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"Multi-Stage Voting and Sequential Elimination with Productive Players"

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Multi-Stage Voting and Sequential Elimination with Productive Players*

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Abstract

This paper analyzes a sequential voting mechanism that eliminates at each round one candidate, until only one of them is left (the winner). The candidates are the voters and they only differ across their skill level. The payoff allocated to the winner depends on the sequence of elimination of the players' skills, the rest of the players receiving a payoff of zero. We fully characterize the equilibria of the game with two skills. The winner must be a high-skilled player if there is an initial majority of strong types. On the contrary, a high-skilled player might win with an initial majority of weak players *independently* of the size of the majority. For an arbitrary number of types, if some type of candidates form a strict majority at the first stage, the winner belongs either to the majoritarian type or to a more skilled one.

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1 Introduction

Since Farquharson [2], the concept of sophisticated voting has been applied to analyze many different electoral dynamic settings. A good example of dynamic voting procedures

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are the so-called “binary voting agendas” which are among the most fundamental voting procedures. The alternatives are paired together to be voted on in a tree, where each node in the tree represents a majority vote between two alternatives. The vote at any point in the tree may be decided by simultaneous voting or sequential voting. In the case of simultaneous voting, in which we focus, the tree represents a finite imperfect information extensive game[1].

Our focus is on rules that at each stage remove one of the alternatives. These rules have been recently proved to exhibit appealing properties, mainly as far as Condorcet Consistency is concerned (Bag et al. [1]). Indeed, it is well-known that simultaneous strategic voting under most voting rules fails to satisfy a number of minimal requirements such as Condorcet-consistency (the election of the Condorcet Winner[2]) or Pareto optimality[3]. This problem is mostly related to the theory of instrumental voting: indeed in a Nash equilibrium, if a candidate is more than 2 votes ahead of the rest of the candidates, then no player can change the outcome of the election (provided that the maximal score a player can assign is normalized to one). This leads, inevitably, to a multiplicity of equilibria. Bag et al. [1] prove that most of the common scoring rules lead to the election of the Condorcet Winner if applied in a dynamic setting. More formally, they prove the previous statement when the voting follows what they call the weakest link procedure[4]: “voting occurs in rounds with all the players simultaneously casting their votes for one candidate in each successive round. In any round the candidate with minimal votes is eliminated, with any ties broken by a deterministic tie-breaking rule. Continue with this process to pick a winner”.

Building on their framework, we analyze the voting behavior with two major modifications: (i) the players are the candidates and (ii) the players' payoffs depend on the sequence of elimination of the candidates. To do so, we consider that the players are

[1] Using sequential voting defines a finite perfect information extensive game (Farquharson [2], McKelvey and Niemi [7], Moulin [8] and Hummel [5].

[2] A Condorcet Winner is the candidate that beats every other candidate in sincere pairwise comparisons. It is considered as a compelling democratic principle.

[3] This problem exists if the focus is on Nash equilibrium of one-shot voting games with complete information over the players' preferences. This might be partially solved if one focuses for instance on games with population uncertainty as for instance Poisson Games.

[4] There is a related literature that has on the TV show called the weakest link in which a very similar voting rule is used to determine the winner (see for example Levitt [6], Février and Linnemer [3] or, more recently Hann *et al.* [4]). This literature looks for empirical evidence of some phenomena like discrimination or equilibrium selection in the histories of voters' decisions.

divided by their skill, considering that within each group all players are identical. The voting at each round concerns the players' type: that is each player votes for one of the groups and one of the players with the highest number of votes is randomly eliminated. In the event of a tie between both groups, the removed player belongs to the group with the lowest skill level. This elimination proceeds until only player remains in the game: the winner. This implies that there is no Condorcet winner since every player prefers his own victory and is indifferent between the rest of the events. As far as the production is concerned, each player makes at each round a contribution that just depends on his skill level s. These contributions are stocked until the end of the game, and only the winner of the game receives a payoff that depends on the sequence of elimination of the players. One crucial assumption is that the payoff allocated to the winner is larger when removing first a low-skilled player than a high-skilled one, the rest of elimination sequence being unchanged. Note that if each individual contribution is independent of the rest of the contributions at each period (*no complementarities*) and is constant through time, our assumption is satisfied. Moreover, our results are quite general since they hold provided that the arbitrage between removing a high-skilled and low-skilled player remains unchanged through the different periods. Indeed, if one assumes that the payoff becomes larger when removing first a high-skilled player rather than a low-skilled one, the results remain quite similar to those in the current work. However, our assumption seems more realistic since the players' contributions are compulsory and hence keeping as long as possible the most productive players in the game helps to maximize the pie awarded to the winner.

We prove that with two skill levels the equilibrium of the game is unique. If there is a (weak) majority of high-skilled players in the initial stage of the game, the sequence of elimination consists of eliminating first all the low-skilled (weak) players and then the high-skilled ones: the winner is then a high-skilled player. On the contrary, if the weak players are majoritarian, the elimination sequence coincides with the one that maximizes the expected utility of a low-skilled player. Surprisingly, it consists on first eliminating weak players until the number of weak players equals the number of strong players plus one. Then, iteratively it removes strong and weak players in order to ensure that the weak players never lose the strict majority while at the same time maximizing their expected payoff. In this case, the winner is either weak or strong. With more than two types, there is a plethora of equilibria. However, some of the results of the two-type case hold. Indeed, if some group is majoritarian on the first stage of the game, the winner belongs either to this group or to the group with a higher skill level.

The paper is organized as follows. Section 2 presents the basic notation and definitions. Section 3 describes the equilibria in the semifinal and Section 4 presents the results with two different types. Finally, Section 5 discusses the equilibria when players have many different types.

2 The setting

2.1 Players.

The candidates are the voters so that we use *player* to refer to both of them. Each player i has a skill level s with s being an integer in $S = \{s_1, s_2, \ldots, s_m\}$ with $0 < s_1 < s_2 < \ldots < s_m$.

At stage $t \in T = \{1, \ldots, n\}$, the player set is denoted by N^t with $|N^t| = n^t$. At each stage, one of the players is removed (the identity of which will be determined through a vote) until only one of them is left.

Given that at the beginning of the game, there are n players ($n^1 = n$), we write that $n^t = n-(t-1)$. We denote the number of players of type s_c at stage t by n_c^t with $n^t = \sum_{c=1}^m n_c^t$. The set S^t depicts the skill levels present in stage t, i.e. $S^t = \{s_c \in S \mid n_c^t > 0\}$. The highest and lowest skill level present at stage t are respectively denoted by $\max(S^t)$ and $\min(S^t)$
A skill s_c is *majoritarian* at some stage t if $n_c^t \geq \sum_{s_d \in S^t} n_d^t$, with $d \neq c$.
A skill s_c is *strongly majoritarian* at some stage t if $n_c^t > \sum_{s_d \in S^t} n_d^t$, with $d \neq c$.

2.2 Voting.

There are three types of stages:

1. The stage n in which just one candidate survives, the winner. In this stage, payoffs are allocated and no vote is held.

2. The final (stage $n-1$) in which just two players survive.

3. Any stage $t < n-1$ previous to the final is a voting stage in which players vote to remove one of the players.

We now describe the rules of the final and of each of the voting stages.

Voting Stages.

The vote is done on the skill levels of the players and not on their identity. Each player votes for one of the skill levels (i.e. picks one type from the set S^t) and, then, a player is eliminated. This eliminated player is randomly picked from the group s with the most votes. If several groups are tied, the group in which a player is randomly eliminated is the one with the lowest skill among the tied ones. This breaking-tie rule seems to favor the "efficiency" of the outcome of the game. No abstentions are allowed for simplicity.

We now introduce some notations to be used throughout. At stage t,

- $v_i^t \in S^t$ is the player i's choice,
- $v^t = (v_i^t)_{i \in N^t}$ describes the vote profile,
- $w_c^t = w^t(s_c) = \#\{i \in N^t \mid v_i^t = s_c\}$ stands for the number of votes for s_c,
- $w^t = (w_c^t)_{s_c \in S^t}$ denotes the score vector,
- w_{-i}^t is the score vector without the vote of player i,
- $p(s_c \mid w^t)$ is the probability that a player of type s_c is not eliminated,
- $L(w^t) = \{s_c \in S^t \mid w^t(s_c) \geq w^t(s_d) \,\text{for any}\, s_d \neq s_c\}$ denotes the types with the most votes.

Given the score vector w^t, a player with skill s_c is not eliminated at any stage $t < n-1$ and hence is present in stage $t+1$ with probability:

$$p(s_c \mid w^t) = \begin{cases} 1 & \text{if } s_c \notin L(w^t), \\ 1 & \text{if } s_c \in L(w^t) \text{ and } s_c > s_d \text{ for some } s_d \in L(w^t), \\ 1 - 1/n_c^t & \text{if } s_c \in L(w^t) \text{ and } s_c < s_d \text{ for any } s_d \in L(w^t). \end{cases}$$

If every player in the game has the same type, then S^t is a singleton and hence so is $L(w^t)$. Therefore, one of the players is randomly eliminated since the players can only announce their own type.

The Final.

In the final stage $t = n-1$ with just two players, there is a vote even though the tie breaking rule is different from the precedent voting stages. If $L(w^{n-1}) = \{s_c\}$ for some $s_c \in S$, then the s_c-group gets 2 votes. In this case, one of these players is removed with probability 1/2 as in the previous stages. If, on the contrary $L(w^{n-1}) = \{s_c, s_d\}$ for some $s_c, s_d \in S$, then both groups are tied. We assume that

$$p(s_c \,|\, w^{n-1}) = p_{cd},$$

with $p_{cd} + p_{dc} = 1$. Moreover, the probability of winning the final in the event of a tie is higher against a lower type than against a higher one:

$$p_{ce} > p_{cd} \iff d > e,$$

and for any type s_c, $p_{cc} = 1/2$. The previous inequalities imply that $p_{cd} < 1/2$ if $d > c$ and $p_{cd} > 1/2$ when $d < c$.

This tie-breaking rule differs from the one used in the previous stages in which we assumed that $p_{cd} = 1$ as long as $c > d$ since we assume that the tie-breaking rule favors the efficiency of the outcome of the game.

2.3 Payoffs.

Each stage in which each player is present in the game he produces some amount of money to be collected at every stage. Such a contribution just depends on his skill level s. These contributions are stocked until the end of the game, and the winner of the game receives a payoff that depends on the sequence of elimination of the players.

The timing of the game at each stage is as follows:

1. At the beginning of each stage, each player contributes with some amount of money which is collected.

2. The vote is held.

3. One of the players is removed and then a new stage starts.

4. The payoff is allocated to the surviving player at stage n.

Since the contribution of a player only depends on his skill level s (i.e. he cannot choose whether he contributes or not), the players' expected payoffs only depend on the type of the player eliminated at each stage t. In other words, as each player must contribute at each stage, the final payoff just depends on the type of the player eliminated at each stage. In a sense, given that the level of contributions is given (there is no effort so that the contribution just depends on the skills of the players in the game), the vote determines how the contribution in the next stage will be.

The skill level removed at each stage t is denoted by $\alpha^t \in S$. The whole sequence of removals $\alpha^1 \dots \alpha^{n-2}$ is denoted by α. The sequence stops at the semifinal ($t = n-2$) since this is the last stage in which the skill of the removed player matters to determine the players' best responses. In other words, the skill of the eliminated player in the final (α^{n-1}) does not affect the total contribution since the player is removed after the contributions have been done. Furthermore, there are no contributions in stage n.

Let us remark that α is an integer since each skill s_c is an integer.

Example 1: Let $n = 5$ and $\alpha = s_1 s_1 s_3$. Then, the sequence α represents the elimination of a player of type s_1 at stages 1 and 2 and of a player of type s_3 at stage 3.

Definition 1 (The Pie)**.** *Given a sequence of elimination α, we shall denote by $f(\alpha)$ the pie.*

Therefore, given the elimination sequence α, we write that the utility of player i equals

$$u_i(\alpha) = \begin{cases} f(\alpha), & \text{if } i \text{ is the winner} \\ 0, & \text{otherwise.} \end{cases}$$

This payoff scheme is a winner-take-all one in which only the surviving player in stage n gets a positive payoff.

Example 2: Let $n = 4$ with $S = \{s_1, s_2\}$. We set $n_1^1 = 3$ and $n_2^1 = 1$. Let $\alpha = s_2 s_1$. Hence, $f(\alpha)$ corresponds to the pie associated to the sequence of elimination according to which an s_2-player is removed first, followed by the removal of a s_1-player at $t = 2$. Given the sequence of elimination α and the initial set of of players N^1, the final takes place between two players of type s_1.

Example 3: Assume that each s_c-player contributes an amount s_c per stage in which he is present. No complementarities are allowed so that each player contributes the same amount independently of the skills of the rest of the players. Then at each stage t, the money collected equals $m_t = \sum_{s_d \in S^t} s_d n_d^t$. So, at the end of the game, the collected money equals $M = \sum_{t=1}^{n-1} m_t$. Assume that M is the payoff allocated to the winner.

The relation between the payoff allocated to the winner and the elimination sequence is relatively straightforward. Indeed, we have

$$m_{t+1} = m_t - \alpha^t,$$

since the only difference between both contributions is the removed player. It follows that $m_t = m_1 - \sum_{i=1}^{t-1} \alpha^i$ and so that

$$M = (n-1)m_1 - \sum_{t=1}^{n-2}(n-1-t)\alpha^t.$$

In other words, in this example, it seems particularly intuitive to define the payoff of the winner M to depend simply on the elimination sequence $\{\alpha^t\}_{t=1}^{n-2}$ and hence write $M = f(\alpha)$.

In general, we do not assign explicit values to $f(\alpha)$, even though we do need to impose an assumption in order to compare them. This assumption implies that the size of the pie $f(\alpha)$ is strictly decreasing on α. Indeed, a more talented player contributes more than a less talented one. Therefore, as each player gives a contribution at *every stage* until he is eliminated, a player who is eliminated early contributes less than a player of the same type that is eliminated later. Hence, the pie $f(\alpha)$ is larger when type $\underline{s}$ players are eliminated first, followed by the elimination of type $\bar{s}$ players. Formally, this assumption can be stated as follows:

We denote by $(\alpha_1 s_c \alpha_2 s_d)$ the sequence consisting the subsequence α_1 followed by the elimination of an s_c-player, the subsequence α_2 and finally the elimination of an s_d-player.

Assumption 1. *For any two elimination sequences $\alpha_1 s_d \alpha_2 s_c$ and $\alpha_1 s_c \alpha_2 s_d$ and any two types $s_c, s_d \in S$, $f(\alpha_1 s_d \alpha_2 s_c) > f(\alpha_1 s_c \alpha_2 s_d)$ iff $s_c > s_d$.*

Example 3 (continued): Given that $f(\alpha) = (n-1)m_1 - \sum_{t=1}^{n-2}(n-1-t)\alpha^t$, let us prove that Assumption 1 holds. Take any two sequences $\alpha = \alpha_1 s_d \alpha_2 s_c$ and $\beta = \alpha_1 s_c \alpha_2 s_d$ with $s_c > s_d$. It follows that for any pair t, t' with $t < t'$, we have

$$\begin{aligned} f(\alpha) - f(\beta) &= -((n-1-t)s_d + (n-1-t')s_c) + ((n-1-t)s_c + (n-1-t')s_d) \\ &= (n-1-t)(s_c - s_d) + (n-1-t')(s_d - s_c) \\ &= (s_c - s_d)((n-1-t) - (n-1-t')) > 0, \end{aligned}$$

which holds since $t < t'$ and $s_c > s_d$.

A natural implication of Assumption 1 is that no player is indifferent between two different elimination sequences α and β provided that he has a positive probability of being in the final under both sequences.

Some comments are in order:

- The fact that players vote for the types and not for the players corresponds to an anonymity property of the game: only the type of a player can influence his payoffs.
- Note that, given Assumption 1, the elimination of a player in the sequence has more impact on the size of the pie the earlier it arrives. This is in line with the idea that, as each player contributes at each step, the earlier he is removed, the more impact there is on the pie. This assumption is the main culprit of most of our results.
- The goal of each player is to maximize his expected payoff. This means that (i) he wishes to reach the last stage of the game and (ii) he wishes to first eliminate weak players in order to get the largest possible payoff $f(\alpha)$.

2.4 Equilibrium

At each stage, the players simultaneously vote and one candidate is removed. At the end [5] of $n-1$ rounds of voting one candidate survives who is the winner. For any $t \leq n-2$, we let $h^t = (v^1, v^2, \ldots, v^{t-1})$ denote the complete history of the voting decisions up to stage t. We denote by $\mathcal{H}^t$ the set of histories at stage t and $\mathcal{H} = \cup_t \mathcal{H}^t$ be the set of all histories. $\mathcal{H}^0$ stands for the null history.

[5] Note that if the game reaches a stage in which all players have the same type s, there is a vote but players can only vote for their own type.

A pure strategy for a player i is a function $p_i : \mathcal{H} \to \cup_t S^t$ such that $p_i(h) \in \cup_t S^t$ if $h \in \mathcal{H}^t$. The set of pure strategies of player i is denoted by P_i with $P = \prod_i P_i$.

Following Bag et al. [1], we require our equilibrium concept to be subgame perfect and, moreover, to be undominated. Therefore, an equilibrium is a strategy profile for the players that is a subgame perfect equilbrium and is such that at each stage the votes of each player are not weakly dominated given the equilibrium continuation strategies of other players in future stages.

Formally, for any history $h \in \mathcal{H}$, let $\Gamma(h)$ be the subgame at h. For any strategy profile $p \in P$ and any history $h \in \mathcal{H}$, we define the set of strategies for all players other than i that are consistent with p in every subgame after h by

$$\begin{aligned}\tilde{P}_{-i}(h,p) = \{p'_{-i} \in P_{-i} \mid p'_{-i}(h,h') = p_{-i}(h,h') \\ \text{for all non-empty } h' \text{ s.t.} (h,h') \in \mathcal{H}\}.\end{aligned}$$

We denote by $U_i(p,h)$ the expected utility of player i given history h when players play according to the strategy combination p.

Definition 2. *A strategy profile p^* is an equilibrium if for any history h it satisfies the following properties in the subgame $\Gamma(h)$:*

(Nash) For any i, $U_i(p^*,h) \geq U_i(p_i, p^*_{-i}, h)\ \forall p_i \in P_i$,

(Weak Non Domination) For any i, $\nexists\, p_i \in P_i$ s.t.

$$\begin{aligned}&U_i(p_i, p_{-i}, h) \geq U_i(p_i^*, p_{-i}, h)\ \forall p_{-i} \in \tilde{P}_{-i}(h,p^*)\\ &\text{and } U_i(p_i, p_{-i}, h) > U_i(p_i^*, p_{-i}, h) \text{ for some } p_{-i} \in \tilde{P}_{-i}(h,p^*).\end{aligned}$$

3 The Final Stages

We now state some preliminary observations on the final stages of the game that will be key throughout. Indeed, since we focus on subgame perfection, the outcome achieved in the last stages will determine the players' behavior throughout the tree.

3.1 The Final

Due to the equilibrium concept we use, it is simple to see that in the final there is a unique best response for each player. Indeed if there is one skill present in the final, the only possible choice for a player is to vote for him. If there are two types of players, it is weakly undominated to vote for the other type of player.

3.2 The Semifinal

We now focus in the semifinal (the stage $t = n - 2$) in which there are three players. Note that the players' behavior is driven by the distribution of skills. There are few possibilities for the skill distribution S^{n-2}: either each player has a different skill level or there are two skill levels (the case in which all players have the same level requires no analysis). Note that with just two skill levels, there is a majoritarian type (two players have this skill) than can be either higher or lower than the remaining skill.

We now state two propositions (the proofs of which are included in the appendix) that show that the game exhibits a unique equilibrium in the semifinal.

Proposition 1. *There is a unique equilibrium in the semifinal with two types.*

With two types, the results are quite simple to state.

In the presence of two strong players in the semifinal, the weak player is first removed and therefore the winner is a strong player.

If, on the contrary, there are two weak players in the semifinal, the strong player need not be removed. Indeed, the weak players may prefer the removal of a weak player provided that the benefit of removing him at this stage compensates the lower probability of winnning the final.

Proposition 2. *With three types in the semifinal, the equilibrium is unique.*

With three types, the situation is more subtle.

As proved in the appendix, the least-skilled player has no weakly undominated strategy. The two other players have a unique weakly undominated strategy that varies as a function of the pie $f(.)$ and of the probabilities p_{ij} of winning the final. Therefore, there is a unique equilibrium given the pie and the probabilities of winnin the final. The identity of the winner varies accordingly since the three types of players can be removed in the semifinal.

4 A characterization of equilibria with two-types

Within this section, we only consider games with two skill levels. We respectively denote by $\underline{s}$ and $\overline{s}$ the low and the high skill.

In this two-type version of the model, the probabilities of surviving to the next stage simplify. Indeed, for a low-skilled player $\underline{s}$, we write,

$$p(\underline{s}\,|\,w^t) = \begin{cases} 1 & \text{if } w^t(\underline{s}) < w^t(\overline{s}), \\ 1 - 1/\underline{n}^t & \text{if } w^t(\underline{s}) = w^t(\overline{s}), \\ 1 - 1/\underline{n}^t & \text{if } w^t(\underline{s}) > w^t(\overline{s}). \end{cases}$$

On the contrary, the surviving probability of a high-skilled player equals

$$p(\overline{s}\,|\,w^t) = \begin{cases} 1 & \text{if } w^t(\overline{s}) < w^t(\underline{s}), \\ 1 & \text{if } w^t(\overline{s}) = w^t(\underline{s}), \\ 1 - 1/\overline{n}^t & \text{if } w^t(\overline{s}) > w^t(\underline{s}). \end{cases}$$

Lemma 1. *For any stage t, if $\overline{s}$ is weakly majoritarian, then $\overline{s}$-players weakly prefer to vote for $\underline{s}$.*

Proof. The proof is done by backward induction on the stages.

Consider the semifinal (just three players remain in the game) and assume that there is a majority of $\overline{s}$-players. As previously shown by Proposition 1, the $\overline{s}$-players weakly prefer to vote for $\underline{s}$.

Assume now that that the claim holds from some stage $t+1$ onwards. In other words, assume that in any stage j ($j > t+1$) with a majority of $\overline{s}$-players, the $\overline{s}$-players weakly prefer to vote for $\underline{s}$.

Let us now prove that the $\overline{s}$-players weakly prefer to vote for $\underline{s}$ in stage t with a majority of strong players (i.e. $\overline{n}^t \geq \underline{n}^t$).

Indeed, in the event of being pivotal, an $\overline{s}$-player decides whether to remove an $\overline{s}$-player or an $\underline{s}$-player.

If an $\underline{s}$-player is removed, then $\overline{n}^{t+1} > \underline{n}^{t+1}$. Note that we have assumed that from stage $t+1$ onwards in which there is a majority of $\overline{s}$-players, the $\overline{s}$-players weakly prefer to vote for $\overline{s}$. Hence, an $\overline{s}$-player wins and all $\underline{s}$-players are eliminated first. Hence, if an $\overline{s}$-player is removed, the $\overline{s}$-player's expected utility is maximal as the pie is maximal when removing first the low-skilled players.

Conversely, if an $\overline{s}$-player is removed the expected utility of any $\overline{s}$-player is strictly lower. Indeed, in the event that an $\overline{s}$-player is in the final, removing first an $\overline{s}$-player gives a more reduced pie than removing an $\overline{s}$-player.

Hence, the claim follows by induction: whenever $\overline{n}^t \geq \underline{n}^t$, then $\overline{s}$-players weakly prefer to vote for $\underline{s}$.

□

Lemma 2. *For any stage $t < n-2$, if $\underline{n}^t = \overline{n}^t + 1$, then $\underline{s}$-players weakly prefer to vote for $\overline{s}$.*

Proof. Consider the set of stages $\hat{T} = \{t' \in \{1,\ldots,n-2\} \mid \underline{n}^{t'} = \overline{n}^{t'} + 1\}$. For $t = n-2$, the claim is not true as stated by the analysis of the semifinal with a majority of weak players in section 3. Take the last stage prior to the semifinal in the set $\hat{T}$ (the maximum in $\hat{T} \setminus \{n-2\}$). We denote it by t. In the event of being pivotal, an $\underline{s}$-player decides whether to remove an $\overline{s}$-player or an $\underline{s}$-player.

If he votes for $\underline{s}$, then an $\underline{s}$-player is removed. Then, $\underline{n}^{t+1} = \overline{n}^{t+1}$. Therefore in stage $t+1$, all $\overline{s}$-players vote for $\overline{s}$, as stated by Lemma 1. It follows that in stage $t+1$, an $\underline{s}$-player is removed and then Lemma 1 applies in the rest of the stages. Hence, the winner is an $\overline{s}$-player. In other words, in the event of being pivotal, if an $\underline{s}$-player votes for $\overline{s}$, then his expected payoff equals zero.

If he votes for $\overline{s}$, then an $\overline{s}$-player is removed. As the stage t is the last one in $\hat{T} \setminus \{n-2\}$, it follows that in any stage $t' > t$ (posterior to t), the difference between the number of weak and strong players is higher or equal than two. Hence, independently of the path of play, in the semifinal there is a majority of $\underline{s}$-players. Therefore there is at least one $\underline{s}$-player in the final so that the expected payoff for $\underline{s}$-players is strictly positive.

The same claim applies to any stage prior to t in the set $\hat{T}$, concluding the proof. Therefore, an $\underline{s}$-player weakly prefers to vote for $\overline{s}$ in any stage in $\hat{T}$. □

Lemma 3. *For any stage $t < n-2$, if $\underline{n}^t > \overline{n}^t + 1$, then $\underline{s}$-players weakly prefer to vote for $\underline{s}$.*

Proof. Take any stage $t < n-2$ with $\underline{n}^t > \overline{n}^t + 1$. In the event of being pivotal, an $\underline{s}$-player decides whether to remove a weak or a strong player.

Given Lemma 2, we know that there is an $\underline{s}$-player in the final. Hence, an $\underline{s}$-player has a strictly positive expected utility. Moreover, given that an $\underline{s}$-player is in the final, a weak player prefers to remove an $\underline{s}$-player in the event of being pivotal. Hence, the $\underline{s}$-players vote for $\underline{s}$, proving the claim. □

We now introduce two important definitions that describe the ratio between the payoffs allocated to the different skill levels. For any two skills $s_c, s_d \in S$ with $s_c < s_d$, we have:

*Balanced Reward*s: If for any sequence of elimination α, $\frac{f(\alpha s_d)}{f(\alpha s_c)} > p_{cd}$, then the society has balanced rewards.

Unbalanced Rewards: If for any sequence of elimination α, $\frac{f(\alpha s_d)}{f(\alpha s_c)} < p_{cd}$, then the society has unbalanced rewards.

Theorem 1. *1. If $\overline{s}$ is majoritarian at the first stage, then an $\overline{s}$-player wins at equilibrium.*
2. If $\underline{s}$ is strongly majoritarian at the first stage and the rewards are balanced, then an $\underline{s}$-player wins at equilibrium.
3. If $\underline{s}$ is strongly majoritarian at the first stage and the rewards are unbalanced, both types of players can win with positive probability at equilibrium.

Proof. 1. This is a direct consequence of Lemma 1.
2. In this case, Lemmata 2 and 3 apply until the stage $j = n-3$. Then, in the semifinal ($j = n-2$), there is a majority of weak players. Therefore, as the rewards are balanced, the analysis of the semifinal with a majority of weak players proves that an $\underline{s}$-player wins.
3. This case is similar to 2 so that in the semifinal, there is a majority of weak players. Therefore, as the rewards are unbalanced, the analysis of the semifinal proves that both types of players win with positive probability. □

The next corollaries (stated without proof as they are simple consequences of the previous results) depict the sequence of eliminations for each of the cases and the pie in equilibrium.

Corollary 1. *If $\overline{s}$ is majoritarian at the first stage, then all the $\underline{s}$-players are eliminated first. In equilibrium, the payoff allocated to the winner is the maximal possible one in the game.*

Corollary 2. *If $\underline{s}$ is strongly majoritarian at the first stage, then the $\underline{s}$-players are eliminated first until $\underline{n}^j = \overline{n}^j + 1$ for some stage j. From this stage until the semifinal, the sequence of removals consists of an $\overline{s}$-player followed by an $\underline{s}$-player iteratively.*

assume that the player's utility function equals:

5 The Multi-type Game

In this section, we address the situation with more than two-types. A full characterization of equilibria in this more general setting seems unreachable. Indeed, if one considers the semifinal with three different types, any outcome is an equilibrium if one adequately calibrates the rewards for the different types of players.

Nonetheless, we are able to describe the equilibria for polarized initial situations in which there is an absolute majority (i.e. strictly more than half) of players with the same type.

Consider the following strategy ϑ_c that depicts the behavior of s_c-players if they are majoritarian at some stage prior to the final:

1. For any stage prior to the semifinal and any type s_c, if $n_c^t = (\sum_{l \neq c} n_l^t) + 1$, then s_c-players vote for the lowest possible type different from s_c in stage t.

2. For any stage prior to the semifinal and any type s_c, if $n_c^t > (\sum_{l \neq c} n_l^t) + 1$, then s_c-players vote for the lowest possible type in stage t.

3. In the semifinal[6], if for some $s_c > s_l$, $n_c^t = n_l^t + 1$, then s_c-players vote for type s_l in stage t.

4. In the semifinal, if for some $s_c < s_l$, $n_c^t = n_l^t + 1$, then s_c-players vote for type s_l in stage t if rewards are balanced.

5. In the semifinal, if for some $s_c < s_l$, $n_c^t = n_l^t + 1$, then s_c-players vote for type s_c in stage t if rewards are unbalanced.

[6] Note that if there is a majoritarian type in the semifinal (i.e. three players), there is at most two different types.

It is not too difficult to check that, if any group of majoritarian players of type s_c vote according to ϑ_c, they ensure that there is at least an s_c-player in the final. Note that the s_c-players keep their majority at every stage if they all vote according to ϑ_c. Moreover, let us remark that, since preferences over the terminal nodes only depend on the type of the player, every player with the same type has the same preferences. Due to the existence of this strategy, we can prove the next results.

Proposition 3. *Let $s_m = \max(S^1)$. For any $s_c \in S$ majoritarian at some stage t, the strategy ϑ_c is the unique weakly undominated one for s_c-players.*

Proof. By a similar argument to the one explained by Lemma 1, it can be shown that the equilibrium strategies depicted by ϑ_c are the unique weakly undominated ones for the s_m-players. The proof proceeds by backward induction on the stages. □

Winning Types

Theorem 2. *Let $s_m = \max(S^1)$. If $n_m^1 \geq (\sum_{l \neq m} n_l^1)$, then in equilibrium, only s_m-players win.*

Proof. According to ϑ_c, the s_m-players vote at any stage for the lowest possible type, which is different from s_m by definition. Hence, as $n_m^1 \geq (\sum_{l \neq m} n_l^1)$, it follows that first the players with the lowest type are eliminated, followed by the players with the second lowest type and so on. Therefore, the final takes place between two s_m-players and the pie allocated to the winner is maximal. □

Theorem 3. *Let $s_m = \max(S^1)$. If for some $s_c < s_m$, $n_c^1 > (\sum_{l \neq c} n_l^1)$, then in equilibrium there is an s_c-player in the final.*

Proof. Take any game with for some $s_c < s_m$, $n_i^1 > (\sum_{l \neq i} n_l^1)$. As there exists ϑ_c, we know that any path of play that leads to no s_c-player in the final is dominated for the s_i-players and hence cannot be an equilibrium. This proves that in equilibrium there is an s_c-player in the final. □

Theorem 4. *Let $s_m = \max(S^1)$. If for some $s_1 < s_d < s_m$, $n_d^1 > (\sum_{l \neq d} n_l^1)$, then in equilibrium there is no player in the final with type lower than s_d.*

Proof. Assume that there is some player with type s_c with $s_c < s_d$ in the final. Due to Theorems 2 and 3, we know that there is at least an s_d-player in the final so that the probability of winning the final for player of type s_d equals $p_{dc} > 1/2$. Moreover, following ϑ_c, the s_d-players ensure to have a strict majority at every stage. Hence, as there is an s_c-player in the final, in the semifinal, there must be one s_c-player and two s_d players in the semifinal. However, given ϑ_c, in this semifinal the s_d-players vote for the s_c-player and hence the s_c-player is removed. Therefore, the final $s_d s_c$ does not arise. Hence, given ϑ_c, either there is a final between two s_c-players or a final between an s_d-player and a s_l-player with $s_l > s_d$, which proves the claim. □

eliminated player at stage t equals $\min(S^t)$. by ϑ_c. ϑ_c, the weak players (players with a type lower than s_{m-1}) are first removed by increasing order of skill. Once the weak players have been removed, there are only two types of players in the game: s_{m-1} and s_m. Hence, the results concerning two types of players apply, which concludes the proof.

6 Conclusion

Our work considers a dynamic voting procedure in which the candidates are the voters. The originality of our contribution lies in the fact that the order of deletion of the players has an impact on the final payoffs allocated to the players. Each player contributes at each stage some amount of money that just depends on his skill level; the higher the skill level the higher the contribution. With just two types, we prove that a strong player must win if the game starts with a majority of strong players. On the contrary, if the game starts with more than half of the players being low-skilled, the winner might be either high or low-skilled. Note that this result holds even if the game starts with all players but one being low-skilled. With more than two types, the multiplicity of equilibria seems unavoidable. Yet, in the presence of an initial majority, some of the results present in the two-types' case are still valid.

Two assumptions seem to be the main culprits behind our result.

First, the players' contributions are compulsory. That is, to keep things simple, we assume that each player must contribute at each stage and then decides which ballot to cast. Relaxing this assumption and allowing players to choose a certain effort level seems to be a interesting venue of research. One might also assume that the players' type is private information rather than public as in our model and hence allow the players to hide their level of competence.

The second assumption is the payoff structure. Indeed, we have focused on a winner-take-all scheme in which just the winner gets a positive payoff. Understanding whether sharing the profits among the different players (for instance as a function of their removal stage) seems also of potential interest, specially if one allows for endogenous labor supply.

References

[1] P.K. Bag, H. Sabourian, and E. Winter. Multi-stage Voting, Sequential Elimination and Condorcet Consistency. *Journal of Economic Theory*, 144:1278–1299, 2009.

[2] R. Farquharson. *Theory of Voting*. Yale University Press, New Haven, 1969.

[3] P. Février and L. Linnemer. Equilibrium Selection: Payoff or Risk Dominance?: The case of the Weakest Link. *Journal of Economic Behavior and Organization*, 60:164–181, 2006.

[4] M. Hann, B. Los, and Y. Riyanto. Signaling Strength? An Analysis of Economic Decision Making in the Weakest Link. *Theory and Decision*, 71:519–537, 2011.

[5] P. Hummel. Iterative Elimination of Weakly Dominated Strategies in Binary Voting Agendas with Sequential Voting. *Social Choice and Welfare*, 31:257–269, 2008.

[6] S. Levitt. Testing Theories of Discrimination: Evidence from Weakest Link. *Journal of Law and Economics*, 47:431–452, 2004.

[7] R. McKelvey and Niemi. R.G. A Multistage Game Representation of Sophisticated Voting Procedures. *Journal of Economic Theory*, 18:1–22, 1978.

[8] H. Moulin. Dominance Solvable Voting Schemes. *Econometrica*, 47:1337–1351, 1979.

A Proof of Proposition 1

We assume that there are just two skill levels, so that $S^1 = \{\underline{s}, \overline{s}\}$. Hence, we simply write $\overline{n}^t$ and $\underline{n}^t$ to respectively denote the number of low ($\underline{s}$) and high-skilled ($\overline{s}$) players.

Semifinal with a Majority of Strong Players.

Let $n^1 = \{1,2,3\}$ with $\overline{n}^1 = 2$ and $\underline{n}^1 = 1$. That is, there are two strong players and one weak player. There is a unique equilibrium in weakly undominated strategies, in which the weak player is first eliminated and the winner is a strong player.

- Take player 1 and assume w.l.o.g. that 1 is the $\underline{s}$-player.

If the rest of the other players cast the same ballot ($w^1_{-1} \in \{(2,0),(0,2)\}$), he is indifferent between both ballots. If each of the other players cast a different ballot ($w^1_{-1} = \{(1,1)\}$), then player 1 is not anymore indifferent between his two ballots.

If he votes $\overline{s}$, then $w^1 = (1,2)$ so that an $\overline{s}$-player is eliminated. Then the final takes place between one strong player and one weak player. Hence, the expected utility of 1 of playing $\overline{s}$ equals $p_{\underline{s}\overline{s}} f(\overline{s}) > 0$.

Similarly, if he votes $\underline{s}$, then $w^1 = (2,1)$ so that a $\underline{s}$-player is eliminated. Then the final takes place between two strong players so that 1 gets a payoff of 0.

Therefore, player 1 votes $\overline{s}$ as it is weakly undominated.

- Take now a strong player, w.l.o.g we let $i = 2$.

As previously argued, the player is indifferent between his two ballots unless $w^1_{-2} = (1,1)$. In this case, if he votes $\overline{s}$, then the final takes place between one strong player and one weak player. In this case, player 2's expected payoff equals $\frac{1}{2} p_{\overline{s}\underline{s}} f(\overline{s})$.

Following the same reasoning, if he votes $\underline{s}$, his expected payoff equals $\frac{1}{2} f(\underline{s})$.

That is, the player prefers to vote $\underline{s}$ rather than $\overline{s}$ if and only if $\frac{1}{2} p_{\overline{s}\underline{s}} f(\overline{s}) < \frac{1}{2} f(\underline{s})$ which is equivalent to:

$$p_{\overline{s}\underline{s}} f(\overline{s}) < f(\underline{s}). \quad (*)$$

However, by assumption $f(\overline{s}) < f(\underline{s})$ which implies that $(*)$ always holds since $p_{\overline{s}\underline{s}} \leq 1$. Therefore, the strong players vote $\underline{s}$ as it is weakly undominated.

The same reasoning applies for player 3.

In other words, the strong players weakly prefer to vote $\underline{s}$. As the equilibrium concept requires that players use weakly undominated strategies, in the unique equilibrium of this game the weak player is removed and hence the winner has a high type.

Semifinal with a Majority of Weak Players.

Let $n^1 = \{1,2,3\}$ with $\overline{n}^1 = 1$ and $\underline{n}^1 = 2$. That is, there are two weak players and one strong player. Let us study the equilibria in weakly undominated strategies.

- Take player 1 and assume w.l.o.g. that 1 is the $\overline{s}$-player.

As previously argued, the player is indifferent between his two ballots unless $w^1_{-1} = (1,1)$. In this case, if he votes $\overline{s}$, then the final takes place between the two weak players. Hence, his expected payoff equals 0.

Similarly, if he votes $\underline{s}$, then $w^1 = (2,1)$ so that a $\underline{s}$-player is eliminated. In this case, his expected utility equals $p_{\overline{s}\underline{s}} f(\underline{s}) > 0$, which entails that the strong player votes $\underline{s}$ as it is weakly undominated.

- Take now a weak player. We prove the claim for $i = 2$ and the same logic applies to $i = 3$.

As previously argued, the player is indifferent between his two ballots unless $w^1_{-2} = (1,1)$. In this case, if he votes $\overline{s}$, then his expected utility equals $\frac{1}{2} f(\overline{s})$.

Following the same reasoning, if he votes $\underline{s}$, then $w^1 = (2,1)$ so that his expected utility is equal to $\frac{1}{2} p_{\underline{s}\overline{s}} f(\underline{s})$.

That is, he prefers to vote $\underline{s}$ rather than $\overline{s}$ if and only if $\frac{1}{2} f(\overline{s}) < \frac{1}{2} p_{\underline{s}\overline{s}} f(\underline{s})$ which is equivalent to:

$$f(\overline{s}) < p_{\underline{s}\overline{s}} f(\underline{s}) \Longleftrightarrow \frac{f(\overline{s})}{f(\underline{s})} < p_{\underline{s}\overline{s}}. \quad (+)$$

By assumption $f(\overline{s}) < f(\underline{s})$ and $p_{\underline{s}\overline{s}} < 1/2$. We let $f(\underline{s}) = \eta f(\overline{s})$ so that (+) is equivalent to $\frac{1}{\eta} < p_{\underline{s}\overline{s}}$. Moreover, $\eta = \frac{f(\underline{s})}{f(\overline{s})}$ which implies that $\eta \in (1, \infty)$

Hence, as $\eta \to \infty$, then the left side of (+) tends towards zero so that (+) holds. In this case, the weak players vote $\underline{s}$ (for their own type) as the difference between the contributions of strong and weak players grows. On the contrary, as $\eta \to 1$, then there must exist some η for which (+) is violated so that players vote $\overline{s}$.

Therefore, the more alike the contributions of the players are, the more weak players vote for the strong type. On the contrary, when the contribution of a strong player becomes relatively large compared to the one a weak player, weak players eliminate their own type.

Therefore, there are two cases:

Balanced Rewards: If for any sequence of elimination α, $\frac{f(\alpha\overline{s})}{f(\alpha\underline{s})} > p_{\underline{s}\overline{s}}$, then the society has balanced rewards. Then $\underline{s}$-players vote for $\bar{s}$: a strong player is eliminated and the winner is a weak player.

Unbalanced Rewards: If for any sequence of elimination α, $\frac{f(\alpha\bar{s})}{f(\alpha\underline{s})} < p_{\underline{s}\bar{s}}$, then the society has unbalanced rewards. Then $\underline{s}$-players vote for $\underline{s}$: a weak player is eliminated and the final takes place between a strong and a weak player.

Since we require that players use weakly undominated strategies, the equilibrium is unique given the rewards scheme.

B Proof of Proposition 2

Semifinal with three types of players.

Let $n^1 = \{1, 2, 3\}$ with $n_i^1 = 1$ for each $i \in \{1, 2, 3\}$. That is, there are three players, each of them with a different skill level. We now focus on the equilibria in weakly undominated strategies.

-W.l.o.g we assume that player i has type s_i. The next tables depict the consequences of choosing each of the ballots (s_1, s_2 or s_3) in the different pivotal events $s_{-i} \in \{(1,1,0),(1,0,1),(0,1,1)\}$. Note that if $s_{-i} \notin \{(1,1,0),(1,0,1),(0,1,1)\}$, each voter is indiffferent between his three available ballots.

Strategy	Total Scores when $s_{-i} = (0,1,1)$	Removed Player	Payoffs	Player 1	Player 2	Player 3
s_1	(1,1,1)	s_1		0	$p_{23}f(s_1)$	$p_{32}f(s_1)$
s_2	(0,2,1)	s_2		$p_{13}f(s_2)$	0	$p_{31}f(s_2)$
s_3	(0,1,2)	s_3		$p_{12}f(s_3)$	$p_{21}f(s_3)$	0

Strategy	Total Scores when $s_{-i} = (1,0,1)$	Removed Player	Payoffs	Player 1	Player 2	Player 3
s_1	(2,0,1)	s_1		0	$p_{23}f(s_1)$	$p_{32}f(s_1)$
s_2	(1,1,1)	s_1		0	$p_{23}f(s_1)$	$p_{32}f(s_1)$
s_3	(1,0,2)	s_3		$p_{12}f(s_3)$	$p_{21}f(s_3)$	0

Strategy	Total Scores when $s_{-i} = (1,1,0)$	Removed Player	Payoffs	Player 1	Player 2	Player 3
s_1	(2,1,0)	s_1		0	$p_{23}f(s_1)$	$p_{32}f(s_1)$
s_2	(1,2,0)	s_2		$p_{13}f(s_2)$	0	$p_{31}f(s_2)$
s_3	(1,1,1)	s_1		0	$p_{23}f(s_1)$	$p_{32}f(s_1)$

Weakly Dominated Strategies:

Player 1: Since he prefers to play s_2 when $s_{-i} = (1,1,0)$ and s_3 when $s_{-i} = (1,0,1)$, he has no weakly dominant strategy.

Player 2: Playing s_1 weakly dominates s_2 since the player obtains either the same payoff (when $s_{-i} = (1,0,1)$) or a strictly higher one in the rest of the cases.

If $p_{23}f(s_1) > p_{21}f(s_3)$ then s_1 weakly dominates s_3. On the contrary, if $p_{23}f(s_1) < p_{21}f(s_3)$, then s_3 weakly dominates s_1.

Player 3: Playing s_1 weakly dominates s_3. Moreover, if $p_{32}f(s_1) > p_{31}f(s_2)$, then s_1 weakly dominates s_2. On the contrary, when $p_{32}f(s_1) < p_{31}f(s_2)$, then s_2 weakly dominates s_1.

Since 2 out of the 3 players have a unique weakly undominated strategy, it follows that there are four cases:

1. If both 2 and 3 have s_1 as a unique weakly undominated strategy, then this is an equilibrium since the strategy of 1 is irrelevant.

2. If 2's unique weakly undominated strategy is s_1 and s_2 is the one for 3, then 1 plays s_2 as unique best response.

3. If $2 \rightarrow s_3$ and $3 \rightarrow s_1$, then 1 has s_3 as a unique best response.

4. If $2 \rightarrow s_3$ and $3 \rightarrow s_2$, then 1 has a unique best response (either s_2 or s_3 depending on whether $p_{13}f(s_2) \lesseqgtr p_{12}f(s_3)$) .