Evolutionary Competition between Adjustment Processes in Cournot Oligopoly: Instability and Complex Dynamics

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Abstract

We introduce evolutionary competition between adjustment processes in the Cournot oligopoly model. Our main focus is on rational play versus a general short-memory adaptive adjustment process. We find that, although rational play has a stabilizing influence, a sufficient increase in the number of firms in the market tends to make the Cournot-Nash equilibrium unstable. Moreover, the interaction between adjustment processes naturally leads to the emergence of complicated endogenous fluctuations as the number of firms increases, even when demand and costs are linear.

JEL classification: C72, C73, D43

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1   Introduction

Since its original inception in the nineteenth century the Cournot model has become one of the standard and most widely used models of imperfect competition. Although the Cournot-Nash equilibrium is typically used as a description of firm behavior in that model, the question if and how firms will coordinate on that equilibrium has still not been unambiguously resolved. Classic short-memory adaptive adjustment processes such as best-reply dynamics (see e.g. Theocharis (1960)) and gradient learning (see e.g. Arrow and Hurwicz (1960)) may converge to the Cournot-Nash equilibrium, but instability typically sets in when the number of firms in the market increases (see e.g. Palander (1939) and Theocharis (1960) who show that, with linear demand and constant marginal costs, the Cournot-Nash equilibrium is unstable under best-reply dynamics for more than three firms[1]). An alternative to these intuitive, but not very sophisticated, adaptive processes are models based on introspection, where firms are rational and have full knowledge of the demand function and of their own and their opponents' cost functions. Common knowledge of rationality then allows firms to derive and coordinate on the Cournot-Nash equilibrium through deductive reasoning.[2,3]

Apart from the difference in dynamic behavior induced by these different types of adjustment processes, these processes have other drawbacks as a description of market behavior.

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[1] Although this finding is typically credited to Theocharis (1960), the argument was already made, in Swedish and some 20 years earlier, in Palander (1939). See Puu (2008) for a discussion. Later contributions to the literature on the stability of best-reply dynamics in the Cournot model are Fisher (1961) and McManus (1964), who consider an adaptive response in the direction of the best-reply and/or increasing marginal costs and Hahn (1962) who considers the *continuous* adjustment process towards the best-reply. For related models, see e.g. Okuguchi (1970) and Szidarovszky, Rassenti, and Yen (1994).

[2] However, coordination problems may emerge when the Cournot-Nash equilibrium is not unique.

[3] The Cournot-Nash equilibrium is also supported by relatively sophisticated *long-memory* adjustment processes. For example, *fictitious play* (see Brown (1951)), which asserts that each player best-replies to the empirical distribution of the opponents' past record of play, converges to the Cournot-Nash equilibrium for a large set of demand-cost structures (see e.g. Deschamp (1975) and Thorlund-Petersen (1990)).

On the one hand, the assumption underlying conventional processes, such as best-reply dynamics and gradient learning, that rivals will not revise their output from the last period, is continuously invalidated outside equilibrium (see e.g. Seade (1980), Al-Nowaihi and Levine (1985)). On the other hand, more sophisticated processes, such as rational play, put very high demands on the cognitive capacities of the players. It seems reasonable that in a market where all firms use the same adjustment process a tendency exists for some firms to change to another type of behavior – either to avoid structural decision-making errors in an unstable environment, or to save on cognitive efforts in a stable environment. In this paper we therefore introduce a model that presents a middle ground between adaptation and introspection by allowing firms to use different adjustment processes and switch between those on the basis of past performance, as in e.g. Brock and Hommes (1997) and Droste, Hommes, and Tuinstra (2002).

To some extent our approach is supported by findings from laboratory experiments with human subjects. In particular, the predictions of the rational model as well as those of less sophisticated short-memory adjustment processes fail to describe data from these experiments convincingly. Rassenti, Reynolds, Smith, and Szidarovszky (2000), for example, present an experiment on a Cournot oligopoly with linear demand, constant (but asymmetric) marginal costs and five firms, implying that the Cournot-Nash equilibrium is unstable under best-reply dynamics. Indeed, they find that aggregate output persistently oscillates around the equilibrium and does not converge. Individual behavior, however, is not explained very well by best-reply dynamics. Huck, Normann, and Oechssler (2002) discuss a linear (and symmetric) Cournot oligopoly experiment with four firms. Instead of diverging quantities, as predicted by best-reply dynamics, they find that the time average of quantities converges to the Cournot-Nash equilibrium quantity, although there is substantial volatility around this equilibrium throughout the experiment. Interestingly, Huck, Normann, and Oechssler (2002) find that a process where participants mix between best-replying and imitating the previous period's average quantity describes participants' behavior best.

In this paper we focus in particular on the interaction between a single short-memory adjustment process and rational play. We find that the presence of rational firms increases the threshold number of players that triggers instability, although the dynamics may still be unstable if the number of firms is sufficiently large. Moreover, evolutionary competition between adjustment processes may lead to complicated dynamics, characterized by perpetual, but bounded, fluctuations in production levels. As in the experiments discussed above, these fluctuations have a smaller amplitude than the fluctuations that would emerge when all firms use the short-memory adjustment process, but they are more erratic and less predictable.

These complicated dynamics arise naturally from the interaction of two opposing forces. If the fraction of rational firms is sufficiently high the Cournot-Nash equilibrium will be stable. This induces firms to switch to a short-memory adjustment process that gives similar market profits, but does not require as much cognitive effort. As a large enough fraction of the population of firms uses this short-memory adjustment process the Cournot-Nash equilibrium becomes unstable and quantities start fluctuating. When these fluctuations are sufficiently large, firms are attracted to rational play, which stabilizes the dynamics again, and so on.

Our paper contributes to the literature on complicated dynamics and endogenous fluctuations in Cournot oligopoly. This literature typically considers Cournot duopolies with non-monotonic reaction curves that are postulated ad hoc (Rand (1978)), derived from iso-elastic demand functions together with substantial asymmetries in marginal costs (Puu (1991)) or derived from cost externalities (Kopel (1996)), and shows that best-reply dynamics might result in periodic cycles and chaotic behavior. For these models with non-monotonic reaction curves complicated behavior might also arise for other adjustment processes (see e.g. Agiza, Bischi, and Kopel (1999), Bischi, Naimzada, and Sbragia (2007)). Although non-monotonic reaction curves cannot be excluded on economic grounds[4] complicated behavior in our model

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[4]Corchon and Mas-Colell (1996) show that any type of behavior can emerge for continuous time gradient (or best-reply) dynamics in heterogeneous oligopoly, although Furth (2009) argues that for homogeneous Cournot oligopoly there are certain restrictions as to what behavior can arise. Relatedly, Dana and Montrucchio (1986) show that in a duopoly model where firms maximize their discounted stream of future profits and play Markov perfect equilibria – and therefore are rational – any behavior is possible for small discount factors.

emerges in a much more natural fashion and perpetual but bounded fluctuations occur even for linear demand and cost curves. Finally, our work is closely related to Droste, Hommes, and Tuinstra (2002) who investigate evolutionary competition – modelled by the replicator dynamics – between best-reply dynamics and rational play in a Cournot duopoly with linear demand and quadratic costs.[5] They find that complicated dynamics may be possible when evolutionary pressure is high and marginal costs are *decreasing* sufficiently fast. The latter implies the existence of multiple Cournot-Nash equilibria (a symmetric interior equilibrium and two asymmetric boundary equilibria where one of the firms produces nothing) as well as a perverse comparative static effect: an exogenous increase in demand *reduces* the symmetric Cournot-Nash equilibrium price. In the present paper, which features a model with a unique Cournot-Nash equilibrium and intuitive comparative statics effects, instability and complicated dynamics emerge more naturally as the market size increases.

The rest of the paper is organized as follows. Section 2 briefly reviews short-memory adjustment processes in the general symmetric n-player Cournot model. Section 3 introduces a Cournot population game where firms can choose between rational play and a general short-memory adjustment process and Section 4 illustrates the global dynamics of this model for the Cournot oligopoly game with rational play versus best-reply dynamics for linear demand and constant marginal costs. Section 5 provides a short discussion.

2   Short-memory adjustment processes in Cournot oligopoly

Consider a Cournot oligopoly with n firms supplying a homogeneous commodity.[6] The inverse demand function $P(Q)$ is non-negative, nonincreasing and, whenever it is strictly positive, twice continuously differentiable. Here $Q = \sum_{i=1}^{n} q_i$ is aggregate output, with q_i production

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[5]See Ochea (2010) for an analysis of this model with a larger selection of adjustment processes.

[6]For a thorough treatment of the Cournot oligopoly game under general demand and cost structures we refer the reader to Bischi, Chiarella, Kopel, and Szidarovszky (2010).

of firm i. The cost function $C(q_i)$ is twice continuously differentiable and the same for every firm. Moreover, $C(q_i) \geq 0$ and $C'(q_i) \geq 0$ for every q_i.

Each firm wants to maximize instantaneous profits $P(Q_{-i} + q_i) q_i - C(q_i)$, where $Q_{-i} = \sum_{j \neq i} q_j = Q - q_i$. This gives the following first order condition for an interior solution

$$P(Q_{-i} + q_i) + q_i P'(Q_{-i} + q_i) - C'(q_i) = 0, \tag{1}$$

with second order condition for a local maximum given by $2P'(Q_{-i} + q_i) + q_i P''(Q_{-i} + q_i) - C''(q_i) \leq 0$.

The first order condition (1) implicitly defines the *best-reply correspondence* or *reaction curve*:

$$q_i = R(Q_{-i}). \tag{2}$$

We assume that a symmetric Cournot-Nash equilibrium q^*, that is, the solution to $q^* = R((n-1) q^*)$, exists and is strictly positive and unique.[7] Aggregate equilibrium production is then given by $Q^* = n q^*$.

The key question is: how do firms learn to play q^*? One approach is to assume complete information, rational firms, and common knowledge of rationality. Then firms derive the Cournot-Nash equilibrium by introspection and coordinate on that equilibrium. Note, however, that rational players may deviate from the Cournot-Nash equilibrium if not all firms are rational (see the discussion of rational play in a heterogeneous environment in Section 3).

As an alternative to rational play we consider short-memory adaptive adjustment processes with the following general structure

$$q_{i,t} = F(q_{i,t-1}, Q_{-i,t-1}). \tag{3}$$

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[7]Sufficient conditions for the existence and uniqueness of the Cournot-Nash equilibrium are that $P(\cdot)$ is twice continuously differentiable, nonincreasing and concave on the interval where it is positive, and that $C(\cdot)$ is twice continuously differentiable, nondecreasing and convex, see Szidarovszky and Yakowitz (1977). For more general conditions on existence and uniqueness, see e.g. Novshek (1985) and Kolstad and Mathiesen (1987), respectively.

That is, the firm's current production decision depends upon its own choice and the aggregate choices of the other firms from the previous period. We make the following assumption on the adjustment process (3), where $F_q^* = \frac{\partial F(q,Q_{-i})}{\partial q}\Big|_{(q^*,(n-1)q^*)}$ and $F_Q^* = \frac{\partial F(q,Q_{-i})}{\partial Q_{-i}}\Big|_{(q^*,(n-1)q^*)}$ denote the partial derivatives of F, evaluated at the Cournot-Nash equilibrium.

Assumption A *The adjustment process (3) satisfies (i) $F\left(q^*,(n-1)q^*\right) = q^*$, (ii) $\left|F_q^*\right| < 1$, $F_Q^* \in (-1,-\delta)$, where $\delta > 0$ is a strictly positive constant, and $F_q^* - F_Q^* < 1$.*

Part (i) of Assumption A ensures that the Cournot-Nash equilibrium quantity corresponds to a steady state of the adjustment process. Part (ii) puts some natural restrictions on the partial derivatives of F which facilitate stability of adjustment process (3). In particular, note that either $\left|F_q^*\right| > 1$ or $F_Q^* < -1$ would make the adjustment process inherently unstable: a small change in q or Q_{-i} in the previous time period, respectively, would then bring about a larger change in q in the current period. Similarly, $F_q^* - F_Q^* > 1$ would imply that a redistribution of production from Q_{-i} to q in the current period additionally increases next period's output q by more than that redistribution. The assumption that F_Q^* is negative and bounded away from zero makes sense because quantities are strategic substitutes.

A number of well-known adjustment processes can be represented by (3).[8] Probably best-known is the *best-reply dynamics* (see e.g. Theocharis (1960)) which assumes that firms best-reply to the aggregate quantity of the other firms from the previous period, that is

$$F\left(q,Q_{-i}\right) = R\left(Q_{-i}\right).$$

Note that we have $F_q^* = 0$ and $F_Q^* = R'\left(Q_{-i}^*\right)$, which is indeed typically negative.[9] The closely related *adaptive best-reply dynamics* (see e.g. Fisher (1961)), where firms move *in*

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[8]Bischi, Chiarella, Kopel, and Szidarovszky (2010) provide a systematic analysis of a variety of adjustment processes in Cournot oligopoly games.

[9]From the first order condition (1) we find that

$$\frac{dq_i}{dQ_{-i}} = R'\left(Q_{-i}\right) = -\frac{P'\left(Q\right) + q_i P''\left(Q\right)}{2P'\left(Q\right) + q_i P''\left(Q\right) - C''\left(q_i\right)}. \tag{4}$$

the direction of their best reply, can be written as $F(q, Q_{-i}) = \alpha R(Q_{-i}) + (1 - \alpha) q_i$, with $\alpha \in (0, 1]$ and where $F_q^* = 1 - \alpha$ and $F_Q^* = \alpha R'(Q_{-i}^*)$. Another variation is suggested in Huck, Normann, and Oechssler (2002), where it is found that participants to a laboratory experiment use a weighted average of best-reply and *imitation*.

Another famous adjustment process is *gradient learning* (see e.g. Arrow and Hurwicz (1960) and Bischi, Chiarella, Kopel, and Szidarovszky (2010)) where firms adapt their decision in the direction of increasing profits, that is

$$F(q_i, Q_{-i}) = q_i + \lambda \frac{\partial \pi(q_i, Q_{-i})}{\partial q_i},$$

with $\lambda > 0$ the *speed of adjustment* parameter. Here $F_Q^* = \lambda [P'(Q^*) + q^* P''(Q^*)]$ and $F_q^* = 1 + \lambda [2P(Q^*) + q^* P''(Q^*) - C''(q^*)]$, where $F_q^* < 1$ follows from the second order condition for a local maximum and $F_Q^* < 0$ holds under the familiar condition that the inverse demand function is "not too convex" (see footnote 9).

Besides these benchmark adjustment processes many other processes obey the general form (3), such as *local monopolistic approximation*[10] or *imitating the average* (although the latter does not satisfy part (i) of Assumption A). Some other adjustment processes, such as fictitious play and *least squares learning* (see e.g. Anufriev, Kopányi, and Tuinstra (2013)), cannot be represented by (3).

The next proposition characterizes when the Cournot-Nash equilibrium is stable, given

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Note that the second order condition for a local maximum implies that the denominator, evaluated at the Cournot-Nash equilibrium, is negative. Typically the numerator is also negative (although this is not necessarily the case if the inverse demand function is sufficiently convex), and therefore we generally have $R'(Q_{-i}^*) < 0$.

[10] The idea behind local monopolistic approximation is that every firm estimates a linear demand curve on the basis of his last observed price-quantity combination and the slope of the inverse demand function at that quantity in the last period. It then uses this estimated demand function to determine its perceived profit maximizing quantity. For constant marginal costs c this gives rise to adjustment process

$$F(q, Q_{-i}) = \frac{1}{2} q - \frac{1}{2} \frac{P(q + Q_{-i}) - c}{P'(q + Q_{-i})}.$$

For details, see Tuinstra (2004) and Bischi, Naimzada, and Sbragia (2007).

that all firms use the same adjustment process (3).[11]

Proposition 1 *Let all firms use adjustment process (3). The symmetric Cournot-Nash equilibrium* $(q^*, \ldots, q^*)$ *is locally stable if*

$$\left| F_q^* + (n-1) F_Q^* \right| < 1. \tag{5}$$

For n large enough the Cournot-Nash equilibrium is unstable.

Proof. The dynamics of quantities is governed by a system of n first order difference equations, given by (3) for $i = 1, \ldots, n$. All of the diagonal elements of the corresponding $n \times n$ Jacobian matrix $\mathbf{J}^*$, evaluated at the Cournot–Nash equilibrium, are equal to F_q^* and all of its off-diagonal elements are equal to F_Q^*. It follows that $\mathbf{J}^*$ has eigenvalues $\lambda_1 = F_q^* - F_Q^*$, with multiplicity $n - 1$, and $\lambda_2 = F_q^* + (n-1) F_Q^*$. Because the Cournot-Nash equilibrium is locally stable if all eigenvalues of $\mathbf{J}^*$ lie within the unit circle, and since by Assumption A we have $|\lambda_1| < 1$, a sufficient condition for local stability is $|\lambda_2| = \left| F_q^* + (n-1) F_Q^* \right| < 1$. Because $F_Q^* \leq -\delta < 0$ and $F_q^* \in (-1, 1)$, condition (5) will not be satisfied for n sufficiently large. $\blacksquare$

Proposition 1 shows that the Cournot-Nash equilibrium becomes unstable, under adjustment process (3), if the number of firms increases sufficiently. In particular, condition (5) gives the following instability threshold

$$n > 1 - \frac{1 + F_q^*}{F_Q^*}. \tag{6}$$

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[11] Recall that the local stability properties of the fixed point of a nonlinear dynamical system are qualitatively the same as those of the linearized system, provided that the fixed point is *hyperbolic* (that is, the Jacobian matrix has no eigenvalues on the unit circle), see e.g. Kuznetsov (1995). Such a fixed point is locally stable (a *sink*) if all eigenvalues of the Jacobian matrix (evaluated at that fixed point) lie within the unit circle, and the fixed point is unstable either when all eigenvalues lie outside the unit circle (the fixed point is then called a *source*) or when at least one eigenvalue lies outside the unit circle, and at least one eigenvalue lies inside the unit circle (the fixed point is then called a *saddle*).

The intuition is that individual firms, who choose their production level partly on the basis of last period's aggregate production of the other firms, do not take into account that those other firms also adjust their production level. Obviously, disregarding other firms' adjustments will have a larger effect when there are more firms in the market (or when $\left|F_Q^*\right|$ is higher) and eventually destabilizes the Cournot-Nash equilibrium. For example, with linear demand and costs, the slope of the resulting linear reaction curve equals $-\frac{1}{2}$. This means that if one firm deviates from the equilibrium by producing one additional unit, under best-reply dynamics *every* other firm responds by decreasing its own production by half a unit. Consequently, for $n > 3$ the aggregate reduction in production is larger than the earlier increase in production, which renders the dynamics unstable. Similarly, for gradient learning with a speed of adjustment λ low enough to induce convergence to the Cournot-Nash equilibrium when the number of firms is small, a sufficient increase in the number of firms will destabilize the dynamics.

Since F_Q^* typically depends upon n through q^*, in principle a market structure could exist with the property that F_Q^* decreases in n faster than $\frac{1}{n}$, meaning that (3) may converge to the Cournot-Nash equilibrium for any number of firms. However, such a market structure seems unlikely and, to the best of our knowledge, has not been considered in the literature.[12] The assumption that F_Q^* is bounded away from zero therefore seems innocuous.

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[12] For the specification of Theocharis (1960), with linear inverse demand function and constant marginal costs the reaction curve is linear with a constant slope that is independent of n. For an iso-elastic inverse demand function and constant marginal costs the slope of the reaction curve, evaluated at the Cournot-Nash equilibrium does depend upon n. In this case the Cournot-Nash equilibrium is unstable under best-reply dynamics for $n \geq 5$ (see Ahmed and Agiza (1998) and Puu (2008)). Puu (2008) provides an example for which the best-reply dynamics do remain stable when n increases, but he assumes that the cost function of each firm depends directly upon the number of firms n: as the number of firms increases the capacity of each individual firm is reduced, increasing its marginal costs.

3 Evolutionary competition between adjustment processes

Proposition 1 establishes that dynamic behavior under adjustment processes of the form (3) is quite different from more sophisticated adjustment processes, such as rational or fictitious play, particularly when the number of firms in the market is large. However, the latter typically require more cognitive effort. In this section we introduce an evolutionary competition between the different adjustment processes. For this we model our Cournot oligopoly as a population game. That is, we consider a large population of firms from which in each period groups of n firms are sampled randomly to play the one-shot n-firm Cournot oligopoly. Firms may use different adjustment processes and they switch between these processes according to a general, monotone selection dynamic, capturing the idea that an adjustment process that performs better is more likely to spread through the population of firms. In this paper we focus on the interaction between rational play and a single short-memory adjustment process of the form (3).[13] Denote by $\rho_t \in [0, 1]$ the fraction of firms in the population that is rational in period t, with a fraction $1 - \rho_t$ using the short-memory adjustment process – from here on we will refer to the latter as F-firms. After each period, the fraction ρ_t is updated and the random matching procedure is repeated.

First consider the decision of a rational firm that knows the fraction of rational firms in the population and the production decision of the F-firms, but does not know the exact composition of firms in its market (or it has to make a production decision before observing this). This firm forms expectations over all possible mixtures resulting from independently drawing $n - 1$ other players from a large population, each of which is either a rational or a F-firm. Rational firm i therefore chooses quantity q_i such that the objective function

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \rho_t^k (1 - \rho_t)^{n-1-k} \left[P \left((n - 1 - k) q_t + k q^r + q_i \right) q_i - C \left(q_i \right) \right],$$

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[13] See Ochea (2010) for more examples, with similar qualitative results.

is maximized. Here q^r is the (symmetric) output level of each of the other rational firms, and q_t is the output level of each F-firm. The first order condition for an optimum is characterized by equality between marginal cost and *expected* marginal revenue. We assume that, given the value of q_t, all rational firms coordinate on the same output level q^r.[14] The first order condition, with $q_i = q^r$, reads

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \rho_t^k (1 - \rho_t)^{n-1-k} \times$$

$$[P((n-1-k)q_t + (k+1)q^r) + q^r P'((n-1-k)q_t + (k+1)q^r) - C'(q^r)] = 0. \quad (7)$$

Let the solution to (7) be given by $q^r = H(q_t, \rho_t)$.[15] Note that if the F-firms play the Cournot-Nash equilibrium quantity q^*, or if all firms are rational, then rational firms will produce q^* as well, that is $H(q^*, \rho_t) = q^*$, for all ρ_t and $H(q_t, 1) = q^*$ for all q_t. Moreover, a rational firm that is certain it will only meet F-firms plays a best-reply to *current* aggregate output of these F-firms, that is $H(q_t, 0) = R((n-1)q_t)$, for all q_t.

We assume that F-firms know the *average* quantity $\bar{q}_{t-1}$ played across the population of firms in period $t - 1$. We therefore obtain

$$q_t = F\left(q_{t-1}, (n-1)\bar{q}_{t-1}\right) = F\left(q_{t-1}, (n-1)\left(\rho_{t-1}H(q_{t-1}, \rho_{t-1}) + (1 - \rho_{t-1})q_{t-1}\right)\right), \quad (8)$$

with the output of a rational firm in period t given by $q_t^r = H(q_t, \rho_t)$.

The evolutionary competition between adjustment processes is driven by the profits they generate. Taking into account that a rational firm meets between 0 and $n - 1$ other rational

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[14] In Droste, Hommes, and Tuinstra (2002) the terminology 'Nash firms' is used instead of 'rational firms' to stress that these firms do not only make a rational decision given their beliefs, but that they also successfully coordinate on the appropriate quantity. In the present paper we will stick to the terminology 'rational firms', assuming implicitly that these firms also coordinate on the output level.

[15] Note that in general the solution to (7) does not necessarily have to be unique, although it will be unique under the standard assumptions of nondecreasing marginal costs and concave inverse demand. If, for some q_t and ρ_t, there are multiple solutions to (7) we assume that the rational firms are able to identify which of these solutions corresponds to the global maximum of their profit function and coordinate on this solution, which we then refer to as $H(q_t, \rho_t)$.

firms, expected profits for a rational firm are given by

$$\Pi_R\left(q^r, q, \rho\right) = \sum_{k=0}^{n-1} \binom{n-1}{k} \rho^k \left(1-\rho\right)^{n-1-k} \left[P\left((k+1)\,q^r + (n-1-k)\,q\right) q^r - C\left(q^r\right)\right]. \quad (9)$$

Expected profits $\Pi_F\left(q^r, q, \rho\right)$ for an F-firm can be determined in a similar manner. If the population of firms and the number of groups of n firms drawn from that population are large enough average profits will be approximated quite well by these expected profits, which we will use as a proxy for average profits from now on. In addition, because the information requirements for rational play are substantially higher than those for short-memory adjustment processes, we allow for differences in *information* or *deliberation* costs $\kappa_R, \kappa_F \geq 0$ required to implement these types of behavior. Performance of rational and F-firms is then evaluated according to $V_i = \Pi_i - \kappa_i$ where $i = R, F$.

The fraction ρ_t of rational firms evolves endogenously according to a dynamic which is an increasing function of the performance differential between the two adjustment processes, that is

$$\rho_t = G\left(V_{R,t-1} - V_{F,t-1}\right) = G\left(\Pi_{R,t-1} - \Pi_{F,t-1} - \kappa\right), \quad (10)$$

where $\kappa \equiv \kappa_R - \kappa_F$ is the difference in deliberation costs, which we – given the information requirements for rational play in a heterogeneous environment – assume to be nonnegative.[16] The map $G : \mathbb{R} \to [0,1]$ is a continuously differentiable, monotonically increasing function with $G\left(0\right) = \frac{1}{2}$, $\lim_{x\to-\infty} G\left(x\right) = 0$ and $\lim_{x\to\infty} G\left(x\right) = 1$.[17] Note that it is straightforward to generalize this approach to allow for other (and more than two) adjustment processes, or to let it depend upon performance of these processes from earlier periods.

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[16] Note that κ does not necessarily only represent the difference in information costs; it could also capture a predisposition towards (or away from) rational play.

[17] The well-known discrete choice model, which is very popular in *heterogeneous agent models* (see e.g. Brock and Hommes (1997)) and in the literature on *quantal response equilibria* (see e.g. McKelvey and Palfrey (1995)) satisfies these properties. We will use this specification in Section 4. Alternatively, one might model the switching mechanism by an evolutionary process such as the replicator dynamics, as was done in Droste, Hommes, and Tuinstra (2002). The replicator dynamics does not satisfy all the properties that we impose upon $G\left(\cdot\right)$. However, simulations of our model with the replicator dynamics lead to similar stability results and qualitatively the same type of dynamics, although the precise global dynamics may be somewhat different.

The dynamics of the quantities and fractions are governed by equations (8) and (10). The steady state of this dynamic system is (q^*, ρ_κ), where q^* is the Cournot-Nash equilibrium quantity, and $\rho_\kappa = G(-\kappa)$ is the fraction of rational players at the steady state. Because market profits are the same in equilibrium, this fraction depends only on the difference in information costs. We have the following stability result:

Proposition 2 *The equilibrium (q^*, ρ_κ) of the model with evolutionary competition between rational play and short-memory adjustment process (3) is locally stable if:*

$$\frac{(1 - \rho_\kappa)(n-1)}{1 - \rho_\kappa (n-1) R'\left(Q_{-i}^*\right)} < -\frac{1 + F_q^*}{F_Q^*}. \tag{11}$$

Proof. The variables q_t and ρ_t evolve according to

$$q_t = \Phi^1\left(q_{t-1}, \rho_{t-1}\right) \equiv F\left(q_{t-1}, (n-1)\left(\rho_{t-1} H\left(q_{t-1}, \rho\right) + (1 - \rho_{t-1}) q_{t-1}\right)\right), \tag{12}$$

$$\rho_t = \Phi^2\left(q_{t-1}, \rho_{t-1}\right) \equiv G(\Pi_{R,t-1} - \Pi_{F,t-1} - \kappa).$$

Local stability of (q^*, ρ_κ) is determined by the Jacobian matrix of (12), evaluated at (q^*, ρ_κ).

First, we determine the partial derivatives of Φ^2 with respect to q_{t-1} and ρ_{t-1}, respectively. To that end, note that we can write the profit differential as

$$\triangle \Pi_R = \Pi_{R,t-1} - \Pi_{F,t-1} = \sum_{k=0}^{n-1} A_k\left(\rho_{t-1}\right) D_k\left(q_{t-1}, \rho_{t-1}\right)$$

with $A_k\left(\rho_{t-1}\right) = \binom{n-1}{k} \rho_{t-1}^k \left(1 - \rho_{t-1}\right)^{n-1-k}$, which does not depend upon q_{t-1}, and

$$D_k\left(q_{t-1}, \rho_{t-1}\right) = P\left((n - 1 - k) q_{t-1} + (k + 1) q_{t-1}^r\right) q_{t-1}^r - C\left(q_{t-1}^r\right)$$
$$- \left[P\left((n - k) q_{t-1} + k q_{t-1}^r\right) q_{t-1} - C\left(q_{t-1}\right)\right],$$

which only depends upon ρ_{t-1} through $q_{t-1}^r = H\left(q_{t-1}, \rho_{t-1}\right)$. Note that $D_k\left(q^*, \rho_\kappa\right) = 0$. Moreover, the partial derivatives of $D_k\left(q_{t-1}, \rho_{t-1}\right)$, evaluated at the equilibrium (q^*, ρ_κ) are

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given by

$$\left.\frac{\partial D_k\left(q_{t-1}, \rho_{t-1}\right)}{\partial q_{t-1}}\right|_{\left(q^*, \rho_\kappa\right)} = \left[P\left(Q^*\right) + q^* P'\left(Q^*\right) - C'\left(q^*\right)\right]\left(\left.\frac{\partial H\left(q_{t-1}, \rho_{t-1}\right)}{\partial q_{t-1}}\right|_{\left(q^*, \rho_\kappa\right)} - 1\right) = 0,$$

$$\left.\frac{\partial D_k\left(q_{t-1}, \rho_{t-1}\right)}{\partial \rho_{t-1}}\right|_{\left(q^*, \rho_\kappa\right)} = \left[P\left(Q^*\right) + q^* P'\left(Q^*\right) - C'\left(q^*\right)\right]\left.\frac{\partial H\left(q_{t-1}, \rho_{t-1}\right)}{\partial \rho_{t-1}}\right|_{\left(q^*, \rho_\kappa\right)} = 0,$$

where we use the fact that at the Cournot-Nash equilibrium the individual firm's first order condition (1) is satisfied. We now have

$$\left.\frac{\partial \Phi^2}{\partial q_{t-1}}\right|_{\left(q^*, \rho_\kappa\right)} = G'\left(-\kappa\right)\left.\frac{\partial \triangle \Pi_R}{\partial q_{t-1}}\right|_{\left(q^*, \rho_\kappa\right)} = G'\left(-\kappa\right)\sum_{k=0}^{n-1} A_k\left(\rho_\kappa\right)\left.\frac{\partial D_k\left(q_{t-1}, \rho_{t-1}\right)}{\partial q_{t-1}}\right|_{\left(q^*, \rho_\kappa\right)} = 0$$

and

$$\left.\frac{\partial \Phi^2}{\partial \rho_{t-1}}\right|_{\left(q^*, \rho_\kappa\right)} = G'\left(-\kappa\right)\left.\frac{\partial \triangle \Pi_R}{\partial \rho_{t-1}}\right|_{\left(q^*, \rho_\kappa\right)}$$

$$= G'\left(-\kappa\right)\sum_{k=0}^{n-1}\left[\left.\frac{\partial A_k\left(\rho_{t-1}\right)}{\partial \rho_{t-1}}\right|_{\rho_\kappa} D_k\left(q^*, \rho_\kappa\right) + A_k\left(\rho_\kappa\right)\left.\frac{\partial D_k\left(q_{t-1}, \rho_{t-1}\right)}{\partial \rho_{t-1}}\right|_{\left(q^*, \rho_\kappa\right)}\right]$$

$$= 0.$$

The Jacobian matrix of (12), evaluated at $\left(q^*, \rho_\kappa\right)$, therefore has the following structure

$$\begin{pmatrix} \left.\frac{\partial \Phi^1}{\partial q_{t-1}}\right|_{\left(q^*, \rho_\kappa\right)} & \left.\frac{\partial \Phi^1}{\partial \rho_{t-1}}\right|_{\left(q^*, \rho_\kappa\right)} \\ 0 & 0 \end{pmatrix},$$

with eigenvalues $\lambda_1 = \left.\frac{\partial \Phi^1}{\partial q_{t-1}}\right|_{\left(q^*, \rho_\kappa\right)}$ and $\lambda_2 = 0$. Hence, $\left(q^*, \rho_\kappa\right)$ is locally stable when

$$|\lambda_1| = \left|\left.\frac{\partial \Phi^1}{\partial q_{t-1}}\right|_{\left(q^*, \rho_\kappa\right)}\right| = \left|F_q^* + (n-1)\left(\rho H_q^* + (1-\rho)\right) F_Q^*\right| < 1, \tag{13}$$

where $H_q^* = \left.\frac{\partial H\left(q_{t-1}, \rho_{t-1}\right)}{\partial q_{t-1}}\right|_{\left(q^*, \rho_\kappa\right)}$. To determine H_q^* we totally differentiate first order condition

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(7):

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \rho^k (1-\rho)^{n-1-k} (n-1-k) \left[P'(Q^*) + q^* P''(Q^*) \right] dq_t +$$

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \rho^k (1-\rho)^{n-1-k} \left[k \left(P'(Q^*) + q^* P''(Q^*) \right) + 2P'(Q^*) + q^* P''(Q^*) - C''(q^*) \right] dq^r.$$

Using $\sum_{k=0}^{n-1} \binom{n-1}{k} \rho^k (1-\rho)^{n-1-k} = 1$ and $\sum_{k=0}^{n-1} \binom{n-1}{k} \rho^k (1-\rho)^{n-1-k} k = \rho(n-1)$ and rearranging we find that

$$
\begin{aligned}
H_q^* = \frac{dq^r}{dq_t} &= -\frac{(1-\rho)(n-1)\left(P'(Q^*) + q^* P''(Q^*)\right)}{\rho(n-1)\left(P'(Q^*) + q^* P''(Q^*)\right) + 2P'(Q^*) + q^* P''(Q^*) - C''(q^*)} \\
&= \frac{(1-\rho)(n-1) R'\left(Q_{-i}^*\right)}{1 - \rho(n-1) R'\left(Q_{-i}^*\right)},
\end{aligned}
\tag{14}
$$

where the last equality follows from the fact that from (1) the slope of the best-reply function equals

$$\frac{dq_i}{dQ_{-i}} = -\frac{P'(Q^*) + q P''(Q^*)}{2P'(Q^*) + q^* P''(Q^*) - C''(q^*)}.$$

Substituting (14) into condition (13) and rearranging gives condition (11). ∎

Note that it follows from condition (11) that for a sufficiently large fraction of rational players the Cournot-Nash equilibrium will be stable. On the other hand, from rearranging condition (11) we find that a sufficient condition for *instability* is

$$\frac{n - \rho_\kappa (n-1)\left[1 + R'\left(Q_{-i}^*\right)\right]}{1 - \rho_\kappa (n-1) R'\left(Q_{-i}^*\right)} > 1 - \frac{1 + F_q^*}{F_Q^*} \tag{15}$$

Note that the right-hand sides of conditions (15) and (6) are the same, but that the left-hand side of (15) is smaller than n (the left-hand side of (6)), provided $-1 \leq R'\left(Q_{-i}^*\right) \leq 0$. Introducing rational firms in an environment with F-firms therefore has a stabilizing effect. In the next section we will see that instability is still possible, and that the model with

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interaction between rational play and a short-memory adjustment process may actually give rise to complicated and unpredictable dynamics.

4 Rational play versus best-reply: global dynamics and perpetual bounded fluctuations

In this section we study the global dynamical behavior of the model discussed in Section 3 where for the short-memory adjustment process we take the best-reply dynamics, $F(q_i, Q_{-i}) = R(Q_{-i})$. This choice is supported by evidence from laboratory experiments that suggests that best-reply dynamics is relevant in human decision making. Cox and Walker (1998), for example, present an experiment on Cournot duopoly with linear demand and quadratic costs where participants' quantity choices fail to converge to the (interior) Cournot–Nash equilibrium when that equilibrium is unstable under best-reply dynamics. Also Rassenti, Reynolds, Smith, and Szidarovszky (2000) and Huck, Normann, and Oechssler (2002) find that a Cournot-Nash equilibrium that is unstable under best-reply dynamics will not be reached by human subjects.

Applying Proposition 2 to best-reply dynamics (that is, $F_q^* = 0$ and $F_Q^* = R'\left(Q_{-i}^*\right) < 0$) and using $\rho_0 = \frac{1}{2}$ we find that the Cournot-Nash equilibrium is locally stable for any number of firms if there are no information costs for rational play:

Corollary 3 *Let $P'(Q^*) + q^* P''(Q^*) < 0$. Then the equilibrium (q^*, ρ_κ) of the model of endogenous switching between rational play and best-reply dynamics is locally stable if*

$$(1 - 2\rho_\kappa)(n-1)R'\left(Q_{-i}^*\right) > -1. \tag{16}$$

Moreover, in the absence of a difference in information costs, $\kappa = 0$, the equilibrium (q^*, ρ_0) is locally stable for all $n \geq 2$.

To investigate global dynamics we need to specify the demand and cost structure, as well as the switching mechanism. We will use linear demand $P(Q) = a - bQ$, and costs, $C_i(q_i) = cq$, with $a > c \geq 0$ and $b > 0$. The reaction curve then becomes

$$q_i = R_i(Q_{-i}) = q^* - \frac{1}{2}(Q_{-i} - (n-1)q^*), \tag{17}$$

with $q^* = \frac{a-c}{b(n+1)}$ the unique Cournot-Nash equilibrium. Furthermore, given q_t and ρ_t, rational firms in period t coordinate on the solution to equation (7) which is

$$q_t^r = H(q_t, \rho_t) = q^* - \frac{(1-\rho_t)(n-1)}{2+(n-1)\rho_t}(q_t - q^*). \tag{18}$$

It can be easily checked that $q_t = R((n-1)(\rho_{t-1}H(q_{t-1}, \rho_{t-1})) + (1-\rho_{t-1})q_{t-1}) = H(q_{t-1}, \rho_{t-1}) = q_{t-1}^r$, that is, in each period best-reply firms produce the quantity that rational firms produced in the period before, illustrating the information advantage of the latter. From equation (18) we see that rational firms respond to best-reply firms by choosing a high (low) production level when production of best-reply firms is low (high) in that period.[18] Rational firms therefore partially neutralize the instability created by best-reply firms. However, if the equilibrium fraction ρ_κ of rational firms in the population is too small, or the number of firms n in a market sufficiently large, the Cournot-Nash equilibrium will still be unstable, as can be seen by condition (16) which, for the current specification, reduces to

$$(1 - 2\rho_\kappa)(n-1) < 2. \tag{19}$$

We model evolutionary competition by the *discrete choice dynamics* (see e.g. Brock and Hommes (1997)):

$$G(\Pi_{R,t-1} - \Pi_{F,t-1} - \kappa) = \frac{1}{1 + \exp[-\beta(\Pi_{R,t-1} - \Pi_{F,t-1} - \kappa)]}. \tag{20}$$

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[18]In fact, the production level of rational firms will lie between q^* and $R((n-1)q_t)$. To be specific, for $\rho \in (0,1)$ and $q_t \neq q^*$ we either have $R((n-1)q_t) < H(q_t, \rho) < q^* < q_t$ or $R((n-1)q_t) > H(q_t, \rho) > q^* > q_t$.

The parameter $\beta \geq 0$ measures the *intensity of choice*: for a higher value of β firms are more likely to switch to the more successful adjustment process from the previous period. A straightforward computation shows that the profit difference is given by

$$\Pi_{R,t} - \Pi_{F,t} = b \left(\frac{n+1}{2 + (n-1)\rho_t} \right)^2 (q_t - q^*)^2.$$

Note that, abstracting from information costs κ, average profits of rational firms are always higher than those of the best-reply firms. The difference increases with the deviation of q_t from its equilibrium value and decreases with the fraction of rational firms. The full model with endogenous switching between rational and best-reply behavior is

$$q_t = q^* - \frac{(1 - \rho_{t-1})(n-1)}{2 + (n-1)\rho_{t-1}} (q_{t-1} - q^*), \tag{21}$$

$$\rho_t = \frac{1}{1 + \exp \left[-\beta \left(b \left(\frac{n+1}{2+(n-1)\rho_{t-1}} \right)^2 (q^* - q_{t-1})^2 - \kappa \right) \right]},$$

with the equilibrium given by $(q^*, \rho_\kappa) = \left(\frac{a-c}{b(n+1)}, [1 + \exp[\beta\kappa]]^{-1} \right)$. This equilibrium is locally stable when condition (19) holds. This condition is always satisfied for $\rho_\kappa \geq \frac{1}{2}$ or $n \leq 3$ but for $n > 3$ the Cournot-Nash equilibrium becomes unstable if the fraction of rational firms in equilibrium is too low, with the critical value for ρ given by

$$\rho_t < \overline{\rho} = \frac{1}{2} \frac{n-3}{n-1}. \tag{22}$$
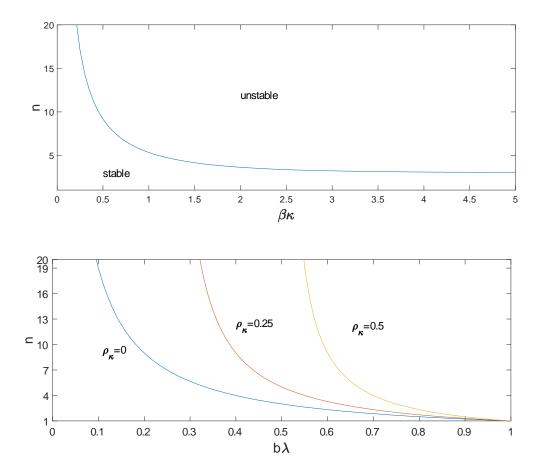
Figure 1: Upper panel: stability curve for rational vs. best-reply firms in $(\beta\kappa,\, n)$ space. When the stability curve is crossed from below the interior Cournot-Nash equilibrium loses stability and a two-cycle is born. Lower panel: stability curves for rational play versus gradient learning, for different values of ρ_κ

As is already clear from Corollary 3 the equilibrium is always locally stable in the absence of information costs, $\kappa = 0$ (note that $\overline{\rho} < \rho_0 = \frac{1}{2}$ for all n). However, for any $n > 3$ there exist an intensity of choice β and information costs κ such that the equilibrium becomes unstable, because the fraction of rational firms in equilibrium is too small. In fact, the equilibrium is unstable for all $n \geq 4$ when $\rho_\kappa < \frac{1}{6}$, that is, whenever $\beta \times \kappa > \ln 5 \approx 1.609$.

The trade-off between evolutionary pressure and the number of firms n in the market for which the equilibrium is stable is illustrated in the upper panel of Figure 1. This figure plots the period-doubling *bifurcation curve*, where, for convenience, we interpret n as a continuous

variable.[19] For combinations of $\beta\kappa$ and n to the north-east of the curve the equilibrium is unstable.

The dynamics can become quite complicated when the equilibrium is unstable. Figure 2 shows the results of some representative numerical simulations of the model with $a = 17$, $b = 1$, $c = 10$, $\beta = 5$ and $\kappa = \frac{1}{2}$. Note that in this case $\rho_\kappa = \left[1 + \exp\left[\frac{5}{2}\right]\right]^{-1} \approx 0.076$ and the equilibrium will be unstable for any $n > 3$. Panel (a) shows a bifurcation diagram for $n = 2$ to $n = 8$, establishing that a stable period two cycle exists for $n = 4$ and more complicated behavior emerges for larger values of n. Panels (b)-(d) show the dynamics of quantities, profit differences and fractions for $n = 8$, respectively.[20] Note that close to the equilibrium (in fact, when $|q_t - q^*| < \frac{1}{9}\sqrt{2}\left(1 + \frac{7}{2}\rho_t\right)$) best-reply firms do better than rational firms because they do not have to pay information costs and the difference in average market profits is relatively small. This decreases the fraction of rational firms, which destabilizes the quantity dynamics. As the dynamics moves away from the equilibrium, eventually rational firms outperform best-reply firms and more firms become rational again, increasing ρ_t. Now, when $\rho_t > \overline{\rho} = \frac{5}{14}$ (the horizontal dashed line in panel (d)) the quantity dynamics stabilizes again and quantities converge to their Cournot-Nash equilibrium level, and the whole story repeats. Panel (f) shows that, for $n = 8$, the largest Lyapunov exponent is strictly positive if the intensity of choice β is high enough, indicating chaotic behavior.

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[19]For a discussion on these period-doubling thresholds for more general learning rules, i.e. adaptive expectations and fictitious play, see Chapter IV in Ochea (2010).

[20]Observe that the dynamics of quantities have a smaller amplitude and are much less regular than they would be under pure best-reply dynamics. In that case (under symmetric initial conditions) individual quantities would fluctuate in a period-two cycle between 0 and $\frac{1}{2}(n+1)q^*$.
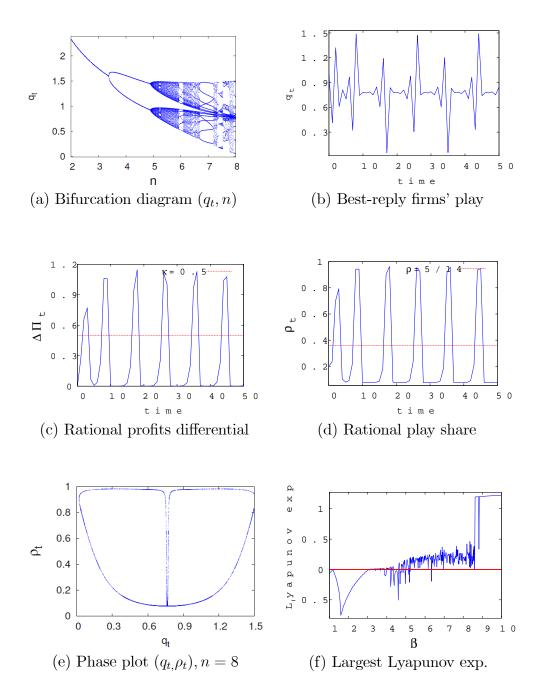
(a) Bifurcation diagram (q_t, n)

(b) Best-reply firms' play

(c) Rational profits differential

(d) Rational play share

(e) Phase plot $(q_t, \rho_t), n = 8$

(f) Largest Lyapunov exp.

Figure 2: Linear n-player Cournot game with rational vs. best-reply firms. Panel (a) depicts a sequence of period-doubling bifurcations as the number of players n increases. Instability sets in already for the triopoly game. Panel (b)-(d) display oscillating time series of the quantity chosen by the best-reply firm, the profit differential (net of information costs $\kappa = 0.5$) and the fraction of rational firms, respectively. The threshold fraction of rational firms $\rho = 5/14$ for which the dynamics become stable is also marked in Panel (d). A typical phase portrait is shown in Panel (e) while Panel (f) plots the largest Lyapunov exponent for increasing β. Game and behavioral parameters: $n = 8, a = 17, b = 1, c = 10, \kappa = 0.5, \beta = 5$.

5 Discussion

In this paper we introduced a model of evolutionary competition between different adjustment processes in Cournot oligopoly. We focused on the interaction between rational play and a single adaptive adjustment process. The availability of rational play stabilizes the dynamics: although the Cournot-Nash equilibrium will typically still be unstable if the number of firms is sufficiently high, the stability threshold increases. For the special case of rational play versus best-reply dynamics we find that the Cournot-Nash equilibrium is locally stable for any number of firms if, in the equilibrium of the evolutionary model, at least half of the population of firms uses rational play.[21] However, this does not generalize to other adjustment processes. The lower panel of Figure 1 shows stability curves for rational play versus gradient learning (for the case of linear demand and costs) where the horizontal axis shows the normalized speed of adjustment parameter $b\lambda$ and the vertical axis shows market size n.[22] The lowest curve demarcates the stability region when all firms use gradient learning (for combinations of $b\lambda$ and n to the north-east of this curve the Cournot-Nash equilibrium is unstable) and the highest curve characterizes stability in the case where, in equilibrium, half of the population of firms is rational. It follows immediately that the stability region increases with ρ_κ, although, even for $\rho_\kappa = \rho_0 = \frac{1}{2}$ (and $b\lambda > \frac{1}{2}$) one can always find a market size n such that the Cournot-Nash equilibrium is unstable.

For the case of rational play versus best-reply dynamics the dynamics of the evolutionary

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[21] One would expect the number of rational firms to be lower in equilibrium however, since in equilibrium best-reply firms can free ride on the rational firms: they produce the same quantity, but do not incur the high information costs.

[22] Gradient learning, for $P(Q) = a - bQ$ and $C(q) = cq$, is given by

$$q_{i,t+1} = (1 - 2b\lambda) q_{i,t} + \lambda [a - c - bQ_{-i}].$$

Note that $\left| F_q^* \right| < 1$, from Assumption A, requires $b\lambda < 1$. The critical value for n implied by stability condition (11) then becomes

$$n^{GD} = \frac{2 - b\lambda - \rho_\kappa}{b\lambda - \rho_\kappa}.$$

The equilibrium (q^*, ρ_κ) is locally stable as long as $b\lambda \leq \rho_\kappa$. For any $\rho_\kappa < 1$ and $b\lambda \in (\rho_\kappa, 1)$ we can always find n large enough such that the equilibrium is unstable. In particular, in the absence of information costs for rational play, the equilibrium will be unstable for $n > (3 - 2b\lambda)/(2b\lambda - 1)$ and $b\lambda > 1/2$.

model can give highly irregular, perpetual but bounded fluctuations, even if demand and costs are linear. Complicated dynamics have been established in Cournot models before, but typically require non-monotonic reaction curves, which are not standard. In our model the bounded fluctuations are created naturally by the interaction of different adjustment processes and the increase in the number of firms.

The analysis provided in this paper can be extended by considering other adjustment processes, although this will lead to qualitatively similar results.[23] In addition, our results are robust against changing the specifics of the switching mechanism. For example, replacing the discrete choice model (20) by the well-known replicator dynamics (as in Droste, Hommes, and Tuinstra (2002)), the dynamics remains qualitatively the same. Finally, continuous-time processes typically generate stable equilibria for a wide array of adjustment processes[24], at least for Cournot oligopoly with linear demand and costs and an arbitrary number of firms. It remains an open question whether continuous-time processes with evolutionary competition between adjustment processes can generate complicated dynamics in such an environment.

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[23]For example, we could use our framework to study evolutionary competition between imitation and best-reply dynamicsc (which is the combination of types of behavior found in the laboratory experiment presented in Huck, Normann, and Oechssler (2002)). When a fraction ρ_t of the population imitates last period's average and a fraction of $1 - \rho_t$ uses best-reply, the average quantity produced evolves as $\overline{q}_t = \rho_t \overline{q}_{t-1} + (1 - \rho_t) R\left((n-1)\overline{q}_{t-1}\right)$. The equilibrium (q^*, ρ_κ) is stable if and only if $\left|\rho_\kappa + (1 - \rho_\kappa)(n-1) R'\left((n-1)q^*\right)\right| < 1$. In absence of information costs ($\kappa = 0$ and $\rho_0 = \frac{1}{2}$) and with linear demand and costs we obtain that the Cournot-Nash equilibrium is localy stable in this setting for $n \le 7$.

[24]See Bischi, Chiarella, Kopel, and Szidarovszky (2010) p.90 for the stability analysis of two continuous-time adjustment processes: *best-reply* with adaptive expectations and *partial adjustment towards the best-reply* with naive expectations.

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