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Learning and convergence in networks*

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Abstract

We study the convergence properties of learning in social and economic networks. We characterize the effect of network structure on the long-run convergent behavior and on the time of convergence to steady state. Agents play a repeated game governed by two underlying behavioral rules; they are myopic and boundedly rational. Under this setup, the long-run limit of the adoptive process converges to a unique equilibrium. We treat the dynamic process as a Markov chain and derive the bound for the convergence time of the chain in terms of the stationary distribution of the initial and steady state population configurations, and the spectrum of the transition matrix. We in turn differentiate between the two antagonistic effects of the topology of the interaction structure on the convergence time; that through the initial state and that through the asymptotic behavior. The effect through the initial state favors local interactions and sparsely connected network structures, while that through the asymptotic behavior favors densely and uniformly connected network structures. The main result is that the most efficient network topologies for faster convergence or learning are those where agents belong to subgroups in which the inter-subgroup interactions are "weaker" relative to within subgroup interactions. We further show that the inter-subgroup interactions or long-ties should not be "too weak" lest the convergence time be infinitely long and hence slow learning.

Keywords: Learning, Local interactions, Coordination game, Strategic complementarity, Convergence time, Convergence rate, Markov chain.
JEL Codes: C73, D83

1 Introduction

In a range of economic and social environments, individuals have to make decisions without complete information about the rewards from the alternatives they have to choose from. In such environments individuals rely on their past experience and the experience of others; colleagues,

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friends, neighbors or relatives when taking a decision. The dynamic process thus involves feedback effects in which individuals generally form expectations based on their experience and in turn make decisions based on their expectations. The various situations in which *learning* from the experience of others matters range from simple pure coordination games to general decision problems with complementarity to adoption as well as collective action problems.

The examples of pure coordination problems include social norms such as punctuality, the side of the road to drive on, equity norms or legal contracts. Young [1996, 1998b] shows that these social and cultural traits can be viewed as an equilibrium state resulting from evolutionary processes of repeated interaction among individuals. The examples of strategic complementarity include the decision problems where one has to take into account compatibility issues; adoption of information technologies, for example when making a choice between mobile phone providers, one has to consider the choice made by the persons whom he wishes to communicate with, and the same applies to an individual deciding whether to adopt a fax machine. Economides and Himmelberg [1995] empirically show that interaction effects are present in the adoption of such technologies.[1] Other examples of complementarity problems include adoption of consumer products and durable goods with positive returns to adoption, which can range from simple conformity effects [Arthur and Lane, 1993, Bernheim, 1994, Banerjee, 1992, Brock and Durlauf, 2001], to making choices based on the accessibility of post-purchase services, specially in the case of durable goods [Katz and Shapiro, 1985]. Knowing the experience of others and hence expectations of their choices or actions is also vital in the problems of collective political actions such as protests and revolutions, and participation in crime [Chwe, 2000, Glaeser et al., 1995].

In all the above examples, since the experience of others matters in decision making, it naturally follows that the structure of the interactions or the topology of the social network matters; that is it should influence the equilibrium outcomes and the time and speed of convergence of the dynamic process to the steady state(s).

In this paper, we study the convergent behavior and the convergence time of learning dynamics in social and economic networks in general. We shall follow the traditional learning models in the line of Kandori et al. [1993], Blume [1993], Anderlini and Ianni [1996], Young [1993], Ellison [1993], in which individual interactions are modeled in a stochastic evolutionary manner.[2] The convergence results in these models have been established, and of particular interest is that of Anderlini and Ianni [1996] who prove the existence of equilibrium distribution for learning in generalized network topologies. The prediction from the evolutionary stochastic models is that if the level of noise is kept as minimal as possible, then the emergent long-run population state is that in which all agents take the same choice (*conform*), but this outcome depends on the topology of the social network in that it is possible to find a situation in which the coexistence of the choices within the population persists for very long periods of time. The later result has been derived by considering specific types of topologies and a small tractable number of agents, which leaves some questions about the effect of *general topologies* of the network structure on

[1] A related theoretical paper is by Oren and Smith [1981].

[2] There is also a large set of literature on Bayesian learning which aims to model environments with information sharing problems. Unlike in these models where an agent's payoff from an action depends only on the action chosen by him, we shall mostly consider the case in which others' actions have a direct impact on ones payoff. For more on Bayesian learning, see Bala and Goyal [1998], Acemoglu et al. [2011] and the references therein.

the convergent behavior of the learning process unanswered. The reason for this problem is that one is always forced to make a trade-off between capturing the details of the learning process and capturing the complexities involved in the structure of the networks of interactions. Here, we point out some of the questions that remain largely open. The first concerns the identification of the general features of the topologies of the social network that lead to conformism and the ones that sustain diversity of strategies. The second question concerns the efficiency in terms of the time and the speed of convergence to the steady state and/or transition between steady states in the case of multiple steady states. The convergent behavior is normally derived for asymptotic time scales, so it is necessary to determine if such time-scale is within the limits of economic interest. The third and closely related question pertains to the possibility of learning as compared to the speed of learning. In other words, do the same network structures that exhibit higher probability of learning support faster convergence to steady state. In some economic situations the relevant question would be, given the topology of the network, what level of initial adoption leads to faster global cascade of adoption?

We aim to answer these questions in a generalized form by first placing a *bound* on the time it takes the learning process to converge, in terms of the model and the network parameters, and the initial state distribution. From the bounds on the convergence time and convergence rates, we can then answer multiple questions by holding respective variables and parameters constant. We shall address the above questions for general network topologies while at the same time making sure that we do not compromise on the details of the learning process.

Our work builds on the previous literature both methodologically and in terms of results. First, the literature on coordination games, and in particular the three papers we discuss below. The first is by Ellison [1993], who explored the convergence rate of learning in coordination games and compares two extremes of the topology of the interaction structure; the purely local interactions around a circle, and the uniform or global interactions. Ellison [1993] shows that starting from the worst possible population state, if the interaction structure is governed by local interactions then learning dynamics ensures that a *risk-dominant* strategy gets spread or played by the rest of the population. That is given a binary choice A and B, and that coordinating on A is risk-dominant, if we start from an all B population state, then it requires only a few mutations for strategy A to gain a minimum level of initial adoption above which the strategy can spread to the rest of the population. For the case of global interactions on the other hand, it takes longer for the minimum level of adoption to be attained and the learning process will be stuck at the state where all agents prefer coordinating on B. In the same lines but with a different learning rule, Young [1998a, 2011] shows that when the topology of the interaction structure consists of subgroups that are close-knit, then learning is *fast*.[3] That is given that we start from an all B population state, the waiting time until A is adopted by a target fraction of the population is bounded if the interaction structure is made up of close-knit subgroups. The third paper is by Morris [2000], whose result is closely related to that of Young [1998a]. Morris [2000] employs a deterministic analysis and identifies the network property of cohesiveness as a

[3] A close-knit subgroup is one in which the number of interactions between members of the subgroup is more than that with members outside.

key factor for a strategy to spread faster to the rest of the population.[4] However, this result is conditional on the assumption that a minimum level of initial adoption is met. Ellison [1993] derives bounds for the waiting time but for the specific topologies considered and not in terms of any particular general network parameters. Young [1998a, 2011] and Morris [2000] on the other hand identify a particular network property but do not place a bounds on the waiting time in terms of this property. The lack of a particular bound on the time and speed of convergence leaves the questions we asked above unanswered.[5]

Secondly, our work and in particular the results we find, is also related to a broad range of literature in sociology that studies the role of social networks as a medium of information sharing or communication between individuals; and specifically the role played by "weak" and "strong" ties in information diffusion. Granovetter [1973] made the distinction between weak and strong ties, where strong ties are regarded as those connecting individuals within a subgroup (friends, family members, colleagues or firms in the same product subspace). Through strong links information diffuses fast within subgroup members. The weak ties on the other hand, create "bridges" between subgroups and act as a means of information exchange between subgroups. The implications of this categorization is that when connecting a large society, weak links are more vital than the strong links [Granovetter, 1973, Montgomery, 1991].

The notion that weak ties are good for diffusion of information ("the strength of weak ties") has too often been generalized, but as Centola and Macy [2007] note, it is not always the case especially when the diffusion process is complex and involves thresholds. Centola and Macy [2007] show that if the adoption process is complex then the "strength of weak ties" can turn out to be the "weakness of weak ties". In the case of collective political action for example, Chwe [2000] shows that weak links play a lesser role in inserting influence to participate. Our result is consistent with Centola and Macy [2007] and Chwe [2000]. We find that networks with topologies that are intermediate between purely local and purely global are the most efficient for faster adoption of an action or diffusion of a behavior, but if the bridges connecting the subgroups are "too narrow", then it will take an infinitely long period of time for the population to converge to full adoption or conformism.

The implication of this result extends to problems of complementarity with positive externalities or returns to adoption. For example the diffusion of technologies within national boarders and across international boarders. Bassanini and Dosi [1998] show that the prevalence of monopolies in a single market even when the survival of more than one technology maybe socially optimal, and the existence of different standards of technological products across international levels, can be explained by rates of convergence of the market structure to a stable state.

The remainder of the paper is organized as follows. In section 2 we introduce the model of

[4] A cohesive subgroup according to Morris [2000] has the same meaning with close-knit subgroups according to Young [1998a, 2011] with the difference of 0.5 rescaling factor.

[5] Bramoullé [2001] extends a similar approach as in Young [1998a, 2011] to complementarity games with negative externalities, such as adoption of goods related to social distinctiveness for example fashions and fads, or pollution problems of curbing levels of CO_2 emissions at say regional and international levels. The other papers that explore local interaction effects on coordination but for specifically regular network structures include Anderlini and Ianni [1996], Blume [1995], Ellison [2000]

learning under evolutionary setting, explicitly stating the behavioral rules, matching rule and and the associated payoff functions for both pure coordination games and games with strategic complementarity. We provide the convergence result and specifically for the case with reciprocal interactions. Section 3 introduces a method of reducing and linearizing the detailed but complex adoptive process to get rid of the high order correlations among agents' states, but in turn we get to work with a computationally less expensive, analytically tractable adoptive system. We additionally prove the convergent behavior of this system. In section 4, we define and provide results for the convergence time of the learning process in general networks. Extra measures related to the convergence of the learning process are introduced, namely the *convergence rate* and the *waiting time*. Both measures are explicitly defined, and the bounds provided. In section 5, we derive the conditions under which the interaction structure leads to diversity of strategies across the population for long periods of time. We also revisit the findings of Young [1998a, 2011] and Morris [2000], and redefine their result in terms of the eigenstructure of the matrix of the interaction structure. Section 6 is devoted to a model for *non-observational learning*, where we provide the convergent results and explore the network effect and in particular individual connectivity on the impact of evolutionary forces versus that due to historical factors. We give concluding remarks and areas for future research in section 7.

2 The model

The number of agents is allowed to be very large and will be indexed by $\{1, \cdots, i, \cdots, n\}$. The interactions are local with the interaction structure defined in a graph theoretic way; let $g(n, E)$ be a graph with n vertices, representing the number of agents and E edges linking different pairs of agents such that a graph g_{ij} defines the connection between i and j. If $g_{ij} = 1$ then an undirected link exist between i and j, and zero implies otherwise. We thus have an undirected network g describing the relationship of any one agent with every other agent in the population. The *adjacency matrix* G of an interaction structure with a network topology given by g is basically an $n \times n$ matrix with entries being the elements of g (that is $g_{ij} = 1$ for all i and j). The neighbors of agent i, $\mathcal{N}_i$, is defined as $\mathcal{N}_i = \{j \in n | g_{ij} = 1\}$, and gives the set of players to which i is linked to. The cardinality $\#\mathcal{N}_i = k_i$, the *degree* of i.

We consider a binary action set, and both pure coordination games and games of complementarity with positive externalities or increasing returns to adoption.

2.1 Payoff structure and matching rule

In the case of a pure coordination game, the payoff structure takes the form given in Table 1, where $a > c$, $b > d$; which implies that both (A, A) and (B, B) are Nash equilibria for an encounter between any pair of agents i and j.

Our interest is in the case where coordinating on one of the actions is *risk dominant*, in the context of Harsanyi and Selton [1988]; that is $a \neq b$, such that if $a - d > b - c$ then (A, A) is risk dominant.

Let $x = \{A, B\}$, such that $x_{i,t} \in \mathbf{x}_t$ is i's action at period t, where $\mathbf{x}_t \in \mathbf{X}$ is the realized population state or configuration at period t and $\mathbf{X} = \{A, B\}^n$. Denote by by $x_{\mathcal{N}_i,t}$ as the state

player i \ player j	A	B
A	a , a	d , c
B	c , d	b , b

Table 1: Payoff structure for the pure coordination game between i and j

of i's neighbors. Let $v_i(x_{i,t}, x_{j,t})$ be the reward i gets after playing a coordination game with his opponent j, then i's payoff from playing a coordination game with each of his k_i neighbors at any period t becomes

$$U_i(x_{i,t}, x_{\mathcal{N}_i,t}) = \sum_{j \in \mathcal{N}_i} J_{ij} v_i(x_{i,t}, x_{j,t}), \tag{1}$$

where J_{ij} is the strength of interaction or influence agent j exerts on i, and it defines the matching rule of the game. In particular, we shall consider the case of *local random matching*, in which each player is uniformly matched with a fixed set of other players (the neighborhood). That is

$$J_{ij} = \begin{cases} \frac{1}{k_i} & j \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases}$$

The second and more general representation of the payoff structure involves additively separable intrinsic and social rewards or utilities. That is

$$U_i(x_{i,t}, x_{\mathcal{N}_i,t}) = u_i(x_{i,t}) + S_i(x_{i,t}, x_{\mathcal{N}_i,t}), \tag{2}$$

where $u_i(x_{i,t})$ is i's intrinsic utility and we assume it to be linear in x; $u_i(x_{i,t}) = h_i x_{i,t} + c$ such that h_i is i's valuation of the choice x_i, and c is a constant. The social utility $S_i(x_{i,t}, x_{\mathcal{N}_i,t})$, depends on i's action and the configuration of i's neighborhood, and the form it assumes depends on how the choices of i's opponents directly influence his choice; for example positive or negative externalities.[6] This payoff structure is also suitable for modeling the situations in which individuals do not observe the past actions of their neighbors and can only make decisions based on their prior or subjective beliefs. Let $\mathbb{E}_{i,t}[x_{\mathcal{N}_i}] = \sum_{j \in \mathcal{N}_i} J_{ij} \mathbb{E}_{i,t}[x_j]$ be the prior beliefs or the conditional probability measure that i places on the choices of his neighbors at the time of making a decision. If we consider the case in which the expected social utility exhibits strategic complementarity, then

$$\mathbb{E}[S_i(x_{i,t}, x_{\mathcal{N}_i,t})] = S_i(x_{i,t}, \mathbb{E}_{i,t}[x_{\mathcal{N}_i}]) = x_{i,t} \sum_{j \in \mathcal{N}_i} J_{ij} \mathbb{E}_{i,t}[x_j] \tag{3}$$

[6] This payoff structure has been considered before by Schelling [1971], Brock and Durlauf [2001], Manski [2000] to model social interactions

and the corresponding expected utility becomes

$$\mathbb{E}[U_i(x_{i,t}, x_{\mathcal{N}_i,t})] = u_i(x_{i,t}) + x_{i,t} \sum_{j \in \mathcal{N}_i} J_{ij} \mathbb{E}_{i,t}[x_j]. \tag{4}$$

Under the setup in (4), we would mainly be interested to study how the topology of the interaction structure affects the evolution of individual beliefs.

2.2 Updating process

Given the interaction and payoff structures above, agents play a repeated game which is governed by two main behavioral rules; they are *myopic* and *boundedly rational*. The myopia assumption means that the agents update their states such that the state of the next period is attained by choosing an action that optimizes the payoff given the state of ones neighbors in the current period. Formally, $x_{i,t+1}$ satisfies

$$x_{i,t+1} \in \arg\max_{x_i} U_i(x_i, x_{\mathcal{N}_i,t}). \tag{5}$$

The assumption of bounded rationality implies that the optimization given by (5) is only true to a given probability, that is i chooses the best response action with probability $1 - \varepsilon$, and with a "small" probability ε, i makes an error (mistake). This randomness accounts for the fact that though i has knowledge of what his opponents played at the end of period t, he will not be absolutely certain about what they will exactly play at period $t + 1$. We shall assume that the agents follow a *log-linear* update rule employed by Blume [1995], rather than the *uniform error model* in Ellison [1993], Young [1993] and Kandori et al. [1993]. The log-linear update rule implies that the probability that i chooses action A at period $t+1$ given the population state at the end of period t, and the interaction structure G is given by

$$\mathbb{P}(x_{i,t+1} = A | x_{\mathcal{N}_i,t}, G) = \frac{\exp(\beta U_i(A, x_{\mathcal{N}_i,t}))}{\exp(\beta U_i(A, x_{\mathcal{N}_i,t})) + \exp(\beta U_i(B, x_{\mathcal{N}_i,t}))}. \tag{6}$$

The implication of (7) is that if the optimal choice defined by (5) is A, then i will choose action B with probability proportional to the exponent of the difference between payoff of choosing A and that of choosing[7] B; $\varepsilon^{\Delta U_i(x_i, x_{\mathcal{N}_i,t})}$, where $\Delta U_i(x_i, x_{\mathcal{N}_i,t}) = U_i(A, x_{\mathcal{N}_i,t}) - U_i(B, x_{\mathcal{N}_i,t})$.

Our choice to adopt the log-linear learning rule over the uniform error follows from the findings of Bergin and Lipman [1996], where they show that convergence results of learning dynamics depend on the specific assumptions made about the mutation or experimentation process. They find that the uniform error model adopted by Young [1993] and Kandori et al. [1993], may not always converge to a unique (risk-dominant) equilibrium. By endogenizing mistakes through control costs, van Damme and Weibull [1999] show that the learning process converges to or

[7] That is

$$\mathbb{P}(x_i = B) = \frac{\exp[-\beta \Delta U_i(x_i, x_{\mathcal{N}_i})]}{1 + \exp[-\beta \Delta U_i(x_i, x_{\mathcal{N}_i})]},$$

where for large β

$$\mathbb{P}(x_i = B) \cong \exp[-\beta \Delta U_i(x_i, x_{\mathcal{N}_i})]$$

selects the risk-dominant equilibrium. The log-linear learning rule in (7) is an alternative way of endogenizing the mistakes procedurally by perturbing individual payoffs. To see this, notice that (7) can be derived from

$$\mathbb{P}(\arg\max_{x_i} U_i(x_i, x_{\mathcal{N}_i,t}) + \varepsilon_i(x_i) = A), \tag{7}$$

the probability that the optimal strategy is $x_i = A$. The variable $\varepsilon_i(x_i)$ is the random payoff term. Equation (7) can then be derived from (7) by assuming that the ε's are independent and identically distributed type I extreme-value distributed random variables. See McKelvey and Palfrey [1995] and Chen et al. [1997] for more details concerning the relationship between (7) and (7).

The implication of the relationship between (7) and (7) is that β is a parameter that measures the rationality of an agent. The smaller β the more an agent experiments or the more prone an agent is to mistakes.

To fully describe the dynamic properties of the population state, first notice that at each period t, there are 2^n possible realizations of $\mathbf{x}_t$; $\mathbf{x}_t \in \{A, B\}^n$. The transition of the population state from $\mathbf{x}_t = x'$ to $\mathbf{x}_{t+1} = x''$ is define by the joint probability p_{jk} given by

$$p_{jk} = \prod_{i=1}^{n} \mathbb{P}(x_{i,t+1} = x''_i | x_{\mathcal{N}_i,t} = x'_{\mathcal{N}_i}) \tag{8}$$

The evolution of the population state for a given interaction structure, can thus be treated as a Markov chain defined on the finite set of states $\{A, B\}^n$, with a fixed $2^n \times 2^n$ transition matrix P_β whose elements are defined by (8). The rows of P_β correspond to $1 \leq j \leq 2^n$ and the columns to $1 \leq k \leq 2^n$ such that p_{jk} is the probability of being in a state corresponding to column k at $t+1$ given that the population state at t corresponds to that defined by row j. We note that P_β is a stochastic matrix.

Let $\mathbf{s}_t$ denote a column vector of length 2^n with entries all zero except a one at a position corresponding to the population state at t. The evolution of states can be expressed in a compact form as follows

$$\left.\begin{aligned} \mathbf{q}_{t+1}^T &= \mathbf{s}_t^T P_\beta \\ \mathbf{s}_{t+1} &= \text{Realize}(\mathbf{q}_{t+1}) \end{aligned}\right\} \tag{9}$$

where $\mathbf{q}_t$ is the joint *probability mass function vector* (PMF), from which the state of period $t+1$ is defined through Realize($\mathbf{q}_{t+1}$), and the superscript T denotes the transpose. Since the elements of $\mathbf{s}_{t+1}$ are binary random variables, the elements of $\mathbf{q}_{t+1}^T$ are thus expectations of each of the corresponding elements of $\mathbf{s}_{t+1}$; that is $\mathbf{q}_{t+1} = \mathbb{E}(\mathbf{s}_{t+1})$. The implication is that if the initial distribution of strategies is known, denote it by $\mathbf{s}_0$, then (9) can be written in expectation form as follows

$$\mathbb{E}(\mathbf{s}_t^T | \mathbf{s}_0) = \mathbf{q}_0^T P_\beta^t \tag{10}$$

where $\mathbb{E}(\mathbf{s}_0) \equiv \mathbf{q}_0$. It should also be noted that from the definition in (9), each row of P_β is

a joint PMF describing the expected population state in the next period given the considered state.

2.3 Equilibrium distribution

We consider G to be an undirected network such that $g_{ij} = g_{ji}$. If G is irreducible (connected), then the resulting P_β is also irreducible. The specification of the update process defined by (7) implies that the induced Markov chain is *reversible* and *aperiodic* (since self-loops are allowed through errors). The irreducibility of P_β means that the Markov chain defined in (10) has a unique invariant distribution ν_β (defined below), for any irreducible undirected G. Before explicitly specifying the structure of ν_β, we first elaborate on the concept of *stochastic stability* as defined by Foster and Young [1990] and Young [1993].

For a stochastic game defined by (10) in which agents occasionally make mistakes, we would expect that the mistakes (or mutations) will lead to an evolution of the system (back and forth) between steady states (or Nash Equilibria) depending on the size of the mutations allowed. A stochastically stable equilibrium is the Nash Equilibrium of pure strategies that will be observed with higher probability in a long-run.[8] The corresponding distribution is the *stochastically stable distribution*, which can also be obtained by taking limits on β, that is $\nu_\infty = \lim_{\beta\to\infty} \nu_\beta$. The stationary distribution of the log-linear learning rule dynamics assumes the same form as the Gibbs distribution, a well documented concept in statistical mechanics [Gibbs, 1960].

Corollary 1 *A stochastic game on a connected network graph G in which the learning dynamics assumes a log-linear rule with a stochastic parameter $\beta > 0$, has a unique stationary distribution $\nu_\beta(\mathbf{x})$;*

$$\nu_\beta(\mathbf{x}) = \frac{e^{\beta \bar{H}(\mathbf{x})}}{\sum_{\mathbf{y}\in\mathbf{X}} e^{\beta \bar{H}(\mathbf{y})}} \tag{11}$$

where

$$\bar{H}(\mathbf{x}) = \begin{cases} \sum_{i=1}^{n} \sum_{j\in\mathcal{N}_i} J_{ij} v_i(x_i, x_j) & \text{for a pure coordination game} \\ \sum_i u_i(x_i) + \sum_i S_i(x_{i,t}, x_{\mathcal{N}_i,t}) & \text{for complementarity games,} \end{cases}$$

with $\bar{H}$ being the potential associated with the configuration $\mathbf{x}$.

Proof. See Appendix B.2 ■

The equilibrium or the stochastically stable state is therefore the one that maximizes $\bar{H}$. Young [1998a, Corollary 6.1] exploits the potential function property of the stationary distribution in relation to potential games (a concept due to Monderer and Shapley [1996]) to show that in the case of 2×2 pure coordination games, if the interaction structure is connected then the stochastically stable states of the adoptive process governed by P_β are those in which all agents coordinate on a risk-dominant equilibrium. This argument also follows directly from the expression of $\bar{H}$ above, in that if A is the risk-dominant equilibrium for an encounter between any two agents such that $U_i(A, A) > U_i(B, B)$ for all i, then $\bar{H}$ must be maximum when all

[8] In a literal sense, given two possible states to which a system of interacting agents can converge to, s and s', if the system has converged to one state s, a movement from state s to s' will require a part of the agents to make mistakes, and vise versa for s' to s. A particular state s is said to be *stochastically stable* if it takes relatively more mutations to evolve the system from s to s' than from s' to s.

agents conform to strategy A. This argument generalizes to even complementarity problems with positive externalities.

In the next section we introduce the reduced and linearized form of the adoptive process and the motivation for its relevance.

3 Reduced Markov chain

The master Markov chain (P_β) captures a lot of detail about the learning process but does not give a direct insight on the role played by the topology of the interaction structure. We note that some of the details contained in P_β can be forgone if we are mainly interested in the convergent behavior and the time or speed of convergence to the steady state. One particular of such detail is the high order correlation of agents' states. Here, we aim to linearize the master Markov chain in such a way that we get rid of the high order correlations but in return we get to work with a computationally less expensive, analytically tractable Markov chain, and gives direct insight on the role of the interaction structure on the evolution of individual strategies specially the convergence rates.

To linearize P_β we introduce an operator, the *event matrix* and its inverse. Denote by Ψ the linearizing operator and Ψ^{-1} as its inverse, then P_β can be factorized in the form

$$P_\beta = \Psi \Pi_\beta \Psi^{-1} \tag{12}$$

where Π_β is the linearized form of P_β. The operator Ψ will be formally defined below but in a literal sense, for a binary choice set it is a $2^n \times 2n$ matrix obtained by stacking all the possible realizations of the population states into rows.

3.1 Event matrix

To define the event matrix, we first define an *event vector* for the state of an agent, as vector with all zeros except one, which is a one at a position corresponding to the state/action of an agent at a given time.

Definition 1 *Given an action set $x = \{A, B\}$, let e_t be an event vector for any agent $i \in n$ at time t, then $e_t = (1, 0)$ if i has chosen action A, and $e_t = (0, 1)$ if i chooses B.*

The structure of the event matrix results from stacking all possible event vectors of all n. Formally,

Definition 2 *Asavathiratham [2001]. Given a binary action set, define a recursive matrix Ψ_i from $i = 1$ to $i = n$ as*

$$\begin{aligned} \Psi_1 &= I_2 \\ \Psi_i &= (\Psi_{(i-1)} \otimes \mathbf{1}_2 | \mathbf{1}_{\mu_{i-1}} \otimes I_2) \end{aligned} \tag{13}$$

where $\mu_i = 2^i$, and $\otimes$ is the Kronecker product operation. The event matrix is $\Psi = \Psi_n$.

Example 1

Consider the case of $n = 3$ and the binary action set, then

$$\Psi = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \tag{14}$$

Note that since Ψ is a singular matrix, its inverse Ψ^{-1} can only be approximated by its Moore-Penrose pseudoinverse.

3.2 The reduced adoptive process

The reduced adoptive process is obtained by substituting (12) into (10) such that we have

$$\mathbf{q}_t^T \Psi = (\mathbf{q}_0^T \Psi)\Pi_\beta^t \tag{15}$$

Let $\mathbf{v}_t = \mathbf{q}_t^T \Psi$ and $\mathbf{v}_0 = \mathbf{q}_0^T \Psi$. Each $\mathbf{v}_t$ is the marginal PMF derived from the joint PMFs $\mathbf{q}_t$ of the master Markov chain. To see the validity of this statement, first notice that $\mathbf{v}_t$ can be partition into $\mathbf{v}_t = (v_{t,1}, \cdots, v_{t,n})$, where for a binary action set each $v_{t,i}$ for all i is a two dimensional vector of choice probabilities for agent i. That is, if the action set $X = \{A, B\}$, then the first element of $v_{t,i}$ is the probability that i chooses A at time t and the second element is the probability that i chooses B. Let v_{t,i_X} be the probability that i chooses action X at time t.

Now, note that each element of P_β is the joint probability describing the status of every player simultaneously. Let Ψ_{i_X} denote the column of Ψ corresponding to the realization of the possible number of times i chooses action X. Then each $v_{t,i_X} = \mathbf{q}_t \Psi_{i_X}$, which is basically the sum of the joint probabilities of the population action realizations that include agent i choosing action X, is a marginal probability of action X for agent i. So we have for each i, $v_{t,i} = (v_{t,i_A}, v_{t,i_B})$, and consequently $\mathbf{v}_t$ is a $2n$-dimensional vector consisting of n different marginal PMFs governing the state of each $i \in n$. The reduced and linearized form of the Markov chain describing the evolution of the population state then becomes,

$$\mathbf{v}_t^T = \mathbf{v}_0^T \Pi_\beta^t \tag{16}$$

where the matrix Π_β is of order $2n \times 2n$ unlike P_β which is of order $2^n \times 2^n$.
The direct implication of the above discussion is that the rows of the transition matrix Π_β are made up of n-marginal PMFs governing the state of each $i \in n$. The indirect implication is that Π_β can be expressed as the *Kronecker product* of the matrix of interaction influences among agents (normalized adjacency matrix), and the matrix of *individual transition or choice probabilities* (which we shall denote by Σ). We state this result more formally in Lemma 1 below

Lemma 1 *Let Ω denote the weighted or normalized adjacency matrix of the topology of the*

interaction structure, the transition matrix of the reduced adoptive process Π_β can be factorized into the Kronecker product of the Ω^T and the matrix of individual choice probabilities Σ.

$$\Pi_\beta = \Omega^T \otimes \Sigma \tag{17}$$

where

$$\Sigma = \begin{pmatrix} \mathbb{P}(x_i = A|x_j = A) & \mathbb{P}(x_i = A|x_j = B) \\ \mathbb{P}(x_i = B|x_j = A) & \mathbb{P}(x_i = B|x_j = B) \end{pmatrix}, \tag{18}$$

and the quantities $\mathbb{P}(x_i = A|x_j)$ are the probabilities that i chooses $x_i \in \{A, B\}$ given the choice of his opponent j in the previous period.

Proof. We demonstrate the validity of the lemma with an example in Appendix B.3 ■

3.3 The eigenstructure of Π_β

Understanding the eigenstructure of Π_β will prove to be very useful in determining the convergence rates as well as the convergent behavior of the learning process, so we shall elaborate on some of the main properties next. We shall give the relationship between the eigenstructures of Π_β and P_β shortly, but first the following notations will be used:

Notations

(a) The dimension of the master Markov chain state vector is 2^n and will be denoted η. The dimension of the reduced Markov chain state vector is $2n$ and will be denoted by γ.

(b) The eigenvalue spectrum of P_β and Π_β will be denoted by $\rho(P_\beta) = \mu_1, \cdots, \mu_\eta$ and $\rho(\Pi_\beta) = \lambda_1, \cdots, \lambda_\gamma$ respectively, ordered such that $\mu_1 > \mu_2 \geq \cdots \geq \mu_\eta$ and $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_\gamma$.

(c) The right and left eigenvectors corresponding to the eigenvalues λ_i will be denoted by $\mathbf{r}_i$ and $\mathbf{z}_i$ respectively. Similarly for μ_i let the right and left eigenvectors be denoted by $\tilde{\mathbf{r}}_i$ and $\tilde{\mathbf{z}}_i$ respectively

The following corollary gives the relationship between P_β and Π_β.

Corollary 2 *(a) If μ_1 and λ_1 are the unique largest eigenvalues of P_β and Π_β respectively, then $\mu_1 = \lambda_1 = 1$.*

(b) For any $\lambda \in \rho(\Pi_\beta)$ and $\mu \in \rho(P_\beta)$, if $\lambda = \mu$ and $\mathbf{r}$ is the right eigenvector of λ, then $\tilde{\mathbf{r}}_i = \Psi \mathbf{r}_i$. Similarly, if $\tilde{\mathbf{z}}$ is the left eigenvector corresponding to μ, then $\mathbf{z}_i^T = \tilde{\mathbf{z}}_i^T \Psi$.

Proof. See Appendix B.4 ■

The second vital property of the eigenstructure of Π_β concerns its relation to the eigenstructure of the adjacency matrix and the individual transition matrix Σ. Denote the eigenvalue spectra of Ω and Σ by $\rho(\Omega) = (\theta_1, \cdots, \theta_n)$ and $\rho(\Sigma) = (\vartheta_1, \vartheta_2)$ respectively. The following lemma holds.

Lemma 2 *If Π_β is factorisable according to* (17)*, then $\rho(\Pi_\beta) = \theta_i \vartheta_j \ \forall \theta_i \in \rho(\Omega),\ \vartheta_j \in \rho(\Sigma)$*

Proof. The proof is provided in Horn and Johnson [1990, page 245, Theorem 4.2.12] ■

The third useful relation that will be useful in the analysis that follow, is between the eigenvectors of Π_β and that of the interaction structure Ω. Formally,

Lemma 3 *Given the relation between Π_β and Ω in Proposition 1, if $\mathbf{w}$ is the left eigenvector corresponding to the eigenvalue θ, then $\mathbf{w} \otimes \mathbf{1}_2$ is the right eigenvector of Π_β with the same eigenvalue.*

Proof. An exact proof can be found in Asavathiratham [2001]. ■

It follows from Lemma 2 that if Ω is nearly decomposable then so is Π_β, but the reverse may not be true. The eigendecomposition of a nearly decomposable matrix in section A equation (A.3), together with Lemma 2 gives the following corollary.

Corollary 3 *Let the learning process be governed by the transition matrix Π_β, if the topology of the interaction structure is nearly decomposable, then given the set of model parameters β and h, the transition matrix is also nearly decomposable. That is*

$$\Pi_\beta^t = \sum_{k=1}^{2} \theta_k^t \mathbf{r}_{1_1 k} \mathbf{z}_{1_1 k}^T + \sum_{k=1}^{2} \sum_{l=2}^{L} (\theta_{1_l} \vartheta_k)^t \mathbf{r}_{1_l k} \mathbf{z}_{1_l k}^T + \sum_{k=1}^{2} \sum_{l=1}^{L} \sum_{j=2}^{n_l} (\theta_{j_l} \vartheta_k)^t \mathbf{r}_{j_l k} \mathbf{z}_{j_l k}^T. \tag{19}$$

where $\lambda_1 \equiv \lambda_{1_1 1} = \theta_{1_1} \vartheta_1, \cdots, \lambda_\gamma \equiv \lambda_{n_L 2} = \theta_{n_L} \vartheta_2$. The eigenvectors $\mathbf{r}_{j_l k}$ and $\mathbf{z}_{j_l k}$ correspond to the partitioning of the eigenvalues $\lambda_{j_l k} = \theta_{j_l} \vartheta_k$

The following proposition states the nature of the convergent behavior of the learning process under the reduced Markov chain.

Proposition 1 *For an adoptive process with a transition matrix Π_β, if the underlying interaction structure is strongly connected, then the process converges and there exists a unique distribution associated with the first eigenvector of Π_β. That is $\lim_{t\to\infty} \mathbf{v}_0 \Pi_\beta^t$ exists for all vectors $\mathbf{v}_0 \in \Gamma = \{A, B\}^\gamma$, and that*

$$\lim_{t\to\infty} \mathbf{v}_0 \Pi_\beta^t = \mathbf{z}_1 \tag{20}$$

where $\mathbf{z}_1$ is the unique left eigenvector of Π_β corresponding to the first eigenvalue.

Proof. See Appendix B.5 ■

4 Convergence

In this section we derive bounds on the convergence time of the learning process in terms of the model and interaction structure parameters. Knowing the bounds for the convergence time will on one hand tell us if the learning process converges, and if it does so in the time limits of economic interest. On the other hand, from the convergence time we can determine the topologies of the interaction structure that are most efficient in attaining full conformism, or lead to persistent

coexistence of strategies over long periods of time. We shall define the convergence time following the standard convergence measures for Markov chains, and in particular in relation to our interest is the *hitting time* and the *mixing time*. Hitting time is the time it takes the Markov chain to transit from one state (the *initial state*) to another (target or *final state*), while the mixing time is the time for the Markov chain to converge close to its stationary distribution given that the process starts from the worst possible state.

We shall first give a generalized definition for the convergence time, then in Theorem 1 we provide its bound for the case of general learning dynamics (the master Markov chain P_β). From Theorem 1 we derive insights on the nature of the learning dynamics, from which we develop a motivation to introduce another (and more intuitive) related measure for convergence, the *waiting time*.

4.1 Convergence time

The main objective of our analysis is to determine if a new behavior or strategy (say A) can spread to a large fraction if not to the rest of the population through learning, given that there exists a dominant behavior (say B) that everyone conforms to initially. If this strategy diffuses, how long will it take to do so given the initial state, the learning rule and topology of the interaction structure. Recall the expression for stationary distribution of the master Markov chain, $\nu(\mathbf{x})$ given by Corollary 1. Recall also that the stochastically stable population state or configuration is that which maximizes the stationary distribution. Let it be the all A state. We are interested in the time for the learning dynamics to converge to an all A state given that we start from the worst possible case, the all B state.

Let $\mathbf{s}_{t_0}$ be the initial state of the adoptive process P_β, which corresponds to the initial population state or configuration $\mathbf{x}_{t_0}$. We shall sometimes write $\mathbf{s}(\mathbf{x}_{t_0})$ to imply the state of P_β corresponding to configuration $\mathbf{x}_{t_0}$, or generally $\mathbf{s}(\mathbf{x})$ for any $\mathbf{x} \in \mathbf{X}$. We first define a measure that is used in defining the convergence time and the rest of the convergence measures that will follow, namely the *total variation distance*.

Definition 3 *The total variation distance between any two probability distributions* $\mathbf{v}$ *and* $\mathbf{v}'$ *as defined on the state space* $\mathbf{X}$ *is*

$$\left|\left|\mathbf{v}_t - \mathbf{v}'_t\right|\right| = \max_{x \in \mathbf{X}} |\mathbf{v}(x) - \mathbf{v}'(x)| = \frac{1}{2} \sum_{x \in \mathbf{X}} |\mathbf{v}(x) - \mathbf{v}'(x)| \tag{21}$$

The convergence time will be defined as the time starting from $\mathbf{s}_{t_0}$ till the adoptive process approaches its stationary distribution ν.

Definition 4 *The convergence time of the adoptive process* P_β*, denoted by* T_c *is defined as*

$$T_c(G, \beta, n) = \min \left\{ t \middle| \left|\left| \mathbf{s}_{t_0} P_\beta^{t'} - \nu \right|\right| \leq \varpi; \forall t' > t \right\}, \tag{22}$$

The following theorem places an upper bound on the convergence time.

Theorem 1 *Let the stationary distribution of the adoptive process* P_β *be* ν*. If the learning process starts from the configuration* $\mathbf{x}_{t_0}$*, the convergence time* T_c *to a configuration* $\mathbf{x}_*$*, for*

$\mathbf{x}_{t_0}, \mathbf{x}_* \in \mathbf{X}$ *has an upper bound given by*

$$T_c(G, \beta, n) \leq \frac{\ln\left(\frac{\varpi\nu(\mathbf{x}_{t_0})}{\nu(\mathbf{x}_*)}\right)}{\ln|\mu_2|} \approx \frac{\ln\left(\frac{\nu(\mathbf{x}_*)}{\varpi\nu(\mathbf{x}_{t_0})}\right)}{1-|\mu_2|}, \tag{23}$$

where the approximation is valid for large values of μ_2.

Proof. See Appendix B.6 ■

The upper bound for the convergence time consists of two components, which will be the focus of a large part of the discussions that follow.

The first component is the numerator; the ratio of the stationary distributions of the target or final configuration to that of the initial configuration $\frac{\nu(\mathbf{x}_*)}{\nu(\mathbf{x}_{t_0})}$. We shall elaborate on the properties of this ratio (and its implications on the dynamics of the learning process in general) by considering a special case of the coordination game with payoff structure in Table 1 for which $a = 1 + h$, $b = 1$ and $c = d = 0$.[9] Let $\vec{A}$ and $\vec{B}$ denote an all A and an all B population states respectively. Consider the case of $h > 0$ such that $\vec{A}$ is the stochastically stable configuration. Denote by τ the fraction $\frac{\nu(\vec{A})}{\nu(\mathbf{x}_{t_0})}$, where τ is a function of h, the parameter of randomness β and the stationary distribution of initial state, that is $\tau := \tau(\beta, h, \nu(\mathbf{x}_{t_0}))$. We first note that β must be large enough (but finite) if the process has to converge to one (stochastically stable) configuration in a long-run. Too see this, notice that as $\beta \to 0$, $\tau \to 1$, and this leads to a situation in which the convergence time and hence the learning process in general is independent of the initial and target configurations; that is every configuration is equally likely and the process never settles at a particular state.

Let $n(A)$ denote the number of agents choosing A in a given configuration $\mathbf{x}$ such that $n(A) = 0$ for $\vec{B}$, the following corollary holds.

Corollary 4 *For the adoptive process* P_β*, given the values of* h *and* β*, there exists an initial configuration* $\mathbf{x}_{t_0}$ *for which* $n(A) > 0$ *such that* τ *is maximum.*

Proof. See Appendix B.7 ■

We denote the maximum τ by τ_{max}, and the corresponding configuration by $\mathbf{x}_{t_0,max}$ and the $n(A)$ by $n_{max}(A)$. The direct implication of Corollary 4 is that if $\vec{A}$ is the target configuration, and given the model and network parameters, the convergence time is maximum when the initial configuration is $\mathbf{x}_{t_0,max}$. Consequently for any $\mathbf{x}_{t_0}$ for which $n(A) < n_{max}(A)$ or $n(A) > n_{max}(A)$, the convergence time is less than that for $n_{max}(A)$. This also implies that if at any time the adoptive process is in a configuration $\mathbf{x}_t$ for which the number of agents playing A is less than $n_{max}(A)$, then it will converge to an all B state. Similarly, if the configuration is such that $n(A) > n_{max}(A)$ then the adoptive process converges to an all A . We can therefore interpret $n_{max}(A)$ as a threshold level of initial adoption formally defined as follows.

Definition 5 *Given that we start from the all* B *population state, the threshold level of initial*

[9] We consider this special case just for illustrative reasons but the implications derived are valid for general coordination and complementarity games.

adoption is the number of agents required to initially adopt strategy A, after which the population converges to full adoption of A.

The threshold level of initial adoption is an inherent characteristic of all complex adoption problems, be it in pure coordination games or in general complementarity games. This concept is also seen to play a great role in sociological problems of collective behavior. It was originally formalized by Granovetter [1978] and Schelling [1971], in which they define it as the number or proportion of other actors who must make a decision before a given actor does.

The main question in relation to the effect of the initial state on the convergence time therefore remains to determine how the model parameters, and in particular h, and the interaction structure affects the threshold level of initial adoption. For the parameter h it generally depends on the structure of the payoff but for the example of the coordination game considered above, and for complementarity problems with positive externality, the threshold level of initial adoption decreases with h (given that h is positive). To fully characterize the effect of the topology of the interaction structure on the threshold level of initial adoption, we shall employ a more intuitive measure, the waiting time. That is starting from an all B state we compute the waiting time till at least $n_{max}(A)$ of the individuals simultaneously choose A.

Before embarking on the concept of waiting time, we shall first highlight on the implications of the second component of the convergence time; the denominator $1-\mu_2$. This quantity is the spectral gap of the transition matrix P_β, which can be shown to correspond to the *asymptotic convergence rate*. By asymptotic convergence rate we mean the long-run dynamics (or simply the dynamics close to the steady state) of the adoptive process, which also means that it is independent of the initial state. Formally,

Proposition 2 *Given any initial state $\mathbf{s}_{t_0}$ of adoptive process P_β, define the asymptotic convergence rate as*

$$r = \limsup_{t\to\infty} \left|\left|\mathbf{s}_{t_0}P_\beta^t - \nu\right|\right|^{\frac{1}{t}}, \tag{24}$$

then

$$1 - r = 1 - \mu_2 \tag{25}$$

Proof. See Appendix B.8 ■

Proposition 2 directly implies that the asymptotic convergence rate is high for μ_2 small. Note that μ_2 is a function of the model parameters and the parameter of the interaction structure θ_2, that is $\mu_2 := \mu_2(h, \beta, \theta_2)$. The proportionality of μ_2 with β is linear, in that when $\beta \to \infty$, $\mu_2 \to 1$. What may not be directly intuitive is the proportionality with θ_2 and h. For this reason, let us consider the reduced adoptive process with transition matrix Π_β, which enables us to obtain direct insight on the role of θ_2. The following proposition provides an equivalent result for the asymptotic convergence rate for adoptive process Π_β.

Proposition 3 *Given any initial state $\mathbf{v}_{t_0}$ for Π_β, define the asymptotic convergence rate as*

$$r_\pi = \limsup_{t\to\infty} \left|\left|\mathbf{v}_{t_0}\Pi^t - \pi\right|\right|^{\frac{1}{t}}, \tag{26}$$

where π is the stationary distribution of Π_β, then

$$1 - r_\pi \geq \frac{1}{2}[(1-\theta_2) + (1-\vartheta_2)]. \tag{27}$$

Proof. See Appendix B.9. ■

The convergence rate is therefore directly proportional to the spectral gaps of the interaction structure and the individual transition matrix Σ, that is $(1-\theta_2)$ and $(1-\vartheta_2)$ respectively. From the composition of Σ it is easy to see that $\vartheta_2 := f(\beta, h)$, where f is monotonically increasing in its arguments. For the case of $(1-\theta_2)$, there are two conditions for which the spectral gap of the interaction structure is small, conversely for which θ_2 is large; when the structure is sparsely connected and when it is made up of non-disjoint subgroups that are weakly connected to each other. In the case of sparsely connected networks, it follows straightforwardly from the principle of *interlacing eigenvalues*, while in the case of non-disjoint subgroups it follows from *near decomposability principle*, both of which are elaborated on in the section of preliminary concepts. In the following example, we give the spectral gap for selected network topologies.

Example 2

The bounds for the spectral gap of a graph (or network) G can be derived from the Cheeger inequality and in particular the Cheeger constant $\phi(G)$, where $1-\theta_2 \leq 2\phi(G)$. We shall hereafter denote $\theta_2(G)$ for the second largest eigenvalue of normalized adjacency matrix of the graph G. For more details about the Cheeger constant and proofs for the following network types see Appendix B.10.

(i) Complete network (G_{com}): It is a network structure in which every vertex is connected to every other vertex. The corresponding spectral gap is bounded above by unity, $1 - \theta_2(G_{com}) \leq 1$.

(ii) $1-D$ cyclic network (G_{cyc}): It is a network in which vertices are arranged in a circle and every vertex is connected to two other neighboring vertices. The spectral gap for such network structures is given by $1 - \theta_2(G_{cyc}) \leq \frac{4}{n}$

(iii) $2D$ $n \times n$ lattice network (G_{2D}): If the lattice is constructed with periodic boundary conditions then it will be a 4-regular network (each agent is connected to 4 neighbors). For this network arrangement, the spectral gap is approximately of order $\frac{4}{n}$, that is $1 - \theta_2(G_{2D}) \leq \frac{4}{n}$.

(iv) Random d-regular network (G_{d-r}): Is a network structure in which each of the n vertices is connected to d other vertices chosen at random. In such networks, the spectral gap is independent of the size of the network n, that is $1 - \theta_2(G_{d-r}) \leq 1$.

(v) Newman's small world network (G_{nsw}): Is a network structure in which the mean shortest-path between nodes increases sufficiently slowly (logarithmically) as a function of the number of nodes in the network. For more detail about Newman's small world network see

Newman [2003]. The spectral gap of such network structures is approximated to be $1-\theta_2(G_{nsw}) = O(\frac{c}{\ln n})$, where c is a constant.

The implication is that given the model parameters β and h, the asymptotic convergence rate is higher for densely and/or randomly connected than for local and/or clustered network structures. Since the convergence time has an inverse relation with the spectral gap, it follows that the effect of the interaction structure on convergence time through the asymptotic dynamics is such that the convergence time is shorter for densely connected networks.

4.2 Expected waiting time

The expected waiting time will be defined as the special case of the convergence time in Definition 22. For the case of the binary action set we have considered, if A is the risk-dominant alternative, then we would like to determine the expected waiting time for which *at least* a fraction of agents α simultaneously play strategy A given that they start from an all B state. Let $n(A_t)$ denote the number of agents playing A at time t, we can thus define the expected waiting time as (a similar concept is employed by Ellison [1993])

$$T_w(G,\beta,\alpha,n) = \mathbb{E}\left[\min\{t|n(A_t)\geq \alpha n; \mathbf{v}_0 = \vec{B}\}\right], \tag{28}$$

Let $\mathbb{P}(n(A_{t_0+T})\geq \alpha n)$ denote the probability that at least αn of the agents simultaneously play A after T time steps starting from t_0. Then the expected waiting time will be equivalent to,

$$T_w(G,\beta,\alpha,n) = \frac{T}{\mathbb{P}(n(A_{t_0+T})\geq \alpha n)} \tag{29}$$

To see the validity of relation (29) note that if the denominator is unity, then $T_w = T$, moreover if the probability that at least αn agents simultaneously playing A tends to zero then the expected waiting time will become infinitely long. The following theorem holds.

Theorem 2 *Let the definition of expected waiting time be as in* (29)*, then for* $0 < p \leq \alpha < 1$

$$T_w(G,\beta,\alpha,n) \geq e^{2n(\alpha-p)^2} \tag{30}$$

where $p = \frac{1}{n}\sum_{i=1}^{n} p_i$, $p_i = \mathbb{P}(x_i = A|x_{\mathcal{N}_i} = \vec{B}_{\mathcal{N}_i})$ *and* $\vec{B}_{\mathcal{N}_i}$ *implies the all* B *configuration of* i*'s neighborhood.*

Proof. See Appendix B.11 ■

The waiting time depends on the size of the population one is sampling from, n, the target fraction of the population α of which we are mainly interested in the threshold fraction α_{max} corresponding to $n_{max}(A)$, and the transition probabilities p_i for all i (the probability that i plays A given that all his neighbors play B). In the case of adoptive process P_β, p_i is given by (7) and for Π_β it will be equivalent to $\sum_{j\in\mathcal{N}_i}\frac{1}{k_i}\mathbb{P}(A|B)$. It is easy to check that under both adoptive processes, $p_i = \mathbb{P}(A|B)$ and hence $p = \mathbb{P}(A|B)$, a quantity that is very small for large values of

β. Since p is constant for a given β and h, our interest is then to analyze how $\alpha = \alpha_{max}$ depends on the topology of interaction structure. Let Δ denote the maximum degree in the network. We shall approximate the upper bound for the threshold level $n_{max}(A)$ by $\alpha_{max}\Delta$. That is the minimum number of agents required to initially or simultaneously adopt A within the population (subgroup) after which the population (subgroup) can converge to full adoption of A, is bounded above by the number of agents required to simultaneously adopt A in the neighborhood of the most connected player.[10]

In the case of the coordination game with payoff structure in Table 1, it can be shown using best response argument that an agent will play A if at least

$$\alpha_{max,i} = \frac{b-d}{(a-c)+(b-d)}$$

of his neighbors play A. For example if $a = 2$, $b = 1$ and $c = d = 0$, then any of the agents will play A if at least $1/3$ of his neighbors play A.

Consider two extreme topologies of the network structure; the cyclic or ring (each agent has only two neighbors) and the complete interactions. In the case of complete interactions, if n is the population size then $k_1 = \cdots = k_n = n-1$. If each agent requires at least one third of his neighbors to play A for him to also play A, then the threshold will be $\frac{1}{3}(n-1)$ and the corresponding threshold fraction $\alpha_{max} = \frac{(n-1)}{3n} \approx \frac{1}{3}$ for large n. In the case of cyclic interactions on the other hand where $k_1 = \cdots = k_n = 2$, we have that $\alpha_{max} = \frac{1}{n}$. For a large but finite population size n, we have $(\frac{1}{n} - p) < (\frac{1}{3} - p)$. Implying that the waiting time until the threshold level of adoption is reached is shorter for local interactions than for complete interactions. This also implies that for the complete interactions, if the population size is infinitely large then so will be the expected waiting time.

The expected waiting time also has implications in relation to the *close-knit* and *cohesive* subgroups results of Young [1998a] and Morris [2000]. Young [1998a] shows that if the topology of the network structure governing agents' interactions is made up of r-close-knit subgroups, then *learning is fast*. That is given that the initial state is $\vec{B}$, the waiting time for the adoption of A is bounded (though a specific bound on the waiting time is not specified as we have done above). A formal definition of a r-close-knit subgroup will be given in the next section, but in a literal sense, a subgroup $l \subset n$ is close-knit or cohesive if at least half of its interactions (links) are between members of the subgroup than with those outside. The main characteristic of a cohesive subgroup is that once its members converge to adoption of the (risk-dominant) strategy, this strategy gains a stronghold within the subgroup and will be resistant to external mutations, at least for a considerably long period of time. The computation of the expected waiting time will therefore be at subgroup level. That is the threshold level of initial adoption for each subgroup will be determined by the largest degree within the subgroup. If the subgroup size and hence the maximum degree is bounded, then the expected waiting time for the subgroup

[10]It is possible that $n_{max}(A)$ is less than $\alpha_{max}\Delta$. For example in the case of a star network (in which each agent is connected to only one, and the same player at the center), if the central player changes his strategy to A then all the peripheral players will do so, but on the other hand the central player may have less incentive to change until at least α_{max} of his neighbors do so.

is also bounded. More over if we assume that the agents update their strategies synchronously, then the waiting time will be bounded for even an infinitely large population size.

The argument by Young [1998a] holds if we assume that the probability of mutations is not so small. In general we consider that β is very large such that the possibility of attaining $n_{max}(A)$ simultaneous mutations within the subgroup or population is very small and the only way to attain the threshold levels is through a combination of mutations and diffusion (influence from other agents or subgroups that already play A). Under this setup it is not obvious that existence of subgroups ensures that learning is fast. For example if β is very large and the topology of the network is such that diffusion is not efficient, then we might end up with a situation in which the convergence time is infinitely large and hence learning is slow. In the next section we elaborate on this aspect, specially the conditions under which the network structure leads to slow learning leading to a situation in which the strategies coexists in the population in the long-run.

Following from the implications of the first and the second components of the convergence time, we can therefore categorically differentiate between the two roles played by the topology of the interaction structure on the learning process. The two effects counteract each other; the effect through the initial configuration favors local interactions while that through the asymptotic convergence rate favors global or complete and random interactions. To get a systematic understanding of how the two effects operate, we can think of the learning process as the local adjustment process consisting of local and global minima. In the case of the binary coordination game we have considered above, one immediate local minimum would be the all B state and the all A state is the global minimum.[11] The main characteristic of the dynamic system with local and global minima is that if we start from a local minimum, there exists a *threshold level*, the *tipping point*, which corresponds to the threshold level of initial adoption in this case. Below the threshold, the adjustment process stays trapped within or will always tend towards the local minimum and above the threshold the process will converge to the global minimum. The first component of the convergence time implies that it will take longer for the adjustment process to get out of the local minima in the case of densely and randomly connected than for locally connected network structure. The second component on the other hand says that once the adjustment process has reached the tipping point, then the speed at which it will converge to the global minimum is higher in the case of densely connected than locally connected social networks.

We conclude that interaction structure with the topology which is intermediate between purely local and purely global leads to a faster convergence and hence efficient diffusion of a behavior. It should consist of (cohesive) subgroups within which the connectivity is high enough to ensure faster asymptotic convergence, but these subgroups should also be small enough that the maximum degree is bounded and small enough to ensure that the waiting time is short. This characteristic of the network structure is consistent with that of the *small world networks* introduced by Granovetter [1973] and formalized by Watts and Strogatz [1998]. Granovetter [1973] referred to the interaction within subgroups as *strong ties* and interactions between subgroups as *weak or long-range ties*. The role of the small-world networks and particularly the weak

[11] Depending on the interaction structure, there will of course be other state configurations that will be local minima, but the main two are the all B and all A configurations.

ties in the diffusion of behavior has been documented in both economic and social literature, to mention but a few Granovetter [1973], Watts and Strogatz [1998] for the diffusion of social behavior and information, Cowan and Jonard [2004] for the diffusion of knowledge, Centola and Macy [2007] for a generalized view on the environments in which small-world concept actually works.[12] Though weak ties may be "good" for simple diffusion, the question that remains is if it is generally true even for the case of complex diffusion problems that involve thresholds. In other words, to what extent can weak ties facilitate the diffusion of behavior or influence across subgroups? Put in another way, how cohesive or weakly connected should the subgroups be for intergroup diffusion to occur? We explore this concept in the next section.

5 Cohesiveness and coexistence of strategies

In this section we explore two aspects of subgroup cohesiveness. The first concerns conditions under which cohesiveness fails to ensure fast convergence to conformism leading to persistence of diversity of strategies across the population in a long-run. This analysis is also motivated by observational evidence across social and economic environments with network externalities. For example the existence of two competing technologies regionally or nationwide even if one technology is superior to the other. This extends to coexistence of social norms regionally for example in Europe where some countries drive in the left while others drive on the right, or the case of punctuality. The second aspect concerns the relation between the measure of cohesiveness and the general network parameters and in particular the eigenvalues. We revisit the definition of cohesiveness according to Young [1998a] and Morris [2000], and establish its relation to the eigenvalue spectrum of the network. Defining cohesiveness in terms of the eigenvalues is advantageous in that the latter are generally well defined and computationally "cheap" measures, such that irrespective of the morphology and size of the network, it will still be possible to "roughly" determine if subgroups do exists and if they do how interconnected they are, hence deductions if learning will be fast or slow.

5.1 Coexistence of strategies

We explore this problem by employing the *eigendecomposition* technique. As noted in section 3 Corollary 3, if the topology of the social network is nearly decomposable, then so should be the linearized transition matrix Π_β. The weak inter-subgroup interactions or narrow bridges will correspond to the situation in which the topology of the social network is near-completely decomposable (in which the first eigenvalues θ_{1_l}'s are very close to unity), while wide bridges would correspond to the weak case of near decomposability. Recall that the local adjustment process can be expressed in the eigendecomposition form, and in particular consider the case in which the topology of the social network is nearly decomposable into L non-disjoint subgroups as in (19), that is

[12] There is also a related literature on social capital for example Burt [2004] and the references therein. Also check Ruef [2002] for the extension of small-world concept to organizational innovation.

$$\mathbf{v}_t = \sum_{k=1}^{2} Z_{1_1 k}\theta_k^t \mathbf{r}_{1_1 k} + \sum_{k=1}^{2}\sum_{l=2}^{L} Z_{1_l k}(\theta_{1_l}\vartheta_k)^t \mathbf{r}_{1_l k} + \sum_{k=1}^{2}\sum_{l=1}^{L}\sum_{j=2}^{n_l} Z_{j_l k}(\theta_{j_l}\vartheta_k)^t \mathbf{r}_{j_l k}. \tag{31}$$

where $Z_{j_l k} = \mathbf{z}_{j_l k}^T \mathbf{v}_0$ are scalars for each j, l and k. Note that $\mathbf{r}_{j_l k}$ is the right eigenvector of the eigenvalue of Π corresponding to $\lambda_{j_l k} = \theta_{j_l}\vartheta_k$. Now consider the state of a single agent i belonging to subgroup l, and we shall write $r_{j_l k}(i)$ to imply the ith element of vector $\mathbf{r}_{j_l k}$, and also i_l for an agent i in group l, (31) becomes

$$\begin{aligned} v_t(i_l) = &\sum_{k=1}^{2} Z_{1_1 k}\theta_k^t r_{1_1 k}(i) + \sum_{k=1}^{2} Z_{1_l k}(\theta_{1_l}\vartheta_k)^t r_{1_l k}(i) + \sum_{k=1}^{2}\sum_{l'\neq l'} Z_{1_{l'} k}(\theta_{1_{l'}}\vartheta_k)^t r_{1_{l'} k}(i) \\ &+ \sum_{k=1}^{2}\sum_{j=2}^{n_l} Z_{j_l k}(\theta_{j_l}\vartheta_k)^t r_{j_l k}(i) + \sum_{k=1}^{2}\sum_{l'\neq l}\sum_{j=2}^{n_{l'}} Z_{j_{l'} k}(\theta_{j_{l'}}\vartheta_k)^t r_{j_{l'} k}(i). \end{aligned} \tag{32}$$

Equation (32) is the evolution of the state of each $i \in n$. The long-run population state is determined by the state of each i for $t \to \infty$, which corresponds to $v_\infty(i) \to Z_{1_1 1} r_{1_1 1}(i)$.

Corollary 5 *Assuming that the social network is near-completely decomposable into non-disjoint subgroups, there exists an arbitrary integer τ_1 such that for $t < \tau_1$,*

$$r_{j_l k}(i) \gg r_{j_{l'} k}(i) \qquad \text{for } j_l, j_{l'} \geq 2 \tag{33}$$

where the relation is weak for weak near decomposability (when θ_{1_l}'s are greater than 0.5 but not very much close to unity).

Proof. The proof follows from Proposition 6. ■

The first implication that can be drawn from (32) and Corollary 5 is that in a short-run $t < \tau_1$, the fourth term of (32) dominates the dynamics since the first and the third terms remain fairly unchanged in that θ_{1_l}'s are close to unity; unless of course if β is small and less than unity in which case ϑ_2 will be very small, and the level of stochasticity is high. Nevertheless, our interest is in the behavior of the system for low levels of randomness. Since the fourth term correspond to the lth partitioning of the interaction structure, it implies that in a short-run, the dynamics of the learning process is limited within the subgroups. That is for all $l \in L$, as $t \to \tau_1$, the subgroup dynamics converges to the behavior dictated by the first three terms and the rate of convergence within subgroup l is determined by the fourth term and particularly θ_{2_l} and ϑ_2. Comparing this to Proposition 3, it implies that if the subgroup is densely connected then the convergence rate is high, and the reverse is true for the waiting time.

The second implication is that there exists an arbitrary integer τ_2 such that for $\tau_1 < t < \tau_2$, we have the long-run dynamics that is governed by the second and third terms. The third term is the contribution of other groups to the dynamics of group l, where for every $l' \neq l$, the quantity $Z_{j_{l'} k}$ is determined by the level of interaction between the subgroup l' and l, and particularly the interaction between the peripheral players of the two connected subgroups. This measure of inter-subgroup interaction is what Centola and Macy [2007] referred to as the width of the bridge. It would be interesting to explore this idea in detail to quantify the role of peripheral players in the diffusion of strategies across bridges but it is beyond the scope of this paper to do

so. What we can note though, is that since the rate of convergence of the entire population to a steady state is determined by the second and third terms for all $i \in n$, if θ_{1_l}'s are very much close to unity, then it will take extremely long period of time for the system to converge to its prescribed long-run behavior; where all agents play the risk-dominant strategy. The implication is that if the topology of the social network is made up of weakly connected subgroups, it is possible to sustain coexistence of strategies or behavior for a considerably long period of time.

5.2 Cohesive and close-knit subgroups

What we have considered above is an extreme case of near decomposability. We can still have a weaker form of near decomposability where subgroups exist but are strongly interconnected, such that θ_{1_l} for all l are not necessarily close to unity. We shall adopt the definitions of Young [1998a] and Morris [2000], and show that the topology of the social network can be divided into at least two subgroups if the partitioning through eigendecomposition gives at least two eigenvalues of the normalized adjacency matrix that are greater than 0.5. A close-knit subgroup according to Young [1998a] is defined as follows.

Definition 6 *A subgroup $l \subseteq n$ is said to be r-close-knit if*

$$\forall l' \subseteq l \subseteq n, \quad \frac{d(l', l)}{d(l')} \geq r, \quad 0 < r < \frac{1}{2} \tag{34}$$

where $d(l) = \sum_{i \in l} k_i$, and $d(l', l)$ is the number of edges from l' to l.

A subgroup $l \in L$ is thus r-close-knit if at least r of its links are within members of the subgroup. Note that since r-close-knit means $2r$-cohesive, the above statement is equivalent to saying that not more than $1 - 2r$ of l's interactions should be with members outside of l. That is, for a subgroup $l \in L$, let $d(l, n - l)$ denote the number of interactions between the members of l and its complement $n - l$, then r-close-knit or $2r$-cohesive implies that

$$\frac{d(l, n - l)}{d(l)} \leq 1 - 2r, \quad 0 < r < \frac{1}{2} \tag{35}$$

The following proposition holds.

Proposition 4 *Let Ω denote the normalized adjacency matrix the social network, if the second largest eigenvalue of Ω, $\theta_2 \in \rho(\Omega)$ is such that $\frac{1}{2} < \theta_2 < 1$, then the social network consists of two non-disjoint cohesive subgroups.*

Proof. See Appendix B.12 ■

Though Proposition 4 is true for binary subgroups partitioning through eigenvalues spectrum, the result can be extended to higher order partitioning through multilevel partitions.[13]

Putting the results together, the conclusion is that there exists a limit to which cohesiveness can lead to fast learning and convergence time, specially when the width of the bridges between

[13] A useful piece of work on higher-order eigenvalue partitioning is Miclo [2008].

subgroups is too narrow. A combination of cohesiveness and intermediate level of inter-subgroup interactions is necessary for fast learning and diffusion of a behavior through the social network.

6 Non-observational learning

In this section we consider the general case of strategic complementarity between agents' choices but more specifically we shall focus on case in which agents do not directly observe the past actions of their opponents. They make decision based on their prior subjective beliefs about their opponents. The action set will still be a binary choice but this time we consider $X = \{-1, 1\}$. In the previous sections we focused on the network properties in general but in this section we shall adopt a much simplified framework that will enable us examine the contributions of individual players contingent on their level of connectivity or simply the degree. The framework below will also give a different way of viewing the learning dynamics problem, and in particular the impact of evolutionary forces versus historical factors in shaping the dynamics of learning. By historical factors we mean the initial population configuration or strategy profile, since this is assumed to be determined by some arbitrary historical factors. Except for these specifications, the rules of the game are as stated in section 2.

The payoff structure in expectation terms becomes

$$\mathbb{E}\left[U_i(x_{i,t}, \mathbf{x}_{\mathcal{N}_i,t})\right] = h_i x_{i,t} + x_{i,t} \sum_{j \in \mathcal{N}_i} J_{ij} \mathbb{E}_{i,t}[x_j], \tag{36}$$

where h_i is i's valuation of or preference for x_i, and in what follows we shall consider $h_i = h$ for all i (note that h is also used in section 4 but in that case it refers to the extent to which the payoff of coordinating on action A is better than that of coordinating on B). As before we shall take $J_{ij} = 1/k_i$, but its sign will depend on whether the game is of positive or negative externalities, and we shall consider positive externalities below. Following from (7), the probability that i takes action 1 at period $t+1$ given his subjective beliefs at the end of period t, and the interaction structure G is given by

$$\mathbb{P}(x_{i,t+1} = 1|G) = \frac{\exp(\beta(h + \frac{1}{k_i}\sum_{j \in \mathcal{N}_i} \mathbb{E}_{i,t}[x_j]))}{\exp(\beta(h + \frac{1}{k_i}\sum_{j \in \mathcal{N}_i} \mathbb{E}_{i,t}[x_j])) + \exp(\beta(-h - \frac{1}{k_i}\sum_{j \in \mathcal{N}_i} \mathbb{E}_{i,t}[x_j]))} \tag{37}$$

The corresponding conditional objective expectation of i's choice at $t+1$ is given by

$$\begin{aligned} \mathbb{E}[x_{i,t+1}|G] &= (-1)\mathbb{P}(x_{i,t+1} = -1) + (1)\mathbb{P}(x_{i,t+1} = 1) \\ &= \tanh\left(\beta\left[h + \frac{1}{k_i}\sum_{j \in \mathcal{N}_i} \mathbb{E}_{i,t}[x_j]\right]\right) \end{aligned} \tag{38}$$

We make an assumption of consistency of beliefs; the agents are rational such that the objective expectations coincide with their subjective expectations at equilibrium, that is $\mathbb{E}[x_i|G] = \mathbb{E}_j[x_i] = m_i$ for all $j \in \mathcal{N}_i$. Let $\mathbf{k}$ denote the row vector with its elements as the degree of each agent k_i's for all i, and let $\mathbf{k}^{-1}$ be the row vector with $1/k_i$'s as its elements. Recall that $\Omega \equiv \mathbf{k}^{-1}G$, where G is the adjacency matrix. The entire learning process can then be character-

ized by a set of n-equations describing the non-linear relations between expectations of agents, that is

$$m_{i,t+1} = \tanh\left(\beta h + \beta \Omega_i \mathbf{m}_t\right) \quad i = 1, \cdots, n, \tag{39}$$

where $\mathbf{m}$ denote the n-dimensional column vector ($n \times 1$ matrix) with its elements being the expected actions for all agents, and Ω_i is the ith element of Ω, that is $\Omega_i = \mathbf{k}_i^{-1}\mathbf{G}_i$.

Definition 7 *We define a self-consistent equilibrium of the non-observation game for a given network structure with topology G, as a set of values of m_i, denoted by $\mathbf{m}^* = (m_1^*, \cdots, m_n^*)$ for which m_i^* for all i is a limiting solution to $m_{i,t+1}$ described by* (39)*. It follows $\mathbf{m}^* = (m_1^*, \cdots, m_n^*)$ must be a solution to*

$$m_i^* = \tanh\left(\beta h + \beta \Omega_i \mathbf{m}^*\right) \quad i = 1, \cdots, n, \tag{40}$$

Proposition 5 *For a game of non-observational learning for which the network of interactions is strongly connected, there exists at least one set of values of $\mathbf{m}^*$ that solves equation* (40) *and is the self-consistent equilibrium of the adoptive process.*

Proof. Appendix B.13 ■

As predicted by stochastic stability, the self-consistent equilibrium for a strongly connected network structure is that in which all players conform, and hence posses identical beliefs about each other.

6.1 Equilibrium features

We shall consider two extreme cases; the uniform interaction case in which $k_1 = \cdots = k_n = n$ and $m_1 = \cdots = m_n$ and the local interaction on a cyclic network where each agent has only two neighbors. In the uniform interaction, it follows $\Omega_1 \mathbf{m} = \cdots = \Omega_n \mathbf{m} = m$. Substituting into and taking the average of (40), we have

$$m = \frac{1}{n}\sum_{i=1}^{n} m_i = \tanh\left(\beta h + \beta m\right) \tag{41}$$

Equation (41) is the mean field model of Brock and Durlauf [2001], and has the following properties:

(i) When $h = 0$, and $\beta < 0$, a unique root exist, $m_0^* = 0$; *a symmetric equilibrium.*

(ii) For $h = 0$ and $\beta > 0$, three roots exist; one symmetric equilibrium and the two *asymmetric equilibria* that take on equal magnitude but opposite sign, m_-^* and m_+^*.

(iii) For $h \neq 0$ and $\beta > 0$, there exists a threshold on βh, h_c such that; when $|\beta h| < h_c$, multiple equilibria exist all of which are asymmetric, m_-^*, m_m^*, m_+^*, and when $|\beta h| > h_c$ the equilibrium is unique and takes on the sign of h.

In the case of cyclic interactions we have that each $k_i = 2$ for all $i \in n$, such that

$$m_i = \tanh\left(\beta h + \beta \frac{1}{2}(m_{i-1} + m_{i+1})\right) \quad i = 1, \cdots, n \tag{42}$$

with the boundary conditions $m_{n+1} = m_1$ and $m_{1-1} = m_n$. The existence of the anisotropic equilibria for this system is detailed in Ioannides [2006, Proposition 3].

6.2 Local adjustment of beliefs

To get a better understanding of how individual beliefs evolve under stochastic dynamics or perturbation, and the role of network structure, we linearize the system about an arbitrary initial population configuration of beliefs.

Denote by $\mathbf{m}_{t_0} = (m_{1,t_0}, \cdots, m_{n,t_0})$ as an arbitrary initial belief configuration at t_0. The perturbation around m_{i,t_0} is defined as $\Delta m_{i,t} = m_{i,t_0} - m_{i,t}$. By linearizing about $\mathbf{m}_{t_0}$, we obtain a system of n equations of perturbations around $\mathbf{m}_{t_0}$.

$$\Delta m_{i,t+1} = \operatorname{sech}^2 (\beta h + \beta \Omega_i \mathbf{m}_{t_0}) \, \beta \Omega_i \Delta \mathbf{m}_t \quad i = 1, \cdots, n \tag{43}$$

Let $\zeta_i = \operatorname{sech}^2 (\beta h + \beta \Omega_i \mathbf{m}_{t_0})$, and $\boldsymbol{\zeta}$ be a vector of ζ_i's. Then (43) can be written in a compact form as

$$\Delta \mathbf{m}_{t+1} = \beta \boldsymbol{\zeta} \Omega \Delta \mathbf{m}_t. \tag{44}$$

If we let $\Delta \mathbf{m}_{t_0}$ denote the vector of initial levels of perturbations for all i, then (44) can be generally expressed in the following form.

$$\Delta \mathbf{m}_t = (\beta \boldsymbol{\zeta})^t \Omega^t \Delta \mathbf{m}_{t_0} \tag{45}$$

By expressing Ω in its eigendecomposition form as described in section A, we shall have

$$\Delta \mathbf{m}_t = (\beta \boldsymbol{\zeta})^t \left[\mathbf{y}_1 \mathbf{w}_1^T \Delta \mathbf{m}_{t_0} + \sum_{i=2}^{n} \mathbf{y}_i \theta_i^t \mathbf{w}_i^T \Delta \mathbf{m}_{t_0} \right] \tag{46}$$

The second summation term in the square bracket of (46) tends to zero in a long-run, if we assume that the level of mutations (the value of β) is such that $\beta \zeta_i \theta_2 < 1$ for all $i \in n$, then in a long-run the only vital component of (46) is $(\beta \zeta_i)^t \mathbf{y}_1 \mathbf{w}_1^T \Delta \mathbf{m}_{t_0}$ for all $i \in n$. We shall denote this component by Δm_i, and the corresponding vector by $\Delta \mathbf{m}_i$. What matters for the adjustment process in particular is whether $\beta \zeta_i$ is less or greater than unity. If for any agent i, $\beta \zeta_i < 1$ then the initial state of that agent is stable, otherwise it is unstable and the perturbation quantity Δm_i grows over time until i adjusts to a stable state. If i's initial state is unstable, then the compositions of the vector $\mathbf{w}_1$, the left hand eigenvector of Ω corresponding to the first eigenvalue, will determine the state that i will adjust to. The vector $\mathbf{w}_1$ gives the degree of influence by each player within the network, such that the ith element $\mathbf{w}_{1,i}$ corresponds to influence of player i and also gives an indication of the level of connectedness of i. The more connected i the higher the value of $\mathbf{w}_{1,i}$, moreover it is even higher if i is connected to players that are themselves well connected.

To illustrate these points further we shall consider an example of a topology with both well connected and less connected players. This will also enable us to demonstrate the impact of evolutionary forces as opposed to historical factors in shaping the dynamics of the learning process at individual level.

Example 3

Consider an interaction structure with the topology in Figure 1

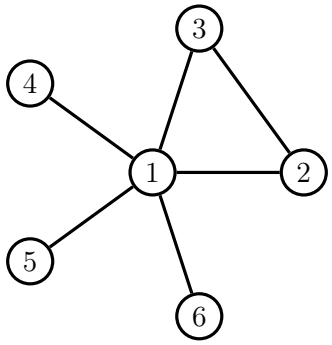

Figure 1: Star-like structure with an extra link

The central player which is also the most connected is labeled by 1, followed by 2 and 3, and the rest have the minimum possible connectivity. The normalized adjacency matrix of this network is

$$\Omega = \begin{pmatrix} 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The eigenvalue spectrum of Ω is

$$\rho(\Omega) = (1, -8.5 \times 10^{-1}, -5.0 \times 10^{-1}, 3.5 \times 10^{-1}, 4.9 \times 10^{-33}, 5.8 \times 10^{-51}),$$

and the respective right and left eigenvectors corresponding to the first eigenvalue are

$$y_1 = (-0.4082, -0.4082, -0.4082, -0.4082, -0.4082, -0.4082)$$

and

$$w_1 = (0.8333, 0.3333, 0.3333, 0.1667, 0.1667, 0.1667).$$

It can be seen from the composition of w_1 that the most influential player is the central player, followed by second and third. As noted above, the terms in (46) containing θ_i for $i \geq 2$ (if $|\beta\zeta_i\theta_2| < 1$) will tend to zero in a long run, such that the only vital component for large values of t can be expressed as follows.

$$\begin{pmatrix} \Delta m_1 \\ \Delta m_2 \\ \Delta m_3 \\ \Delta m_4 \\ \Delta m_5 \\ \Delta m_6 \end{pmatrix} = \begin{pmatrix} \beta\zeta_1 \\ \beta\zeta_2 \\ \beta\zeta_3 \\ \beta\zeta_4 \\ \beta\zeta_5 \\ \beta\zeta_6 \end{pmatrix}^t \begin{pmatrix} -0.3402 & -0.1361 & -0.1361 & -0.068 & -0.068 & -0.068 \\ -0.3402 & -0.1361 & -0.1361 & -0.068 & -0.068 & -0.068 \\ -0.3402 & -0.1361 & -0.1361 & -0.068 & -0.068 & -0.068 \\ -0.3402 & -0.1361 & -0.1361 & -0.068 & -0.068 & -0.068 \\ -0.3402 & -0.1361 & -0.1361 & -0.068 & -0.068 & -0.068 \\ -0.3402 & -0.1361 & -0.1361 & -0.068 & -0.068 & -0.068 \end{pmatrix} \begin{pmatrix} \Delta m_{1,t_0} \\ \Delta m_{2,t_0} \\ \Delta m_{3,t_0} \\ \Delta m_{4,t_0} \\ \Delta m_{5,t_0} \\ \Delta m_{6,t_0} \end{pmatrix} \tag{47}$$

The direct implication of the product of the last two matrices on the right hand side of (47) is

that given the initial perturbations $\Delta \mathbf{m}_{t_0}$, each player's choice of adjustment is influenced most by the most connected player in the network.

Consider an initial state of beliefs in which for some arbitrary historical factors, player 3 is in belief state $m_3 = 1$ while the rest of the players are in state -1. We would like to determine the level of mutations (value of β) required to destabilize the state of players such that $\beta\zeta > 1$, in which case the player regards his current state undesirable. Consider the case for $h = 0.2$, the players 4, 5 and 6 will have the same value of ζ such that $\beta\zeta = \beta \operatorname{sech}^2(-0.8\beta)$ for the three players. In this case there does not exist a value of β for which $\beta\zeta > 1$, implying that the state of players 4, 5 and 6 are stable and will remain so until player 1 adjusts his state.

In the case of player 2, $\beta\zeta_2 = \beta \operatorname{sech}^2(0.2\beta)$, which is greater than unity for $1.05 \leq \beta \leq 8.7$. This implies that the minimum level of mutations required to destabilize the current stat of player 2 is that corresponding to $\beta = 8.7$. Player 1 on the other hand will have $\beta\zeta_1 = \beta \operatorname{sech}^2(-0.4\beta)$, which is greater than unity for $1.3 \leq \beta \leq 2.7$. Implying that the minimum level of randomness corresponds to $\beta = 2.7$.

Comparing player 1 and 2, though they both have one player in common in state 1 while they both are in state -1, the level of randomness required to destabilize the state of each contingent on their connectivity implies that evolutionary forces tend to act more efficiently on less connected players as compared to those with high connectivity. This observation may not be true for all values of h, especially for large values which imply that the difference in level of valuation that agents attach to the two actions is very high such that the network externality effect has little contribution compared to private tastes. For values of h small enough, the above observation is consistent, and as pointed out before just like densely connected subgroups, highly connected players can be a hindrance to learning since their long-run behavioral outcomes are largely determined by historical factors and less by evolutionary forces.

7 Conclusion

We have discussed the dynamic process of learning in social and economic networks for a class of pure coordination games and games with strategic complementarity. We adopted an evolutionary approach as in Kandori et al. [1993], Young [1993], Ellison [1993], Blume [1993], in which agents are assumed to be myopic and boundedly rational. We also provide an alternative dynamic framework in which we linearize the general adoptive process to a form that gives direct insight on the role of the interaction structure. The general adoptive process maintains the high order correlation among players' states and therefore captures more informational detail. In the case of the linearized adoptive process, such information is not readily captured but in return we get to work with a more simplified form from which we can derive more intuitive results.

The convergence results are provided for both general and linearized adoptive processes and for both observation and non-observational learning. We prove the existence of a unique equilibrium in both cases conditional on the network of interactions being strongly connected.

We have discussed some of the questions concerning learning dynamics in networks that are left largely open in the existing literature. The first question we focus on relates to the time and

speed of convergence of the learning process and how the topology of the network of interactions influences both. A related question concerns the probability of learning as compared to the speed of learning; which can be interpreted as the impact of evolutionary forces versus that of historical factors. We derive the bound for the convergence time showing that it depends on the initial state (historical factors) and the asymptotic behavior of the learning process. The asymptotic property of learning is independent of the initial state configuration, and the effect of the interaction structure through the asymptotic behavior is antagonistic to that through the initial state configuration.

We further derive a bound for the waiting time until a prescribed fraction of players simultaneously choose a "new" strategy, a result that helps elaborate on the impact of evolutionary forces versus historical factors. The concept of waiting time has been used before by Ellison [1993] and Young [1998a] but unlike in these works, we place a specific bound for general network structures and model parameters. We have been able to show that evolutionary forces are more efficient and hence the probability of learning is higher in sparsely and locally connected network structure than in densely or uniformly connected networks. In the same light we also show that highly connected players and densely connected subgroups can be a hindrance to learning, but in circumstances where evolutionary forces are strong enough to ensure that learning occurs, then it happens more rapidly than in less connected networks or networks that do not contain highly connected player.

The second question we have tackled in this paper concerns the identification of the general features of the network of interactions that lead to conformism and those that sustain diversity of strategies over long period of time. We employ the eigendecomposition technique to show that though the equilibrium of learning dynamics predicts full conformism for connected networks, diversity of strategies across the population can prevail over long periods if the network is made up of densely connected subgroups but with "weak" inter-subgroup interactions. In such structures the inter-subgroup interactions are too weak to act as a means of diffusion of strategies across subgroups, such that the only way of learning to occur is through evolutionary forces. But since evolutionary forces are not efficient in densely connected groups or subgroups then we expect that learning in such interaction structures will be slow and it will be possible to sustain diversity over long periods of time.

The framework and findings in this paper can be extended in several ways. The first includes considering endogenous network formation. We have considered in the whole of the above analysis that the network is exogenously given, but in the light of Baccara and Yariv [2010] our framework can be extended to incorporate endogenous formation of links between players.

Secondly, we have focused on learning problems in which the payoff of an individual directly depends on the actions of his opponents. Though we find similarities in some of our findings with that of the models of network externalities in information sharing problems, (that is where the network of interactions is simply a medium of communication and information sharing) it would be interesting to establish the generality of our findings particularly with models like Bala and Goyal [1998] and Acemoglu et al. [2011]. The work by Bala and Goyal [1998] and Acemoglu

et al. [2011] provide a framework for this kind of learning problems but their models focus on the details of learning at the expense of capturing the complexities of the interaction structure. Golub and Jackson [2010] and Demarzo et al. [2003] on the other hand employ a framework that focuses on the details of the role played by the interaction structure at the expense of the details of the learning process. The framework we provide in section 3 can be developed into an intermediate model between the more detailed model of Bala and Goyal [1998] and Acemoglu et al. [2011], and the much simplified model of Golub and Jackson [2010] and Demarzo et al. [2003].

The reduced Markov chain frame we provide in section 3 can also be applied to model various environments with diffusion or cascades, such as the global cascade of idiosyncratic shocks in say within or across industrial sectors, in which case the nodes would represent firms or different industries.

The final aspect for future research concerns the procedure in which players update their strategies. We have considered the case in which updating is done simultaneously, but it is not straight forward that similar findings can be obtained if players update asynchronously. There exists a well documented literature in probability theory and statistical mechanics that provides computational algorithms for sequential Markov chain dynamics that not only can give more insights in relation to asynchronous learning dynamics, but could also be useful for computational advancement of models of learning in networks. A good survey on these models can be found in Levin et al. [2009] and the references therein.

Appendix

A Preliminary concepts

For a given adjacency matrix G, we denote the corresponding normalized adjacency matrix by Ω. It follows that Ω is a stochastic matrix; its rows sum up to unity. Following from the eigendecomposition theorem, Ω can be expressed as,

$$\Omega = \theta_1 \mathbf{y}_1 \mathbf{w}_1^T + \sum_{i=2}^{n} \theta_i \mathbf{y}_i \mathbf{w}_i^T \tag{A.1}$$

where θ_i are the eigenvalues of Ω, and $\mathbf{y}_i$ and $\mathbf{w}_i$ are the corresponding right and left eigenvectors respectively. Note that $\theta_1 = 1$ for stochastic matrices

In what will follow, we shall consider the case in which G and Ω are *irreducible* or *not completely decomposable*. We introduce the concept of nearly decomposable matrices (according to Simon and Ando [1961] and hence network structures that will be useful in later analysis. First define a completely decomposable network with adjacency matrix G^* as follows.

Definition 8 *The matrix G^* is completely decomposable if after an appropriate permutation of*

rows and columns, it can be expressed in the form

$$G^* = \begin{pmatrix} G_1^* & & & & \\ & \ddots & & & \\ & & G_l^* & & \\ & & & \ddots & \\ & & & & G_L^* \end{pmatrix}$$

where G_l^*'s are square block matrices or submatrices, and $L < n$. The rest of the undisplayed elements are zeros.

In the context of interaction topology, it would imply that the agents belong to disjoint subgroups; which could actually be studied separately. For a completely decomposable G^*, the corresponding Ω^* is also completely decomposable. Let θ_{il}^* denote the ith eigenvalue in block l of Ω^*, such that $\left(\theta_{1_1}^*, \theta_{1_2}^*, \cdots, \theta_{1_L}^*\right)$ are the largest eigenvalues in blocks 1 to L, and $\left(\theta_{2_1}^*, \theta_{2_2}^*, \cdots, \theta_{2_L}^*\right)$ are the second largest eigenvalues for the L blocks. Index by n_l as the maximum number of columns in block l. We can therefore categorize the spectrum of G^* denoted by $\rho(\Omega^*)$, in the form

$$\rho(\Omega^*) = \left(\theta_{1_1}^*, \theta_{2_1}^*, \cdots, \theta_{n_{11}}^*, \cdots, \theta_{1_2}^* \cdots, \theta_{1_l}^*, \cdots, \theta_{n_{ll}}^*, \cdots, \theta_{1_L}^*, \cdots, \theta_{n_{LL}}^*\right).$$

The eigendecomposition of Ω^* will thus be

$$\Omega^* = \sum_{l=1}^{L} \mathbf{y}_{1_l}^* \mathbf{w}^{*T}_{1_l} + \sum_{l=1}^{L} \sum_{j=2}^{n_l} \theta_{j_l}^* \mathbf{y}_{j_l}^* \mathbf{w}^{*T}_{j_l}. \tag{A.2}$$

where $\mathbf{y}_i^*$ and $\mathbf{w}_i^*$ are the right and left eigenvectors respectively. Note that since Ω^* is a stochastic matrix, we have that $\theta_{1_1}^* = \cdots = \theta_{1_L}^* = 1$.

Consider the situation in which the network is irreducible, but can be categorized into non-disjoint subgroups $l = 1, \cdots, L$, such a network referred to as *nearly decomposable* network with the following definition.

Definition 9 *The matrix G is nearly decomposable if it can be expressed as $G = G^* + \epsilon G'$. Where G^* is a completely decomposable matrix, ϵ is a very small real number, and G' is an arbitrary $n \times n$ matrix.*

Following the similar categorization of $\rho(\Omega)$ as that for $\rho(\Omega^*)$, the eigendecomposition of the corresponding Ω will be

$$\Omega = \mathbf{y}_{1_1} \mathbf{w}_{1_1}^T + \sum_{l=2}^{L} \theta_{1_l} \mathbf{y}_{1_l} \mathbf{w}_{1_l}^T + \sum_{l=1}^{L} \sum_{j=2}^{n_l} \theta_{j_l} \mathbf{y}_{j_l} \mathbf{w}_{j_l}^T \tag{A.3}$$

for $\theta_{1_1} > \theta_{1_l} > \theta_{j_l}$ for all j and l.

The relationship between the eigenvalues and eigenvectors of Ω^* and Ω is through the parameter ϵ. Define $\delta_i(\epsilon) = |\theta_i(\epsilon) - \theta_i^*|$, a small positive real number such that $\lim_{\epsilon \to 0} \delta_i(\epsilon) \to 0$. If $\theta_{1_l}^* = 1$ is the leading eigenvalue for block submatrix Ω_l^*, then it must be that

$$\theta_{1_l}(\epsilon) = 1 - \delta_{1_l}(\epsilon). \tag{A.4}$$

The following proposition will be useful in the later analysis.

Proposition 6 *Let Ω be nearly decomposable as in* (A.3)*, let y_{ij_l} denote ith element of the eigenvector corresponding to eigenvalue θ_{j_l} for all j_l, and l. There exists a value of ϵ, say ϵ' and an arbitrary real positive number ξ, such that for $\epsilon < \epsilon'$,*

$$\max_i |y_{ij_l} - y^*_{ij_l}| < \xi \tag{A.5}$$

for $j_l = 2, \cdots, n_l$ and $l = 1, \cdots, L$.

Proof. See Appendix B.1 ■

The following lemma concerns the second largest eigenvalue of a network graph; the property of *interlacing eigenvalues*

Lemma 4 *Let G and G' be two networks graphs with the same set of vertices, and that θ_2 and θ'_2 are the second largest eigenvalues of G and G' respectively:*

1. *If $G' \subseteq G$ then*
$$\theta'_2 \leq \theta_2 \tag{A.6}$$
2. *Let G, G' be edge-disjoint, if we denote by θ^u_2 as the second largest eigenvalue of $G \cup G'$, then*
$$\theta_2 + \theta'_2 \leq \theta^u_2 \tag{A.7}$$

Proof. The proof follows from Fiedler [1973]. ■

Definition 10 *[Monderer and Shapley [1996]] A non-cooperative game $\mathcal{G}$ with utility function $U : X \mapsto \mathbb{R}$ is a potential game if there exists a real valued function $H(x) : X \mapsto \mathbb{R}$, the potential function, such that for any deviations by any $i \in n$, the change in the payoffs equals the change in potential, with a rescaling in utility functions when it applies. That is, let $u_i(\mathbf{x})$ be the utility function of i when i chooses x_i and the opponents choose x_{-i}, and let $u_i(\mathbf{x}')$ be the utility function of i when i chooses x'_i and the opponents choose x_{-i}, then*

$$u_i(\mathbf{x}) - u_i(\mathbf{x}') = H(\mathbf{x}) - H(\mathbf{x}') \tag{A.8}$$

B Proofs

B.1 Proof of Proposition 6

The following proof is adopted from Simon and Ando [1961]. Let Ω^t_{ij} and Ω^{*t}_{ij} denote the ith row and jth column elements of the tth power of the matrices Ω and Ω^* respectively. Each Ω^t_{ij} is the probability of ending up in Ω_{ij} after t steps starting from any of the ith elements of Ω. Implying that all Ω^t_{ij}s in Ω^t are continuous functions of the Ω_{ij}s in Ω.

Recall that for Ω nearly decomposable, then $\Omega = \Omega^* + \epsilon\Omega'$. This implies that for any positive real number ξ', there exist an ϵ_1 and a T_1 such that for $\epsilon < \epsilon_2$ and $t < T_1$

$$\max_{i,j} |\Omega^t_{ij} - \Omega^{*t}_{ij}| < \xi' \tag{B.1}$$

Recall the structure of the eigendecomposition of Ω,

$$\Omega^t = \mathbf{y}_{1_1}\mathbf{w}_{1_1}^T + \sum_{l=2}^{L}\theta_{1_l}^t\mathbf{y}_{1_l}\mathbf{w}_{1_l}^T + \sum_{l=1}^{L}\sum_{j=2}^{n_l}\theta_{j_l}^t\mathbf{y}_{j_l}\mathbf{w}_{j_l}^T \tag{B.2}$$

Let us denote $\mathbf{y}_{j_l}\mathbf{w}_{j_l}^T$ by $B(j_l)$, such that the $B_{ik}(j_l)$ are the ith row and kth column elements of $B(j_l)$. We can therefore write the elements of Ω as follows

$$\Omega_{ik}^t = B_{ik}(1_1) + \sum_{l=2}^{L}\theta_{1_l}^t B_{ik}(1_l) + \sum_{l=1}^{L}\sum_{j=2}^{n_l}\theta_{j_l}^t B_{ik}(j_l) \tag{B.3}$$

Substituting into (B.1) gives

$$\max_{i,k}|\Omega_{ik}^t - \Omega_{ik}^{*t}| = \left|B_{ik}(1_1) + \sum_{l=2}^{L}\theta_{1_l}^t B_{ik}(1_l) - \sum_{l=1}^{L}\theta_{1_l}^{*t}B_{ik}^*(1_l) + \sum_{l=1}^{L}\sum_{j=2}^{n_l}\theta_{j_l}^t B_{ik}(j_l) - \sum_{l=1}^{L}\sum_{j=2}^{n_l}\theta_{j_l}^{*t}B_{ik}^*(j_l)\right| \tag{B.4}$$

Denote the sum of the first three terms on the right hand side of (B.4) by Q, that is

$$Q_{ik}(t) = B_{ik}(1_1) + \sum_{l=2}^{L}\theta_{1_l}^t B_{ik}(1_l) - \sum_{l=1}^{L}\theta_{1_l}^{*t}B_{ik}^*(1_l) \tag{B.5}$$

The quantity $Q_{ik}(t)$ is nearly independent of t for θ_{i_l}s large, that is for $t < T_2$, $\theta_{i_l}^t \to 1$ for all l. Applying algebraic manipulations yields

$$\max_{i,k}|\Omega_{ik}^t - \Omega_{ik}^{*t}| = \left|Q_{ik}(t) + \sum_{l=1}^{L}\sum_{j=2}^{n_l}(\theta_{j_l}^t - \theta_{j_l}^{*t})B_{ik}(j_l) + \sum_{l=1}^{L}\sum_{j=2}^{n_l}\theta_{j_l}^{*t}(B_{ik}(j_l) - B_{ik}^*(j_l))\right| \tag{B.6}$$

Note that for ϵ sufficiently small, $\theta_{j_l} - \theta_{j_l}^*$ is also small, such that $\theta_{j_l}^t - \theta_{j_l}^{*t} \to 0$

$$\max_{i,k}|\Omega_{ik}^t - \Omega_{ik}^{*t}| = \left|Q_{ik}(t) + \sum_{l=1}^{L}\sum_{j=2}^{n_l}\theta_{j_l}^{*t}(B_{ik}(j_l) - B_{ik}^*(j_l))\right| < \xi' \tag{B.7}$$

Since $Q_{ik}(t)$ is independent of t for θ_{i_l}s close to unity, and that $\theta_{j_l}^*$ for all $j_l \geq 2$ and for all l are distinct, it follows (B.8) is only true if there exists a positive real number ξ and an ϵ_2 such that for $\epsilon < \epsilon_2$ and $t < T_1$

$$\max_{ik}|B_{ik}(j_l) - B_{ik}^*(j_l)| < \xi \tag{B.8}$$

This completes the proof, where $B_{ik}(j_l) = y_{ij_l}w_{kj_l}$.

B.2 Proof of Corollary 1

To prove that a distribution is the stationary distribution of the Markov chain, it suffices to show that it satisfies the detailed balance condition. That is if $\nu(\mathbf{x})$ is the stationary distribution then

$$\nu(\mathbf{x})P(\mathbf{x},\mathbf{y}) = \nu(\mathbf{y})P(\mathbf{y},\mathbf{x}) \quad \text{for } \mathbf{x},\mathbf{y} \in \mathbf{X} \tag{B.9}$$

where

$$\sum_{\mathbf{x}\in\mathbf{X}} \nu(\mathbf{x})P(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{x}\in\mathbf{X}} \nu(\mathbf{y})P(\mathbf{y},\mathbf{x}) = \nu(\mathbf{y}) \sum_{\mathbf{x}\in\mathbf{X}} P(\mathbf{y},\mathbf{x}) = \nu(\mathbf{y}) \tag{B.10}$$

So we basically need to show that

$$\nu_\beta(\mathbf{x}) = \frac{\exp[\beta\bar{H}(\mathbf{x})]}{\sum_{\mathbf{x}'\in\mathbf{X}} \exp[\beta\bar{H}(\mathbf{x}')]} \tag{B.11}$$

satisfies (B.9). Note that $\bar{H}$ is a *potential function* and is derived from the sum of individual payoffs

$$\bar{H}(\mathbf{x}) = \sum_{i=1}^{n} U_i(x_i, x_{-i}), \tag{B.12}$$

which can be expressed in alternative ways depending on the kind of interactions one is interested in. For example, for

$$U_i(x_i, x_{-i}) = \sum_{j\in n} J_{ij} v_i(x_i, x_j), \tag{B.13}$$

if $J_{ij} = J_{ji}$, given the payoff structure for the coordination game in Table 1, $\bar{H}(\mathbf{x})$ will be equivalent to

$$\bar{H}(\mathbf{x}) = (a-d)J_{AA} + (b-c)J_{BB} \tag{B.14}$$

where J_{AA} and J_{BB} are the sum of the weights on all edges $\{i,j\}$ such that $x_i = x_j = A$ and $x_i = x_j = B$ respectively. This is the case considered by Young [1998a].

In general we can rewrite $\bar{H}(\mathbf{x})$ as

$$\bar{H}(x_i, x_{-i}) = U_i(x_i, x_{\mathcal{N}_i}) + \sum_{j\neq i}^{n} U_j(x_i, x_{-i}), \tag{B.15}$$

and it directly follows that

$$U_i(x_i, x_{\mathcal{N}_i}) - U_i(y_i, x_{\mathcal{N}_i}) = \bar{H}(x_i, x_{-i}) - \bar{H}(y_i, x_{-i}). \tag{B.16}$$

Since the change in one agent's choice results in an equivalent (but possibly rescaled) change in the potential, we can thus consider an asynchronous adoptive process in which an agent updates at a time. If i is randomly chosen with a uniform probability $1/n$, then

$$\nu(\mathbf{x})P(\mathbf{x},\mathbf{y}) = \frac{1}{n} \frac{\exp[\beta\bar{H}(\mathbf{x})]}{\sum\limits_{\mathbf{x}'\in\mathbf{X}} \exp[\beta\bar{H}(\mathbf{x}')]} \frac{\exp[\beta U_i(y_i, x_{\mathcal{N}_i})]}{\sum\limits_{y_i'\in\{A,B\}} \exp[\beta U_i(y_i', x_{\mathcal{N}_i})]}. \tag{B.17}$$

Denote by Z the normalizing factor

$$Z = \frac{1}{n \sum\limits_{\mathbf{x}'\in\mathbf{X}} \exp[\beta\bar{H}(\mathbf{x}')] \sum\limits_{y_i'\in\{A,B\}} \exp\left[\beta U_i(y_i', x_{\mathcal{N}_i})\right]} \tag{B.18}$$

Applying (B.16) to (B.18) gives

$$\begin{aligned}\nu(\mathbf{x})P(\mathbf{x},\mathbf{y}) &= Z\exp[\beta\bar{H}(\mathbf{x})+U_i(y_i,x_{\mathcal{N}_i})]\\ &= Z\exp[\beta\left(\bar{H}(x_i,x_{-i})+U_i(x_i,x_{\mathcal{N}_i})-\bar{H}(x_i,x_{-i})+\bar{H}(y_i,x_{-i})\right)]\\ &= Z\exp[\beta\left(U_i(x_i,x_{\mathcal{N}_i})+\bar{H}(y_i,x_{-i})\right)], \qquad \text{(B.19)}\end{aligned}$$

where for $x_{-i}=y_{-i}$, the R.H.S of (B.19) will be equivalent to $\nu(\mathbf{y})P(\mathbf{y},\mathbf{x})$; which completes the proof.

B.3 Proof of Lemma 1

We prove proposition 1 by consider an example of the 3-agent interaction network in Figure 2, with the adjacency matrix

$$G=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$$

The master Markov chain transition matrix for the coordination game with payoff structure in

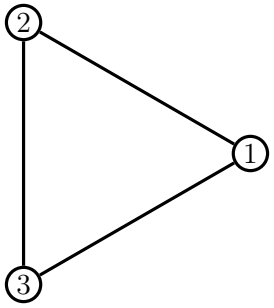

Figure 2: A complete 3-agent network

Table 1 where $a=1.5$, $b=1$, and $c=d=0$, and β the parameter of randomness, is given

$$P_\beta=\begin{pmatrix}0.9983 & 0.0006 & 0.0006 & 3.1\times10^{-7} & 0.0006 & 3.1\times10^{-7} & 3.1\times10^{-7} & 1.7\times10^{-10}\\ 0.6039 & 0.0003 & 1.73 & 9.6\times10^{-5} & 1.73 & 9.6\times10^{-5} & 0.0496 & 2.7\times10^{-5}\\ 0.6039 & 0.173 & 0.0003 & 9.6\times10^{-5} & 0.173 & 0.0496 & 9.6\times10^{-5} & 2.7\times10^{-5}\\ 0.0040 & 0.0012 & 0.0012 & 0.0003 & 0.6002 & 0.1719 & 0.1719 & 0.0493\\ 0.6039 & 0.173 & 0.173 & 0.0496 & 0.0003 & 9.6\times10^{-5} & 9.6\times10^{-5} & 2.7\times10^{-5}\\ 0.0040 & 0.0012 & 0.6002 & 0.1719 & 0.0012 & 0.0003 & 0.1719 & 0.0493\\ 0.0040 & 0.6002 & 0.0012 & 0.1719 & 0.0012 & 0.1719 & 0.0003 & 0.0493\\ 2.0\times10^{-7} & 4.4\times10^{-5} & 4.4\times10^{-5} & 0.0066 & 4.4\times10^{-5} & 0.0066 & 0.0066 & 9.801\end{pmatrix}$$

Applying the operator Ψ on the P_β gives $\Pi_\beta=\Psi^{-1}P_\beta\Psi$ as

$$\Pi_\beta=\begin{pmatrix}0.2134 & 0.1199 & 0.4616 & -0.1283 & 0.4616 & -0.1283\\ 0.2134 & 0.1199 & -0.0348 & 0.3681 & -0.0348 & 0.3681\\ 0.4616 & -0.1283 & 0.2134 & 0.1199 & 0.4616 & -0.1283\\ -0.0348 & 0.3681 & 0.2134 & 0.1199 & -0.0348 & 0.3681\\ 0.4616 & -0.1283 & 0.4616 & -0.1283 & 0.2134 & 0.1199\\ -0.0348 & 0.3681 & -0.0348 & 0.3681 & 0.2134 & 0.1199\end{pmatrix}$$

Notice that Π_β consists of square block submatrices which are identical and can be identified with the topology of the interaction structure. The rows of Π_β can be rescaled without loss of generality to a new transition matrix say Π'_β, which plays the same role as Π_β.

$$\Pi'_\beta = \begin{pmatrix} 0 & 0 & 0.5683 & -0.0683 & 0.5683 & -0.0683 \\ 0 & 0 & 0.0719 & 0.4281 & 0.0719 & 0.4281 \\ 0.5683 & -0.0683 & 0 & 0 & 0.5683 & -0.0683 \\ 0.0719 & 0.4281 & 0 & 0 & 0.0719 & 0.4281 \\ 0.5683 & -0.0683 & 0.5683 & -0.0683 & 0 & 0 \\ 0.0719 & 0.4281 & 0.0719 & 0.4281 & 0 & 0 \end{pmatrix}$$

The transition matrix Π'_β, can be factorized into

$$\Pi'_\beta = \Omega^T \otimes \Sigma \tag{B.20}$$

where

$$\Omega = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

and

$$\Sigma = \begin{pmatrix} 0.9994 & 0.0006 \\ 0.0067 & 0.9933 \end{pmatrix}.$$

B.4 Proof of Corollary 2

From (12), we have $P_\beta \Psi = \Psi \Pi_\beta$. Multiplying by $\mathbf{r}_i$, $P_\beta \Psi \mathbf{r}_1 = \Psi \Pi_\beta \mathbf{r}_1$. Since P_β is a stochastic matrix, $\mu_1 = 1$, and it follows that $P_\beta \Psi \mathbf{r} = \Psi \mathbf{r}_1$, which is true if and only if $\Pi_\beta \mathbf{r}_1 = \mathbf{r}_1$; that is $\lambda_1 = \mu_1 = 1$. It also follows that if $\lambda_i = \mu_i$ and $\Psi \mathbf{r}_i \neq 0$, then $\tilde{\mathbf{r}}_i = \Psi \mathbf{r}_i$.

Similarly, multiplying $P_\beta \Psi = \Psi \Pi_\beta$ by the transpose of $\tilde{\mathbf{z}}_i$ such that $\tilde{\mathbf{z}}_i^T P_\beta \Psi = \tilde{\mathbf{z}}_i^T \Psi \Pi_\beta$, then

$$\tilde{\mathbf{z}}_i^T P_\beta \Psi = \mu_i \tilde{\mathbf{z}}_i^T \Psi = \tilde{\mathbf{z}}_i^T \Psi \Pi_\beta \tag{B.21}$$

Implying that if $\lambda_i = \mu_i$ and $\tilde{\mathbf{z}}_i^T \Psi \neq 0$ then it is also true that $\mathbf{z}_i^T = \tilde{\mathbf{z}}_i^T \Psi$

B.5 Proof of Proposition 1

If the interaction structure is strongly connected, and additionally that $\beta > 0$, then the adoptive process with the transition matrix Π_β is finite, irreducible, and aperiodic. This also implies that no multiple eigenvalues of Π_β exists. From the eigendecomposition of Π_β,

$$\mathbf{v}_0 \Pi^t = \mathbf{v}_0 \mathbf{r}_1 \mathbf{z}_1^T + \sum_{i=2}^{\gamma} \lambda_i^t \mathbf{v}_0 \mathbf{r}_i \mathbf{z}_i^T \tag{B.22}$$

where we treat $\mathbf{v}_0$ as a row vector, the right eigenvectors ($\mathbf{r}_i$) of Π_β are column vectors. As $t \to \infty$, only the first term on the right hand side of (B.23) remains such that

$$\mathbf{v}_0 \Pi^t = \mathbf{v}_0 \mathbf{r}_1 \mathbf{z}_1^T \tag{B.23}$$

Let the eigenvectors of Π_β be normalized so that $\mathbf{z}_i^T \mathbf{r}_i = 1$, and recall from Lemma 3 that $\mathbf{r}_1 = \mathbf{w}_1 \otimes \mathbf{1}_2$, where $\mathbf{w}_1$ is the left eigenvector of Ω. Denote the ith element of $\mathbf{v}_0$ by $v_0(i)$. It follows that

$$\begin{aligned}\mathbf{v}_0\mathbf{r}_1 &= \mathbf{v}_0(\mathbf{w}_1 \otimes \mathbf{1}_2) \\ &= [v_0(1), \cdots, v_0(\gamma)][w_1(1) \otimes \mathbf{1}_2, \cdots, w_1(n) \otimes \mathbf{1}_2]^T \\ &= w_1 + \cdots + w_1 \\ &= 1 \end{aligned} \tag{B.24}$$

where the third equality follows from the fact that $\mathbf{v}_0$ comprises of event vectors (e_i for all $i \in n$) of size m, such that $\mathbf{v}_0 = [e_1, \cdots, e_n]$ and that for each $i \in n$, $e_i(w_i \otimes \mathbf{1}_2) = w_i$. It therefore follows that the steady state or stationary distribution of the adoptive process with transition matrix Π_β and initial state $\mathbf{v}_0$ is

$$\lim_{t\to\infty} \mathbf{v}_0 \Pi_\beta^t = \mathbf{v}_0 \mathbf{r}_1 \mathbf{z}_1^T = \mathbf{z}_1^T \tag{B.25}$$

B.6 Proof of Theorem 1

We derive insights for the following proof from the general proof for the mixing time of Markov chains in Levin et al. [2009]. Most importantly we make adaptations and extensions to derive results for our analytical objectives. Recall that the eigenvalue spectrum of P_β is $\rho(P_\beta) = (\mu_1, \cdots, \mu_\eta)$. To avoid notational clutter we shall write P instead of P_β, $\mathbf{s}_0$ for $\mathbf{s}_{t_0}$, $\mathbf{x}_0$ for $\mathbf{x}_{t_0}$ and $\mathbf{s}(\mathbf{x}_0)$ for $\mathbf{s}(\mathbf{x}_{t_0})$.

We first take note of the following lemma.

Lemma 5 *Let $P(\mathbf{x}, \mathbf{y})$ imply the transition from configuration $\mathbf{x}$ to $\mathbf{y}$, the total variation distance has the following equivalence.*

$$\big|\big|\mathbf{s}_0 P^t - \nu\big|\big| = \|P^t(\mathbf{x}_0, \mathbf{y}) - \nu(\mathbf{y})\| \tag{B.26}$$

for $\mathbf{x}_0, \mathbf{y} \in \mathbf{X}$.

Proof. Recall the definition of the total variation distance,

$$\big|\big|\mathbf{s}_0 P^t - \nu\big|\big| = \frac{1}{2} \sum_{\mathbf{y}\in\mathbf{X}} |\mathbf{s}_0 P^t(\mathbf{y}) - \nu(\mathbf{y})| \tag{B.27}$$

where $\mathbf{s}_0 P^t = P^t(\mathbf{x}_0, .)$, hence

$$\begin{aligned}\big|\big|\mathbf{s}_0 P^t - \nu\big|\big| &= \frac{1}{2} \sum_{\mathbf{y}\in\mathbf{X}} |P^t(\mathbf{x}_0, \mathbf{y}) - \nu(\mathbf{y})| \\ &= \|P^t(\mathbf{x}_0, \mathbf{y}) - \nu(\mathbf{y})\| \end{aligned} \tag{B.28}$$

■

From the definition of convergence time, what is required is to place a bound on the left hand side of (5), which reduces to obtaining the bound for the right hand side of (5). Note that

if we are interested in the convergence time from the known initial configuration say $\mathbf{x}_0$ to the stochastically stable configuration $\mathbf{x}_*$, then we simply need to obtain the bound for the quantity $\|P^t(\mathbf{x}_0, \mathbf{x}_*) - \nu(\mathbf{x}_*)\|$. The worst possible initial state (configuration) in the binary choice case we consider is the all B configuration $\vec{B}$, and the all A, $\vec{A}$ is the stochastically stable configuration, such that

$$\left|\left|\mathbf{s}_0 P^t - \nu\right|\right| = \|P^t(\vec{B}, \vec{A}) - \nu(\vec{A})\| = \max_{\mathbf{x}\in\mathbf{X}} \|P^t(\mathbf{x}, \mathbf{y}) - \nu(\mathbf{y})\| \tag{B.29}$$

To keep the proof as general as possible, we shall carry on working with $P^t(\mathbf{x}, \mathbf{y})$ rather than $P^t(\mathbf{x}_0, \mathbf{y})$ then simply deduce the results for the latter towards the end of the proof. Recall that P is reversible. That is given the stationary distribution ν,

$$\nu(\mathbf{x})P(\mathbf{x}, \mathbf{y}) = \nu(\mathbf{y})P(\mathbf{y}, \mathbf{x}) \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbf{X} \tag{B.30}$$

We can define an equivalent symmetric matrix S such that

$$S(\mathbf{x}, \mathbf{y}) = \sqrt{\frac{\nu(\mathbf{x})}{\nu(\mathbf{y})}} P(\mathbf{x}, \mathbf{y}) \tag{B.31}$$

The reason for introducing S is to be able to exploit the properties of symmetric matrices, particularly the spectral theorem stating that, for symmetric matrices, there exists a set of orthonormal basis $\{\mathbf{u}_i\}_{i=1}^{|\mathbf{X}|}$, such that $\mathbf{u}_i$ is an eigenfunction corresponding to the real eigenvalue μ_i.

Now, let us denote the diagonal matrix with elements $\nu(\mathbf{x})$ by D, then we have

$$S = D^{\frac{1}{2}} P D^{-\frac{1}{2}} \tag{B.32}$$

Let $f_i = D^{-\frac{1}{2}}\mathbf{u}_i$, (where $\mathbf{u}_i$'s are eigenfunctions of S) it follows that f_i is an eigenfunction of P corresponding to eigenvalue μ_i, and that f_i's are orthonormal with respect to ν. That is

$$Pf_i = PD^{-\frac{1}{2}}\mathbf{u}_i = D^{-\frac{1}{2}}\left(D^{\frac{1}{2}} P D^{-\frac{1}{2}}\right)\mathbf{u}_i = D^{-\frac{1}{2}} S \mathbf{u}_i = D^{-\frac{1}{2}}\mu_i \mathbf{u}_i = \mu_i f_i. \tag{B.33}$$

To prove orthonormality of the functions f_i's with respect to ν, its useful to take note of the following definition of the *inner product*.

Let $\langle ., . \rangle$ denote the inner product on $\mathbb{R}^{\mathbf{X}}$, that is

$$\langle f, g \rangle = \sum_{\mathbf{x}\in\mathbf{X}} f(\mathbf{x})g(\mathbf{x}), \tag{B.34}$$

then we can define the inner product with respect to the distribution ν as

$$\langle f, g \rangle_\nu = \sum_{\mathbf{x}\in\mathbf{X}} f(\mathbf{x})g(\mathbf{x})\nu(\mathbf{x}). \tag{B.35}$$

Let $\delta_{i,j}$ denote the Dirac delta function (that is $\delta_{i,j} = 1$ if and only if $i = j$), then

$$\delta_{i,j} = \langle \mathbf{u}_i, \mathbf{u}_j \rangle = \left\langle D^{\frac{1}{2}} f_i, D^{\frac{1}{2}} f_j \right\rangle = \sum_{\mathbf{x}\in\mathbf{X}} \nu(\mathbf{x})^{\frac{1}{2}} f(\mathbf{x}) \nu(\mathbf{x})^{\frac{1}{2}} g(\mathbf{x}) = \sum_{\mathbf{x}\in\mathbf{X}} f(\mathbf{x})\nu(\mathbf{x})g(\mathbf{x}) = \langle f, g \rangle_\nu .$$

Note that $P^t(\mathbf{x}, \mathbf{y})$ is the $\mathbf{x}, \mathbf{y}$ element of P^t, implying that $P^t(\mathbf{x}, \mathbf{y}) = (P^t \delta_{\mathbf{y}})(\mathbf{x})$; where $\delta_{\mathbf{y}}(\mathbf{x})$ is a Dirac function assuming the value of unity for $x = y$ and zero otherwise. Notice also that $\delta_{\mathbf{y}}$ belongs to the inner product space $\mathbb{V} = (\mathbb{R}^{\mathbf{X}}, \langle ., . \rangle_\nu)$, and since the set $\{f_1, \cdots, f_{|\mathbf{X}|}\}$ is an orthonormal basis of $\mathbb{V}$, then $\delta_{\mathbf{y}}$ can be written via basis decomposition as

$$\delta_{\mathbf{y}} = \sum_{i=1}^{\eta} \langle \delta_{\mathbf{y}}, f_i \rangle_\nu f_i = \sum_{i=1}^{\eta} f_i(\mathbf{y})\nu(\mathbf{y}) f_i \tag{B.36}$$

Substituting (B.36) and $P^t f_i = \mu_i^t f_i$ gives $P^t(\mathbf{x}, \mathbf{y})$ as

$$P^t(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\eta} f_i(\mathbf{y})\nu(\mathbf{y})\mu_i^t f_i(\mathbf{x}) = \nu(\mathbf{y}) + \sum_{i=2}^{\eta} f_i(\mathbf{y})\nu(\mathbf{y})\mu_i^t f_i(\mathbf{x}) \tag{B.37}$$

Taking the absolute values of (B.37) yields

$$\begin{aligned} \left| P^t(\mathbf{x}, \mathbf{y}) - \nu(\mathbf{y}) \right| &= \sum_{i=2}^{\eta} \left| f_i(\mathbf{y})\nu(\mathbf{y}) f_i(\mathbf{x}) \mu_i^t \right| \leq \sum_{i=2}^{\eta} \left| f_i(\mathbf{y})\nu(\mathbf{y}) f_i(\mathbf{x}) \right| |\mu_2|^t \\ &\leq \nu(\mathbf{y}) \left[\sum_{i=2}^{\eta} f_i^2(\mathbf{y}) \sum_{i=2}^{\eta} f_i^2(\mathbf{x}) \right]^{\frac{1}{2}} |\mu_2|^t \end{aligned} \tag{B.38}$$

$$\tag{B.39}$$

where the last inequality follows from Cauchy-Schwarz inequality. Following from the definition of inner product together with that of $\delta_{\mathbf{y}}(\mathbf{x})$ in (B.36), and relying on the orthonormality of the set f_i, we have

$$\nu(\mathbf{y}) = \langle \delta_{\mathbf{y}}, \delta_{\mathbf{y}} \rangle = \left\langle \sum_{i=1}^{\eta} f_i(\mathbf{y})\nu(\mathbf{y}) f_i, \sum_{i=1}^{\eta} f_i(\mathbf{y})\nu(\mathbf{y}) f_i \right\rangle = \nu(\mathbf{y})^2 \sum_{i=1}^{\eta} f_i(\mathbf{y})^2 \tag{B.40}$$

The implication is that $\sum_{i=2}^{\eta} f_i(\mathbf{y})^2 \leq \frac{1}{\nu(\mathbf{y})}$, and consequently

$$\left| P^t(\mathbf{x}, \mathbf{y}) - \nu(\mathbf{y}) \right| \leq \frac{\nu(\mathbf{y})}{\sqrt{\nu(\mathbf{x})\nu(\mathbf{y})}} |\mu_2|^t \tag{B.41}$$

If $\nu(\mathbf{x}_{t_0})$ and $\nu(\mathbf{x}_*)$ are the stationary distribution for the initial and final configuration, then it follows from Lemma 5 and the above discussion that

$$\left|\left| \mathbf{s}_0 P^t - \nu \right|\right| = \frac{1}{2} \sum_{\mathbf{y}\in\mathbf{X}} |P^t(\mathbf{x}_{t_0}, \mathbf{y}) - \nu(\mathbf{y})| \leq \frac{1}{2} \sum_{\mathbf{y}\in\mathbf{X}} \frac{\nu(\mathbf{y})}{\sqrt{\nu(\mathbf{x}_{t_0})\nu(\mathbf{y})}} |\mu_2|^t \leq \frac{\nu(\mathbf{x}_*)}{\nu(\mathbf{x}_{t_0})} |\mu_2|^t$$

where the last inequality follows from the fact that if we start from the worst possible case then $\nu(\mathbf{x}_{t_0}) \leq \nu(\mathbf{x}_*)$.

For $\big|\big|\mathbf{s}_0 P^t - \nu\big|\big| \leq \varpi$, then $\frac{\nu(\mathbf{x}_*)}{\nu(\mathbf{x}_{t_0})}|\mu_2|^t \leq \varpi$. Consequently, the convergence time is

$$T_c(G,\beta,n) \leq \frac{\ln\left(\varpi\nu(\frac{\mathbf{x}_{t_0})}{\nu(\mathbf{x}_*)}\right)}{\ln|\mu_2|} \tag{B.42}$$

Note that if $|\mu_2|$ is large and positive, then $\ln|\mu_2| \approx 1-\mu_2$. Substituting into (B.42) yields

$$T_c(G,\beta,n) \leq \frac{\ln\left(\frac{\nu(\mathbf{x}_*)}{\varpi\nu(\mathbf{x}_{t_0})}\right)}{1-\mu_2} \tag{B.43}$$

And for the binary action set we have considered,

$$T_c(G,\beta,n) \leq \frac{\ln\left(\frac{\nu(\vec{A})}{\varpi\nu(\vec{B})}\right)}{1-\mu_2} \tag{B.44}$$

B.7 Proof of Corollary 4

First note that for $n(A)=0$ corresponding to $\mathbf{x}_{t_0}=\vec{B}$, we have that $\tau(\beta,h,\nu(\vec{B})) = \nu(\vec{A})/\nu(\vec{B}) > 1$.

Now consider the values of $n(A)$ in the neighborhood of zero, say some positive integer ε. Denote the $\mathbf{x}_{t_0}$ for which $n(A)=\varepsilon$ by $\mathbf{x}_{t_0}(\varepsilon)$. It is easy to see from eqrefeq:distn that the quantity $\nu(\mathbf{x}_{t_0}(\varepsilon)) < \nu(\vec{B})$, hence $\tau(\beta,h,\nu(\mathbf{x}_{t_0}(\varepsilon))) > \tau(\beta,h,\nu(\vec{B}))$.

Since $\tau(\beta,h,\nu(\vec{A})) = 1 < \tau(\beta,h,\nu(\vec{B})) < \tau(\beta,h,\nu(\mathbf{x}_{t_0}(\varepsilon)))$, there must exists a configuration $\mathbf{x}_{t_0}$ for which $n(A)>0$ such that τ is maximum.

B.8 Proof of Proposition 2

For any initial state $\mathbf{s}_0 = \mathbf{s}(\mathbf{x}_0)$ and following from the proof of 1, we have that

$$\begin{aligned}\big|\big|\mathbf{s}_0 P^t - \nu\big|\big| = \|P^t(\mathbf{x}_0,.) - \nu(.)\| &= \frac{1}{2}\sum_{\mathbf{y}\in\mathbf{X}} |P^t(\mathbf{x}_0,\mathbf{y}) - \nu(\mathbf{y})| \\ &= \sum_{i=2}^{\gamma} \big|f_i(\mathbf{y})\nu(\mathbf{y})f_i(\mathbf{x}_0)\mu_i^t\big| \end{aligned} \tag{B.45}$$

Taking the lim sup and $1/t$-th power gives

$$\begin{aligned}\limsup_{t\to\infty} \big|\big|\mathbf{s}_0 P^t - \nu\big|\big|^{1/t} &= \limsup_{t\to\infty}\left[\sum_{i=2}^{\gamma}\big|f_i(\mathbf{y})\nu(\mathbf{y})f_i(\mathbf{x}_0)\mu_i^t\big|\right]^{1/t} \\ &= |\mu_2|\limsup_{t\to\infty}\left[|f_2(\mathbf{y})\nu(\mathbf{y})f_2(\mathbf{x}_0)| + \frac{1}{|\mu_2|^t}\sum_{i=3}^{\gamma}\big|f_i(\mathbf{y})\nu(\mathbf{y})f_i(\mathbf{x}_0)\mu_i^t\big|\right]^{1/t} \\ &= |\mu_2| \end{aligned} \tag{B.46}$$

where the result is valid for any $\mathbf{y}\in\mathbf{X}$ and for $\mathbf{x}\neq\mathbf{y}$; hence

$$1 - r = 1 - |\mu_2| \tag{B.47}$$

B.9 Proof of Proposition 3

Let π denote the stationary distribution of Π_β or Π in short form. We showed in Proposition 1 that $\pi = \mathbf{v}_0\mathbf{r}_1\mathbf{z}_1^T = \mathbf{z}_1$, where $\mathbf{z}_1$ is the unique left eigenvector of Π. This implies that,

$$\mathbf{v}_0\Pi^t - \pi = \sum_{i=2}^{\gamma} \lambda_i^t \mathbf{v}_0\mathbf{r}_i\mathbf{z}_i^T \tag{B.48}$$

such that

$$\begin{aligned} |\mathbf{v}_0\Pi^t - \pi| &= \sum_{i=2}^{\gamma} |\lambda_i^t \mathbf{v}_0\mathbf{r}_i\mathbf{z}_i^T| \\ &= |\lambda_2^t \mathbf{v}_0\mathbf{r}_2\mathbf{z}_2^T| + |\lambda_3^t \mathbf{v}_0\mathbf{r}_3\mathbf{z}_3^T| + \sum_{i=4}^{\gamma} |\lambda_i^t \mathbf{v}_0\mathbf{r}_i\mathbf{z}_i^T| \\ &\leq \left(|\lambda_2|^t + |\lambda_3|^t\right) |\mathbf{v}_0\mathbf{r}_2\mathbf{z}_2^T| + \sum_{i=4}^{\gamma} |\lambda_i^t \mathbf{v}_0\mathbf{r}_i\mathbf{z}_i^T| \end{aligned} \tag{B.49}$$

The triangular inequality for higher powers is as follows: $\forall x_1, \cdots, x_\gamma \in \mathbb{R}^\gamma$; and for $\gamma \in \mathbb{N}$,

$$\left| \sum_{i=1}^{\gamma} x_i \right|^p \leq \gamma^{p-1} \sum_{i=1}^{\gamma} |x_i|^p \tag{B.50}$$

The triangular inequality can be extended to scalars , such that for two scalars λ_2 and λ_3, we have

$$|\lambda_2|^t + |\lambda_3|^t \geq \frac{1}{2^{t-1}} \left(|\lambda_2| + |\lambda_3|\right)^t \tag{B.51}$$

Note that if $\lambda_2 = \lambda_3$ then the inequality in (B.51) becomes an equality, and if λ_2 and λ_3 are comparable or both close to unity then the inequality becomes an approximation. Since we shall mostly be interested in the properties of the learning process for which β is large ($\lambda_3 = \vartheta_2$ is close to unity), and for G with local interactions we shall assume that λ_2 and λ_3 are fairly comparable. That is

$$|\lambda_2|^t + |\lambda_3|^t \approx 2 \left[\frac{1}{2}(|\lambda_2| + |\lambda_3|)\right]^t \tag{B.52}$$

substituting into (B.49) yields

$$\begin{aligned} |\mathbf{v}_0\Pi^t - \pi| &\leq 2 \left[\frac{1}{2}(|\lambda_2| + |\lambda_3|)\right]^t |\mathbf{v}_0\mathbf{r}_2\mathbf{z}_2^T| + \sum_{i=4}^{\gamma} |\lambda_i^t \mathbf{v}_0\mathbf{r}_i\mathbf{z}_i^T| \\ &= \left[\frac{1}{2}(|\lambda_2| + |\lambda_3|)\right]^t \left[2|\mathbf{v}_0\mathbf{r}_2\mathbf{z}_2^T| + \frac{1}{\left[\frac{1}{2}(|\lambda_2| + |\lambda_3|)\right]^t} \sum_{i=4}^{\gamma} |\lambda_i^t \mathbf{v}_0\mathbf{r}_i\mathbf{z}_i^T|\right] \end{aligned} \tag{B.53}$$

Consequently the lim sup of the total variation distance becomes

$$\limsup_{t\to\infty} \left|\left|\mathbf{v}_0\Pi^t - \pi\right|\right|^{\frac{1}{t}} \leq \frac{1}{2}(|\lambda_2| + |\lambda_3|) \limsup_{t\to\infty} \left[2|\mathbf{v}_0\mathbf{r}_2\mathbf{z}_2^T| + \frac{1}{\left[\frac{1}{2}(|\lambda_2| + |\lambda_3|)\right]^t} \sum_{i=4}^{\gamma} |\lambda_i^t \mathbf{v}_0\mathbf{r}_i\mathbf{z}_i^T|\right]^{\frac{1}{t}}$$
$$= \frac{1}{2}(|\lambda_2| + |\lambda_3|) \tag{B.54}$$

Substituting for $\lambda_2 = \theta_2$ and $\lambda_3 = \vartheta_2$ we get

$$1 - r \geq \frac{1}{2}\left[(1 - |\theta_2|) + (1 - |\vartheta_2|)\right] \tag{B.55}$$

B.10 Proofs for Example 2

Let $G = (n, E)$ be a graph or network of n vertices. Denote by S a subset of n and by $e(S, n-S)$ as the number of interactions or edges between S and its complement $n-S$. Also let $d(S)$ denote the total degree of subset S. For regular network graphs (in which all vertices posses the same degree size), it is shown by [Alon and Milman, 1985] that

$$\phi(G) = \min_{S,\#S\leq\frac{n}{2}} \frac{e(S, n-S)}{d(S)} \geq \frac{1 - \theta_2(G)}{2} \tag{B.56}$$

where $\#S$ denotes the size of the set S.

For a complete graph, since every vertex is connected to every other vertex, we have that every vertex in S is connected to all other vertices in $n - S$. This implies that $e(S, n - S) = \#S \times \#(n - S) = \#S \times (n - \#S)$, and $d(S) = n \times \#S$ such that

$$\frac{1 - \theta_2(G_{com})}{2} \leq \min_{S,\#S\leq\frac{n}{2}} \frac{\#S \times (n - \#S)}{n \times \#S} \leq \frac{1}{2}, \tag{B.57}$$

where the last inequality follows from the fact that $\#S \leq \frac{n}{2}$. We thus have that

$$1 - \theta_2(G_{com}) \leq 1 \tag{B.58}$$

In the case of a $1 - D$ cyclic network, $e(S, n - S) = 2$, and $d(S) = 2 \times \#S$ such that

$$1 - \theta_2(G_{cyc}) \leq 2 \min_{S,\#S\leq\frac{n}{2}} \frac{2}{2 \times \#S} \leq \frac{4}{n}. \tag{B.59}$$

$2D$ network: Let the composition of S be chosen in such a way that the peripheral vertices (vertices at the perimeter or boundary of S) contain approximately one edge each connecting it to the set $n - S$. Since it is a 2-dimensional structure there should be approximately $\sqrt{\#S}$ vertices forming such a boundary. This implies that $e(S, n - S) \approx \sqrt{\#S}$, and $d(S) = 4 \times \#S$ such that

$$1 - \theta_2(G_{2D}) \leq 2 \min_{S,\#S\leq\frac{n^2}{2}} \frac{\sqrt{\#S}}{4 \times \#S} \leq \frac{4}{n}. \tag{B.60}$$

where the last inequality follows from the fact that $\sqrt{\#S} \leq \sqrt{\frac{n^2}{2}} \leq \sqrt{\frac{1}{2}n^2}$.

Random d-regular network: Since for each vertex the vertices to which it ic connected to are chosen at random, and that the maximum size of S is $\frac{n}{2}$, then a typical vertex in S is connected to approximately $\frac{d\times\#(n-S)}{n}$ other vertices in $n-S$ such that $e(S,n-S)\approx\frac{d\times\#S\#(n-S)}{n}$. We thus have

$$1-\theta_2(G_{d-r})\leq 2\min_{S,\#S\leq\frac{n}{2}}\frac{\frac{d\times\#S\#(n-S)}{n}}{d\times\#S}\leq 1. \tag{B.61}$$

For Newman's small world networks see Durrett [2006].

B.11 Proof of Theorem 2

We aim to derive a bound on $\mathbb{P}(n(A_{t_0+T})\geq\alpha n)$, the probability that at least αn of the agents simultaneously play A after T time steps starting from t_0, and given n and β. We proceed with the derivation by performing a random walk on the network of agents connections.

We shall let p_i denote the probability that an agent i will choose A given the choice of his neighbors in the current period. Specifically, since we are interested in the case of an all B initial configuration such that $x_{t_0}=\vec{B}$, and if we let $\vec{B}_{\mathscr{N}_i}$ denote the all B configuration of $\mathscr{N}_i$ then $x_{\mathscr{N}_i,t_0}=\vec{B}_{\mathscr{N}_i}$. We shall therefore let

$$p_i=\mathbb{P}(x_{i,t_0+T}=A|x_{\mathscr{N}_i,t_0}=\vec{B}_{\mathscr{N}_i})$$

. Recall that the expected waiting time is given by,

$$T_w(G,\beta,\alpha,n)=\frac{T}{\mathbb{P}(n(A_{t_0+T})\geq\alpha n)} \tag{B.62}$$

We shall consider the case of $T=1$, such that we simply compute the bound for $\mathbb{P}(n(A_{t_0+1})\geq\alpha n)$. Under this consideration, $p_i=\mathbb{P}(x_{i,t_0+1}=A|x_{\mathscr{N}_i,t_0}=\vec{B}_{\mathscr{N}_i})$ for all i; the probability that i plays A in the next period given that all his neighbors currently play B.

Define a parameter $w\in[0,1]$ such that if $p_i\geq w$ agent i chooses A or else he chooses B. This leads to a random variable denoted by I_i, which is equal to one if i chooses A and zero otherwise. Let $I=(I_1,\cdots,I_n)$ be the realization of I_i for all $i\in n$. We then rephrase our problem as the case of bounding $\mathbb{P}(\sum_{i=1}^n I_i\geq\alpha n)$.

Now define a binomial sampling over the vector I, $Bin(n,b)$, such that with probability b, I_i is picked and with $1-b$ it is not. Denote the n-dimensional vector generated by $Bin(n,b)$ by $\mathbf{u}=(u_1,\cdots,u_n)$, where $\mathbb{P}(u_i=1)=b$ and $\mathbb{P}(u_i=0)=1-b$. Let S be a subset generated by the binomial sampling such that $S\subseteq\mathbf{u}$ containing all ones. This enables us to define an event that $\forall_{i\in S}I_i=1$; that is all members of S choose A, and consequently $\mathbb{P}(\forall_{i\in S}I_i=1)$ is the probability that all $i\in S$ choose A.

We can thus define the following conditional relation,

$$\mathbb{P}\left(\sum_{i=1}^{n} I_i \geq \alpha n\right) = \frac{\mathbb{E}\big[\forall_{i\in S} I_i = 1\big]}{\mathbb{E}\big[\forall_{i\in S} I_i = 1 | \sum_{i=1}^{n} I_i \geq \alpha n\big]}, \tag{B.63}$$

where the expectations are taken over the vector $\mathbf{u}$.

Since the elements of $\mathbf{u}$ are a result of independent sampling, we have

$$\mathbb{E}[\forall_{i\in S} I_i = 1] = \mathbb{E}_{\forall u_i \in \mathbf{u}}\left[\prod_{i=1}^{n} (p_i|u_i)\right], \tag{B.64}$$

where $(p_i|u_i)$ is the probability that $I_i = 1$ given the vector $\mathbf{u}$. Equation (B.64) can be simplified as follows

$$\mathbb{E}\big[\forall_{i\in S} I_i = 1\big] = \prod_{i=1}^{n} \mathbb{E}_{u_i}\big[p_i|u_i\big] = \prod_{i=1}^{n} \big(bp_i + 1 - b\big) \tag{B.65}$$

If we define $p = \frac{1}{n}\sum_{i=1}^{n} p_i$ to be the average of the probability of choosing A in population of size n, then convexity dictates that

$$\prod_{i=1}^{n} \big(bp_i + 1 - b\big) \leq \big(bp + 1 - b\big)^n \tag{B.66}$$

Note that if G is a regular network graph (in which each node has the same number of neighbors), then it is easy to see from the properties of Π_β that $p_1 = p_2 = \cdots = p_n = p$, such that

$$\prod_{i=1}^{n} \big(bp_i + 1 - b\big) = \big(bp + 1 - b\big)^n \tag{B.67}$$

To obtain the bound for $\mathbb{E}\big[\forall_{i\in S} I_i = 1 | \sum_{i=1}^{n} I_i \geq \alpha n\big]$, recall that $1-b$ is the probability that $u_i = 0$. We also note that if at least αn of the elements of $\mathbf{u}$ are ones, then there are at most $n - \alpha n$ zeros, that is at most $n - \alpha n$ agents are not in set S. It follows that

$$\mathbb{E}\big[\forall_{i\in S} I_i = 1 | \sum_{i=1}^{n} I_i \geq \alpha n\big] \geq (1-b)^{(1-\alpha)n} \tag{B.68}$$

Equations (B.67) together with (B.68) when substituted into (B.63) yield,

$$\mathbb{P}\left(\sum_{i=1}^{n} I_i \geq \alpha n\right) \leq \left(\frac{\big(bp + 1 - b\big)}{(1-b)^{(1-\alpha)}}\right)^n, \tag{B.69}$$

If we choose b that optimizes the quantity $g = \frac{\big(bp+1-b\big)}{(1-b)^{(1-\alpha)}}$ (by equating the derivative to zero and solving for b) and substituting back gives

$$\mathbb{P}\left(\sum_{i=1}^{n} I_i \geq \alpha n\right) \leq \left(\left(\frac{p}{\alpha}\right)^{\alpha}\left(\frac{1-p}{1-\alpha}\right)^{1-\alpha}\right)^n \tag{B.70}$$

By re-expressing (B.69) in exponential form and recalling that
$\mathbb{P}(n(A_{t_0+1}) \geq \alpha n) = \mathbb{P}\left(\sum_{i=1}^{n} I_i \geq \alpha n\right)$, we have

$$\mathbb{P}(n(A_{t_0+1}) \geq \alpha n) \leq e^{-n(\alpha(\ln \alpha - \ln p) + (1-\alpha)(\ln(1-\alpha) - \ln(1-p)))} \tag{B.71}$$

By approximating the ln of p, α, $1-p$ and $1-\alpha$ to the second order gives

$$\begin{aligned} n(\alpha(\ln \alpha - \ln p) + (1-\alpha)(\ln(1-\alpha) - \ln(1-p))) &\geq n\left(\frac{7\alpha^2}{2} - \alpha - 4\alpha p + p + \frac{p^2}{2}\right) \\ &\geq 2n(\alpha - p)^2 \end{aligned} \tag{B.72}$$

where the last inequality is true for $0 \leq p \leq \alpha$. Substituting into (B.71) yields

$$\mathbb{P}(n(A_{t_0+1}) \geq \alpha n) \leq e^{-2n(\alpha-p)^2} \tag{B.73}$$

From (B.62) with $T = 1$, we therefore have that

$$T(G, \beta, \alpha, n) \geq e^{2n(\alpha-p)^2} \tag{B.74}$$

B.12 Proof of Proposition 4

The proof follows from the edge expansion property of the graph corresponding to the social network. Given a symmetric adjacency matrix G of the topology of the social network, the edge boundary of a subset l of n, $l \subset n$ is defined as $d(l, n-l)$; the number of interactions originating from l to its complement $n-l$. Note that since G is symmetric, that is the network is undirected, then $d(l, n-l)$ will be the number of interactions between l and $n-l$.

The edge expansion ratio of l, $\phi(l)$ is defined as The *edge expansion Ratio* of l, denoted $\phi(l)$, is defined as:

$$\phi(l) = \frac{d(l, n-l)}{\min\{d(l), d(n-l)\}}. \tag{B.75}$$

where $d(l) = \sum_{i \in l} k_i$, sum of degrees of all members of subgroup l. If G is a k regular graph then $d(l) = k|l|$.

The *edge expansion Ratio* of G, denoted $\phi(G)$, is defined as the minimum of $\phi(l)$, that is

$$\phi(G) = \min_{l, d(l) \subset d(n)} \frac{d(l, n-l)}{\min\{d(l), d(n-l)\}} \tag{B.76}$$

where $\phi(G)$ is also sometimes referred to as the *isoperimetric constant* [Cheeger, 1970, Alon and Milman, 1985, Buser, 1982]. Note that if we are partitioning G into two subgroups d and $n-l$, then either $d(l)$ or $d(n-l)$ is less than half of $d(n)$, such that if $d(l) \leq \frac{d(n)}{2}$ then (B.76) becomes

$$\phi(G) = \min_{l, d(l) \leq \frac{d(n)}{2}} \frac{d(l, n-l)}{d(l)} \tag{B.77}$$

The upper bound for $\phi(G)$ is given by Cheeger's inequality [Sinclair, 1992]

$$\min_{l, d(l) \leq \frac{d(n)}{2}} \frac{d(l, n-l)}{d(l)} \leq \sqrt{2(1-\theta_2)} \tag{B.78}$$

Recall that a a subgroup $l \subset n$ is cohesive if

$$\frac{d(l, n-l)}{d(l)} \leq 1 - 2r \qquad 0 < r < \frac{1}{2} \tag{B.79}$$

If we assume that the partitioning of n into the subgroups l and $n-l$ is performed by cutting through the minimum inter-subgroup connectivity as in (B.78), then we have that

$$r = \frac{1}{2}\left(1 - \sqrt{2(1-\theta_2)}\right). \tag{B.80}$$

Finally, note that when $r = 0$, $\theta_2 = 0.5$ and when $r = 0.5$, $\theta_2 = 1$. Implying that G consists of two cohesive subgroup if the second largest eigenvalue of the corresponding normalized adjacency matrix θ_2 is such that $0.5 < \theta_2 < 1$.

B.13 Proof of Proposition 5

First note that the system of equations (40) for all $i = 1, \cdots, n$ defines a mapping

$$\mathcal{M} : \mathbb{R}^n \to \underbrace{(-1, 1) \times \cdots \times (-1, 1)}_{n}.$$

It follows from Banach fixed-point theorem that at least one fixed point $\mathbf{m}^*$ exists, which can be obtained by successive iterations of the mapping of a sequence derived from $\mathcal{M}$.

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