Dichotomous multi-type games: Shapley-Shubik and Banzhaf-Coleman power indices

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Abstract

This work focuses on multi-type games in which there are a number of non-ordered types in the input, while the output consists of a single real value. When considering the dichotomous case, we extend the Shapley-Shubik and Banzhaf-Coleman indices. We also provide a full characterization of both extensions. Our results generalize the literature on classical cooperative games.

KEYWORDS: Game theory, Multi-type games, Simple games, Shapley-Shubik index, Banzhaf-Coleman index

JEL Classification Numbers: C71 ;D71.

1 Introduction

This paper is devoted to the study of multi-type games, which are a generalization of the classical cooperative game. This framework models games for which the following conditions are met: (i) Several, say r non-ordered types (not levels) of support are allowed in the input. Each player chooses one of the r types of support. Note that a player can also make no choice. The players' choices then lead to a choice configuration. (ii) The characteristic function maps each choice configuration to a real number, the value of the configuration.

By their nature, classical cooperative games allow two levels of inputs. For example, in a transferable utilities game, each player chooses either "to participate" or "not", and a characteristic function maps each choice configuration to a real value. In a simple voting game, when a single alternative, such as a bill or an amendment, is pitted against the status quo, the alternative is approved only on the basis of the votes cast by those who are in favor. In other words, voting "yes" and "no" are the only feasible alternatives.

It has been pointed out by many scholars that a number of interesting questions of economics, politics and more generally social sciences cannot be described by a classical cooperative game. For example, in some real voting systems such as the United Nations Security Council, the United States federal system and the Council of Ministers of the European Union, "abstention" plays a key role. However, these voting systems cannot be modeled by such games. This goes some way towards explaining the introduction of many different models of such games in recent years.

Felsenthal and Machover [1997], Felsenthal and Machover [1998] and Felsenthal and Machover [2001] introduced ternary voting games where "abstention" is permitted as a distinct third option of a voter, intermediate between a "yes" and a "no" vote. Thus, each individual voter expresses one of the three possible levels of input support, and the output consists of either collective acceptance or collective rejection. A similar model is the bicooperative game of Bilboa et al. [2000] in which the third alternative is "no participation" . Clearly, these three input supports are totally ordered in the sense that a "yes" vote is more favorable to the collective acceptance than an "abstention" vote (or a "not partipating" vote), which in turn is more favorable than a "no" vote. Ternary voting games or $(3,2)$ games have been directly extended to (j,k) games by Freixas and Zwicker [2003]. In the (j,k) simple games, each player expresses one of j ordered possible levels of input support, and the out-

put consists not of a real value but of one of k possible levels of collective support. Another model of game with ordered inputs is the so-called multichoice game introduced by Hsiao and Raghavan [1993]. In these games, each player is allowed to have a given number of effort levels, each of which is assigned a nonnegative weight. The weight assigned to an effort level leads to an ordering on the set of effort levels. Any choice configuration is then associated with a real value.

An example of a model in which alternatives in the inputs are not totally ordered is the one formalized by Laruelle and Valenciano [2012]. They study quaternary voting rules in which the four possible alternatives are "yes", "no", "abstention" and "non-participation". The collective decision is dichotomous, *i.e.*, either the proposal is accepted or rejected. Levels of support here are not totally ordered, since the "non-participation" and the "abstention" options are not ranked. Indeed, in some situations, the "abstention" option may be more favorable to the rejection of the proposal than the "non-participation" option, while in other situations the converse is observed. However, the "yes" alternative is always more favorable than any other alternative while the "no" alternative is less favorable than any alternatives. This model and the other models above are particular cases of the more general framework of games on lattices developed by Grabisch and Lange [2007]. In this model, each player i has a set L_i of possible actions and this set is endowed with a lattice, *i.e.*, a partial order such that any pair of actions possesses a least upper bound and a greatest lower bound.

A model of a game that does not feature among the class of games on lattices is that developed by Bolger [1986], Bolger [1993], Bolger [2000] and Bolger [2002], called games with n players and r alternatives, or simply r–games. In such games, there are r possible input alternatives that are not ordered. Each alternative j attracts its own coalition of supporting voters. A configuration, which is a partition of the set of players into r subsets (some of which might be empty), is then associated with an r-tuple of cardinal values. The component j represents the value of the coalition of the configuration that has chosen the input j. This model is related to ours in the sense that the set of inputs is not ordered. No alternative is a priori more favorable than another. However, the models differ in their outputs. Our output consists of a single value. It is worth noting that our model is less general than that of Monroy and Fernandez [2009], who introduced multi-criteria simple games. Indeed, in their framework, no ordering is considered in the input or in the output, and furthermore more than one alternative can be chosen by each player.

As usual, a central concern in game theory is to define a value or solution concept for a game. The prominent value is the well-known Shapley value. This value for multichoice cooperative games is due to Hsiao and Raghavan [1993]. More generally, Grabisch and Lange [2007] extend the Shapley value to all games on lattices. The corresponding value for r-games was developed by Bolger [1993], and later extended by Amer et al. [1998] and Magana [1996].

In the particular context of simple games, different theories of power have been proposed. The most famous are those of the Shapley-Shubik (Shapley and Shubik [1954]) and the Banzhaf-Coleman (Banzhaf [1965] and Coleman [1971]) indices. These indices have been extended to the context of multiple alternatives in various games. They were defined for ternary voting games by Felsenthal and Machover [1997]. For (j,k) games the extensions are due to Freixas [2005a] and Freixas [2005b].

Our main contribution is to extend and fully characterize the most commonly used voting power indices, namely the Shapley-Shubik and Banzhaf indices, when "dichotomous" multi-type games (*DMG*) are considered. *DMG* are particular cases of multi-type games where the output is dichotomous.

This work is structured as follows. Section 2 introduces the general framework of multi-type games and some examples. Section 3 defines and characterizes the Shapley-Shubik power index for *DMG*, while Section 4 is devoted to the definition and the characterization of the Banzhaf power index for *DMG*. Finally, Section 5 concludes the paper.

2 General framework of multi-type games

Classical cooperative game

A finite set of players is denoted by $N = \{1, 2, ..., n\}$ and 2^N is the set of all nonempty subsets of N. We refer to any subset S of N as a coalition. A classical cooperative n-person game in characteristic form is defined by a function $v : 2^N \rightarrow \mathbb{R}$, such that $v(\emptyset) = 0$. In this paper we consider monotonic cooperative n-person games, *i.e.* cooperative n-person games satisfying $v(S) \leq v(T)$ if $S \subseteq T \subseteq N$.

Now let $\beta = \{0, 1\}$ and $\beta^n = \{(x_1, ..., x_n) : x_i \in \beta, \forall i \in N\}$. Given $S \in 2^N$, define $b(S) = (x_1, ..., x_n)$, where $x_i = \begin{cases} 1 & if \ i \in S \\ 0 & otherwise \end{cases}$.

We can thus identify $2^N \simeq \beta^n$.

Let us call β^n the action space of the n persons $\{1, 2, ..., n\}$. Thus the characteristic function of the cooperative game can also be defined by $v : \beta^n \to \mathbb{R}$. $\beta = \{0, 1\}$ can be thought of as the set of types of support of players where 1 means "participation" and 0 means "non-participation". [1]

We will now generalize classical cooperative games to multi-type games.

Multi-type games

Let $R = \{a_1, ..., a_k, ..., a_r\}$, where r is a non-null integer, be the set of types of support the players can choose. [2]

A configuration is a sequence $F = (F_k)_{k \in R}$, such that for all $k \in R$, $F_k \subseteq N$ and for all $k, j \in R$, $k \neq j \implies F_k \cap F_j = \emptyset$. F can be seen as a repartition of the voters according to their support, while F_k is the set of voters who choose the support $k \in R$. We denote by R^N the set of all configurations and by $\emptyset^r$ the empty configuration, *i.e.* for all $k \in R$, $\emptyset^r_k = \emptyset$. Let $N_F = \underset{k \in R}{\cup} F_k$ be the set of voters who choose one of the supports. Respectively, $N - N_F$ represents all the voters who do not support the proposal.

A multi-type game is a pair (N, V) where $V : R^N \to \mathbb{R}$, such that $V(\emptyset^r) = 0$ and V is monotonic, *i.e.* for all $F, G \in R^N$, $F \subseteq G \implies V(F) \leq V(G)$. [3]

Clearly a multi-type game reduces to a classical cooperative game when there is only one type of support, *i.e.* when $|R| = 1$.

Examples

The first example is inspired by Example 2, page 320 in Bolger [1993].

Example 1. *Consider a society of n individuals. This society can be divided into three categories. The first one consists of those who are working. The second one concerns those who are currently unemployed. And finally, in Category 3 we find the individuals who leave the unemployment system. If a player i is employed, he can earn (annually) w_i euros. If player i is in Category 2, then he receives, as an unemployment benefit, 70 percent of his potential earning, namely 0.7 w_i. Finally individuals in Category 3 receive a minimal income,* i.e., *30 percent of their potential income.*

A configuration F can be seen as a couple $F = (F_1, F_2)$ where: F_1 is the set of individuals in category 1; F_2 is the set of individuals in category 2; and $N \backslash (F_1 \cup F_2)$ is the set of

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[1] This formulation is due Hsiao and Raghavan [1993].
[2] For the sake of simplicity and when there is no ambiguity, we write $k \in R$ for an element $a_k \in R$.
[3] Note that $F \subseteq G$ if for all $k \in R$, $F_k \subseteq G_k$.

individuals in category 3. The value of a given configuration can be thought of as the "wage income" of the society, that is the sum of incomes of each category.

The second example deals with non-hierarchical multitask organizations, inspired by Pongou et al. [2015].

Example 2. *Consider an organization in which there is a nonempty set of workers and a collection of finite sets of tasks. A real-valued production function maps each allocation of workers across the tasks to a real number which measures the aggregate output or productivity of the organization under that particular configuration.*

Note that the configuration of workers may lead to a product of good or bad quality. In these special situations the output is dichotomous, as in the following example.

The third example concerns the particular case of dichotomous multi-type games and is adapted from Freixas [2005b] and Hsiao and Raghavan [1993].

Example 3. *Consider a committee that must decide on the promotion of a certain junior colleague in an economics department. There are three full professors (namely 1, 4 and 5) and two associate professors (namely 2, 3) on this committee. Each of the professors may express a Research Support (RS), a Teaching Support (TS) or an Administrative Support (AS). The candidate will be selected only when both of the following conditions are met:*

1. *At least two full professors express a Research Support.*

2. *One or more associate professors expresses at least one of the supports.*

The multi-type game associated with this situation is such that $R = \{RS, TS, AS\}$.

We will now focus on dichotomous games. In the classical framework, dichotomous game are represented by simple games.

Simple voting games

A simple (voting) game (SG) consists of a pair (N, v) such that: (i) $v(S) \in \{0, 1\}$ for all $S \in 2^N$; (ii) $v(N) = 1$; and (iii) v is monotonic. A coalition S is said to be winning in $v \in 2^N$ if $v(S) = 1$, and losing otherwise. An SG is said to be anonymous if there exists a non-null integer q such that for all $S \in 2^N$: $v(S) = 1$ iff $|S| \geq q$. In an anonymous SG the outcome depends solely on the number of the "yes" votes rather than on their individual identities.

In the following, for simplicity an SG (N, v) will be denoted by v, $S \cup \{i\}$ will be written $S + i$, while $S \setminus \{i\}$ represents $S - i$.

Dichotomous multi-type games

As a natural extension of *SG*, a dichotomous multi-type (*DMG*) is given by a map $V : R^N \to \mathbb{R}$ such that: (i) $V(F) \in \{0,1\}$ for all $F \in R^N$; (ii) $V(F) = 1$ for at least one $F \in R^N$; and (iii) V is monotonic. A configuration F is a winning configuration for $V(F) = 1$ while it is a losing configuration for $V(F) = 0$.

A *DMG* is anonymous if there exists a vector of integers $(q_k)_{k \in R}$ such that for all $F \in R^N$: $V(F) = 1 \Leftrightarrow$ for all $k \in R, |F_k| \geq q_k$. This means that the number of voters who choose each support k is sufficient to determine the outcome.

Once again, the dichotomous multi-type game reduces to a simple game when there is only one type of support.

The main problem is to define a value for multi-type games. However, in this paper we restrict ourselves to *DMG*. In this context, a value refers to a power index.

Power index: *SG*

A power index is a mapping φ assigning to each *SG* an n-dimensional real-valued vector $\varphi(v) = (\varphi_1(v), ..., \varphi_n(v))$. $\varphi_i(v)$ can be interpreted as the voting power of player $i \in N$ in game v.

In general, the classical power indices can be summarized by the following formula: [4]

$$\varphi_i(v) = \sum_{S \in 2^N, i \in S} \alpha(i, S)[v(S) - v(S - i)].$$

The marginal contribution of a player i belonging to a coalition S is given by $v(S) - v(S - i)$. If $v(S) - v(S - i) = 1$, player i is said to be decisive in S.

$\alpha(i, S)$ can be seen as a weight which depends on the coalition $S \in 2^N$ and on the player i. However, in most of these indices, $\alpha(i, S)$ does not depend on i but solely on S. This is the case, for example, with:

- Johnston (Johnston [1978]), for which $\alpha(i, S) = \frac{1}{d(S)}$ with $d(S)$ being the number of decisive players in S;

- Shapley-Shubik (Shapley and Shubik [1954]), for which $\alpha(i, S) = \frac{(|S|-1)!(n-|S|)!}{n!}$;

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[4] There is a large literature on the many notions of power indices (see Andjiga et al. [2003] and Laruelle and Valenciano [2008] for a detailed description of these different notions).

- Banzhaf-Coleman (Banzhaf [1965] and Coleman [1971]), for which $\alpha(i, S) = \frac{1}{2^{n-1}}$.

We call this class of power indices weight-dependent power indices. In what follows, a weight-dependent power index for a player i in a SG v is then given by:

$$\varphi_i(v) = \sum_{S \in 2^N} \alpha_S [v(S) - v(S - i)],$$

with α_S the weight of coalition S.

Power index in *DMG*

We first propose a natural extension of a decisive player to *DMG*.

A player i is decisive in F if $[V(F) - \bar{V}(F - i)] = 1$ with $\bar{V}(F) = \underset{Z \in R^N}{Max}\{V(Z), N_Z = N_F\}$ and $F - i = (F_k - i)_{k \in R}$. In words, a player i is decisive in a configuration F if F is a winning configuration and whatever the types of support of the other voters, when he decides to not support the proposal, the winning configuration becomes a losing one.

From this definition, we derive a generalization of weight-dependent power indices, in the context of *DMG*. A weight-dependent power index for a player i in a *DMG* is in the form :

$$\Psi_i(V) = \sum_{F \in R^N} \alpha_F [V(F) - \bar{V}(F - i)]$$

where α_F is referred to as the weight of the configuration F.

Note that for $|R| = 1$, a weight-dependent power indice for a *DMG* is equivalent to a weight-dependent power index for an *SG*.

In the following, we focus on the most important power indices, *i.e.* the Shapley-Shubik and Banzhaf power indices.

3   Shapley-Shubik power index for *DMG*

In this section, we outline an axiomatic approach for the Shapley-Shubik power index both for *SG* and *DMG*.

8

Characterization: *SG*

As mentioned above, for any *SG* v, the Shapley-Shubik index of player i is defined by:

$$SS_i(v) = \sum_{S \in 2^N} \frac{(|S|-1)! \, (n-|S|)!}{n!} [v(S) - v(S-i)].$$

There is a large literature on the characterization of this index. Below, we provide an alternative characterization of the Shapley-Shubik power index in the class of weight-dependent power indices. Let us first remark that if a power index φ satisfies the symmetric axiom (that is, for all $i, j \in N$, all $S \in 2^N$, $v(S + i) = 1 \Leftrightarrow v(S + j) = 1$) and the efficiency axiom ($\sum \varphi_i(v) = 1$) then the following property holds.[5]

Axiom 1. *Anonymity:*

φ satisfies Anonymity if for all anonymous SG *v and all $i \in N$, $\varphi_i(v) = \frac{1}{n}$.*

We introduce below another property for weight-dependent power indices.

Axiom 2. *Equal Size:*

φ satisfies Equal Size if for all $S, T \in 2^N$, if $|S| = |T|$ then $\alpha_S = \alpha_T$.

According to the Equal Size axiom, two coalitions with the same size must be equally weighed. We can now state another characterization of the Shapley-Shubik power index.

Proposition 1. *There exists one and only one weight-dependent power index satisfying the Anonymity and Equal Size axioms, which is the Shapley-Shubik power index.*

Proof. $\Rightarrow$) Let φ be a weight-dependent power index that satisfies the Anonymity and the Equal Size axioms.

We shall show that for all $S \in 2^N$, $\alpha_S = \frac{(|S|-1)!(n-|S|)!}{n!}$.

Let $S \in 2^N$: consider the anonymous *SG* v defined by:

for all $T \in 2^N$, $v(T) = 1 \Longleftrightarrow |T| \geq |S|$.

Let $i \in N$: set $W_i = \Big\{T \in 2^N, i \in T : v(T) - v(T - i) = 1\Big\}$.

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[5]Even though in the literature there exist power indices that do not satisfy the efficiency property (as can easily be understood) it is reasonable to consider that a power index is efficient at least in the class of anonymous games where the status of a coalition (winning or losing) depends solely on the number of players in that coalition.

Then,

$$
\begin{aligned}
W_i &= \left\{ T \in 2^N, i \in T : v(T) - v(T - i) = 1 \right\} \\
&= \left\{ T \in 2^N, i \in T : v(T) = 1 \text{ and } v(T - i) = 0 \right\} \\
&= \left\{ T \in 2^N, i \in T : |T| \geq |S| \text{ and } |T - i| < |S| \right\} \\
&= \left\{ T \in 2^N, i \in T : |T| = |S| \right\}
\end{aligned}
$$

Moreover, $\varphi_i(v) = \sum_{T \in 2^N} \alpha_T [v(T) - v(T - i)] = \sum_{T \in W_i} \alpha_T$ and for all $T \in W_i$, $|T| = |S|$. By the Equal Size axiom, it holds that $\alpha_T = \alpha_S$.

Thus,

$$
\begin{aligned}
\varphi_i(v) \quad &= \quad \sum_{T \in W_i} \alpha_S \quad = \quad \alpha_S |W_i| \\
&= \quad \alpha_S \left| \left\{ T \in 2^N, i \in T : |T| = |S| \right\} \right| \\
&= \quad \alpha_S C_{n-1}^{|S|-1} \\
&= \quad \alpha_S \frac{(n-1)!}{(|S|-1)!(n-1-|S|+1)!} \\
&= \quad \alpha_S \frac{(n-1)!}{(|S|-1)!(n-|S|)!}
\end{aligned}
$$

Finally, by Anonymity, $\forall i \in N$, $\varphi_i(v) = \frac{1}{n}$. Hence, $\alpha_S \frac{(n-1)!}{(|S|-1)!(n-|S|)!} = \frac{1}{n}$, which implies that $\alpha_S = \frac{(|S|-1)!(n-|S|)!}{n(n-1)!} = \frac{(|S|-1)!(n-|S|)!}{n!}$.

$\Longleftarrow$) Conversely, the weight-dependent power index for which the weight of a coalition S is given by $\alpha_S = \frac{(|S|-1)!(n-|S|)!}{n!}$ is clearly the Shapley-Shubik power index, which obviously satisfies the Anonymity and Equal Size axioms. $\qquad\square$

Characterisation: *DMG*

We will now extend the result above to *DMG*. The Anonymity and Equal Size axioms defined above for *SG* can be extended as follows.

Let Ψ be a weight-dependent power indice for *DMG* with α_F the weight of configuration F.

Axiom 3. *Anonymity for* DMG *:*

Ψ *satisfies Anonymity for* DMG *if for all anonymous* DMG V *and all* $i \in N$, $\Psi_i(V) = \frac{1}{n}$.

Axiom 4. *Equal Size for* DMG *:*

Ψ *satisfies Equal Size for* DMG *if for all* $F, G \in R^N$, *and all* $k \in R$, $|F_k| = |G_k|$ *implies* $\alpha_F = \alpha_G$.

In any anonymous *DMG*, any player has exactly the same power which is given by $\frac{1}{n}$. According to the Equal Size condition, all configurations must be equally weighted, provided that only the number of players in a given type of support matters. The following theorem generalizes the Shapley-Shubik power index to *DMG*.

Theorem 1. *There exists exactly one weight-dependent power index satisfying the Anonymity and Equal Size axioms for* DMG, *the weight of which for a configuration F is given by:*

$$\alpha_F = \frac{\left(n - \sum\limits_{k \in R} |F_k|\right)! \left(\prod\limits_{k \in R} |F_k|!\right)}{n! \left(\sum\limits_{k \in R} |F_k|\right)}$$

In order to prove Theorem 1 above, the following results will be useful.

The first result is straightforward from combinatorial analysis

Lemma 1. *Let N be a set of n players and* (n_t) *a sequence of r integers such that* $\sum\limits_{t=1}^{r} n_t = n$. *The number of partitions of N into r coalitions* $S_1, ..., S_r$ *such that for all* $t = 1, 2, ..., r$ $|S_t| = n_t$ *is given by the formula* $\frac{n!}{n_1! \times n_2! \times ... \times n_r!}$.

The next result characterizes a decisive player in an anonymous *DMG*.

Lemma 2. *Let V be an anonymous* DMG *in which for all* $F \in R^N$, $V(F) = 1$ *iff for all* $k \in R, |F_k| \geq q_k$. *Let* $F \in R^N$: *a voter* $i \in N$ *is decisive in F if and only if* $i \in N_F$ *and for all* $k \in R, |F_k| = q_k$.

Proof. Let V be an anonymous *DMG* in which for all $F \in R^N$, $V(F) = 1 \Leftrightarrow$ for all $k \in R, |F_k| \geq q_k$.

Let $F \in R^N$ and a voter $i \in N$.

$\Rightarrow$) Assume that i is decisive in F. Then by definition, $i \in N_F$.

Assume now that there exists $j \in R$ such that $|F_j| > q_j$.

- If $i \in F_j$ then $|F_j - i| = |F_j| - 1 \geq q_j$; furthermore, $\forall k \in R - j, |F_k - i| = |F_k| \geq q_k$. It follows that $V(F - i) = 1$ and consequently $\overline{V}(F - i) = 1$. This is a contradiction since i is decisive in F.

11

- If $i \notin F_j$; consider a given player $i' \in F_j$, set $i \in F_p$ with $p \neq j$ and consider the configuration G defined as follows :

$$\text{for all } k \in R,\ G_k = \begin{cases} F_k \text{ if } k \notin \{p, j\} \\ F_k - i' \text{ if } k = j \\ (F_k - i) + i' \text{ if } k = p \end{cases}$$

It is straightforward that $\forall k \in R, |G_k| \geq q_k$, that is, $V(G) = 1$ and $\overline{V}(F - i) = 1$ since $N_G = N_F - i$. Once again, this is a contradiction since i is decisive in F. It then follows that $\forall k \in R, |F_k| = q_k$.

$\Longleftarrow$) Let $F \in R^N$ with $i \in N_F$ and $\forall k \in R, |F_k| = q_k$.

Since $|F_k| = q_k$, $V(F) = 1$ and $|N_F| = |\bigcup\limits_{k \in R} F_k| = \sum\limits_{k \in R} q_k$. Let $G \in R^N$ such that $N_G = N_F - i$; then $|N_G| = |N_F| - 1$ and $|N_G| < |N_F|$. Thus, $\sum\limits_{k \in R} |G_k| - \sum\limits_{k \in R} q_k < 0$, which implies that $\sum\limits_{k \in R} (|G_k| - q_k) < 0$, and therefore there exists a $j \in R$ such that $|G_j| < q_j$. This leads to $V(G) = 0$ and $\overline{V}(F - i) = 0$. Finally, i is decisive in F, because $V(F) = 1$ and $\overline{V}(F - i) = 0$.

$\square$

We can now prove Theorem 1.

Proof. $\Longrightarrow$) Let Ψ be a weight-dependent power index for *DMG* associated with the weights (α_F) satisfying the Anonymity and Equal Size axioms. Let $F \in R^N$.

We shall prove that $\alpha_F = \dfrac{\left(n - \sum\limits_{k \in R} |F_k|\right)! \left(\prod\limits_{k \in R} |F_k|!\right)}{n! \left(\sum\limits_{k \in R} |F_k|\right)}$.

Define the *DMG* V as follows: $\forall G \in R^N$, $V(G) = 1 \Leftrightarrow \forall k \in R, |G_k| \geq |F_k|$. Clearly, V is an anonymous *DMG*.

We have $\Psi_i(V) = \sum\limits_{G \in R^N} \alpha_G \left[V(G) - \overline{V}(G - i) \right] = \sum\limits_{G \in W_R^i} \alpha_G$

where $W_R^i = \left\{ G \in R^N : V(G) - \overline{V}(G - i) = 1 \right\}$. By Lemma 2, if i is decisive in G then $i \in N_G$ and $\forall k \in R, |G_k| = |F_k|$, that is,

$V(G) - \overline{V}(G - i) = 1 \implies \alpha_G = \alpha_F$, thanks to the Equal Size axiom.

Therefore, $\Psi_i(V) = \sum\limits_{G \in W_R^i} \alpha_F = \alpha_F \left| W_R^i \right|$.

We have

$$\begin{aligned}
W_R^i &= \left\{ G \in R^N : V(G) - \overline{V}(G - i) = 1 \right\} \\
&= \left\{ G \in R^N : i \in N_G \text{ and } \forall k \in R, |G_k| = |F_k| \right\} \\
&= \left\{ G \in R^N : i \in \left(\underset{j \in R}{\cup} G_j \right) \text{ and } \forall k \in R, |G_k| = |F_k| \right\} \\
&= \underset{j \in R}{\cup} \left\{ G \in R^N : i \in G_j \text{ and } \forall k \in R, |G_k| = |F_k| \right\} \\
&= \underset{j \in R}{\cup} W_R^{i,j}, \text{ where } W_R^{i,j} = \left\{ G \in R^N : i \in G_j \text{ and } \forall k \in R, |G_k| = |F_k| \right\}
\end{aligned}$$

Thus, $\Psi_i(V) = \alpha_F \left| W_R^i \right| = \alpha_F \left| \underset{j \in R}{\cup} W_R^{i,j} \right| = \alpha_F \sum_{j \in R} \left| W_R^{i,j} \right|$ since

$\forall k, j \in R, k \neq j \implies W_R^{i,k} \cap W_R^{i,j} = \varnothing$. We have $\Psi_i(V) = \alpha_F \sum_{j \in R} \left| W_R^{i,j} \right|$.

$\left| W_R^{i,j} \right|$ is exactly the number of partitions of $N - i$ into $r + 1$ coalitions G_1, G_2, ..., G_r and G_{r+1} such that $|G_1| = |F_1|$, ..., $|G_{j-1}| = |F_{j-1}|$, $|G_j| = |F_j| - 1$, $|G_{j+1}| = |F_{j+1}|$, ..., $|G_r| = |F_r|$ and $|G_{r+1}| = (n - \sum_{k \in R} |F_k|)$. Thanks to Lemma 1 we obtain :

$$\begin{aligned}
\left| W_R^{i,j} \right| &= \frac{(n-1)!}{\left(n - \sum_{k \in R} |F_k| \right)! \left(\prod_{k \in R, k \neq j} |F_k|! \right) \left(|F_j| - 1 \right)!} \\
&= \frac{|F_j| \times (n-1)!}{\left(n - \sum_{k \in R} |F_k| \right)! \left(\prod_{k \in R} |F_k|! \right)}
\end{aligned}$$

and hence,

$$\Psi_i(V) = \alpha_F \sum_{j \in R} \left| W_R^{i,j} \right|$$

$$= \alpha_F \sum_{j \in R} \left[\frac{\left| F_j \right| \times (n-1)!}{\left(n - \sum\limits_{k \in R} |F_k| \right)! \left(\prod\limits_{k \in R} |F_k|! \right)} \right]$$

$$= \alpha_F \frac{(n-1)! \left(\sum\limits_{j \in R} \left| F_j \right| \right)}{\left(n - \sum\limits_{k \in R} |F_k| \right)! \left(\prod\limits_{k \in R} |F_k|! \right)}$$

Thanks to the Anonymity of the index Ψ, we have $\Psi_i(V) = \frac{1}{n}$ since V is anonymous.

Finally, $\alpha_F \dfrac{(n-1)! \left(\sum\limits_{j \in R} |F_j| \right)}{\left(n - \sum\limits_{k \in R} |F_k| \right)! \left(\prod\limits_{k \in R} |F_k|! \right)} = \frac{1}{n}$ which yields $\alpha_F = \dfrac{\left(n - \sum\limits_{k \in R} |F_k| \right)! \left(\prod\limits_{k \in R} |F_k|! \right)}{n(n-1)! \left(\sum\limits_{j \in R} |F_j| \right)} = \dfrac{\left(n - \sum\limits_{k \in R} |F_k| \right)! \left(\prod\limits_{k \in R} |F_k|! \right)}{n! \left(\sum\limits_{j \in R} |F_j| \right)}.$

This concludes the first part of the proof.

$\Longleftarrow$) Conversely, let Ψ be a power index such that $\forall F \in R^N$, $\alpha_F = \dfrac{\left(n - \sum\limits_{k \in R} |F_k| \right)! \left(\prod\limits_{k \in R} |F_k|! \right)}{n! \left(\sum\limits_{k \in R} |F_k| \right)}.$

a) It is obvious that Ψ satisfies the Equal Size condition.

b) Let us now prove that Ψ satisfies Anonymity.

Consider an anonymous DMG for which $\forall F \in R^N$, $V(F) = 1 \Leftrightarrow \forall k \in R, |F_k| \geq q_k$.

We have $\Psi_i(V) = \sum\limits_{F \in R^N} \alpha_F (V(F) - \overline{V}(F-i)) = \sum\limits_{F \in W_R^i} \alpha_F$

where $W_R^i = \left\{ F \in R^N : V(F) - \overline{V}(F-i) = 1 \right\}$.

We know from Lemma 2 that for all $F \in W_R^i$, i is decisive in F and therefore $i \in N_F$ and $\forall k \in R, |F_k| = q_k$, which implies that α_F does not depend on F and is given by $\alpha_F = \alpha = \dfrac{(n - \sum\limits_{k \in R} q_k)! (\prod\limits_{k \in R} q_k)!}{n! (\sum\limits_{k \in R} q_k)}$; thus, $\Psi_i(V) = \alpha |W_R^i|$.

As we did in the first part of the proof, let us remark that:

$W_R^i = \bigcup\limits_{j \in R} X_R^{i,j}$, where $X_R^{i,j} = \left\{ F \in W_R^i : i \in F_j \right\}$.

$|X_R^{i,j}|$ is the number of partitions of $N - i$ into $r+1$ coalitions $F_1, F_2, ..., F_r$ and F_{r+1} such that $|F_1| = q_1, ..., \left| F_{j-1} \right| = q_{j-1}, |F_j| = q_j - 1$, (by assuming that $i \in F_j$), $|F_{j+1}| = q_{j+1}$,

..., $|F_r| = q_r$ and $|F_{r+1}| = (n - \sum_{k \in R} |F_k|)$.

Thanks to Lemma 1 we obtain:

$|X_R^{i,j}| = \frac{(n-1)!}{(n-\sum_{k \in R} q_k)!(\prod_{k \in R, k \neq j} q_k!)(q_j-1)!}$ and hence $|W_R^i| = \sum_{j \in R} \frac{q_j(n-1)!}{(n-\sum_{k \in R} q_k)!(\prod_{k \in R} q_k!)} = \frac{(n-1)!}{(n-\sum_{k \in R} q_k)!(\prod_{k \in R} q_k!)} \sum_{j \in R} q_j$.

Finally

$$
\Psi_i(V) = \alpha \times \frac{(n-1)!\left(\sum_{j \in R} q_j\right)}{\left(n - \sum_{k \in R} q_k\right)!\left(\prod_{k \in R} q_k!\right)}
$$

$$
= \frac{\left(n - \sum_{k \in R} q_k\right)!\left(\prod_{k \in R} q_k!\right)}{n!\left(\sum_{k \in R} q_k\right)} \times \frac{(n-1)!\left(\sum_{j \in R} q_j\right)}{\left(n - \sum_{k \in R} q_k\right)!\left(\prod_{k \in R} q_k!\right)}
$$

$$
= \frac{1}{n}.
$$

We conclude that $\forall i \in N, \Psi_i(V) = \frac{1}{n}$, and Ψ satisfies Anonymity.

The proof is now complete. □

Let us mention that when $|R| = 1$ our formula clearly reduces to that of Shapley-Shubik for SG. The Shapley-Shubik index of a player in a DMG V is then defined by:

$$
SS_i^r(V) = \sum_{F \in R^N} \frac{\left(n - \sum_{k \in R} |F_k|\right)!\left(\prod_{k \in R} |F_k|!\right)}{n!\left(\sum_{k \in R} |F_k|\right)} \left[V(F) - \bar{V}(F - i)\right]
$$

4 Banzhaf-Coleman power index for *DMG*

Characterization: *SG*

We provide in this section a characterization of the class of weight-dependent power indices based on the famous Banzhaf score (Banzhaf [1965]). Given an *SG* v, the Banzhaf score of a player i in v, denoted by $\eta_i(v)$, is the number of coalitions in which he is decisive in an *SG*. More formally, for all $i \in N$, $\eta_i(v) = \sum_{S \in 2^N} [v(S) - v(S - i)]$.

The non-normalized Banzhaf power index (Dubey and Shapley [1979]) is the quantity denoted $Bz_i(v)$ and given by

$$Bz_i(v) = \frac{\eta_i(v)}{2^{n-1}}$$

In order to provide an alternative characterization of the Banzhaf power index, let us introduce the following axiom, refered to as the Constant-Weight axiom

Axiom 5. *Constant-Weight:*

A weight-dependent power index satisfies the Constant-Weight axiom if there exists $\lambda > 0$ such that for all $S \in 2^N$, $\alpha_S = \lambda$.

If it is the case then the power index is said to be a λ-constant-weight power index.

We obtain the following proposition.

Proposition 2. *A weight-dependent power index φ is λ-constant-weight if and only if for all SG v, for all $i \in N$, $\varphi_i(v) = \lambda \eta_i(v)$.*

Proof. $\Rightarrow$) Let φ be a weight-dependent power index. Assume further that φ is λ-constant-weight. Then, $\forall S \subseteq N$, $\alpha_S = \lambda$.

Let v be an SG and $i \in N$, we have:

$$
\begin{aligned}
\varphi_i(v) &= \sum_{S \in 2^N} \alpha_S [v(S) - v(S-i)] \\
&= \sum_{S \in 2^N} \lambda [v(S) - v(S-i)] \\
&= \lambda \sum_{S \in 2^N} [v(S) - v(S-i)] \\
&= \lambda \eta_i(v)
\end{aligned}
$$

$\Leftarrow$) Now, assume that for all SG v, and all $i \in N$, $\varphi_i(v) = \lambda \eta_i(v)$. Then clearly, φ is λ-constant-weight.

Then $\varphi_i(v) = \lambda \sum_{S \in 2^N} [v(S) - v(S-i)] = \sum_{S \in 2^N} \lambda [v(S) - v(S-i)]$. $\qquad \square$

Characterization: *DMG*

We will now extend the Banzhaf scores to *DMG*.

The Banzhaf score is defined as follows:

Let V be a *DMG*, $i \in N$, then the Banzhaf score of i is given by $\eta_i^r(V) = \sum_{F \in R^N} \left[V(F) - \bar{V}(F-i) \right]$.

The constant-weight axiom is extended in *DMG* as follows

Axiom 6. *Constant-Weight axiom for* DMG:

A weight-dependent power index satisfies the Constant-Weight axiom for DMG *if there exists a constant $\delta > 0$ such that for all $F \in R^N$, $\alpha_F = \delta$.*

If this is the case then the power index is said to be a δ-constant-weight power index.

The theorem below is a generalization of Proposition 2.

Theorem 2. *A weight-dependent power index Ψ is δ-constant-weight if and only if for all* DMG *V, for all $i \in N$, $\Psi_i(V) = \delta\eta_i^r(V)$.*

Proof. Let Ψ be a weight-dependent power index for a *DMG*. If Ψ is δ-constant-weight, then for all $i \in N$,

$$
\begin{aligned}
\Psi_i(V) &= \sum_{F \in R^N} \alpha_S \left[V(F) - \overline{V}(F - i) \right] \\
&= \sum_{F \in R^N} \delta \left[V(F) - \overline{V}(F - i) \right] \\
&= \delta \sum_{F \in R^N} \left[V(F) - \overline{V}(F - i) \right] \\
&= \delta\eta_i^r(V).
\end{aligned}
$$

$\Longleftarrow$) The converse is straighforward. $\qquad\square$

Once again, for $\delta = \frac{1}{(r+1)^{n-1}}$, we obtain a generalization of the non-normalized Banzhaf power index.

Therefore the non-normalized Banzhaf power index of a player i is the quantity

$$
Bz_i^{\ r}(V) = \frac{\sum\limits_{F \in R^N} \left[V(F) - \bar{V}(F - i) \right]}{(r + 1)^{n-1}}
$$

To conclude, let us evaluate the Shapley-Shubik and non-normalized Banzhaf for the *DMG* defined in Example 3 dealing with the promotion of a junior professor.

Example 4. *(example 3 continued)*

i) In an SG *context, the professors only have to say if they are "for" or "against" the promotion. The candidate will be selected when at least two full professors and at least one associate professor vote "yes". Therefore with Shapley Shubik, $SS(v) = (\frac{7}{30}, \frac{9}{60}, \frac{9}{60}, \frac{7}{30}, \frac{7}{30})$; while with the non-normalized Banzhaf, $Bz(v) = (\frac{3}{8}, \frac{1}{4}, \frac{1}{4}, \frac{3}{8}, \frac{3}{8})$.*

ii) Consider two configurations in the DMG *context: $F^1 = (\{1, 5\}, \{23\}, \{4\})$ and $F^2 = (\{1\}, \{23\}, \{\varnothing\})$. In F^1, 1 and 5 support RS, 2 and 3 choose TS and 4 chooses AS; whereas*

in F^2 the individuals 4 and 5 make no choice and 1, 2 and 3 continue to select the same support. Therefore $V(F^1) = 1$ and $V(F^2) = 0$.

Is individual 1 decisive in F^1? The answer is No. Indeed if we consider the configuration $G = (\{4, 5\}, \{23\}, \{\varnothing\})$, it is easy to show that $\bar{V}(G) = 1$. Therefore 1 is not decisive in F^1. In G, player 4 changes his support from AS to RS, and this leads to G, which is a winning configuration.

Therefore in DMG, Shapley Shubik, $SS^r(V) = (\frac{7}{30}, \frac{9}{60}, \frac{9}{60}, \frac{7}{30}, \frac{7}{30})$; while the non-normalized Banzhaf, $Bz^r(V) = (\frac{17}{128}, \frac{12}{128}, \frac{12}{128}, \frac{17}{128}, \frac{17}{128})$.

5 Conclusion

This paper was devoted to games in which r non-ordered levels of input supports are available to players. In the particular case of a dichotomous game, the output consists of 0 or 1. Thus any choice configuration is either winning or losing. Thanks to alternative characterizations of the Shapley-Shubik and Banzhaf-Coleman power indices in the classical context, we provide full characterizations of these two famous indices in our framework.

There are several ways in which (dichotomous) multi-type games could be explored in further research. We can study, first, the generalization of the Shapley value to multi-type games. Secondly, we can extend two closely related solution concepts for these games, the stable sets and the core. Another interesting direction is the extension to multi-type games of the different a priori relationships between players in the spirit of Owen [1977] and Owen [1981]. Following the works of Tchantcho et al. [2008], we can also derive the desirability relation to qualitatively compare the a priori influence of voters in a dichotomous multi-type game.

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