\nu^2}{\sigma_i^2}\right]^2}$$

Hence,

$$\zeta_i\left(S_j\right)\simeq\frac{1}{n_j}-\frac{\sum_{k\in S_j}\sigma_k^2}{\sigma_i^2 n_j N}$$

which leads to

$$\zeta_i\left(S_j\right)\simeq\frac{1}{n_j}.$$

The expression of β_i is obtained using a similar reasoning.